A Model in Which $\mathfrak{hom} < \mathfrak{c}$

We first define a forcing notion which satisfies *ccc* and then show that an ω_1-iteration with countable support of this forcing notion, starting in a model in which $\mathfrak{c} > \omega_1$ yields a model in which $\omega_1 = \mathfrak{hom} < \mathfrak{c}$. The forcing notion we define is called **Laver forcing** restriced to some ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$, denoted $\mathbb{L}_{\mathscr{U}}$.

Before we introduce restricted Laver forcing $\mathbb{L}_{\mathscr{U}}$, we first fix some terminology. We shall identify $\text{seq}(\omega)$ (the set of finite sequences of ω) with $\bigcup_{n\in\omega} {}^n\omega$. Consequently, for $s \in \text{seq}(\omega)$ with $|s| = n+1$ we can write $s = \langle s(0), \ldots, s(n)\rangle$. Furthermore, for $s, t \in \text{seq}(\omega)$ with $|s| \leq |t|$ we write $s \preccurlyeq t$ if $t|_{|s|} = s$ (*i.e.*, s is an initial segment of t). Now, a set $T \subseteq \text{seq}(\omega)$ is a **tree**, if it is closed under initial segments, *i.e.*, $t \in T$ and $s \preccurlyeq t$ implies $s \in T$. For an $s \in \text{seq}(\omega)$ and an $n \in \omega$, $s^\frown n$ denotes the concatenation of the sequences s and $\langle n\rangle$. Furthermore, for a tree T and an $s \in T$ let

$$next_T(s) = \left\{n \in \omega : s^\frown n \in T\right\}.$$

A tree T is called a **Laver tree**, if there exists an $s \in T$, called the **stem** of T, such that for every $t \in T$ we have $t \preccurlyeq s$ or $s \preccurlyeq t$, and for every $t \in T$ with $s \preccurlyeq t$, the set $next_T(t)$ is infinite. Let now $\mathscr{U} \subseteq [\omega]^\omega$ be an ultrafilter. Then a tree T is called a **Laver tree restricted to** $\mathscr{U}$, if T is a Laver tree and for all $t \in T$ with $s \preccurlyeq t$, where s is the stem of T, we have $next_T(t) \in \mathscr{U}$. In addition we require that the sequences t in an $\mathbb{L}_{\mathscr{U}}$-condition T are strictly increasing, *i.e.*, if $m, n \in \text{dom}(t)$ and $m < n$, then $t(m) < t(n)$.

Now, for an arbitrary but fixed ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$, the conditions of restricted Laver forcing $\mathbb{L}_{\mathscr{U}}$ are Laver trees restricted to $\mathscr{U}$. For $\mathbb{L}_{\mathscr{U}}$-conditions T_s and $T_{s'}$ with stem s and s' respectively, we define

$$T_s \leq T_{s'} :\iff T_{s'} \subseteq T_s\,.$$

Notice that $T_s \leq T_{s'}$ implies that for all $t \in T_{s'}$ we have $next_{T_{s'}}(t) \subseteq next_{T_s}(t)$; in particular we get $s \preccurlyeq s'$.

As a first result we show that $\mathbb{L}_{\mathscr{U}}$ is σ-centred.

FACT 1.12. *Restricted Laver forcing $\mathbb{L}_{\mathscr{U}}$, where $\mathscr{U} \subseteq [\omega]^\omega$ is an ultrafilter, is σ-centred.*

Proof. Let T_s and T'_s be two $\mathbb{L}_{\mathscr{U}}$-conditions with the same stem s. Then, since $\mathscr{U}$ is an ultrafilter, $T_s \cap T'_s$ is an $\mathbb{L}_{\mathscr{U}}$-condition which is stronger than both, T_s and T'_s. Hence, any two $\mathbb{L}_{\mathscr{U}}$-conditions with the same stem are compatible, which implies, since the set of stems $s \in \text{seq}(\omega)$ is countable, that $\mathbb{L}_{\mathscr{U}}$ is σ-centred. ⊣

Now we show that forcing with $\mathbb{L}_{\mathscr{U}}$ adds a real, which is almost homogeneous with respect to all colourings $\pi : [\omega]^2 \to 2$ in the ground model.

