Chapter 3
Semantics: Making Sense of the Symbols

There are two different views to a given set of formulae $\mathbf{\Phi}$, namely the *syntactical* view and the *semantical* view.

From the syntactical point of view (presented in the previous chapters), we consider the set $\mathbf{\Phi}$ just as a set of well-formed formulae—regardless of their intended sense or meaning—from which we can prove some formulae. So, from a formal point of view there is no need to assign real objects (what ever this means) to our strings of symbols.

In contrast to this very formal syntactical view, there is also the semantical point of view from which we consider the intended meaning of the formulae in $\mathbf{\Phi}$ and then seeking for a *model* in which all formulae of $\mathbf{\Phi}$ become true. For this, we have to explain some basic notions of Model Theory like *structure* and *interpretation*, which we will do in an natural, informal language. In this language, we will use words like "or", "and", or phrases like "if. . .then". These words and phrases have the usual meaning. Furthermore, we assume that in our normal world, which we describe with our informal language, the basic rules of *common logic* apply. For example, a statement φ is true or false, and if φ is true, then $\neg\varphi$ is false; and vice versa. Hence, the statement "φ or $\neg\varphi$" is always true, which means that we tacitly assume the LAW OF EXCLUDED MIDDLE, also known as TERTIUM NON DATUR, which corresponds to the logical axiom L_0. Furthermore, we assume DEMORGAN'S LAWS and we apply MODUS PONENS as inference rule.

Structures & Interpretations

In order to define structures and interpretations, we have to assume some notions of NAIVE SET THEORY like *subset*, *cartesian product*, or *relation*, which shall be defined properly in Part IV. On this occasion we also make use of the set theoretical symbol "$\in$", which stands for the binary *membership relation*.

Let $\mathscr{L}$ be an arbitrary but fixed language. An **$\mathscr{L}$-structure** $\mathbf{M}$ consists of a non-empty set A, called the **domain** of $\mathbf{M}$, together with a mapping which assigns to each constant symbol $c \in \mathscr{L}$ an element $c^{\mathbf{M}} \in A$, to each n-ary relation symbol $R \in \mathscr{L}$ a set of n-tuples $R^{\mathbf{M}}$ of elements of A, and to each n-ary function symbol $F \in \mathscr{L}$ a function $F^{\mathbf{M}}$ from n-tuples of A to A. In other word, the constant symbols denote elements of A, n-ary relation symbols denote subsets of A^n (*i.e.*, subsets of the n-fold cartesian product of A), and n-ary functions symbols denote n-ary functions from A^n to A.

The interpretation of variables is given by a so-called assignment: An **assignment** in an $\mathscr{L}$-structure $\mathbf{M}$ is a mapping j which assigns to each variable an element of the domain A.

Finally, an **$\mathscr{L}$-interpretation $\mathbf{I}$** is a pair $(\mathbf{M}, j)$ consisting of an $\mathscr{L}$-structure $\mathbf{M}$ and an assignment j in $\mathbf{M}$. For a variable ν, an element $a \in A$, and an assignment j in $\mathbf{M}$ we define the assignment $j\frac{a}{\nu}$ by stipulating

$$j\tfrac{a}{\nu}(\nu') = \begin{cases} a & \text{if } \nu' \equiv \nu, \\ j(\nu') & \text{otherwise.} \end{cases}$$

Furthermore, for $a, a' \in A$ and variables ν, ν' we shall write $j\frac{a}{\nu}\frac{a'}{\nu'}$ instead of $\big(j\frac{a}{\nu}\big)\frac{a'}{\nu'}$.

