Preservation Theorems for Proper Forcing Notions

Below, we state some preservation theorems—most of them *without proofs*—for countable support iteration of proper forcing notions. These preservation theorems will be crucial in the following chapters, where we consider countable support iterations of length ω_2 of various proper forcing notions—usually starting with a model in which CH holds.

The first of these preservation theorems is concerned with ordinals (see also Bartoszyński and Judah [2, Lemma 1.4.16 & 1.4.18]).

THEOREM 1.3. *Let* $\mathbf{V} \vDash \mathsf{ZFC}$, *let* $\mathbb{P} = (P, \leq)$ *be a proper forcing notion, and let* χ *be some uncountable cardinal.*

(a) *Let* $\kappa \in \chi$, *let* $\underset{\sim}{\alpha}$ *be a* $\mathbb{P}$*-name for an ordinal, and let* $\mathbf{N} = (N, \in)$ *be a countable elementary submodel of* $(\mathrm{H}_\chi, \in)$ *which contains* $\mathbb{P}$, κ, $\underset{\sim}{\alpha}$, *et cetera. Furthermore, assume that*

$$\mathbf{N} \vDash \text{“}\mathbf{0} \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in \kappa\text{”}.$$

Then for every $\mathbf{N}$*-generic* $\mathbb{P}$*-condition* q *we have*

$$\mathbf{V} \vDash \text{“}q \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in \underset{\cdot}{N}\text{”}.$$

(b) *If* G *is* $\mathbb{P}$*-generic over* $\mathbf{V}$ *and* $A \in \mathbf{V}[G]$ *is a countable subset of* κ, *then there is a countable set of ordinals* $B \subseteq \kappa$, $B \in \mathbf{V}$, *such that* $\mathbf{V}[G] \vDash A \subseteq B$.

Proof. (a) Since $\underset{\sim}{\alpha} \in N$ and $\mathbf{N} \vDash$ “$\mathbf{0} \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in \kappa$”, the set

$$D := \big\{ r \in P : \exists \beta \in \kappa \big(\text{“}r \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} = \beta\text{”}\big)\big\}$$

is an open dense subset of $P \cap N$. Let $f \in N$ be a function with domain D, such that for each $r \in D$,

$$r \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} = f(r)\,.$$

Now, by definition of $\mathbf{N}$-generic, for every $\mathbf{N}$-generic condition q we have

$$q \Vdash_{\mathbb{P}} \exists r \in (D \cap N \cap \underset{\cdot}{G})\,.$$

Hence, $q \Vdash_{\mathbb{P}} \exists r \in D\big(\underset{\sim}{\alpha} = f(r)\big)$, which shows that $q \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in \underset{\cdot}{N}$.

(b) Let $\underset{\sim}{A}$ be a $\mathbb{P}$-name for A and without loss of generality let us assume

$$\mathbf{V} \vDash \text{“}\mathbf{0} \Vdash_{\mathbb{P}} \underset{\sim}{A} = \{\underset{\sim}{\alpha}_n : n \in \omega\} \wedge \forall n \in \omega\, (\underset{\sim}{\alpha}_n \in \kappa)\text{”}$$

for some countable set $\{\underset{\sim}{\alpha}_n : n \in \omega\}$ of $\mathbb{P}$-names. Let $\mathbf{N} = (N, \in)$ be a countable elementary submodel of $(\mathrm{H}_\chi, \in)$ containing $\mathbb{P}$ and $\underset{\sim}{\alpha}_n$ for each $n \in \omega$. Now, in $\mathbf{V}$ let $B := N \cap \kappa$. Then B is a countable subset of κ which belongs to $\mathbf{V}$ and by (a), for every $\mathbf{N}$-generic condition q and each $n \in \omega$ we have

$$q \Vdash_{\mathbb{P}} \underset{\sim}{\alpha}_n \in B$$

which shows that $\mathbf{V}[G] \vDash A \subseteq B \wedge B \subseteq \kappa \wedge |B| = \omega$. ⊣

As a consequence we get the following

COROLLARY 1.4. *If $\mathbb{P}$ is proper, then forcing with $\mathbb{P}$ does not collapse ω_1.*

Proof. Assume towards a contradiction that $\mathbb{P}$ collapses ω_1. Then, in $\mathbf{V}[G]$, $\omega_1^{\mathbf{V}}$ is a countable set of ordinals A. Now, by THEOREM 1.3.(b), A is contained in some countable set $B \in \mathbf{V}$. Hence, in $\mathbf{V}$, ω_1 is contained in some countable set, which is obviously a contradiction. ⊣

The following preservation theorem states that properness is preserved under countable support iteration of proper forcing notions (for proofs see Goldstern [6, Corollary 3.14] and Shelah [9, III.§3]).

