Chapter 15
Models of Finite Fragments of Set Theory

In this chapter we provide the model-theoretical tools which will be crucial to understand independence proofs. The main result in this chapter is the construction of a countable transitive set-model of a finite fragment of ZFC within some model of Set Theory.

Basic Model-Theoretical Facts

Let $\mathscr{L}$ be an arbitrary but fixed language. Two $\mathscr{L}$-structures $\mathbf{M}$ and $\mathbf{N}$ with domain A and B, respectively, are called **isomorphic** if there is a bijection $f : A \to B$ between A and B which has the following properties:

- for each constant symbol $c \in \mathscr{L}$:

$$f\big(c^{\mathbf{M}}\big) = c^{\mathbf{N}}$$

- for n-ary relation symbols $R \in \mathscr{L}$:

$$R^{\mathbf{M}}(a_1, \dots, a_n) \iff R^{\mathbf{N}}\big(f(a_1), \dots, f(a_n)\big)$$

- for n-ary function symbols $F \in \mathscr{L}$:

$$f\big(F^{\mathbf{M}}(a_1, \dots, a_n)\big) = F^{\mathbf{N}}\big(f(a_1), \dots, f(a_n)\big)$$

If the $\mathscr{L}$-structures $\mathbf{M}$ and $\mathbf{N}$ are isomorphic and $f : A \to B$ is the corresponding bijection, then for all $a_1, \dots, a_n \in A$ and each formula $\varphi(x_1, \dots, x_n)$ we have

$$\mathbf{M} \vDash \varphi(a_1, \dots, a_n) \quad \iff \quad \mathbf{N} \vDash \varphi\big(f(a_1), \dots, f(a_n)\big).$$

This shows that isomorphic $\mathscr{L}$-structures are essentially the same, except that their elements have different "names", and therefore, isomorphic structures are usually identified. For example the *dihedral group* of order six and S_3 (*i.e.*, the *symmetric group* of order six) are isomorphic; whereas C_6 (*i.e.*, the *cyclic group* of order six) is not isomorphic to S_3 (*e.g.*, consider $\varphi(x_1, x_2) \equiv x_1 \circ x_2 = x_2 \circ x_1$).

If $\mathbf{N}$ and $\mathbf{M}$ are $\mathscr{L}$-structures and $B \subseteq A$, then $\mathbf{N}$ is said to be an **elementary substructure** of $\mathbf{M}$, denoted $\mathbf{N} \prec \mathbf{M}$, if for every formula $\varphi(x_1, \ldots, x_n)$ and every $b_1, \ldots, b_n \in B$:

$$\mathbf{N} \vDash \varphi(b_1, \ldots, b_n) \iff \mathbf{M} \vDash \varphi(b_1, \ldots, b_n).$$

For example the linearly ordered set $(\mathbb{Q}, <)$ is an elementary substructure of $(\mathbb{R}, <)$. On the other hand, $(\mathbb{Z}, <)$ is not an elementary substructure of $(\mathbb{Q}, <)$, *e.g.*, the formula $\exists z(0 < z < 1)$ is false in $(\mathbb{Z}, <)$ but true in $(\mathbb{Q}, <)$.

The key point in construction of elementary substructures of a given structure $\mathbf{M}$ with domain A is the following fact: A structure $\mathbf{N}$ with domain $B \subseteq A$ is an elementary substructure of $\mathbf{M}$ if and only if for every formula $\varphi(u, x_1, \ldots, x_n)$ and all $b_1, \ldots, b_n \in B$:

$$\exists a \in A : \ \mathbf{M} \vDash \varphi(a, b_1, \ldots, b_n) \iff \exists b \in B : \ \mathbf{M} \vDash \varphi(b, b_1, \ldots, b_n).$$

Notice that the implication from the right to the left is obviously true (since $B \subseteq A$). Equivalently we see that $\mathbf{N} \prec \mathbf{M}$ if for every formula $\varphi(u, x_1, \ldots, x_n)$ and all $b_1, \ldots, b_n \in B$:

$$\forall a \in A : \ \mathbf{M} \vDash \varphi(a, b_1, \ldots, b_n) \iff \forall b \in B : \ \mathbf{M} \vDash \varphi(b, b_1, \ldots, b_n).$$

Notice that in this case, the implication from the left to the right is obviously true.

