Chapter 3
Soundness & Completeness

In this chapter we investigate the relationship between syntax and semantic. In particular, we investigate the relationship between a formal proof of a formula from a theory T and the truth-value of that formula in a model of T. In this context, two questions arise naturally:

- Is each formula φ, which is provable from some theory T, valid in every model $\mathbf{M}$ of T?
- Is every formula φ, which is valid in each model $\mathbf{M}$ of T, provable from T?

In the following section we give an answer to the former question; the answer to the latter is postponed to Part II.

Soundness Theorem

A logical calculus is called *sound*, if all what we can prove is valid (*i.e.*, true), which implies that we cannot derive a contradiction. The following theorem shows that First-Order Logic is sound.

THEOREM 3.1 (SOUNDNESS THEOREM). *Let T be a set of $\mathscr{L}$-formulae and $\mathbf{M}$ a model of T. Then for every $\mathscr{L}$-formula φ_0 we have:*

$$\mathsf{T} \vdash \varphi_0 \quad \Longrightarrow \quad \mathbf{M} \vDash \varphi_0$$

Somewhat shorter we could say:

$$\forall \varphi_0 : \ \mathsf{T} \vdash \varphi_0 \quad \Longrightarrow \quad \forall \mathbf{M} \big(\mathbf{M} \vDash \mathsf{T} \Longrightarrow \mathbf{M} \vDash \varphi_0 \big)$$

Proof. First we show that all logical axioms are valid in $\mathbf{M}$. For this we have to define truth-values of composite statements in the metalanguage.
In the previous chapter we defined for example:

$$\underbrace{\mathbf{M} \vDash \varphi \wedge \psi}_{\Theta} \iff \underbrace{\mathbf{M} \vDash \varphi}_{\Phi} \ \text{AND}\ \underbrace{\mathbf{M} \vDash \psi}_{\Psi}$$

Thus, in the metalanguage the statement "Θ" is true if and only if the statement "Φ AND Ψ" is true. So, the truth-value of "Θ" depends on the truth-values of "Φ" and "Ψ". In order to determine truth-values of composite statement like "Φ AND Ψ", we introduce so called *truth-tables*, in which "**1**" *stands for* "**true**" and "**0**" *stands for* "**false**":

Φ	Ψ	NOT Φ	Φ AND Ψ	Φ OR Ψ	IF Φ THEN Ψ
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	1

With these truth-tables one can show that all logical axioms are valid in $\mathbf{M}$. As an example we that every instance of L_1 is valid in $\mathbf{M}$: For this, let φ_1 be an instance of L_1, *i.e.*, $\varphi_1 \equiv \varphi \to (\psi \to \varphi)$ for some $\mathscr{L}$-formulae $\varphi \,\&\, \psi$. Then $\mathbf{M} \vDash \varphi_1$ *iff* $\mathbf{M} \vDash \varphi \to (\psi \to \varphi)$:

$$\underbrace{\mathbf{M} \vDash \varphi \to (\psi \to \varphi)}_{\Theta} \iff \text{IF}\ \underbrace{\mathbf{M} \vDash \varphi}_{\Phi}\ \text{THEN}\ \underbrace{\mathbf{M} \vDash \psi \to \varphi}_{\text{IF}\ \underbrace{\mathbf{M} \vDash \psi}_{\Psi}\ \text{THEN}\ \underbrace{\mathbf{M} \vDash \varphi}_{\Phi}}$$

This shows that

$$\Theta \iff \text{IF}\ \Phi\ \text{THEN}\ (\text{IF}\ \Psi\ \text{THEN}\ \Phi\,).$$

Writing the truth-table of "Θ", we see that the statement "Θ" is always true in $\mathbf{M}$:

Φ	Ψ	IF Ψ THEN Φ	IF Φ THEN (IF Ψ THEN Φ)
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Therefore, $\mathbf{M} \vDash \varphi_1$, and since φ_1 was an arbitrary instance of L_1, every instance of L_1 is valid in $\mathbf{M}$.

In order to show that also the logical axioms L_{11}–L_{17} are valid in $\mathbf{M}$, we need somewhat more than just truth-tables:

Let A be the domain of $\mathbf{M}$, let j be an arbitrary assignment, and let $\mathbf{I} = (\mathbf{M}, j)$ be the corresponding $\mathscr{L}$-interpretation.

