Silver-Like Forcing Adds Splitting Reals

LEMMA 22.3. *Silver-like forcing $\mathbb{S}_{\mathscr{E}}$ adds splitting reals.*

Proof. For every function $f \in {}^{\omega}2$ we define the set $\sigma_f \subseteq \omega$ by stipulating:

$$n \in \sigma_f :\iff \sum_{j \in n^+} f(j) \equiv 1 \pmod{2}$$

Similarly, we define σ_f also for partial functions $f : E \to 2$, where $E \subseteq \omega$. Notice that if two functions $f, f' \in {}^{\omega}2$ differ at just a single place, say at $k_0 \in \omega$, then $\sigma_f \cap \sigma_{f'}$ is finite; in fact, $\sigma_f \cap \sigma_{f'} \subseteq k_0$.

Let $g \in {}^{\omega}2$ be a Silver real over $\mathbf{V}$. We are going to show that the set σ_g splits every real in the ground model. For this, it is enough to show that for each real $x \in [\omega]^{\omega}$ in the ground model $\mathbf{V}$ and for every $n \in \omega$, the set

$$D_{x,n} = \bigl\{q \in S_{\mathscr{E}} : q \Vdash_{\mathbb{S}_{\mathscr{E}}} \bigl(|x \cap \underset{\sim}{\sigma_g}| > n \wedge |x \setminus \underset{\sim}{\sigma_g}| > n\bigr)\bigr\}$$

is open dense in $S_{\mathscr{E}}$. It is clear that $D_{x,n}$ is open. In order to prove that $D_{x,n}$ is also dense, let $p \in S_{\mathscr{E}}$ be an arbitrary $S_{\mathscr{E}}$-condition. We have to construct an $S_{\mathscr{E}}$-condition $q \in D_{x,n}$ which is stronger than p. Firstly, let $C := \omega \setminus \mathrm{dom}(p)$ and notice that since $\mathscr{E}$ is a free family, C is infinite. Hence, we can choose two sets $A = \{a_i : i \in 2n\} \subseteq C$ and $B = \{b_i : i \in 2n\} \subseteq x$ such that

$$a_0 < b_0 < a_1 < \cdots < a_{2n-1} < b_{2n-1} \,.$$

Let $E := \mathrm{dom}(p) \cup b_{2n-1}^{+}$; then E contains both sets A and B. Now, we define a condition $q \in D_{n,x}$ with $q \geq p$ by induction on $2n$: Let $q_0 \in S_{\mathscr{E}}$ be such that $\mathrm{dom}(q_0) = E \setminus A$, $q_0|_{\mathrm{dom}(p)} = p$, and for all $m \in \mathrm{dom}(q_0) \setminus \mathrm{dom}(p)$, $q_0(m) = 0$. In particular, $B \subseteq \mathrm{dom}(q_0)$ and for each $k \in 2n$ we have $q_0(b_k) = 0$. If, for some $k \in 2n$, q_k is already defined, let $q_{k+1} \in S_{\mathscr{E}}$ be such that $\mathrm{dom}(q_{k+1}) = \mathrm{dom}(q_k) \cup \{a_k\}$, $q_{k+1}|_{\mathrm{dom}(q_k)} = q_k$, and

$$q_{k+1}(a_k) := \begin{cases} 0 & \text{if } k \in n \text{ and } b_k \in \sigma_{q_{k+1}} \cap b_k^{+}, \\ 0 & \text{if } k \notin n \text{ and } b_k \notin \sigma_{q_{k+1}} \cap b_k^{+}, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, let $q := q_{2n}$. Then by construction we have

$$q \Vdash_{\mathbb{S}_{\mathscr{E}}} \{b_0, \ldots, b_{n-1}\} \subseteq \underset{\sim}{\sigma_g} \ \wedge\ \{b_n, \ldots, b_{2n-1}\} \cap \underset{\sim}{\sigma_g} = \emptyset$$

and since $\{b_0, \ldots, b_{2n}\} \subseteq x$, this implies that $q \in D_{x,n}$. ⊣