The following theorem—which we state without proof—is somewhat similar to COROLLARY 15.4 below, even though it goes beyond ZFC (see RELATED RESULT 86). However, it is not used later, but it is a nice consequence of the characterisation of elementary submodels given above.

THEOREM 15.1 (LÖWENHEIM–SKOLEM THEOREM). *Every infinite model for a countable language has a countable elementary submodel. In particular, every model of* ZFC *has a countable elementary submodel.*

The Reflection Principle

Instead of aiming for a *set model* of all of ZFC, we can restrict our attention to **finite fragments** of ZFC (*i.e.*, to finite sets of axioms of ZFC), denoted by ZFC*.

We will see that for every finite fragment ZFC* of ZFC, in any model $\mathbf{V} \vDash$ ZFC we find a countable *set* which is a model of ZFC*; but before we can state this result we have to give some further notions from model theory.

Let $\mathbf{V} \vDash$ ZFC, let $M \in \mathbf{V}$ be any set, and let $\mathbf{M} = (M, \in)$ be an $\in$-structure with domain M. An $\in$-structure $\mathbf{M} = (M, \in)$, where $M \in \mathbf{V}$ is a set, is called a

set model. Notice that this definition of *model* is slightly different to the one given in Chapter **??**, where we defined models with respect to a set of formulae.

Now, let $\mathbf{V}$ be an arbitrary but fixed model of ZFC and let $\mathbf{M} = (M, \in)$ be a set model. For any formula φ we define $\mathbf{M} \vDash \varphi$ by induction on the complexity of the formula φ:

- $\mathbf{M} \vDash x = y \iff \mathbf{V} \vDash \{x, y\} \in M \wedge (x = y)$
- $\mathbf{M} \vDash x \in y \iff \mathbf{V} \vDash \{x, y\} \in M \wedge (x \in y)$
- $\mathbf{M} \vDash \psi_1 \wedge \psi_2 \iff \mathbf{M} \vDash \psi_1$ *and* $\mathbf{M} \vDash \psi_2$
- $\mathbf{M} \vDash \neg\psi \iff \mathbf{M} \nvDash \psi$
- $\mathbf{M} \vDash \exists x\, \psi \iff \mathbf{V} \vDash \exists x\, (x \in M \wedge \psi)$

From this definition we get that if $\varphi(x_1, \ldots, x_n)$ is a formula and $a_1, \ldots, a_n \in M$, then $\mathbf{M} \vDash \varphi(a_1, \ldots, a_n)$ is the same as $\mathbf{V} \vDash \varphi(a_1, \ldots, a_n)$, except that the bound variables of φ just range over M. Notice that the interpretation of the non-logical symbol "$\in$" remains unchanged for sets in M. Furthermore, notice that also the sets themselves remain unchanged (which will not be the case for example when we apply MOSTOWSKI'S COLLAPSING THEOREM 15.3).

For a set model $\mathbf{M} = (M, \in)$ and a set of formulae Φ, $\mathbf{M} \vDash \Phi$ means $\mathbf{M} \vDash \varphi$ for each formula $\varphi \in \Phi$. If Φ is a set of formulae and for each formula $\varphi \in \Phi$ we have

$$\mathbf{M} \vDash \varphi \iff \mathbf{V} \vDash \varphi,$$

then we say that M **reflects** Φ.

The following theorem shows that if ZFC is consistent, then for any finite fragment $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ there is a set which reflects ZFC^*.

THEOREM 15.2 (REFLECTION PRINCIPLE). *Let* $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ *be an arbitrarily large but finite fragment of* ZFC.

