Almost disjointness is similar to disjointness in name only. It is clear that there can be no family of more than κ disjoint subsets of κ; but, since $|\kappa| = |\kappa \times \kappa|$, there is a family of κ disjoint subsets whose union is all of κ, so that this family is a maximal disjoint family. However, the following theorem holds.

1.2. THEOREM. *Let $\kappa \geq \omega$ be a regular cardinal, then:*

(a) *If $\mathscr{A} \subset \mathscr{P}(\kappa)$ is an* a.d. *family and $|\mathscr{A}| = \kappa$, then $\mathscr{A}$ is not maximal.*

(b) *There is a* m.a.d.f. *$\mathscr{B} \subset \mathscr{P}(\kappa)$ of cardinality $\geq \kappa^+$.*

PROOF. (b) is immediate from (a) and Zorn's Lemma: Let $\mathscr{A} \subset \mathscr{P}(\kappa)$ be any disjoint (or a.d.) family of size κ, and let $\mathscr{B} \supset \mathscr{A}$ be a m.a.d.f., then $|\mathscr{B}| > \kappa$ by (a).

(a) is proved by a diagonal argument. Let $\mathscr{A} = \{A_\xi : \xi < \kappa\}$. Let $B_\xi = A_\xi \setminus \bigcup_{\eta<\xi} A_\eta$. $B_\xi \neq 0$, since $B_\xi = A_\xi \setminus \bigcup_{\eta<\xi}(A_\xi \cap A_\eta)$, $|A_\xi| = \kappa$, and $|\bigcup_{\eta<\xi}(A_\xi \cap A_\eta)| < \kappa$ by regularity of κ. Pick $\beta_\xi \in B_\xi$. The β_ξ are distinct since the B_ξ are disjoint, so $D = \{\beta_\xi : \xi < \kappa\}$ has cardinality κ. $\beta_\xi \in A_\eta \rightarrow \eta \geq \xi$, so $D \cap A_\eta \subset \{\beta_\xi : \xi \leq \eta\}$, so D and A_η are a.d. for each η. □

If one does not assume GCH, Theorem 1.2 suggests a number of questions. First, is there an a.d. family of 2^κ subsets of κ? The answer is yes if $\kappa = \omega$. It is still yes if $\kappa = \omega_1$ if one assumes CH (but 2^{ω_1} can be anything). More generally, the following holds.

1.3. THEOREM. *If $\kappa \geq \omega$ and $2^{<\kappa} = \kappa$, then there is an* a.d. *family $\mathscr{A} \subset \mathscr{P}(\kappa)$ with $|\mathscr{A}| = 2^\kappa$.*

PROOF. Let $I = \{x \subset \kappa : \sup(x) < \kappa\}$. Since $2^{<\kappa} = \kappa$, $|I| = \kappa$. If $X \subset \kappa$, let $A_X = \{X \cap \alpha : \alpha < \kappa\}$. If $|X| = \kappa$, then $|A_X| = \kappa$. If $X \neq Y$, then $|A_X \cap A_Y| < \kappa$, since if we fix β such that $\neg(\beta \in X \leftrightarrow \beta \in Y)$, then

$$A_X \cap A_Y \subset \{X \cap \alpha : \alpha \leq \beta\}.$$

Let $\mathscr{A} = \{A_X : X \subset \kappa \wedge |X| = \kappa\}$; then $|\mathscr{A}| = 2^\kappa$ and is an a.d. family of subsets of I. If we let f be a 1–1 function from I onto κ, $\{f''A : A \in \mathscr{A}\}$ is an a.d. family of 2^κ subsets of κ. □

The hypothesis $2^{<\kappa} = \kappa$ cannot be dropped; the existence of an a.d. family of 2^{ω_1} subsets of ω_1 can be neither proved nor refuted from $2^\omega = 2^{\omega_1} = \omega_3$ (see VIII Exercise B5).

Another question when $2^\kappa > \kappa^+$ is: is there a m.a.d.f. of cardinality κ^+? The answer to this question is independent of the axioms of set theory (see §2 and VIII 2.3).

Another property with "disjoint" in its name is "quasi-disjoint".

