We show that $\mathscr{F}^+_{\mathscr{A}}$ is not a P-family: Let $k_0 := 0$ and let $x_0 := \omega$ be the first move of the MAIDEN, and let s_0 be DEATH's response. In general, if s_n is DEATH's n^{th} move, then the MAIDEN chooses k_{n+1} such that $k_{n+1} \geq \max(s_n)$, $|t_{k_{n+1}}| = n+1$, and $t_{k_n} \subseteq t_{k_{n+1}}$, and then she plays

$$x_{n+1} = \{i \in \omega : t_{k_{n+1}} \subseteq s_i\}.$$

Obviously, for every $n \in \omega$ we have $x_{n+1} \subsetneq x_n$. Moreover, all moves of the MAIDEN are legal:

CLAIM. *For every* $n \in \omega$, $x_n \in \mathscr{F}^+_{\mathscr{A}}$.

PoC Firstly, for every $n \in \omega$, x_n has infinite intersection with infinitely many members of $\mathscr{A}_0$. Indeed, $x_n \cap x_f$ is infinite whenever $f|_n = t_{k_n}$. Secondly, for every $z \in \mathscr{F}_{\mathscr{A}}$ there are finitely many $y_0, \dots, y_k \in \mathscr{A}$ such that $(y_0 \cup \dots \cup y_k)^{\mathsf{c}} \subseteq^* z$. Now, for x_n let $x_f \in \mathscr{A}_0 \setminus \{y_0, \dots, y_k\}$ such that $x_f \cap x_n$ is infinite. Then, since $x_f \cap (y_0 \cup \dots \cup y_k)$ is finite, $x_f \subseteq^* z$. Hence, $x_n \cap z$ is infinite which shows that $x_n \in \mathscr{F}^+_{\mathscr{A}}$. $\dashv_{\text{Claim}}$

By the MAIDEN's strategy, $\bigcup_{n\in\omega} t_{k_n} = f$ for some particular function $f \in {}^\omega\omega$. Moreover, $\bigcup_{n\in\omega} s_n \subseteq x_f \in \mathscr{A}_0$, and since subsets of members of $\mathscr{A}_0$ do not belong to $\mathscr{F}^+_{\mathscr{A}}$, $\bigcup_{n\in\omega} s_n \notin \mathscr{F}^+_{\mathscr{A}}$. Hence, DEATH loses the game, no matter what he is playing, which shows that the MAIDEN has a winning strategy in the game $\mathcal{G}^*_{\mathscr{F}^+_{\mathscr{A}}}$. In other words, the happy family $\mathscr{F}^+_{\mathscr{A}}$ is not a P-family. $\dashv$

The Rudin–Keisler Ordering of Ultrafilters over ω

In this section, we introduce an ordering on the set of all ultrafilters over ω. For this, we first define the image of an ultrafilter under a function $f : \omega \to \omega$.

For $f \in {}^\omega\omega$ and an ultrafilter $\mathscr{V} \subseteq \mathscr{P}(\omega)$, let

$$f(\mathscr{V}) := \big\{x \subseteq \omega : \exists y \in \mathscr{V}\big(f[y] \subseteq x\big)\big\}.$$

We leave it as an exercise to the reader to show that

$$f(\mathscr{V}) = \big\{x \subseteq \omega : f^{-1}[x] \in \mathscr{V}\big\},$$

where $f^{-1}[x] := \{n \in \omega : f(n) \in x\}$.

FACT 11.20. *If* $\mathscr{V} \subseteq \mathscr{P}(\omega)$ *is an ultrafilter over* ω *and* $\mathscr{U} = f(\mathscr{V})$, *then* $\mathscr{U}$ *is also an ultrafilter over* ω.

Proof. Since $f^{-1}[\omega] = \omega$, we get $\omega \in \mathscr{U}$, and since $f^{-1}[\emptyset] = \emptyset$, we get $\emptyset \notin \mathscr{U}$.

If $x \subseteq x'$ and $x \in f(\mathscr{V})$ (*i.e.*, $x \in \mathscr{U}$), then $f[y_0] \subseteq x$ for some $y_0 \in \mathscr{V}$, and therefore $f[y_0] \subseteq x'$, which shows that $x' \in f(\mathscr{V})$ (*i.e.*, $x' \in \mathscr{U}$).

If $x, x' \in f(\mathscr{V})$ (*i.e.*, $x, x' \in \mathscr{U}$), then $f^{-1}[x], f^{-1}[x'] \in \mathscr{V}$, and since $\mathscr{V}$ is an ultrafilter, $\big(f^{-1}[x] \cap f^{-1}[x']\big) \in \mathscr{V}$. Now, since $f^{-1}[x] \cap f^{-1}[x'] = f^{-1}[x \cap x']$, we get $x \cap x' \in f(\mathscr{V})$ (*i.e.*, $x \cap x' \in \mathscr{U}$). ⊣

The so-called **Rudin–Keisler ordering** "$\leq_{RK}$" on the set of ultrafilters over ω is now defined as follows:

$$\mathscr{U} \leq_{RK} \mathscr{V} :\iff \exists f \in {}^{\omega}\omega \big(\mathscr{U} = f(\mathscr{V})\big)$$

Furthermore, for ultrafilters $\mathscr{U}, \mathscr{V} \subseteq \mathscr{P}(\omega)$ we define

$$\mathscr{U} \equiv_{RK} \mathscr{V} :\iff \mathscr{U} = f(\mathscr{V}) \quad \text{for some bijection } f \in {}^{\omega}\omega.$$

FACT 11.21. (a) *The relation "$\leq_{RK}$" is reflexive and transitive.*

(b) *The relation "$\equiv_{RK}$" is an equivalence relation on the set of ultrafilters over ω.*

Proof. (a) For the identity function $\iota : \omega \to \omega$ we obviously have $\iota(\mathscr{U}) = \mathscr{U}$, hence, $\mathscr{U} \leq_{RK} \mathscr{U}$. Furthermore, if $f(\mathscr{W}) = \mathscr{V}$ and $g(\mathscr{V}) = \mathscr{U}$ for some functions $f, g \in {}^{\omega}\omega$, then $g \circ f(\mathscr{W}) = \mathscr{U}$, hence, $\mathscr{U} \leq_{RK} \mathscr{V}$ and $\mathscr{V} \leq_{RK} \mathscr{W}$ implies $\mathscr{U} \leq_{RK} \mathscr{W}$.