THEOREM 1.5. *If $\mathbb{P}_\alpha$ is a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have $\mathbf{0}_\beta \Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is proper", then $\mathbb{P}_\alpha$ is proper.*

The following lemma is in fact just a consequence of COROLLARY 1.4.

LEMMA 1.6. *Let $\mathbb{P}_\alpha$ be a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have $\mathbf{0}_\beta \Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is a proper forcing notion of size $\leq\mathfrak{c}$". If* CH *holds in the ground model and $\alpha \leq \omega_2$, then for all $\beta \in \alpha$, $\mathbf{0}_\beta \Vdash_\beta$* CH.

Since, by LEMMA **??**, no new reals appear at the limit stage ω_2 one can prove the following theorem—a result which we shall use quite often in the forthcoming chapters.

THEOREM 1.7. *Let $\mathbb{P}_{\omega_2}$ be a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \omega_2 \rangle$, where for each $\beta \in \omega_2$ we have*

$$\mathbf{0}_\beta \Vdash_\beta \text{“} \underset{\sim}{\mathbb{Q}}_\beta \text{ is a proper forcing notion of size } \leq\mathfrak{c} \text{ which adds new reals”.}$$

Further, let $\mathbf{V}$ be a model of ZFC + CH *and let G be $\mathbb{P}_{\omega_2}$-generic over $\mathbf{V}$. Then we have*

(a) $\mathbf{V}[G] \vDash \mathfrak{c} = \omega_2$, *and*
(b) *for every set of reals $\mathscr{F} \subseteq [\omega]^\omega \cap \mathbf{V}[G]$ of size $\leq\omega_1$ there is a $\beta \in \omega_2$ such that $\mathscr{F} \subseteq \mathbf{V}[G|_\beta]$.*

Now, let us say a few words concerning preservation of the Laver property and of ${}^\omega\omega$-boundedness: It can be shown that a countable support iteration of proper ${}^\omega\omega$-bounding forcing notions is ${}^\omega\omega$-bounding (for a proof see Section 5 and Application 1 of Goldstern [6]).

THEOREM 1.8. *If $\mathbb{P}_\alpha$ is a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have $\mathbf{0}_\beta \Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is proper and ${}^\omega\omega$-bounding", then $\mathbb{P}_\alpha$ is ${}^\omega\omega$-bounding.*

Further, one can show that the Laver property is preserved under countable support iteration of proper forcing notions which have the Laver property (for a proof see Section 5 and Application 4 of Goldstern [6]).

THEOREM 1.9. *If $\mathbb{P}_\alpha$ is a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have $\mathbf{0}_\beta \Vdash_\beta$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is proper and has the Laver property", then $\mathbb{P}_\alpha$ has the Laver property.*

Another property which is preserved under countable support iteration of proper forcing notions is preservation of P-points: A forcing notion $\mathbb{P}$ is said to **preserve P-points** if for every P-point $\mathscr{U} \subseteq [\omega]^\omega$,

$$\mathbf{0} \Vdash_{\mathbb{P}} \text{“}\mathscr{U} \text{ generates an ultrafilter over } \omega\text{”},$$

i.e., for every set $x \in [\omega]^\omega$ in the $\mathbb{P}$-generic extension there exists a $y \in \mathscr{U}$ such that either $y \subseteq x$ or $y \subseteq \omega \setminus x$. Notice that we do not require that a P-point in the ground model generates a P-point in the extension—we just require that it generates an ultrafilter in the extension. However, in the case when $\mathbb{P}$ is proper, this is equivalent.

LEMMA 1.10. *Let $\mathbb{P}$ be a proper forcing notion. If $\mathbb{P}$ preserves P-points and $\mathscr{U}$ is a P-point in the ground model $\mathbf{V}$, then $\mathscr{U}$ generates a P-point in the $\mathbb{P}$-generic extension $\mathbf{V}[G]$.*