LEMMA 1.13. *Let $\mathbf{V} \vDash$ ZFC, let $\mathscr{U} \subseteq [\omega]^\omega$ be an arbitrary but fixed ultrafilter, and let G be $\mathbb{L}_{\mathscr{U}}$-generic over $\mathbf{V}$. Furthermore, let $g := \bigcup\left\{s \in \text{seq}(\omega) : T_s \in G\right\}$ and* *let $H_g := g[\omega]$. Then for every colouring $\pi : [\omega]^2 \to 2$ in the ground model $\mathbf{V}$, H_g is almost homogeneous for π, i.e., there is an $n \in \omega$ such that $\pi|_{[H_g \setminus n]^2}$ is constant.*

Proof. Let T_s be an arbitrary $\mathbb{L}_{\mathscr{U}}$-condition with stem s and let $\pi : [\omega]^2 \to 2$ be a 2-colouring of $[\omega]^2$ in the ground model $\mathbf{V}$. Furthermore, for $s = \emptyset$ let $\bar{s} := 0$, otherwise, for $s \in {}^{n+1}\omega$ let $\bar{s} := s(n) + 1$. We will construct an $\mathbb{L}_{\mathscr{U}}$-condition $\hat{T}_s$ with the same stem as T_s, such that $\hat{T}_s \geq T_s$ and

$$\hat{T}_s \Vdash_{\mathbb{L}_{\mathscr{U}}} \pi|_{[H_g \setminus \bar{s}]^2} \text{ is constant}.$$

To do this, for every $m \in next_{T_s}(s)$ we define a colouring $\tau_m : next_{T_s}(s^\frown m) \to 2$ by stipulating

$$\tau_m(n) := \pi(\{m, n\}).$$

Recall that by the definition of $\mathbb{L}_{\mathscr{U}}$-conditions, $m < n$. Since $\mathscr{U}$ is an ultrafilter, for every $m \in next_{T_s}(s)$, either

$$H_{m,0} := \big\{n \in next_{T_s}(s^\frown m) : \tau_m(n) = 0\big\}$$

or

$$H_{m,1} := \big\{n \in next_{T_s}(s^\frown m) : \tau_m(n) = 1\big\}$$

belongs to $\mathscr{U}$. Furthermore, again since $\mathscr{U}$ is an ultrafilter, either

$$H_{s,0} := \big\{m \in next_{T_s}(s) : H_{m,0} \in \mathscr{U}\big\}$$

or

$$H_{s,1} := \big\{m \in next_{T_s}(s) : H_{m,1} \in \mathscr{U}\big\}$$

belongs to $\mathscr{U}$. Without loss of generality let us assume $H_{s,0} \in \mathscr{U}$. Then for all $m \in next_{T_s}(s) \cap H_{s,0}$ and $n \in next_{T_s}(s^\frown m) \cap H_{m,0}$ we have $\pi(\{m, n\}) = 0$.

In order to construct the tree $\hat{T}_s$, we just thin out the tree T_s by the following procedure: First, let

$$T_0 := \bigcup \big\{T \subseteq T_s : next_T(s) = H_{s,0} \wedge \forall m \in H_{s,0}\big(next_T(s^\frown m) = H_{m,0}\big)\big\}.$$

Then T is an $\mathbb{L}_{\mathscr{U}}$-condition and for every $s^\frown m^\frown n \in T_0$ we have

$$\pi(\{m, n\}) = 0. \qquad (*)$$

Now, let $A_s := H_{s,0}$ and for every $n_0 \in A_s$, let

$$A_{s^\frown n_0} := next_{T_0}(s^\frown n_0) \cap A_s.$$

Notice that for every $n_0 \in A_s$, $A_{s^\frown n_0} = H_{n_0,0} \cap H_{s,0}$. Furthermore, for every $n_1 \in A_{s^\frown n_0}$ we define

$$A_{s^\frown n_0^\frown n_1} := next_{T_0}(s^\frown n_0{}^\frown n_1) \cap A_{s^\frown n_0} \cap A_s.$$

In general, if $A_{s^\frown n_0^\frown \ldots ^\frown n_k}$ is already defined, then for every $n_{k+1} \in A_{s^\frown n_0^\frown \ldots ^\frown n_k}$ we define

$$A_{s^\frown n_0^\frown \ldots ^\frown n_{k+1}} := next_{T_0}(s^\frown n_0^\frown \ldots ^\frown n_{k+1}) \cap A_{s^\frown n_0^\frown \ldots ^\frown n_k} \cap A_{s^\frown n_0^\frown \ldots ^\frown n_{k-1}} \cap \cdots \cap A_{s^\frown n_0^\frown n_1} \cap A_{s^\frown n_0} \cap A_s \,.$$

Since T_0 is an $\mathbb{L}_{\mathscr{U}}$-condition and $\mathscr{U}$ is an ultrafilter, each set $A_{s^\frown n_0^\frown \ldots ^\frown n_k}$ defined above belongs to $\mathscr{U}$. If, for $t \in \operatorname{seq}(\omega)$, A_t is defined, then we say that t is admissible. Now, let

$$\hat{T}_s := \{t \in \operatorname{seq}(\omega) : A_t \text{ is defined}\} \,.$$

Then $\hat{T}_s$ is an $\mathbb{L}_{\mathscr{U}}$-condition with stem s and for each $t \in \hat{T}_s$ with $s \preccurlyeq t$ we have $next_{\hat{T}_s}(t) = A_t$.