(b) Notice that if $f, g \in {}^{\omega}\omega$ are bijections, then f^{-1}, g^{-1}, and $f \circ g$ are also bijections. From this observation it follows easily that the relation "$\equiv_{RK}$" is reflexive, symmetric, and transitive (*e.g.*, if $f(\mathscr{U}) = \mathscr{V}$, where f is a bijection, then $f^{-1}(\mathscr{V}) = \mathscr{U}$). ⊣

The following lemma will be crucial in the proof of THEOREM 11.23.

LEMMA 11.22. *For any ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ and any function $f \in {}^{\omega}\omega$ we have*

$$f(\mathscr{U}) = \mathscr{U} \longrightarrow \{n \in \omega : f(n) = n\} \in \mathscr{U}.$$

Proof. Let $f \in {}^{\omega}\omega$ be an arbitrary but fixed function and let $\mathscr{U} \subseteq \mathscr{P}(\omega)$ be an ultrafilter such that $f(\mathscr{U}) = \mathscr{U}$. We consider the following three sets:

$$\begin{aligned} D &:= \{n \in \omega : f(n) < n\} \quad \text{(decreasing)} \\ E &:= \{n \in \omega : f(n) = n\} \quad \text{(equal)} \\ I &:= \{n \in \omega : f(n) > n\} \quad \text{(increasing)} \end{aligned}$$

Since $\mathscr{U}$ is an ultrafilter, exactly one of the sets D, E, I belongs to $\mathscr{U}$. If $E \in \mathscr{U}$, then we are done. So, we have to show that neither D nor I belongs to $\mathscr{U}$.

Assume towards a contradiction that $D \in \mathscr{U}$. Then for every $n \in D$ we consider the sequence $\langle f^k(n) : k \in \omega \rangle$ where $f^0(n) := n$ and $f^{k+1}(n) := f\big(f^k(n)\big)$. By the definition of D, for every $n \in D$ there is a least $k_n \in \omega$ such that $f^{k_n}(n) \notin D$. Then D is the disjoint union of the sets $D' := \{n \in D : k_n \text{ is odd}\}$ and $D'' := \{n \in D : k_n \text{ is even}\}$, and since $\mathscr{U}$ is an ultrafilter and by assumption $D \in \mathscr{U}$, exactly one of these two sets belongs to $\mathscr{U}$. Now, since $f(D') = D''$ and $f(D'') = D'$, this is a contradiction to $f(\mathscr{U}) = \mathscr{U}$, which shows that $D \notin \mathscr{U}$.

So, assume towards a contradiction that $I \in \mathscr{U}$. Then for every $n \in I$ we consider again the sequence $\langle f^k(n) : k \in \omega \rangle$. If, for $n \in I$, there is a $k \in \omega$ such that $f^k(n) \notin I$, then let k_n be the least such number; otherwise, let $k_n := \omega$. Then I is the disjoint union of the sets $I_0 := \{n \in I : k_n \in \omega\}$ and $I_\omega := \{n \in I : k_n = \omega\}$. Since $\mathscr{U}$ is an ultrafilter and $I \in \mathscr{U}$ (by assumption), exactly one of the sets I_0 and I_ω belongs to $\mathscr{U}$. If $I_0 \in \mathscr{U}$, then exactly one of the sets $I_0' := \{n \in I_0 : k_n \text{ is odd}\}$ and $I_0'' := \{n \in I_0 : k_n \text{ is even}\}$ belongs to $\mathscr{U}$; but since $f(I_0') = I_0''$ and $f(I_0'') = I_0'$, this is a contradiction to $f(\mathscr{U}) = \mathscr{U}$. So, $I_0 \notin \mathscr{U}$, which implies that $I_\omega \in \mathscr{U}$. Now, for each $n \in I_\omega$ there exists a least number $m_n \in I_\omega$ such that there is a $k \in \omega$ with $f^k(m_n) = n$. Let $I_\omega' := \big\{n \in I_\omega : \exists k \in \omega\big(f^{2k+1}(m_n) = n\big)\big\}$ and $I_\omega'' := \big\{n \in I_\omega : \exists k \in \omega\big(f^{2k}(m_n) = n\big)\big\}$. Since the two sets I_ω' and I_ω'' are disjoint and their union is I_ω, either I_ω' or I_ω'' belongs to $\mathscr{U}$, but not both. Furthermore, we get $f(I_\omega') = I_\omega''$ and $f(I_\omega'') = I_\omega'$, which is again a contradiction to $f(\mathscr{U}) = \mathscr{U}$. So, I_ω also does not belong to $\mathscr{U}$, which shows that $I \notin \mathscr{U}$.

Since $\mathscr{U}$ is an ultrafilter and $D \cup E \cup I$ belongs to $\mathscr{U}$, but neither D nor I belongs to $\mathscr{U}$, we get that E belongs to $\mathscr{U}$, which completes the proof. ⊣

The following result shows that up to "$\equiv_{RK}$-equivalence", the Rudin–Keisler ordering "$\leq_{RK}$" is antisymmetric.