1.4. DEFINITION. A family $\mathscr{A}$ of sets is called a *Δ-system*, or a *quasi-disjoint* family iff there is a fixed set r, called the *root* of the Δ-system, such that $a \cap b = r$ whenever a and b are distinct members of $\mathscr{A}$ (see Figure 1.1). □

The main result on Δ-systems is the so-called "Δ-system lemma", which we state first in the special case most often quoted.

1.5. THEOREM. *If $\mathscr{A}$ is any uncountable family of finite sets, there is an uncountable $\mathscr{B} \subset \mathscr{A}$ which forms a Δ-system.* □

This is an immediate corollary of the following theorem when $\kappa = \omega$ and $\theta = \omega_1$. For an easier direct proof of 1.5, see Exercise 1.

1.6. THEOREM. *Let κ be any infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall \alpha < \theta(|\alpha^{<\kappa}| < \theta)$. Assume $|\mathscr{A}| \geq \theta$ and $\forall x \in \mathscr{A}(|x| < \kappa)$, then there is a $\mathscr{B} \subset \mathscr{A}$, such that $|\mathscr{B}| = \theta$ and $\mathscr{B}$ forms a Δ-system.*

PROOF. By shrinking $\mathscr{A}$ if necessary, we may assume $|\mathscr{A}| = \theta$. Then $|\bigcup \mathscr{A}| \leq \theta$. Since what the elements of $\mathscr{A}$ are as individuals is irrelevant, we may assume $\bigcup \mathscr{A} \subset \theta$. Then each $x \in \mathscr{A}$ has some order type $<\kappa$ as a subset of θ. Since θ is regular and $\theta > \kappa$, there is some $\rho < \kappa$, such that $\mathscr{A}_1 = \{x \in \mathscr{A} : x \text{ has type } \rho\}$ has cardinality θ. We now fix such a ρ and deal only with $\mathscr{A}_1$.

For each $\alpha < \theta$, $|\alpha^{<\kappa}| < \theta$ implies that less than θ elements of $\mathscr{A}_1$ are subsets of α. Thus, $\bigcup \mathscr{A}_1$ is unbounded in θ. If $x \in \mathscr{A}_1$ and $\xi < \rho$, let $x(\xi)$ be the ξ-th element of x. Since θ is regular, there is some ξ such that $\{x(\xi) : x \in \mathscr{A}_1\}$ is unbounded in θ. Now fix ξ_0 to be the least such ξ (ξ_0 may be 0). Let

$$\alpha_0 = \sup\{x(\eta) + 1 : x \in \mathscr{A}_1 \wedge \eta < \xi_0\};$$

then $\alpha_0 < \theta$ and $x(\eta) < \alpha_0$ for all $x \in \mathscr{A}_1$ and all $\eta < \xi_0$.

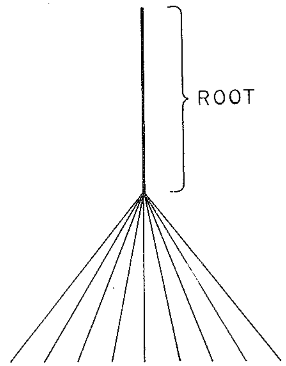

Figure 1.1. A Δ-System.

By transfinite recursion on $\mu < \theta$, pick $x_\mu \in \mathscr{A}_1$ so that $x_\mu(\xi_0) > \alpha_0$ and $x_\mu(\xi_0)$ is above all elements of earlier x_ν; i.e.,

$$x_\mu(\xi_0) > \max(\alpha_0, \sup\{x_\nu(\eta): \eta < \rho \wedge \nu < \mu\}).$$

Let $\mathscr{A}_2 = \{x_\mu: \mu < \theta\}$. Then $|\mathscr{A}_2| = \theta$ and $x \cap y \subset \alpha_0$ whenever x and y are distinct elements of $\mathscr{A}_2$. Since $|\alpha_0^{<\kappa}| < \theta$, there is an $r \subset \alpha_0$ and a $\mathscr{B} \subset \mathscr{A}_2$ with $|\mathscr{B}| = \theta$ and $\forall x \in \mathscr{B}(x \cap \alpha_0 = r)$, whence $\mathscr{B}$ forms a Δ-system with root r. □

We conclude this section with an application of Theorem 1.5 to topology.