Now, we show that every instance of L_{11} is valid in $\mathbf{M}$. For this, let φ_{11} be an instance of L_{11}, *i.e.*, $\varphi_{11} \equiv \forall\nu\varphi(\nu) \to \varphi(\tau)$ for some $\mathscr{L}$-formula φ, where ν is a variable, τ a term, and the substitution $\varphi(\nu/\tau)$ is admissible. We work with $\mathbf{I}$ and show that $\mathbf{I} \vDash \varphi_{11}$.
By definition we have:

$$\mathbf{I} \vDash \forall\nu\varphi(\nu) \to \varphi(\tau) \quad \Longleftrightarrow \quad \text{IF } \mathbf{I} \vDash \forall\nu\varphi(\nu) \text{ THEN } \mathbf{I} \vDash \varphi(\tau)$$

Again by definition we have:

$$\mathbf{I} \vDash \forall\nu\varphi(\nu) \quad \Longleftrightarrow \quad \text{FOR ALL } a \text{ IN } A: \ \mathbf{I}\tfrac{a}{\nu} \vDash \varphi$$

In particular we get:

$$\mathbf{I} \vDash \forall\nu\varphi(\nu) \quad \Longrightarrow \quad \mathbf{I}\tfrac{I(\tau)}{\nu} \vDash \varphi$$

Furthermore, by FACT 2.1.(a) we get:

$$\mathbf{I} \vDash \varphi(\tau) \quad \Longleftrightarrow \quad \mathbf{I}\tfrac{I(\tau)}{\nu} \vDash \varphi(\nu)$$

Hence, we get

$$\text{IF } \mathbf{I} \vDash \forall\nu\varphi(\nu) \text{ THEN } \mathbf{I} \vDash \varphi(\tau)$$

which shows that

$$(\mathbf{M}, j) \vDash \forall\nu\varphi(\nu) \to \varphi(\tau)$$

and since the assignment j was arbitrary, we finally get:

$$\mathbf{M} \vDash \forall\nu\varphi(\nu) \to \varphi(\tau)$$

Therefore, $\mathbf{M} \vDash \varphi_{11}$, and since φ_{11} was an arbitrary instance of L_{11}, every instance of L_{11} is valid in $\mathbf{M}$.

With similar arguments one can show that also every instance of L_{12}, L_{13}, or L_{14} is valid in $\mathbf{M}$ (see EXERCISES 4–6).

Zeigen, dass auch L_{15}–L_{17} *in* $\mathbf{M}$ *gelten.*

Let now $\mathbf{M}$ be a model of T and assume that $\mathsf{T} \vdash \varphi_0$. We shall show that $\mathbf{M} \vDash \varphi_0$. For this, we notice first the following facts:

- As we have seen above, each instance of a logical axiom is valid in $\mathbf{M}$.
- Since $\mathbf{M} \vDash \mathsf{T}$, each formula of T is valid in $\mathbf{M}$.
- By the truth-tables we get

$$\text{IF } \big(\mathbf{M} \vDash \varphi \to \psi \text{ AND } \mathbf{M} \vDash \varphi\big) \text{ THEN } \mathbf{M} \vDash \psi$$

and therefore, every application of MODUS PONENS in the proof of φ_0 from T yields a valid formula (if the premisses are valid).

- Since, by FACT 2.2,

$$\mathbf{M} \vDash \varphi \iff \mathbf{M} \vDash \forall\nu\varphi(\nu)$$

every application of the GENERALISATION in the proof of φ_0 from T yields a valid formula.

From these facts it follows immediately that *each* formula in the proof of φ_0 from T is valid in $\mathbf{M}$. In particular we get

$$\mathbf{M} \vDash \varphi_0$$

which completes the proof. ⊣

The following fact summarises a few consequences of the SOUNDNESS THEOREM.

FACT 3.2.

(a) *Every tautology is valid in each model:*

$$\forall \varphi : \vdash \varphi \implies \forall \mathbf{M} : \mathbf{M} \vDash \varphi$$

(b) *If a theory* T *has a model, then* T *is consistent:*

$$\exists \mathbf{M} : \mathbf{M} \vDash \mathsf{T} \implies \mathrm{Con}(\mathsf{T})$$

(c) *The logical axioms are consistent:*

$$\mathrm{Con}(\mathsf{L}_0\text{-}\mathsf{L}_{17})$$

(d) *If a formula* φ *is not valid in* $\mathbf{M}$*, where* $\mathbf{M}$ *is a model of* T*, then* φ *is not provable from* T*:*

$$\text{IF } \big(\mathbf{M} \nvDash \varphi \text{ AND } \mathbf{M} \vDash \mathsf{T}\big) \text{ THEN } \mathsf{T} \nvdash \varphi$$