THEOREM 11.23. *For all ultrafilters $\mathscr{U}, \mathscr{V} \subseteq \mathscr{P}(\omega)$ we have*

$$(\mathscr{U} \leq_{RK} \mathscr{V} \wedge \mathscr{V} \leq_{RK} \mathscr{U}) \longrightarrow \mathscr{U} \equiv_{RK} \mathscr{V}.$$

Proof. Assume that $\mathscr{U} \leq_{RK} \mathscr{V}$ and $\mathscr{V} \leq_{RK} \mathscr{U}$ and let $f, g \in {}^\omega\omega$ be such that $f(\mathscr{V}) = \mathscr{U}$ and $g(\mathscr{U}) = \mathscr{V}$. Notice that $f \circ g(\mathscr{U}) = \mathscr{U}$. So, by LEMMA 11.22, there is an $x_0 \in \mathscr{U}$ such that for all $n \in x_0$, $f \circ g(n) = n$, *i.e.*, $f \circ g|_{x_0}$ is the identity function. Hence, $g|_{x_0}$ as well as $f|_{g[x_0]}$ is one-to-one, *i.e.*, f and g are both bijections between the sets x_0 and $g[x_0]$. Now, we show that there exists a set $x_0' \subseteq x_0$ in $\mathscr{U}$ such that $g|_{x_0'}$ can be extended to a bijection $\tilde{g} \in {}^\omega\omega$. If $|\omega \setminus x_0| = |\omega \setminus g[x_0]|$, take any bijection h between $\omega \setminus x_0$ and $\omega \setminus g[x_0]$. Then, for $x_0' := x_0$, $\tilde{g} := g \cup h$ has the required properties. Otherwise, the set x_0 must be infinite and we can split x_0 into two disjoint infinite parts x_0' and x_0'' where x_0' belongs to $\mathscr{U}$. In this case, take any bijection h between the two infinite sets $\omega \setminus x_0'$ and $\omega \setminus g[x_0']$ and let $\tilde{g} := g \cup h$.

Since $\tilde{g} \in {}^\omega\omega$ is a bijection, $x_0' \in \mathscr{U}$, $g(\mathscr{U}) = \mathscr{V}$, and $g|_{x_0'} = \tilde{g}|_{x_0'}$, we get that $\tilde{g}[x_0'] \in \mathscr{V}$. It remains to show that this implies $\tilde{g}(\mathscr{U}) = \mathscr{V}$. Since $g(\mathscr{U}) = \mathscr{V}$, we get

$$\bigl\{g[x] : x \in \mathscr{U}\bigr\} \subseteq \mathscr{V} \qquad \text{and} \qquad \bigl\{g^{-1}[y] : y \in \mathscr{V}\bigr\} \subseteq \mathscr{U}\,.$$

Furthermore, by construction of $\tilde{g}$ we have $g|_{x'_0} = \tilde{g}|_{x'_0}$. Now, for every $y \in \mathscr{V}$ let $y' := y \cap \tilde{g}[x'_0]$ and let $x' := \tilde{g}^{-1}[y']$. Then $y' \in \mathscr{V}$, $x' \in \mathscr{U}$, and $\tilde{g}[x'] \subseteq y$, which shows that $\tilde{g}(\mathscr{U}) = \mathscr{V}$. ⊣

For the sake of completeness we give the following

FACT 11.24. *For any ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ and any function $f \in {}^\omega\omega$ we have*

$$f(\mathscr{U}) \equiv_{\scriptscriptstyle RK} \mathscr{U} \;\longrightarrow\; \exists x \in \mathscr{U}\bigl(f|_x \text{ is one-to-one}\bigr).$$

Proof. Assume $f(\mathscr{U}) \equiv_{\scriptscriptstyle RK} \mathscr{U}$, where $f \in {}^\omega\omega$ and $\mathscr{U} \subseteq \mathscr{P}(\omega)$ is an ultrafilter. By definition of "$\equiv_{\scriptscriptstyle RK}$", there exists a bijection $g \in {}^\omega\omega$ such that $g\circ f(\mathscr{U}) = \mathscr{U}$. Hence, by LEMMA 11.22, there is an $x_0 \in \mathscr{U}$ such that $g\circ f|_{x_0}$ is the identity function, and since $g|_{f[x_0]}$ is one-to-one, $f|_{x_0}$ is also one-to-one. ⊣

So far, we have not seen an example of an ultrafilter $\mathscr{W} \subseteq [\omega]^\omega$ which is neither a P-point nor a Q-point. The following result gives now such an example.

THEOREM 11.25. *For any ultrafilters $\mathscr{U}, \mathscr{V} \subseteq [\omega]^\omega$ there is an ultrafilter $\mathscr{W} \subseteq [\omega]^\omega$, which is neither a P-point nor a Q-point, such that*

$$\mathscr{U} \leq_{\scriptscriptstyle RK} \mathscr{W} \quad \text{and} \quad \mathscr{V} \leq_{\scriptscriptstyle RK} \mathscr{W}\,.$$

Proof. In a first step we construct an ultrafilter $\mathscr{W} \subseteq [\omega]^\omega$ which is above $\mathscr{U}$ and $\mathscr{V}$, and in a second step we show that $\mathscr{W}$ is neither a P-point nor a Q-point.