(a) *If* $\mathbf{V} \vDash \mathsf{ZF}$, *then there exists an ordinal* $\gamma \in \Omega^{\mathbf{V}}$, *such that* V_γ *reflects* ZFC^*.

(b) *Let* $\mathbf{V} \vDash \mathsf{ZFC}$ *and let* $M_0 \in \mathbf{V}$ *be a non-empty countable set. Then there exists a countable set* $M \in \mathbf{V}$ *such that* $M \supseteq M_0$ *and* M *reflects* ZFC^*.

Proof. Let $\mathbf{V}$ be a model of ZF, let $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ be finite fragment of ZFC, and let $\psi_0, \ldots, \psi_m$ be an enumeration of the finitely many sentences in ZFC^*. Then

$$\psi \equiv \bigwedge_{j=0}^{m} \psi_j$$

is a single sentence. Since for every sentence there is an equivalent sentence in prenex normal form, by renaming the variables and by adding some auxiliary variables, we can build a sentence

$$\bar{\varphi} \equiv \exists y_0 \forall x_1 \exists y_1 \forall x_2 \cdots \forall x_k \exists y_k\, \varphi(x_1, \ldots, x_k, y_0, \ldots, y_k)$$

which is equivalent to ψ such that $\mathrm{free}(\varphi) \subseteq \{x_1,\dots,x_k,y_0,\dots,y_k\}$ and φ is quantifier free.

The crucial point in the proof of part (a) is to show that for any existential formula $\exists y\tilde{\varphi}(x,y)$ which holds in $\mathbf{V}$ and any set $V \in \mathbf{V}$, there exists a set $V' \supseteq V$ in $\mathbf{V}$ which contains a so-called *witness* for $\exists y\tilde{\varphi}(x,y)$, *i.e.*, there is a set $a \in V'$ such that for all $b \in V$, $(V',\in) \vDash \tilde{\varphi}(a,b)$.

We proceed now as follows: Firstly, by the TRANSFINITE RECURSION THEOREM **??** we define a sequence of ordinals $\langle \alpha_n \in \Omega : n \in \omega\rangle$ in $\mathbf{V}$, where α_0 is an arbitrary ordinal and

$$\alpha_{n+1} := \bigcap \big\{\alpha \in \Omega : \exists y_0 \in V_\alpha\, \forall x_1 \in V_{\alpha_n} \exists y_1 \in V_\alpha \cdots \cdots \forall x_k \in V_{\alpha_n} \exists y_k \in V_\alpha\, \varphi(x_1,\dots,x_k,y_0,\dots,y_k)\big\}\,.$$

Let $\gamma := \bigcup_{n\in\omega} \alpha_n$; then

$$V_\gamma = \bigcup_{n\in\omega} V_{\alpha_n}$$

and by construction we get

$$\mathbf{V} \vDash \exists y_0 \in V_\gamma\, \forall x_1 \in V_\gamma \cdots \exists y_k \in V_\gamma\, \varphi(x_1,\dots,x_k,y_0,\dots,y_k)\,.$$

Therefore, the set V_γ reflects the sentence $\bar{\varphi}$, and since $\bar{\varphi}$ is equivalent to ψ, which is just the conjunction of the sentences in ZFC*, we get that V_γ reflects ZFC*, which completes the proof of part (a).

In order to prove part (b), we first carry out the construction in the proof of part (a) in a model $\mathbf{V} \vDash$ ZFC. By construction, we get a sequence of ordinals $\langle \alpha_n \in \Omega : n \in \omega\rangle$, where α_0 is such that $M_0 \in V_{\alpha_0}$, as well as a set V_γ which reflects $\bar{\varphi}$. By the Axiom of Choice, there is a well-ordering "$<$" of V_γ. For every non-empty set $X \subseteq V_\gamma$ let μX be the $<$-minimal element of X. By induction on ω, for every $n \in \omega$ we define a set M_n as follows: For each i with $0 \le i \le k$ define the function $h_{n,i} : (M_n)^i \to V_{\alpha_{n+1}}$ by stipulating