Proof. Let $\mathscr{U} \in \mathbf{V}$ be a P-point in the ground model $\mathbf{V}$ and let $\hat{\mathscr{U}} \in \mathbf{V}[G]$ be the ultrafilter in the $\mathbb{P}$-generic extension generated by $\mathscr{U}$. We have to show that $\hat{\mathscr{U}}$ is a P-point in $\mathbf{V}[G]$, *i.e.*, we have to show that for every countable set $\{x_n : n \in \omega\} \subseteq \hat{\mathscr{U}}$ there is a $y \in \hat{\mathscr{U}}$ such that for each $n \in \omega$, $y \subseteq^* x_n$. In $\mathbf{V}$, let $f : \mathscr{U} \to \mathfrak{c}$ be a bijection between $\mathscr{U}$ and $\mathfrak{c}$; and in $\mathbf{V}[G]$, let $\{x_n : n \in \omega\} \subseteq \hat{\mathscr{U}}$ be a countable set of elements of $\hat{\mathscr{U}}$ and let

$$A := \big\{ f(x_n) : n \in \omega \big\} .$$

Then $A \subseteq \mathfrak{c}$ is a countable set of ordinals, which is, by THEOREM 1.3.(b), contained in some countable set of ordinals $B \subseteq \mathfrak{c}$, where B belongs to $\mathbf{V}$. Now, let

$$\bar{B} := \big\{ f^{-1}(\beta) : \beta \in B \big\} .$$

Then $\bar{B} \subseteq \mathscr{U}$, $\bar{B} \in \mathbf{V}$, and $\bar{B}$ is countable. Since $\mathscr{U}$ is a P-point in $\mathbf{V}$, there is a $y \in (\mathscr{U} \cap \mathbf{V})$ such that for each $x \in \bar{B}$, $y \subseteq^* x$. By construction, for each $n \in \omega$ we have $y \subseteq^* x_n$. Hence, since $\{x_n : n \in \omega\} \subseteq \hat{\mathscr{U}}$ was arbitrary and $y \in \hat{\mathscr{U}}$, this shows that $\hat{\mathscr{U}}$ is a P-point in $\mathbf{V}[G]$. ⊣

One can show that preservation of P-points is preserved under countable support iteration of proper forcing notions (for a proof see Blass and Shelah [5] or Bartoszyński and Judah [2, Theorem 6.2.6]).

THEOREM 1.11. *If $\mathbb{P}_\alpha$ is a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have $\mathbf{0}_\beta \Vdash_\beta$ " $\underset{\sim}{\mathbb{Q}}_\beta$ is proper and preserves P-points", then $\mathbb{P}_\alpha$ preserves P-points.*

With respect to Ramsey ultrafilters we get similar results:

LEMMA 1.12. *Let $\mathbb{P}$ be a proper forcing notion which is ${}^\omega\omega$-bounding. Furthermore, let $\mathscr{U}$ be a Ramsey ultrafilter in the ground model $\mathbf{V}$ which generates an ultrafilter $\hat{\mathscr{U}}$ in the $\mathbb{P}$-generic extension $\mathbf{V}[G]$. Then $\hat{\mathscr{U}}$ is a Ramsey ultrafilter in $\mathbf{V}[G]$.*

Proof. By LEMMA 1.10, we already know that the ultrafilter $\hat{\mathscr{U}}$ in $\mathbf{V}[G]$ which is generated by $\mathscr{U}$ is a P-point in $\mathbf{V}[G]$. So, it remains to show that the P-point $\hat{\mathscr{U}}$ generated by $\mathscr{U}$ is also a Q-point. For this, we show that in $\mathbf{V}[G]$, for every partition $\{I_n \subseteq \omega : n \in \omega\}$ of ω into finite pieces, there is an $x \in \hat{\mathscr{U}}$ such that for each $n \in \omega$, $|x \cap I_n| \le 1$. By FACT **??** it is enough to consider interval partitions. So, let $\{I_n \subseteq \omega : n \in \omega\}$ be an arbitrary but fixed interval partition in $\mathbf{V}[G]$. For every $n \in \omega$, let $a_n := \max(I_n)$ and define $h \in {}^\omega\omega$ by stipulating $h(n) := a_n$. Notice that h is a function in $\mathbf{V}[G]$. Since $\mathbb{P}$ is assumed to be ${}^\omega\omega$-bounding, there is a function $f \in {}^\omega\omega$ in the ground model $\mathbf{V}$ which dominates h, *i.e.*, for each $n \in \omega$ we have $f(n) > h(n)$. Without loss of generality we may assume that f is strictly increasing. Let $k_0 := f(0)$, and for $i \in \omega$ let $k_{i+1} := f(k_i)$. In $\mathbf{V}$, we define the interval partition $\{J_m : m \in \omega\}$ by stipulating $J_0 := [0, k_0)$, and for each $m \in \omega$, $J_{m+1} := [k_m, k_{m+1})$. Notice that since f is strictly increasing, for all $m \in \omega$ we have $f[J_m] \subseteq J_{m+1}$. Furthermore, since f dominates h, for every $n \in \omega$ there are at most two consecutive integers m and $m+1$ such that the intersections $I_n \cap J_m$ and $I_n \cap J_{m+1}$ are both non-empty. Since $\mathscr{U}$ is a Q-point in $\mathbf{V}$, there is an $x \in \mathscr{U} \cap \mathbf{V}$ such that for each $m \in \omega$, $|x \cap J_m| \le 1$. Let