Firstly, let

$$\mathscr{W}^* = \Bigl\{X \subseteq \omega \times \omega : \bigl\{a \in \omega : \{b \in \omega : \langle a, b\rangle \in X\} \in \mathscr{V}\bigr\} \in \mathscr{U}\Bigr\}.$$

Then $\mathscr{W}^*$ is a non-principal ultrafilter over $\omega \times \omega$. To see this, notice first that $\emptyset \notin \mathscr{W}^*$, that $\omega \times \omega \in \mathscr{W}^*$, that $\mathscr{W}^* \subseteq [\omega \times \omega]^\omega$ (this is because $\mathscr{U}, \mathscr{V} \subseteq [\omega]^\omega$), and that $X \in \mathscr{W}^*$ and $X \subseteq X' \subseteq \omega \times \omega$ implies $X' \in \mathscr{W}^*$. Furthermore, let $X_0 \subseteq \omega \times \omega$ be such that $X_0 \notin \mathscr{W}^*$. Then

$$\bigl\{a \in \omega : \{b \in \omega : \langle a, b\rangle \in X_0\} \in \mathscr{V}\bigr\} \notin \mathscr{U},$$

which implies, since $\mathscr{U}$ is an ultrafilter, that

$$\bigl\{a' \in \omega : \{b \in \omega : \langle a, b\rangle \in X_0\} \notin \mathscr{V}\bigr\} \in \mathscr{U},$$

and consequently, since $\mathscr{V}$ is an ultrafilter, we get

$$\bigl\{a' \in \omega : \{b' \in \omega : \langle a', b'\rangle \notin X_0\} \in \mathscr{V}\bigr\} \in \mathscr{U},$$

which shows that $(\omega \times \omega) \setminus X_0 \in \mathscr{W}^*$. Finally, let $j_0 : \omega \times \omega \to \omega$ be a bijection. Then $\mathscr{W} := \{j_0[X] : X \in \mathscr{W}^*\}$ is an ultrafilter over ω. In order to show that $\mathscr{W}$ is above both ultrafilters $\mathscr{U}$ and $\mathscr{V}$, we work with $\mathscr{W}^*$ and define the projections $\pi_{\mathscr{U}}$ and $\pi_{\mathscr{V}}$ by stipulating

$$\begin{array}{rccc} \pi_{\mathscr{U}}: & \mathscr{P}(\omega \times \omega) & \longrightarrow & \mathscr{P}(\omega) \\ & X & \longmapsto & \{a \in \omega : \exists b \in \omega(\langle a,b\rangle \in X)\} \\ \pi_{\mathscr{V}}: & \mathscr{P}(\omega \times \omega) & \longrightarrow & \mathscr{P}(\omega) \\ & X & \longmapsto & \{b \in \omega : \exists a \in \omega(\langle a,b\rangle \in X)\} \end{array}$$

We leave it as an exercise to the reader to show that $\mathscr{U} = \pi_{\mathscr{U}}[\mathscr{W}^*]$ and that $\mathscr{V} = \pi_{\mathscr{V}}[\mathscr{W}^*]$. Now, we define $f, g \in {}^{\omega}\omega$ by stipulating

$$\begin{aligned} f : \omega &\to \omega \\ n &\mapsto \pi_{\mathscr{U}}\big(\{{j_0}^{-1}(n)\}\big) \\ g : \omega &\to \omega \\ m &\mapsto \pi_{\mathscr{V}}\big(\{{j_0}^{-1}(m)\}\big) \end{aligned}$$

where j_0 is as above. Then, since $\{{j_0}^{-1}[z] : z \in \mathscr{W}\} = \mathscr{W}^*$ and $\mathscr{U} = \{\pi_{\mathscr{U}}(X) : X \in \mathscr{W}^*\}$, for every $x_0 \in \mathscr{U}$ there are $X_0 \in \mathscr{W}^*$ and $z_0 \in \mathscr{W}$, such that $X_0 = {j_0}^{-1}[z_0]$ and $\pi_{\mathscr{U}}(X_0) = x_0$, *i.e.*, $\pi_{\mathscr{U}}\big({j_0}^{-1}[z_0]\big) = x_0$. Hence, $f[z_0] = x_0$ where $z_0 \in \mathscr{W}$, and since $x_0 \in \mathscr{U}$ was arbitrary, we get $f(\mathscr{W}) = \mathscr{U}$. This shows that $\mathscr{U} \leq_{\text{RK}} \mathscr{W}$—the relation $\mathscr{V} \leq_{\text{RK}} \mathscr{W}$ is shown similarly.

It remains to prove that $\mathscr{W}$ is neither a P-point nor a Q-point. We work again with the ultrafilter $\mathscr{W}^* \subseteq [\omega \times \omega]^\omega$ and show that $\mathscr{W}^*$ is neither a P-point nor a Q-point.

$\mathscr{W}^*$ is not a Q-point: Firstly, let

$$D := \{\langle a,b\rangle \in \omega \times \omega : a \leq b\}.$$

Notice that D belongs to $\mathscr{W}^*$. Now, define $\pi : \omega \times \omega \twoheadrightarrow D$ by stipulating

$$\pi(\langle a,b\rangle) = \begin{cases} \langle a,b\rangle & \text{if } a \leq b, \\ \langle a,a\rangle & \text{otherwise,} \end{cases}$$

and for each $m \in \omega$, let

$$u_m := \{\langle a, b \in \omega \times \omega : \pi(\langle a,b\rangle) = \langle a,m\rangle\}.$$