$$h_{n,i}\big(\langle x_1,\dots,x_i\rangle\big) \mapsto \mu\big\{y \in V_{\alpha_{n+1}} : \forall x_{i+1}\exists y_{i+1}\cdots\forall x_k\exists y_k\, \varphi\big(x_1,\dots,x_k,h_{n,0}(\emptyset),\dots \dots h_{n,i-1}(\langle x_1,\dots,x_{i-1}\rangle),y,y_{i+1},\dots,y_k\big)\big\}\,,$$

and define

$$M_{n+1} := M_n \cup \bigcup_{0\le i\le k} h_{n,i}\big[(M_n)^i\big]\,.$$

Finally, let $M := \bigcup_{n\in\omega} M_n$. Then, since for every $n \in \omega$, M_n is a countable set in $\mathbf{V}$, also M is countable in $\mathbf{V}$. Moreover, by construction we get that M reflects the sentence $\bar{\varphi}$. Hence, M reflects ZFC*, which completes the proof of part (b). $\dashv$

The REFLECTION PRINCIPLE 15.2 can be considered as a kind of ZFC-version of the LÖWENHEIM–SKOLEM THEOREM 15.1, and even though it is weaker than that theorem, it has many interesting consequences and important applications, especially for consistency proofs.

Some remarks:

(1) Notice that the set V_γ constructed in part (a) is a transitive set, whereas the countable set constructed in part (b) is in general not transitive: For example if ZFC* is rich enough to define ω_1 as the smallest uncountable ordinal and the countable set M reflects ZFC*, then M cannot be transitive, since otherwise, $\omega_1 \cap M$ would not be countable in $\mathbf{V}$.

(2) Let ZFC* be a finite fragment of ZFC and assume that ZFC* $\vdash \varphi$ (for some sentence φ). Further, assume that M reflects ZFC* and let $\mathbf{M} = (M, \in)$. Then, in the model-theoretic sense, $\mathbf{M} \vDash$ ZFC*, and consequently, $\mathbf{M} \vDash \varphi$.
As we will see later, this is the first step in order to show that a given sentence φ is consistent with ZFC: By the COMPACTNESS THEOREM **??** it is enough to show that whenever $\Phi \subseteq$ ZFC is a finite fragment of ZFC, then $\Phi + \varphi$ has a model. Let Φ be an arbitrary but fixed finite set of axioms of ZFC. Now, let $M \in \mathbf{V}$ be a set which reflects a certain finite fragment ZFC* $\subseteq$ ZFC, where ZFC* makes sure that the model $\mathbf{M} = (M, \in)$ can be extended within $\mathbf{V}$ to a generic set model $\mathbf{M}[X]$ such that $\mathbf{M}[X] \vDash \Phi + \varphi$. Then, since Φ was arbitrary, this shows that φ is consistent with ZFC. This method is explained in great detail in the next chapter.

Countable Transitive Models of Finite Fragments of ZFC

As mentioned above, the model of a finite fragment of ZFC constructed in the proof of the REFLECTION PRINCIPLE 15.2 can be countable or transitive, but in general not both. However, as a consequence of the following result, in any ground model $\mathbf{V} \vDash$ ZFC and for every finite fragment ZFC* $\subseteq$ ZFC there is a countable transitive set model of ZFC*. In order to state the theorem, we first have to introduce the following notion: A set model $\mathbf{M} = (M, \in)$ is called **extensional** if for all $x, y \in M$ we have

$$\mathbf{M} \vDash x = y \iff \mathbf{M} \vDash \forall z(z \in x \leftrightarrow z \in y)\,.$$

In other words, $\mathbf{M}$ is extensional if and only if $\mathbf{M}$ satisfies the Axiom of Extensionality. For an example of a set model $\mathbf{M} = (M, \in)$ which is not extensional, consider the set $M = \big\{\emptyset, \{\{\emptyset\}\}\big\}$. Because there is no $z \in M$ such that $z \in \{\{\emptyset\}\}$, we have $\mathbf{M} \vDash \forall z\big(z \in \emptyset \leftrightarrow z \in \{\{\emptyset\}\}\big)$, but $\mathbf{M} \nvDash \emptyset = \{\{\emptyset\}\}$.

