Shelah's Models of the First Type

The set of atoms of the first type of Shelah's permutation models is built by induction, where every atom encodes certain sets of atoms on a lower level. In this section, we shall give an example of such a model, where the atoms encode finite sequences.

The atoms of the model are constructed as follows:

(α) A_0 is an arbitrary infinite set of atoms.

(β) $\mathscr{G}_0$ is the group of all permutations of A_0.

(γ) $A_{n+1} := A_n \,\dot\cup\, \big\{(n+1, p, \varepsilon) : p \in \bigcup_{k=0}^{n+1} A_n^k \wedge \varepsilon \in \{0,1\}\big\}$.

(δ) $\mathscr{G}_{n+1}$ is the subgroup of the permutation group of A_{n+1} containing all permutations σ for which there are $\pi_\sigma \in \mathscr{G}_n$ and $\varepsilon_\sigma \in \{0,1\}$ such that

$$\sigma(x) = \begin{cases} \pi_\sigma(x) & \text{if } x \in A_n, \\ (n+1, \pi_\sigma(p), \varepsilon_\sigma +_2 \varepsilon) & \text{if } x = (n+1, p, \varepsilon), \end{cases}$$

where for $p = \langle p_0, \ldots, p_{l-1}\rangle \in \bigcup_{0\le k\le n+1} A_n^k$, $\pi_\sigma(p) := \langle \pi_\sigma(p_0), \ldots, \pi_\sigma(p_l)\rangle$ and $+_2$ denotes addition modulo 2.

Let $A := \bigcup\{A_n : n \in \omega\}$ be the set of atoms and let $\mathrm{Aut}(A)$ be the group of all permutations of A. Then

$$\mathscr{G} := \big\{H \in \mathrm{Aut}(A) : \forall n \in \omega\,(H|_{A_n} \in \mathscr{G}_n)\big\}$$

is a group of permutations of A. Let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by $\{\mathrm{fix}_{\mathscr{G}}(E) : E \in \mathrm{fin}(A)\}$ (which happens to be normal) and let $\boldsymbol{\mathcal{V}}_p$ (p for pairs) be the class of all hereditarily symmetric objects.

The proof of the following fact is left as an exercise to the reader.

FACT 7.18. *For each $n \in \omega$, the set A_n belongs to $\boldsymbol{\mathcal{V}}_p$. In particular, the function*

$$\begin{aligned} f : \omega &\to \mathscr{P}(A) \\ n &\longmapsto A_n \end{aligned}$$

is an injective function which belongs to $\boldsymbol{\mathcal{V}}_p$. Moreover, for each atom $a \in A$ there exists a least number $n \in \omega$ such that $a \in A_n$. In particular, there exists a surjection $f : A \twoheadrightarrow \omega$ which belongs to $\boldsymbol{\mathcal{V}}_p$.

Now, we are ready to prove the following

PROPOSITION 7.19. *Let $\mathfrak{m}$ denote the cardinality of the set of atoms A of $\boldsymbol{\mathcal{V}}_p$. Then* $\boldsymbol{\mathcal{V}}_p \vDash \mathrm{seq}(\mathfrak{m}) < [\mathfrak{m}]^2$.

Proof. First we show that $\boldsymbol{\mathcal{V}}_p \vDash \operatorname{seq}(\mathfrak{m}) \leq [\mathfrak{m}]^2$. For this it is sufficient to find a one-to-one function $f \in \boldsymbol{\mathcal{V}}_p$ from $\operatorname{seq}(A)$ into $[A]^2$. We define such a function as follows. For a finite sequence $s = \langle a_0, \ldots, a_{l-1}\rangle \in \operatorname{seq}(A)$ let

$$f(s) := \big\{(n_0 + \ldots + n_{l-1} + 1, s, 0), (n_0 + \ldots + n_{l-1} + 1, s, 1)\big\}\,,$$

where for each $i \in l$, n_i is the smallest number such that $a_i \in A_{n_i}$. For any $\pi \in \mathscr{G}$ and $s = \langle a_0, \ldots, a_{l-1}\rangle \in \operatorname{seq}(A)$ we have $\pi f(s) = f\big(\pi(s)\big)$ and therefore, the function f is as desired and belongs to $\boldsymbol{\mathcal{V}}_p$.