Then $\{u_m : m \in \omega\}$ is a partition of $\omega \times \omega$ where each u_m is finite—in fact, $|u_m| = 2m + 1$. Assume towards a contradiction that $\mathscr{W}^*$ is a Q-point. Then there is a $Y_Q \in \mathscr{W}^*$ such that for each $m \in \omega$, $|Y_Q \cap u_m| \leq 1$. Since $\mathscr{W}^*$ is an ultrafilter, $(Y_Q \cap D) \in \mathscr{W}^*$. Above we have seen that $\mathscr{V} = \pi_{\mathscr{V}}[\mathscr{W}^*]$, so, for $y_Q := \pi_{\mathscr{V}}(Y_Q \cap D)$

we get that $y_Q \in \mathscr{V}$. Furthermore, by definition of $\mathscr{W}^*$ and since $(Y_Q \cap D) \in \mathscr{W}^*$, for each $n_0 \in y_Q$ we get that the set

$$V_{n_0} := \{m \in \omega : \langle n_0, m\rangle \in (Y_Q \cap D)\}$$

belongs to the ultrafilter $\mathscr{V}$. Now, if n_0 and n_0' are distinct members of y_Q, then $V_{n_0} \cap V_{n_0'} \in \mathscr{V}$, in particular, $V_{n_0} \cap V_{n_0'}$ is non-empty. Let m_0 be an element of $V_{n_0} \cap V_{n_0'}$. Then $\langle n_0, m_0\rangle$ and $\langle n_0', m_0\rangle$ are two distinct elements of $Y_Q \cap D$ which both belong to u_{m_0}. So, $|Y_Q \cap u_{m_0}| \geq 2$, which contradicts our assumption and shows that $\mathscr{W}^*$ is not a Q-point.

$\mathscr{W}^*$ is not a P-point: For each $n \in \omega$, let

$$u_n := \{\langle n, m\rangle : m \in \omega\}.$$

Then $\{u_n : n \in \omega\}$ is a partition of $\omega \times \omega$. Assume towards a contradiction that there is an $X_P \in \mathscr{W}^*$ such that for each $n \in \omega$, $X_P \cap u_n$ is finite. Let $x_P := \pi_{\mathscr{U}}(X_P)$ be the projection of X_P. Then, since $X_P \in \mathscr{W}^*$, $x_P \in \mathscr{U}$. Now, since $\mathscr{V}$ contains only infinite sets and $X_P \cap u_n$ is finite for each $n \in \omega$, we get that for each $n_0 \in x_P$, $\{m \in \omega : \langle n_0, m\rangle \in X_P\}$ is finite and therefore does not belong to $\mathscr{V}$. Consequently, $X_P \notin \mathscr{W}^*$, which contradicts our assumption and shows that $\mathscr{W}^*$ is not a P-point. ⊣

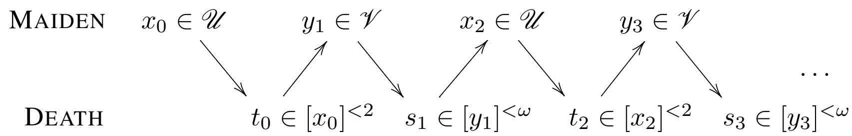
The next result shows that Ramsey ultrafilters are minimal with respect to the Rudin–Keisler ordering.

FACT 11.26. *If $\mathscr{U}, \mathscr{U}' \subseteq [\omega]^\omega$ are ultrafilters, where $\mathscr{U}$ is a Ramsey ultrafilter, then*

$$\mathscr{U}' \leq_{RK} \mathscr{U} \longrightarrow \mathscr{U} \equiv_{RK} \mathscr{U}'.$$

Proof. Assume that $\mathscr{U}' \leq_{RK} \mathscr{U}$, where $\mathscr{U}$ is a Ramsey ultrafilter. By definition of "$\leq_{RK}$", there exists a function $f \in {}^\omega\omega$, such that $f(\mathscr{U}) = \mathscr{U}'$, and since $\mathscr{U}$ is a Ramsey ultrafilter, by PROPOSITION 11.14 (c), there exists an $x \in \mathscr{U}$ such that $f|_x$ is constant or one-to-one. If $f|_x$ is constant, then the ultrafilter $f(\mathscr{U})$ would be principal, which contradicts the fact that $f(\mathscr{U}) = \mathscr{U}'$ and $\mathscr{U}' \subseteq [\omega]^\omega$. So, $f|_x$ is one-to-one. With similar arguments as in the proof of THEOREM 11.23 we find an $x' \subseteq x$ in $\mathscr{U}$ such that $f|_{x'}$ can be extended to a bijection $\bar{f} \in {}^\omega\omega$, such that $\bar{f}(\mathscr{U}) = \mathscr{U}'$, which shows that $\mathscr{U} \equiv_{RK} \mathscr{U}'$. ⊣

In order to state the following lemma—which will play a key role in the construction of a model of ZFC in which there are up to Rudin–Keisler equivalence just finitely many Ramsey ultrafilters (*cf.* PROPOSITION 27.6)—we first have to define a certain game: Let $\mathscr{U}, \mathscr{V} \subseteq [\omega]^\omega$ be two free families. Then the game $\mathcal{G}^{\mathscr{U}}_{\mathscr{V}}$ is the composition of the games $\mathcal{G}_{\mathscr{U}}$ and $\mathcal{G}^*_{\mathscr{V}}$, visualised by the following figure:



The rules for $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$ are as follows: For each $i \in \omega$, $x_{2i} \in \mathscr{U}$, $y_{2i+1} \in \mathscr{V}$, t_{2i} is either the empty set or a singleton $\{a_{2i}\}$ with $a_{2i} \in x_{2i}$, and s_{2i+1} is a finite subset of y_{2i+1}. Finally, DEATH wins the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$ if and only if $\bigcup\{t_{2i} : i \in \omega\} \in \mathscr{U}$ *and* $\bigcup\{s_{2i+1} : i \in \omega\} \in \mathscr{V}$.