1.7. Definition. A topological space X has the *countable chain condition* (c.c.c.) iff there is no uncountable family of pairwise disjoint non-empty open subsets of X. □

The word "chain" usually refers to a total ordering; for an equivalent of c.c.c. in terms of chains, see Exercise 6.

If X is countable, X is trivially c.c.c. If X is uncountable then X may fail to be c.c.c.; for example, if X has the discrete topology, then the singletons form an uncountable disjoint family of open sets.

A non-trivial example of a c.c.c. space is the real numbers, $\mathbb{R}$. $\mathbb{R}$ has c.c.c. since it is *separable*; that is, it has a countable dense subspace. In general, the following holds.

1.8. Lemma. *If X is separable, then X has* c.c.c.

Proof. Let $D \subset X$ be dense and countable. If U_α $(\alpha < \omega_1)$ were open, non-empty, and disjoint, then we could pick $d_\alpha \in U_\alpha \cap D$. Then the d_α would all be distinct, a contradiction. □

A natural question to ask about a topological property is whether it is preserved under products. For example, the famous Tychonov theorem says that compactness is preserved under arbitrary products. The product of two separable spaces is separable since if D is dense in X and E is dense in Y then $D \times E$ is dense in $X \times Y$. However, separability is not preserved under arbitrary products (it is if there are $\leq 2^\omega$ factors, but not if there are $(2^\omega)^+$ factors; see Exercises 3 and 4).

The question of whether the product of any two c.c.c. spaces must be c.c.c. is independent of ZFC; this is true under MA + $\neg$CH (see 2.24), but it fails if CH holds (see VIII Exercise 8) or if Suslin's Hypothesis fails (see Lemma 4.3). The c.c.c. has the strange property that if it is preserved by products with two factors, it is preserved by arbitrary products. To see this, first note that if the product of any two c.c.c. spaces is c.c.c., then by induction we see that the product of any n c.c.c. spaces is c.c.c. $(n \in \omega)$. We now pass immediately to arbitrary products by the following.

1.9. Theorem. *Suppose X_i $(i \in I)$ are spaces such that for every finite $r \subset I$, $\prod_{i \in r} X_i$ is* c.c.c. *Then $\prod_{i \in I} X_i$ is* c.c.c.

Proof. Suppose U_α for $\alpha < \omega_1$ were pairwise disjoint non-empty sets in $\prod_{i \in I} X_i$. By shrinking the U_α if necessary, we may assume that each U_α is a basic open set. Then U_α depends on a finite set of coordinates, $a_\alpha \subset I$. By the Δ-system lemma there is an uncountable $A \subset \omega_1$ such that $\{a_\alpha : \alpha \in A\}$ forms a Δ-system with some root, r. r cannot be empty, since $a_\alpha \cap a_\beta = 0$ implies $U_\alpha \cap U_\beta \neq 0$. Let $\pi(U_\alpha)$ be the projection of U_α onto $\prod_{i \in r} X_i$. Then the $\pi(U_\alpha)$ for $\alpha \in A$ form an uncountable disjoint family in $\prod_{i \in r} X_i$. □

A simple example of Theorem 1.9 is that ${}^\kappa 2$ is c.c.c. for any κ, where 2 is the space $\{0, 1\}$ with the discrete topology. When $\kappa > 2^\omega$, this is an example of a c.c.c. space which is not separable (by Exercise 4).

§2. Martin's Axiom

If CH fails, there arises a large number of questions about the various infinite cardinals $\kappa < 2^\omega$. Some of these questions are of a purely combinatorial nature, such as

Question 1. If $\kappa < 2^\omega$, does $2^\kappa = 2^\omega$?

Question 2. If $\kappa < 2^\omega$, does every a.d. family $\mathscr{A} \subset \mathscr{P}(\omega)$ of size κ fail to be maximal?

Other questions come from measure theory, such as

Question 3. If $\kappa < 2^\omega$, does the union of κ subsets of $\mathbb{R}$, each of Lebesgue measure 0, have measure 0?

There is also the analogous question about category:

Question 4. If $\kappa < 2^\omega$, does the union of κ first-category subsets of $\mathbb{R}$ have first category?

Since these questions all clearly have answer "yes" when $\kappa = \omega$, they are only of interest for $\omega < \kappa < 2^\omega$. For such κ, it is known by the method of forcing that none of the questions can be settled under the axioms ZFC + $\neg$CH.