Now, let $g \in \boldsymbol{\mathcal{V}}_p$ be a function from $[A]^2$ to $\operatorname{seq}(A)$ and let E_g be a finite support of g. We show that g is not one-to-one. Since E_g is finite, there is a number n_g such that $E_g \subseteq A_{n_g}$. By extending E_g if necessary, we may assume that if $(n+1, \langle a_0, \ldots, a_{l-1}\rangle, \varepsilon) \in E_g$, then also $a_0, \ldots, a_{l-1}$ belong to E_g as well as the atom $(n+1, \langle a_0, \ldots, a_{l-1}\rangle, 1-\varepsilon)$ (this assumption will be needed later).

Choose two distinct elements $x, y \in A_0 \setminus E_g$ such that $g(\{x, y\}) \neq \langle\,\rangle$. If there are no such elements, then g is not one-to-one and we are done. So, we may assume that for some positive integer $l \in \omega$:

$$g(\{x, y\}) = \langle a_0, \ldots, a_{l-1}\rangle$$

Now, we are in at least one of the following cases:

(1) $\forall i \in l\,\big(a_i \in E_g\big)$

(2) $\exists i \in l\,\big(a_i \in \{x, y\}\big)$

(3) $\exists i \in l\,\big(a_i \in A_0 \setminus (E_g \cup \{x, y\})\big)$

(4) $\exists i \in l\,\big(a_i \in A \setminus (E_g \cup A_0)\big)$

If we are in case (1), then let $\pi \in \operatorname{fix}(E_g)$ be such that $\pi x \notin \{x, y\}$. To see that such a $\pi \in \operatorname{fix}(E_g)$ exists, recall that by our assumption, E_g has the property that whenever $(n+1, \langle a_0, \ldots, a_{l-1}\rangle, \varepsilon) \in E_g$, also $a_0, \ldots, a_{l-1} \in E_g$. Now, $\pi g(\{x, y\}) = g(\{x, y\})$ (since $\pi \in \operatorname{fix}(E_g)$), but $\pi\{x, y\} \neq \{x, y\}$. Hence, g is not a one-to-one function.

If we are in case (2), then let $\pi \in \operatorname{fix}(E_g)$ be such that $\pi x = y$ and $\pi y = x$. Notice that since $\{x, y\} \cap E_g = \emptyset$, such a permutation π exists. Now, by the choice of π, on the one hand we have $\pi\{x, y\} = \{x, y\}$, *i.e.*, $g(\{x, y\}) = g(\pi\{x, y\})$, but on the other hand, for some $i \in l$ we have $a_i \in \{x, y\}$, which implies $a_i \neq \pi a_i$. To see this, notice that for example $a_i = x$ implies $\pi a_i = y$. Therefore, E_g is not a support of g which contradicts the choice of E_g.

If we are in case (3), then there is an $i \in l$ such that

$$a_i \in A_0 \setminus (E_g \cup \{x, y\})\,.$$

Now, take an arbitrary $b_i \in A_0 \setminus (E_g \cup \{x, y\})$ which is distinct from a_i and let $\pi \in \operatorname{fix}(E_g \cup \{x, y\})$ be such that $\pi a_i = b_i$ and $\pi b_i = a_i$ (notice that such a permutation π exists). By the choice of π, on the one hand we have $\pi\{x, y\} =$

$\{x, y\}$, *i.e.*, $g(\{x, y\}) = g(\pi\{x, y\})$, but on the other hand, $\pi a_i = b_i$ and $b_i \neq a_i$, *i.e.*, $g(\{x, y\}) \neq \pi g(\{x, y\})$. Therefore, E_g is not a support of g which contradicts the choice of E_g.

If we are in case (4), then there is an $i \in l$ such that

$$a_i \in A \setminus (E_g \cup A_0)\,.$$

In particular, $a_i = (n + 1, p, \varepsilon)$ for some $n \in \omega$, $p \in \operatorname{seq}(A)$, and $\varepsilon \in \{0, 1\}$. Furthermore, let $\pi \in \operatorname{fix}(E_g \cup \{x, y\})$ be such that

$$\pi(n + 1, p, \varepsilon) = (n + 1, p, 1 - \varepsilon)\,.$$

To see that such a π exists, recall that by our assumption, E_g has the property that whenever $(n + 1, s, \varepsilon) \in E_g$ for some $s \in \operatorname{seq}(A)$, also $(n + 1, s, 1 - \varepsilon) \in E_g$. Now we have $\pi\{x, y\} = \{x, y\}$ but $\pi g(\{x, y\}) \neq g(\{x, y\})$. Therefore, E_g is not a support of g.

So, in all four cases, either g is not one-to-one or E_g is not a support of g, which completes the proof. $\dashv$