LEMMA 11.27. *Let $\mathscr{U}$ be a Ramsey ultrafilter and $\mathscr{V}$ be a P-point. Then $\mathscr{U} \leq_{RK} \mathscr{V}$ if and only if the* MAIDEN *has a winning strategy in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$.*

Proof. ($\Rightarrow$) First we show that if $\mathscr{U} \leq_{RK} \mathscr{V}$, then the MAIDEN has a winning strategy σ in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. So, assume that $\mathscr{U} \leq_{RK} \mathscr{V}$ and let $f \in {}^{\omega}\omega$ be such that $f(\mathscr{V}) = \mathscr{U}$. Since $\mathscr{V}$ is a P-point, there exists a set $y_0 \in \mathscr{V}$ such that f is finite-to-one on y_0. Let $x_0 := f[y_0]$; then $x_0 \in \mathscr{U}$ and define $\sigma(\emptyset) := x_0$. Assume now that $t_0 \in [x_0]^{<2}$ is the first move of DEATH. Since f is finite-to-one on y_0, $f^{-1}[t_0] \cap y_0$ is finite. Let

$$y_1 := y_0 \setminus \big(\max(f^{-1}[t_0] \cap y_0) + 1\big)$$

and define $\sigma(\emptyset, x_0, t_0) := y_1$. Assume that $s_1 \in [y_1]^{<\omega}$ is the second move of DEATH. Then let

$$x_2 := f[y_1] \setminus \big(\max(f[s_1]) + 1\big)$$

and define $\sigma(\emptyset, x_0, t_0, y_1, s_1) := x_2$. The next moves of the MAIDEN are

$$y_3 := y_1 \setminus \big(\max(f^{-1}[t_2] \cap y_1) + 1\big) \quad \text{and} \quad x_4 := f[y_3] \setminus \big(\max(f[s_3]) + 1\big),$$

respectively. Proceeding in this way we finally get

$$\bigcup_{i\in\omega} t_{2i} \in \mathscr{U} \quad \iff \quad \bigcup_{i\in\omega} s_{2i+1} \notin \mathscr{V},$$

which shows that DEATH loses the game whenever the MAIDEN plays according to the strategy σ—no matter what he plays. Hence, σ is a winning strategy for the MAIDEN.

($\Leftarrow$) By contraposition we show that if $\mathscr{U} \not\leq_{RK} \mathscr{V}$, then no strategy σ for the MAIDEN is a winning strategy. For this we first combine the proofs of THEOREM 11.17 (a) & (b) and then use the premise that $\mathscr{U} \not\leq_{RK} \mathscr{V}$.

Let σ be any strategy for the MAIDEN in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. We have to show that DEATH can win. Let $x_0 := \sigma(\emptyset)$ (*i.e.*, $x_0 \in \mathscr{U}$), let $X_0 := \{x_0\}$, and for positive integers n, $x \in X_n$ if and only if for some $k < n$ there are $t_0, t_2, \ldots, t_{2k} \subseteq n$ and $s_1, s_3, \ldots, s_{2k+1} \subseteq n$ such that $x = \sigma(x_0, t_0, y_1, \ldots, s_{2k+1})$, where for all $i \leq k$ we have:

$$t_{2i} \in [x_{2i}]^{<2} \quad \text{where} \quad x_{2i} = \sigma(x_0, t_0, y_1 \ldots, s_{2i-1}),$$

and

$$s_{2i+1} \in [y_{2i+1}]^{<\omega} \quad \text{where} \quad y_{2i+1} = \sigma(x_0, t_0, y_1 \ldots, t_{2i}).$$

Similarly, for $n \in \omega$ we define Y_n by stipulating $y \in Y_n$ if and only if for some $k \leq n$ there are $t_0, t_2, \ldots, t_{2k} \subseteq n$ and $s_1, s_3, \ldots, s_{2k-1} \subseteq n$ such that $y = \sigma(x_0, t_0, y_1, \ldots, t_{2k})$, where for all $i \leq k$ we have:

$$t_{2i} \in [x_{2i}]^{<2} \quad \text{where} \quad x_{2i} = \sigma(x_0, t_0, y_1 \ldots, s_{2i-1}),$$

and

$$s_{2i-1} \in [y_{2i-1}]^{<\omega} \quad \text{where} \quad y_{2i-1} = \sigma(x_0, t_0, y_1 \ldots, t_{2i-2}).$$

Recall that by the rules of the game, DEATH can always play $\emptyset$. Clearly, for every $n \in \omega$, both sets X_n and Y_n are finite subsets of $\mathscr{U}$ and $\mathscr{V}$, respectively. Hence, for each $n \in \omega$, $\bigcap X_n \in \mathscr{U}$ and $\bigcap Y_n \in \mathscr{V}$. Moreover, since both ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are P-points, there are sets $x^* \in \mathscr{U}$ and $y^* \in \mathscr{V}$, and a strictly increasing function $f \in {}^\omega\omega$ with $f(0) > 0$ such that for all $n \in \omega$,

$$x^* \setminus f(n) \subseteq \bigcap X_n \quad \text{and} \quad y^* \setminus f(n) \subseteq \bigcap Y_n.$$

Let $k_0 := f(0)$, and in general, for $m \in \omega$, let $k_{m+1} := f(k_m)$. Furthermore, for $m \in \omega$, let $u_m := [k_m, k_{m+1})$. Since $\mathscr{U}$ is a Ramsey ultrafilter, there is a set $x = \{a_m : m \in \omega\}$ in $\mathscr{U}$ such that for each $m \in \omega$, $u_m \cap x = \{a_m\}$. Define the two sets $\mathscr{S}, \mathscr{T} \subseteq [\omega]^\omega$ by stipulating

$$S \in \mathscr{S} :\iff \{a_m : m \in S\} \in \mathscr{U},$$

$$T \in \mathscr{T} :\iff \bigcup\{u_m : m \in T\} \in \mathscr{V}.$$

Notice that for any $S, S' \in \mathscr{S}$ we have $S \cap S' \in \mathscr{S}$, in particular, $S \cap S' \in [\omega]^\omega$; similarly for $T, T' \in \mathscr{T}$. In fact, since $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters, $\mathscr{S}$ and $\mathscr{T}$ are ultrafilters, too. We show now that due to the fact that $\mathscr{U} \not\leq_{RK} \mathscr{V}$, the two ultrafilters $\mathscr{S}$ and $\mathscr{T}$ can be separated. For this we prove the following two claims.