$$x_0 := \{x \cap J_{2m} : m \in \omega\} \quad \text{and} \quad x_1 := \{x \cap J_{2m+1} : m \in \omega\}.$$

Since $\mathscr{U}$ is an ultrafilter in $\mathbf{V}$, either x_0 or x_1 belongs to $\mathscr{U}$; let us assume $x_0 \in \mathscr{U}$. Now, in $\mathbf{V}[G]$, for all $n \in \omega$ we have $|x_0 \cap I_n| \le 1$, and since $x_0 \in \hat{\mathscr{U}}$ and $\{I_n : n \in \omega\}$ was an arbitrary interval partition, the ultrafilter $\hat{\mathscr{U}}$ generated by $\mathscr{U}$ is a Q-point in $\mathbf{V}[G]$. ⊣

For a particular Ramsey ultrafilter $\mathscr{U}_0$ we say that a forcing notion $\mathbb{P}$ **preserves $\mathscr{U}_0$**, if

$$\mathbf{0} \Vdash_{\mathbb{P}} \text{“}\mathscr{U}_0 \text{ generates a Ramsey ultrafilter”}.$$

As an immediate consequence of LEMMA 1.12, THEOREM 1.5 & 1.8, and THEOREM 1.11 with respect to $\mathscr{U}_0$, we get the following

COROLLARY 1.13. *Let $\mathscr{U}_0$ be a Ramsey ultrafilter in the ground model $\mathbf{V}$ and let $\mathbb{P}_\alpha$ be a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have*

$$\mathbf{0}_\beta \Vdash_\beta \text{“} \underset{\sim}{\mathbb{Q}}_\beta \text{ is proper, } {}^\omega\omega\text{-bounding, and preserves } \mathscr{U}_0 \text{”},$$

then also $\mathbb{P}_\alpha$ preserves $\mathscr{U}_0$.

As a last result, we show that non-isomorphic Q-points remain non-isomorphic after forcing with an ${}^\omega\omega$-bounding forcing notion.

LEMMA 1.14. *Let $\mathbb{P}$ be a forcing notion, let $\mathbf{V} \vDash \mathsf{ZFC}$, and let G be $\mathbb{P}$-generic over $\mathbf{V}$. Furthermore, let $\mathscr{U}$ and $\mathscr{V}$ be Q-points in $\mathbf{V}$ which generate ultrafilters $\hat{\mathscr{U}}$ and $\hat{\mathscr{V}}$ in $\mathbf{V}[G]$. If $\mathbf{V} \vDash \mathscr{U} \not\equiv_{\scriptscriptstyle RK} \mathscr{V}$ and $\mathbb{P}$ is ${}^\omega\omega$-bounding, then $\mathbf{V}[G] \vDash \hat{\mathscr{U}} \not\equiv_{\scriptscriptstyle RK} \hat{\mathscr{V}}$.*

Proof. By contraposition we show that if $\mathbf{V}[G] \vDash \hat{\mathscr{U}} \equiv_{\scriptscriptstyle RK} \hat{\mathscr{V}}$, then $\mathbf{V} \vDash \mathscr{U} \equiv_{\scriptscriptstyle RK} \mathscr{V}$. So, let $f \in {}^\omega\omega$ be a bijection in $\mathbf{V}[G]$ such that $f(\hat{\mathscr{V}}) = \hat{\mathscr{U}}$. Since $\mathbb{P}$ is ${}^\omega\omega$-bounding, there exists a strictly increasing function $g \in {}^\omega\omega$ in the ground model $\mathbf{V}$ which dominates f as well as f^{-1}, *i.e.*, for all $n \in \omega$ we have $f(n) < g(n) > f^{-1}(n)$. In $\mathbf{V}$, let $k_0 := g(0)$ and for $n \in \omega$ let $k_{n+1} := g(k_n)$. Furthermore, let $I_0 := [0, k_0)$, and for $n \in \omega$, let $I_{n+1} := [k_n, k_n + 1)$. Since $\mathscr{U}$ and $\mathscr{V}$ are Q-points in $\mathbf{V}$, there are sets $x \in \mathscr{U}$ and $y \in \mathscr{V}$, such that for all $n \in \omega$, $x \cap I_n = \{a_n\}$ and $y \cap I_n = \{b_n\}$. By construction of the interval partition $\{I_n : n \in \omega\}$—in fact since g dominates f and f^{-1}—for each $b_n \in y''$ we have