Since set models which are not extensional do not satisfy the Axiom of Extensionality, and since the Axiom of Extensionality is essential in the concept of sets, non-extensional set models are not of any use. For non-transitive set models, the situation is different. On the one hand, every model of Set Theory must be transitive, on the other hand, as we have seen above, there are set models of arbitrarily large finite fragments of ZFC which are not transitive. However, as will see now, every extensional set model is isomorphic to a transitive model.

THEOREM 15.3 (MOSTOWSKI'S COLLAPSING THEOREM). *Let* $\mathbf{V} \vDash$ ZFC, *let* $M \in \mathbf{V}$ *be a set, and let* $\mathbf{M} = (M, \in)$ *be an extensional set model of a finite fragment* ZFC* $\subseteq$ ZFC *(i.e., the* Axiom of Extensionality *belongs to* ZFC**). Then there exists a unique mapping* $\pi : M \to \mathbf{V}$ *in* $\mathbf{V}$, *such that the set* $N := \pi[M]$ *is transitive (in* $\mathbf{V}$*) and the set models* $\mathbf{N} = (N, \in)$ *and* $\mathbf{M}$ *are isomorphic, i.e., the mapping* $\pi : M \to N$ *is bijective and for all* $x, y \in M$, $y \in x \leftrightarrow \pi(y) \in \pi(x)$.

Proof. Notice first that if $M = \emptyset$, then M is already transitive and we are done. So, let us assume that M is a non-empty set in $\mathbf{V}$. Since $\pi[M]$ must be transitive and for all $x, y \in M$ we must have $y \in x \leftrightarrow \pi(y) \in \pi(x)$, we get that for every $x \in M$, $\pi(x) = \{\pi(y) : y \in x \cap M\}$. By the Axiom of Foundation (which holds in $\mathbf{V}$), there is an $x_0 \in M$ such that $x_0 \cap M = \emptyset$, and since $\mathbf{M}$ is extensional, x_0 is unique; let $A_0 = \{x_0\}$. By the properties of π, $\pi(x_0) = \emptyset$, *i.e.*, we do not have any other options to define $\pi(x_0)$. If, for some $\alpha \in \Omega$, A_α is already defined and $M \setminus A_\alpha \neq \emptyset$, then let

$$X_\alpha := \big\{x \in M \setminus A_\alpha : x \cap (M \setminus A_\alpha) = \emptyset\big\}\,,$$

and let $A_{\alpha+1} := A_\alpha \cup X_\alpha$. By the properties of π, for each $x \in X_\alpha$, $\pi(x) = \{\pi(y) : y \in x \cap M\}$ (again, there are no other options to define $\pi(x)$ for $x \in X_\alpha$). Finally, if, for some limit ordinal α, A_β is already defined for each $\beta \in \alpha$, then let $A_\alpha := \bigcup_{\beta \in \alpha} A_\beta$. Now, for some $\lambda \in \kappa^+$, where $\kappa = |M|$, $M = \bigcup_{\alpha \in \lambda} A_\alpha$ and we define $N := \pi[M]$.