CLAIM 1. *There are $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $S \cap T = \emptyset$.*

Proof of Claim 1. If there are $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $S \cap T$ is finite, then $S' = S \setminus (S \cap T)$ is in $\mathscr{S}$ and $S' \cap T = \emptyset$. So, assume towards a contradiction that for all $S \in \mathscr{S}$ and $T \in \mathscr{T}$ we have $|S \cap T| = \omega$.

First we show that this implies that for all $S \in \mathscr{S}$ and $T \in \mathscr{T}$, $S \cap T \in \mathscr{S} \cap \mathscr{T}$, and consequently we get $\mathscr{S} = \mathscr{T}$. Indeed, if $S_0 \cap T_0 \notin \mathscr{S}$ for some $S_0 \in \mathscr{S}$ and $T_0 \in \mathscr{T}$, then $S_0' := \omega \setminus (S_0 \cap T_0)$ belongs to $\mathscr{S}$, and since $\mathscr{S}$ is a filter, $S_0' \cap S_0 \in \mathscr{S}$. Hence, $(S_0' \cap S_0) \cap T_0 = S_0' \cap (S_0 \cap T_0) = \emptyset$ and for $S := S_0' \cap S_0$ in $\mathscr{S}$ and $T = T_0$ in $\mathscr{T}$ we have $S \cap T = \emptyset$, which contradicts our assumption.

Now we show that $\mathscr{S} = \mathscr{T}$ implies $\mathscr{U} \leq_{RK} \mathscr{V}$, which contradicts the fact that $\mathscr{U} \nleq_{RK} \mathscr{V}$: Let $g \in {}^{\omega}\omega$ be such that for all $m \in \omega$ we have $g[u_m] := \{a_m\}$. Then for each $y \in \mathscr{V}$ we get $g[y] \in \mathscr{U}$. To see this, notice that the set $\{m \in \omega : y \cap u_m \neq \emptyset\}$ belongs to $\mathscr{T}$ and therefore, by the definition of g and since $\mathscr{S} = \mathscr{T}$, we get $g[y] \in \mathscr{U}$. So, $g(\mathscr{V}) = \mathscr{U}$, which implies that $\mathscr{U} \leq_{RK} \mathscr{V}$. $\dashv_{\text{Claim 1}}$

CLAIM 2. *There are $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $S \cap T = \emptyset$ and for all distinct $m, m' \in S \cup T$, $|m - m'| \geq 2$, where $|m - m'|$ denotes the absolute value of the difference $m - m'$.*

Proof of Claim 2. By CLAIM 1 there are $\tilde{S} \in \mathscr{S}$ and $\tilde{T} \in \mathscr{T}$ such that $\tilde{S} \cap \tilde{T} = \emptyset$. Let $A := \{2k : k \in \omega\}$ and $B := \{2k+1 : k \in \omega\}$. Then either the set $\tilde{S} \cap A$ or the set $\tilde{S} \cap B$ belongs to $\mathscr{S}$; similarly, either the set $\tilde{T} \cap A$ or the set $\tilde{T} \cap B$ belongs to $\mathscr{T}$. Without loss of generality, let us assume $\tilde{S} \cap A \in \mathscr{S}$.

If $\tilde{T} \cap A \in \mathscr{T}$, let $S_0 := \tilde{S} \cap A$ and $T_0 := \tilde{T} \cap A$. Then $S_0 \in \mathscr{S}$, $T_0 \in \mathscr{T}$, and because S and T are disjoint, S_0 and T_0 are disjoint subsets of A and for all distinct $m, m' \in S_0 \cup T_0$ we have $|m - m'| \geq 2$.

If $\tilde{T} \cap A \notin \mathscr{T}$, then $\tilde{T} \cap B \in \mathscr{T}$. Now, by the definition of $\mathscr{S}$ and $\mathscr{T}$, and since $\mathscr{U}$ and $\mathscr{V}$ are filters, the sets

$$x_A := \{a_{2k} : k \in \omega\} \qquad \text{and} \qquad y_B := \bigcup\{u_{2k+1} : k \in \omega\}$$

belong to $\mathscr{U}$ and $\mathscr{V}$, respectively. Let $g_+, g_- \in {}^{\omega}\omega$ be functions such that for all $k \in \omega$ we have: $g_+[u_{2k+1}] := \{a_{2k+2}\}$, $g_-[u_{2k+1}] := \{a_{2k}\}$, and $g_+[u_{2k+1}] = g_-[u_{2k}] := \{0\}$. In particular, we get $g_+[y_B] = x_A \setminus \{a_0\}$ and $g_-[y_B] = x_A$, *i.e.*, both sets $g_+[y_B]$ and $g_-[y_B]$ belong to $\mathscr{U}$. On the other hand, since $\mathscr{U} \nleq_{RK} \mathscr{V}$, we have that neither $g_+(\mathscr{V}) = \mathscr{U}$ nor $g_-(\mathscr{V}) = \mathscr{U}$. Hence, there are $y_+, y_- \in [y_B]^{\omega}$ which belong to $\mathscr{V}$ such that neither $g_+[y_+]$ nor $g_-[y_-]$ belongs to $\mathscr{U}$. So, for $\bar{y} := y_+ \cap y_-$ we get that $\bar{y} \subseteq y_B$, $\bar{y} \in \mathscr{V}$, and

$$g_+[\bar{y}] \notin \mathscr{U} \quad \text{and} \quad g_-[\bar{y}] \notin \mathscr{U} \,.$$

