Chapter 0
A Natural Approach to Natural Numbers

In the late 19th and early 20th century, several unsuccessful attempts were made to develop the natural numbers from logic. The most promising approaches were the ones due to Frege and Russell, but also their approaches failed at the end. Even though it seems impossible to develop the natural numbers just from logic, this does not justify Kronecker's ridiculous claim that the natural numbers are given by God.

In fact, the problem with the natural numbers is, that we need the notion of finiteness in order to define them, which presuppose the existence of a kind of infinite list of objects, and it is not clear whether these objects are—in some sense—not already the natural numbers which we would like to define.

However, in our opinion there is subtle distinction between the infinite set of natural numbers and an infinite list of objects, since the set of natural numbers is an *actually infinite* set, whereas an infinite list (in contrast for example to an infinite array) is just *potentially infinite*. The difference between these two types of infinity is, that the actual infinity is something which is completed and definite and consists of infinitely many elements. On the other hand, the potential infinity—introduced by Aristotle—is something that is always finite, even though more and more elements can be added to make it arbitrarily large. For example the set of prime numbers can be considered as an actually infinite set (as Cantor did), or just as a potentially infinite list of numbers without last element which is never completed (as Euclid did).

As mentioned above, it seems that there is no way to define the natural numbers just from logic. Hence, if we would like to define them, we have to make some assumptions which cannot be formalised within logic or mathematics in general. In other words, in order to define the natural numbers we have to presuppose some *metamathematical* notions like for example the notion of F I N I T E N E S S. To emphasise this fact, we shall use a wider letter spacing for the metamathematical notions we suppose.

So, let us assume that we all have a notion of F I N I T E N E S S. Let us further assume that we have two characters, say "0" and "s". With these characters, we build now the following finite strings:

$$0 \quad \mathsf{s}0 \quad \mathsf{ss}0 \quad \mathsf{sss}0 \quad \mathsf{ssss}0 \quad \mathsf{sssss}0 \quad \ldots$$

The three dots "..." mean that we always build the next string by appending on the left the character "s" to the string we just built. Proceeding this way, we get in fact a potentially infinite list $\mathbb{N}$ of different strings which is never completed. Thus, the list $\mathbb{N}$ is of the form

$$\mathbb{N} = [0, \mathsf{s}0\,, \mathsf{ss}0\,, \mathsf{sss}0\,, \mathsf{ssss}0\,, \mathsf{sssss}0\,, \ldots]$$

where each strings in the list $\mathbb{N}$ is a so-called **natural number**. For each natural number ξ in the list $\mathbb{N}$ we have:

$$\text{either} \quad \xi \equiv 0 \quad \text{or} \quad \xi \equiv \underbrace{\mathsf{s} \cdots \mathsf{s}}_{\substack{\text{non-empty}\\ \text{finite string}}}0$$

where "$\equiv$" means identical. To each non-empty finite string which consists just of the symbol s we assign a kind of "length" m and write $\underset{m}{\mathsf{s}}$ instead of $\mathsf{s} \cdots \mathsf{s}$. So, $\underset{m}{\mathsf{s}}$ is just an abbreviation of a finite string of the form $\mathsf{s} \cdots \mathsf{s}$.

REMARK. With this notation we get that each string in $\mathbb{N}$ is either 0 or of the form $\underset{m}{\mathsf{s}}0$. Further we get that for any strings $\underset{m}{\mathsf{s}}$ and $\underset{n}{\mathsf{s}}$ we have for example

$$\underset{m}{\mathsf{s}}\underset{n}{\mathsf{s}}0 \;\equiv\; \underset{n}{\mathsf{s}}\underset{m}{\mathsf{s}}0\,, \qquad \mathsf{s}\underset{m}{\mathsf{s}}\underset{n}{\mathsf{s}}0 \;\equiv\; \mathsf{s}\underset{n}{\mathsf{s}}\underset{m}{\mathsf{s}}0\,, \qquad \mathsf{s}\underset{m}{\mathsf{s}}\underset{n}{\mathsf{s}}0 \;\equiv\; \underset{m}{\mathsf{s}}\mathsf{s}\underset{n}{\mathsf{s}}0\,,$$

and further we get:

$$\underset{m}{\mathsf{s}}0 \equiv \underset{n}{\mathsf{s}}0 \iff \mathsf{s}\underset{m}{\mathsf{s}}0 \equiv \mathsf{s}\underset{n}{\mathsf{s}}0$$
$$\underset{m}{\mathsf{s}}0 \equiv \underset{n}{\mathsf{s}}0 \iff \underset{m}{\mathsf{s}}\underset{k}{\mathsf{s}}0 \equiv \underset{n}{\mathsf{s}}\underset{k}{\mathsf{s}}0$$

All these facts can be deduced from Euclid's first book of *the Elements* in which he writes (see [6, p. 155]):

1. *Things which are equal to the same thing are also equal to one another.*
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*

It is convenient to use arabic numbers for explicitly given natural numbers (*e.g.*, we write "1" for "$\mathsf{s}0$") and Latin letters like $n, m, \ldots$ for non-specified natural numbers. If n and m denote different natural numbers, where n appears earlier than m in the list $\mathbb{N}$, then we write $n < m$ and the expression $n, \ldots, m$ means the natural numbers which belong to the sublist $[n, \ldots, m]$ of $\mathbb{N}$; if n appears later than m in $\mathbb{N}$, then we write $n > m$ and the expression $n, \ldots, m$ denotes the empty set.

We shall use natural numbers frequently as subscripts for finite lists of objects like $t_1, \ldots, t_n$. In this context we mean that for each natural number k in the list $[1, \ldots, n]$, there is an object t_k, where in the case when $n = 0$, the set of objects is empty.

If n is a natural number, then $n+1$ denotes the natural number which appears immediately after n in the list $\mathbb{N}$; and if $n \neq 0$, then $n-1$ denotes the natural number which appears immediately before n in the list $\mathbb{N}$. Furthermore, for $\underset{m}{\mathsf{s}}0$, $\underset{n}{\mathsf{s}}0$ in the list $\mathbb{N}$, we define

$$\underset{m}{\mathsf{s}}0 + 0 :\equiv \underset{m}{\mathsf{s}}0 \quad \textit{and} \quad 0 + \underset{n}{\mathsf{s}}0 :\equiv \underset{n}{\mathsf{s}}0$$

and in general, we define:

$$\underset{m}{\mathsf{s}}0 + \underset{n}{\mathsf{s}}0 :\equiv \underset{m}{\mathsf{s}}\underset{n}{\mathsf{s}}0$$

Finally, by our construction of natural numbers we get the following fact:

> *If* a statement A holds for 0 and *if* A holds for a natural number n in $\mathbb{N}$ then it also holds for $n+1$, *then* the statement A holds for *all* natural numbers n in $\mathbb{N}$.

This fact is known as *Induction Priciple*, which is an important tool in proving statements about natural numbers.