Let now $h \in {}^\omega\omega$ be a branch through $\hat{T}_s$ (*i.e.*, for every $n \in \omega$, $h|_n \in \hat{T}_s$). Let $i, j \in \omega$ be such that $|s| < i < j$, and let $m := h(i)$ and $n := h(j)$. Then $\pi(\{m, n\}) = 0$. To see this, notice that by construction, $m \in next_{\hat{T}_s}(s)$ and $n \in next_{\hat{T}_s}(s^\frown n)$. Hence, $s^\frown m^\frown n \in \hat{T}_s$, and by $(*)$ we get $\pi(\{m, n\}) = 0$. In particular, for $H := h[\omega]$, we get that $\pi|_{[H \setminus \bar{s}]^2}$ is constant. So, by the definition of H_g, this shows that

$$\hat{T}_s \Vdash_{\mathbb{L}_{\mathscr{U}}} \pi|_{[H_g \setminus \bar{s}]^2} \text{ is constant} \,.$$

Since the $\mathbb{L}_{\mathscr{U}}$-condition T_s was arbitrary, we get the set of $\mathbb{L}_{\mathscr{U}}$-conditions

$$\big\{T : \exists k \in \omega \big(T \Vdash_{\mathbb{L}_{\mathscr{U}}} \pi|_{[H_g \setminus k]^2} \text{ is constant}\big)\big\}$$

is open dense, H_g is almost homogeneous for π. Finally, since the colouring π was arbitrary, H_g is almost homogeneous for every colouring $\pi : [\omega]^2 \to 2$ in the ground model $\mathbf{V}$. ⊣

Now we are ready to construct a model in which $\mathfrak{hom} < \mathfrak{c}$.

PROPOSITION 1.14. *$\mathfrak{hom} < \mathfrak{c}$ is consistent with* ZFC.

Proof. Let $\mathbf{V} \vDash \mathsf{ZFC} + \mathfrak{c} > \omega_1$ and let $\mathbb{P}_{\omega_1} = \langle \mathbb{Q}_\alpha : \alpha \in \omega_1 \rangle$ be an ω_1-stage iteration with finite support, where for each $\alpha \in \omega_1$, $\mathbb{Q}_\alpha$ is restricted Laver forcing $\mathbb{L}_{\mathscr{U}}$ for some ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ (*e.g.*, take the first ultrafilter with respect to some well-ordering defined in $\mathbf{V}$). Furthermore, let G be $\mathbb{P}_{\omega_1}$-generic over $\mathbf{V}$.

First recall that, by FACT 1.12, $\mathbb{L}_{\mathscr{U}}$ is σ-centred. Now, since σ-centred forcing notions satisfy *ccc* and since by PROPOSITION 1.8 any finite support iteration of *ccc* forcing notions satisfies *ccc*, we get that $\mathbb{P}_{\omega_1}$ satisfies *ccc*. Hence, by LEMMA **??**, $\mathbb{P}_{\omega_1}$ does not collapse cardinals which implies that $\mathbf{V}[G] \vDash \mathfrak{c} > \omega_1$.

By LEMMA 1.13, each forcing notion $\mathbb{Q}_\alpha$ (for $\alpha \in \omega_1$) adds a real $H_\alpha \in [\omega]^\omega$ which is almost homogeneous for all colourings $\pi : [\omega]^2 \to 2$ in $\mathbf{V}[G_\alpha]$, where G_α is $\mathbb{P}_\alpha$-generic over $\mathbf{V}$. Now, since by LEMMA 1.9 no new reals are added at stage ω_1, we get that in $\mathbf{V}[G]$, the set

$$\mathscr{H} := \bigcup \big\{ H_\alpha \setminus n : \alpha \in \omega_1 \wedge n \in \omega \big\}$$

has the property that for every colouring $\pi : [\omega]^2 \to 2$ in $\mathbf{V}[G]$, there is an $H \in \mathscr{H}$ which is homogeneous for π (*i.e.*, $\pi|_{[H]^2}$ is constant). Finally, since $|\mathscr{H}| = \omega_1$, this shows that $\mathbf{V}[G] \vDash \omega_1 = \mathfrak{hom}$.

Hence, in $\mathbf{V}[G]$ we have $\omega_1 = \mathfrak{hom} < \mathfrak{c}$, which completes the proof. $\dashv$