As mentioned above, the mapping $\pi : M \to N$ is unique, and it remains to show that π is a bijection and that N is transitive: By definition of N, $\pi : M \to N$ is surjective. To see that π is also injective, let $x, y \in M$ be two distinct elements. Then, since $\mathbf{M}$ is extensional, there exists a set $z \in M$ which belongs to either x or y but not both, which implies that also $\pi(z)$ belongs to either $\pi(x)$ or $\pi(y)$ but not both. Hence, $\pi(x) \neq \pi(y)$. To see that N is transitive, take an arbitrary $u \in N$. Since $N = \pi[M]$, there is an $x_u \in M$ such that $\pi(x_u) = u$, and by the properties of π we get that $u = \{\pi(y) : y \in x_u \cap M\}$, hence, $u \subseteq N$. ⊣

As an immediate consequence we get

COROLLARY 15.4. *Let* $\mathbf{V}$ *be a model of* ZFC, *let* ZFC* *be a finite fragment of* ZFC, *and let* $\mathbf{M} = (M, \in)$ *be a countable set model of* ZFC* *where* $M \in \mathbf{V}$. *If* ZFC* *contains the* Axiom of Extensionality, *then there is a countable transitive set* $N \in \mathbf{V}$ *such that* $\mathbf{N} = (N, \in)$ *is isomorphic to* $\mathbf{M}$, *(in particular,* $\mathbf{N} \vDash$ ZFC**).*

Proof. Because Axiom of Extensionality belongs to ZFC* and $\mathbf{M} \vDash$ ZFC*, $\mathbf{M}$ is extensional. Thus, by MOSTOWSKI'S COLLAPSING THEOREM 15.3, there is a transitive set $N \in \mathbf{V}$ such that $\mathbf{M}$ and $\mathbf{N} = (N, \in)$ are isomorphic, and since $\pi : M \to N$ is a bijection, N is countable. ⊣

Let ZFC* be any finite fragment of ZFC and let $\mathbf{V}$ be a model of ZFC. Then, by the REFLECTION PRINCIPLE 15.2**??**, there is a countable set M in $\mathbf{V}$ that reflects ZFC* and for $\mathbf{M} = (M, \in)$ we have $\mathbf{M} \vDash$ ZFC*. Thus, by COROLLARY 15.4, there is a countable transitive set N that reflects ZFC*. In other words, for any finite fragment ZFC$^* \subsetneq$ ZFC there is a **countable transitive model** $\mathbf{N}$ in $\mathbf{V}$ such that $\mathbf{N} \vDash$ ZFC*.

Let us briefly discuss the preceding constructions: We start with a model $\mathbf{V}$ of ZFC and an arbitrary large but finite set of axioms ZFC$^* \subsetneq$ ZFC. By the REFLECTION PRINCIPLE 15.2**??** there is a countable set M in $\mathbf{V}$ such that $\mathbf{M} = (M, \in)$ is a model of ZFC*. By applying MOSTOWSKI'S COLLAPSING THEOREM 15.3 to $(M, \in)$ we obtain a countable transitive model $\mathbf{N} = (N, \in)$ in $\mathbf{V}$ such that the models $\mathbf{N} = (N, \in)$ and $\mathbf{M}$ are isomorphic, and consequently, $\mathbf{N}$ is a model of ZFC*.

It is worth mentioning that the model $\mathbf{M} = (M, \in)$ is a genuine submodel of $\mathbf{V}$ and therefore contains the real sets of $\mathbf{V}$. For example if

$$\mathbf{M} \vDash \text{"}\lambda \text{ is the least uncountable ordinal"}$$

then $\lambda = \omega_1$, *i.e.*, $\omega_1 \in M$. However, since the set M is countable in $\mathbf{V}$, there are countable ordinals in $\mathbf{V}$ which do not belong to the set M, and therefore not to the model $\mathbf{M}$ (which implies that M is not transitive). In other words,

$$\mathbf{V} \vDash \lambda = \omega_1 \wedge \omega_1 \in M \wedge |\lambda \cap M| = \omega.$$