Now, since $\mathscr{U}$ is an ultrafilter and $g_+[\bar{y}] \notin \mathscr{U}$, we get $(\omega \setminus g_+[\bar{y}]) \in \mathscr{U}$, which implies that $x_+ := x_A \cap (\omega \setminus g_+[\bar{y}])$ belongs to $\mathscr{U}$; similarly, we get that $x_- := x_A \cap (\omega \setminus g_-[\bar{y}])$ belongs to $\mathscr{U}$. For $\bar{x} := x_+ \cap x_-$ we get $\bar{x} \subseteq x_A$, $\bar{x} \in \mathscr{U}$, and

$$g_+[\bar{y}] \cap \bar{x} = \emptyset \quad \text{and} \quad g_-[\bar{y}] \cap \bar{x} = \emptyset \,.$$

With respect to $\bar{x}$ and $\bar{y}$, consider the two sets

$$S_0 := \big\{2k \in \omega : a_{2k} \in \bar{x}\big\} \quad \text{and} \quad T_0 := \big\{2k+1 \in \omega : \bar{y} \cap u_{2k+1} \neq \emptyset\big\} \,.$$

By definition, $S_0 \in \mathscr{S}$, $T_0 \in \mathscr{T}$, and $S_0 \cap T_0 = \emptyset$. Furthermore, if $n \in T_0$, then $n = 2k+1$ (for some $k \in \omega$) and $\bar{y} \cap u_{2k+1} \neq \emptyset$. Hence, by definition of g_+ and g_-,

$$a_{2k+2} \in g_+[\bar{y}] \quad \text{and} \quad a_{2k} \in g_-[\bar{y}] \,,$$

which implies that neither a_{2k+2} nor a_{2k} belongs to $\bar{x}$, and consequently neither $2k+2$ nor $2k$ belongs to S_0. In other words, if $n \in T_0$, then neither $n+1$ nor $n-1$ belongs to S_0, which shows that for all $m \in S_0$ and $n \in T_0$, $|m-n| \geq 2$. Furthermore, since $\bar{x} \subseteq x_A$, for any distinct $m, m' \in \bar{x}$ we have $|m-m'| \geq 2$. Similarly, since $\bar{y} \subseteq y_B$, for any distinct $m, m' \in \bar{y}$ we have $|m-m'| \geq 2$. Thus, $S_0 \cap T_0 = \emptyset$ and for all distinct $m, m' \in S_0 \cup T_0$ we have $|m-m'| \geq 2$, as required. $\dashv_{\text{Claim 2}}$

Let $S_0 \in \mathscr{S}$ and $T_0 \in \mathscr{T}$ be such that $S_0 \cap T_0 = \emptyset$ and for all distinct $m, m' \in S \cup T$, $|m-m'| \geq 2$. Consider the run $\langle x_0, t_0^*, y_1, s_1^*, \ldots \rangle$ of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$, where the MAIDEN plays according to her strategy σ and DEATH plays

$$t_{2n}^* := \begin{cases} \{a_{m+1}\} & \text{if } n = k_m,\ m+1 \in S_0, \text{ and } a_{m+1} \in x^*, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$s_{2n+1}^* = \begin{cases} y^* \cap u_{m+1} & \text{if } n = k_m \text{ and } m+1 \in T_0, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that $\bigcup_{n\in\omega} t_{2n}^* \in \mathscr{U}$ and that $\bigcup_{n\in\omega} s_{2n+1}^* \in \mathscr{V}$. In other words, the MAIDEN loses the game if the moves of DEATH satisfy the rules of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. To see this, notice first that for any $m \in \omega$ we have

$$x^* \setminus k_{m+1} = x^* \setminus f(k_m) \subseteq \bigcap X_{k_m} \subseteq \bigcap \{x_0, \ldots, x_{2k_m}\} \subseteq x_{2k_m}\,,$$

where $x_0, y_1 \ldots, x_{2k_m}$ are the moves played by the MAIDEN when DEATH plays $t_0^*, s_1^* \ldots, s_{2k_m-1}^*$; and

$$y^* \setminus k_{m+1} = y^* \setminus f(k_m) \subseteq \bigcap Y_{k_m} \subseteq \bigcap \{y_1, \ldots, y_{2k_m+1}\} \subseteq y_{2k_m+1}\,,$$

where $x_0, y_1 \ldots, x_{2k_m}, y_{2k_m+1}$ are the moves played by the MAIDEN when DEATH plays $t_0^*, s_1^* \ldots, t_{2k_m}^*$. By definition, for all $m \in \omega$, $t_{2k_m}^*$ and $s_{2k_m+1}^*$ are both subsets of k_{m+2}—in fact, they are subsets of $[k_{m+1}, k_{m+2})$. Now, recall that whenever $m+1 \in S_0$ ($m+1 \in T_0$), then $m+1 \notin T_0$ ($m+1 \notin S_0$) and neither $m \in S_0$ nor $m \in T_0$. In particular, if $m' < m$ and $m'+1, m+1 \in S_0 \cup T_0$, then $m' \leq m-2$. Hence, for $n = k_m$, $m' < m$, and $m+1 \in S_0 \cup T_0$, we get that $t_{2k_{m'}}^*$ and $s_{2k_{m'}+1}^*$ are both subsets of n (*e.g.*, if $m' = m-1$, then both sets $t_{2k_{m'}}^*$ and $s_{2k_{m'}+1}^*$ are empty). This shows that the moves of DEATH satisfy the rules of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$, which completes the proof. $\dashv$