$$f(b_n) \in \{a_{n-1}, a_n, a_{n+1}\}\,. \tag{$*$}$$

In $\mathbf{V}[G]$ we have $x \in \hat{\mathscr{U}}$ and $y \in \hat{\mathscr{V}}$, and since $f(\hat{\mathscr{V}}) = \hat{\mathscr{U}}$, we get $f[y] \in \hat{\mathscr{U}}$. In particular we get $f[y] \cap x \in \hat{\mathscr{U}}$ and $f^{-1}\big[f[y] \cap x\big] \in \hat{\mathscr{V}}$. In $\mathbf{V}[G]$, let $x' := f[y] \cap x$ and $y' := f^{-1}[x']$. Since $\hat{\mathscr{V}}$ is generated by $\mathscr{V}$, there is a $y'' \in \mathbf{V}$ such that $y'' \subseteq y'$ and $y'' \in \mathscr{V}$. In $\mathbf{V}[G]$, consider the following three subsets of y'':

$$\begin{aligned} y_- &:= \big\{ b_n \in y'' : f(b_n) = a_{n-1} \big\} \\ y_0 &:= \big\{ b_n \in y'' : f(b_n) = a_n \big\} \\ y_+ &:= \big\{ b_n \in y'' : f(b_n) = a_{n+1} \big\} \end{aligned}$$

By $(*)$ we have $y_- \,\dot\cup\, y_0 \,\dot\cup\, y_+ = y''$ and since $y'' \in \hat{\mathscr{V}}$, exactly one of y_-, y_0, y_+ belongs to $\hat{\mathscr{V}}$, *i.e.*, exactly one of $f[y_-], f[y_0], f[y_+]$ belongs to $\hat{\mathscr{U}}$.

Let us just consider the case when $y_+ \in \hat{\mathscr{V}}$ *i.e.*, $f[y_+] \in \hat{\mathscr{U}}$; the two other cases are similar. In $\mathbf{V}$, define the function

$$\begin{aligned} g_+ : \quad y'' &\to \quad x' \\ b_n &\mapsto a_{n+1} \end{aligned}$$

and extend in $\mathbf{V}$ the function g_+ to a function $g_+^* \in {}^\omega\omega$. Then, since $f[y_+] \in \hat{\mathscr{U}}$ and $\hat{\mathscr{V}}$ is generated by $\mathscr{V}$, $g_+^*(\mathscr{V}) = \mathscr{U}$, which shows that $\mathbf{V} \vDash \mathscr{U} \equiv_{RK} \mathscr{V}$. ⊣

As an immediate consequence of COROLLARY 1.13, THEOREM 1.5 & 1.8, and LEMMA 1.14, we get the following

COROLLARY 1.15. *Let $\mathscr{U}$ and $\mathscr{V}$ be two Ramsey ultrafilters in the ground model $\mathbf{V}$ and assume $\mathbf{V} \vDash \mathscr{U} \not\equiv_{RK} \mathscr{V}$. Furthermore, let $\mathbb{P}_\alpha$ be a countable support iteration of $\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta \in \alpha \rangle$, where for each $\beta \in \alpha$ we have*

$$\mathbf{0}_\beta \Vdash_\beta \text{“ } \underset{\sim}{\mathbb{Q}}_\beta \text{ is proper, } {}^\omega\omega\text{-bounding, and preserves } \mathscr{U} \text{ and } \mathscr{V} \text{”},$$

and let G be $\mathbb{P}$-generic over $\mathbf{V}$. Then $\mathscr{U}$ and $\mathscr{V}$ generate Ramsey ultrafilters $\hat{\mathscr{U}}$ and $\hat{\mathscr{V}}$ in $\mathbf{V}[G]$ and $\mathbf{V}[G] \vDash \hat{\mathscr{U}} \not\equiv_{RK} \hat{\mathscr{V}}$.

There are many more preservation theorems for countable support iterations of proper forcing notions. However, what we presented here is all what we shall use in the forthcoming chapters.