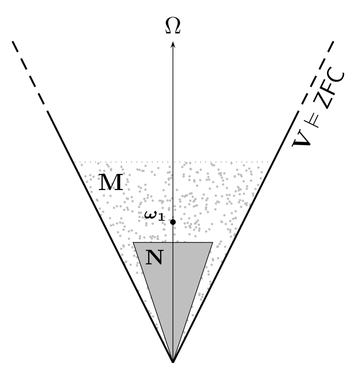
On the one hand, the model $\mathbf{N} = (N, \in)$ is in general not a submodel of $\mathbf{V}$ and just contains a kind of copies of countably many set of $\mathbf{V}$. For example if

$$\mathbf{N} \vDash \text{"}\lambda \text{ is the least uncountable ordinal"}$$

then λ, which corresponds to ω_1 in $\mathbf{N}$, is just a countable ordinal in $\mathbf{V}$. However, since N is transitive, every ordinal in $\mathbf{V}$ which belongs to λ also belongs to the set N, and therefore to the model $\mathbf{N}$. In other words,

$$\mathbf{V} \vDash \lambda \in \omega_1 \wedge \lambda \in N \wedge \lambda \cap N = \lambda.$$

The relationships between the three models $\mathbf{V}$, $\mathbf{M}$, and $\mathbf{N}$, are illustrated by the following figure:



As we shall see in the next chapter, countable transitive models of finite fragments of ZFC play a key role in consistency and independence proofs.

NOTES

For concepts of model theory and model-theoretical terminology we refer the reader to Hodges [**?**] or to Chang and Keisler [**?**]. However, the preceding results (including proofs) can also be found in Jech [**?** , Chapter 12].

The LÖWENHEIM–SKOLEM THEOREM 15.1 was already discussed in the notes of Chapter **??**; the REFLECTION PRINCIPLE 15.2 was introduced by Montague [**?**] (see also Lévy [**?**]); and the transitive collapse was defined by Mostowski [**?**].

RELATED RESULTS

0. *A model of* ZF − Inf *and the consistency of* PA. $V_\omega \vDash$ ZF − Inf, where Inf denotes the Axiom of Infinity, and moreover, we even have $\mathrm{Con}(\mathsf{PA}) \iff \mathrm{Con}(\mathsf{ZF} - \mathsf{Inf})$ (see Jech [**?** , Exercise 12.9] and Kunen [**?** , Chapter IV, Exercise 30]).

1. *Models of* Z. Let Z be ZF without the Axiom Schema of Replacement. For every limit ordinal $\lambda > \omega$ we have $V_\lambda \vDash$ Z (see Jech [**?** , Exercise 12.7] or Kunen [**?** , Chapter IV, Exercise 6]).

For every infinite regular cardinal κ let $H_\kappa := \{x : |\mathrm{TC}(x)| < \kappa\}$. The elements of H_κ are said to be **hereditarily** of cardinality $< \kappa$. In particular, H_ω—which coincides with V_ω—is the set of hereditarily finite sets and H_{ω_1} is the set of hereditarily countable sets.

2. *Models of* ZFC − P. If AC holds in $\mathbf{V}$, then for all cardinals $\kappa > \omega$ we have $H_\kappa \vDash$ Z − P, where P denotes the Axiom of Power Set. Moreover, for regular cardinals $\kappa > \omega$ we even have $H_\kappa \vDash$ ZFC − P (see Kunen [**?** , Chapter IV, Exercise 7] and Kunen [**?** , Chapter IV, Theorem 6.5]).

An uncountable regular cardinal κ is said to be **inaccessible** if for all $\lambda < \kappa$, $2^\lambda < \kappa$. The inaccessible cardinals owe their name to the fact that they cannot be obtained (or accessed) from smaller cardinals by the usual set-theoretical operations. To some extent, an inaccessible cardinal is to smaller cardinals what ω is to finite cardinals and what is reflected by the fact that $H_\omega \vDash$ ZFC − Inf (*cf.* Jech [**?** , Exercise 12.9]). Notice that by CANTOR'S THEOREM **??**, every inaccessible cardinal is a regular *limit* cardinal. One cannot prove in ZFC that inaccessible cardinals exist; moreover, one cannot even prove that uncountable regular limit cardinals exist (see Kunen [**?** , Chapter VI, Corollary 4.13] but also Hausdorff's remark [**?** , p. 131]).