For an interpretation $\mathbf{I} = (\mathbf{M}, j)$ and an element $a \in A$, let

$$\mathbf{I}\tfrac{a}{\nu} := (\mathbf{M}, j\tfrac{a}{\nu}) .$$

We associate with every interpretation $\mathbf{I} = (\mathbf{M}, j)$ and every $\mathscr{L}$-term τ an element $\mathbf{I}(\tau) \in A$ as follows:

- For a variable ν let $\mathbf{I}(\nu) := j(\nu)$.
- For a constant symbol $c \in \mathscr{L}$ let $\mathbf{I}(c) := c^{\mathbf{M}}$.
- For an n-ary function symbol $F \in \mathscr{L}$ and terms $\tau_1, \dots, \tau_n$ let

$$\mathbf{I}\big(F(\tau_1, \dots, \tau_n)\big) := F^{\mathbf{M}}\big(\mathbf{I}(\tau_1), \dots, \mathbf{I}(\tau_n)\big).$$

Now, we are able to define precisely when a formula φ becomes *true* under an interpretation $\mathbf{I} = (\mathbf{M}, j)$; in which case we write $\mathbf{I} \vDash \varphi$ and say that φ is **true** in $\mathbf{I}$ (or that φ **holds** in $\mathbf{I}$). The definition is by induction on the complexity of the formula φ. By the rules (F0)–(F4), φ must be of the form $\tau_1 = \tau_2$, $R(\tau_1, \dots, \tau_n)$, $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \rightarrow \psi_2$, $\exists\nu\psi$, or $\forall\nu\psi$:

$$\begin{aligned}
\mathbf{I} \vDash \tau_1 = \tau_2 \quad &:\Longleftrightarrow \quad \mathbf{I}(\tau_1) \text{ IS THE SAME OBJECT AS } \mathbf{I}(\tau_2) \\
\mathbf{I} \vDash R(\tau_1, \dots, \tau_n) \quad &:\Longleftrightarrow \quad \big\langle \mathbf{I}(\tau_1), \dots, \mathbf{I}(\tau_n) \big\rangle \text{ BELONGS TO } R^{\mathbf{M}} \\
\mathbf{I} \vDash \neg\psi \quad &:\Longleftrightarrow \quad \text{NOT } \mathbf{I} \vDash \psi \\
\mathbf{I} \vDash \psi_1 \wedge \psi_2 \quad &:\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \text{ AND } \mathbf{I} \vDash \psi_2
\end{aligned}$$

$$\mathbf{I} \vDash \psi_1 \vee \psi_2 \quad :\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \text{ OR } \mathbf{I} \vDash \psi_2$$

$$\mathbf{I} \vDash \psi_1 \to \psi_2 \quad :\Longleftrightarrow \quad \text{NOT } \mathbf{I} \vDash \psi_1 \text{ OR } \mathbf{I} \vDash \psi_2$$

$$\mathbf{I} \vDash \exists \nu \psi \quad :\Longleftrightarrow \quad \text{IT EXISTS } a \text{ IN } A: \ \mathbf{I}\frac{a}{\nu} \vDash \psi$$

$$\mathbf{I} \vDash \forall \nu \psi \quad :\Longleftrightarrow \quad \text{FOR ALL } a \text{ IN } A: \ \mathbf{I}\frac{a}{\nu} \vDash \psi$$

Notice that by the logical rules in our informal language, for *every* $\mathscr{L}$-formula φ we have either $\mathbf{I} \vDash \varphi$ or $\mathbf{I} \vDash \neg\varphi$. So, every $\mathscr{L}$-formula is either true or false in $\mathbf{I}$.

The following fact summarises a few immediate consequences of the definitions above:

FACT 3.1. (a) *If φ is a formula and $\nu \notin \text{free}(\varphi)$, then:*

$$\mathbf{I}\frac{a}{\nu} \vDash \varphi \quad \textit{if and only if} \quad \mathbf{I} \vDash \varphi$$

(b) *If $\varphi(\nu)$ is a formula and the substitution $\varphi(\nu/\tau)$ is admissible, then:*

$$\mathbf{I}\frac{\mathbf{I}(\tau)}{\nu} \vDash \varphi(\nu) \quad \textit{if and only if} \quad \mathbf{I} \vDash \varphi(\tau)$$

Models

Let $\mathbf{\Phi}$ be an arbitrary set of $\mathscr{L}$-formulae. Then an $\mathscr{L}$-structure $\mathbf{M}$ is a **model of $\mathbf{\Phi}$** if for every assignment j and for each formula $\varphi \in \mathbf{\Phi}$ we have $(\mathbf{M}, j) \vDash \varphi$, *i.e.*, φ is true in the $\mathscr{L}$-interpretation $\mathbf{I} = (\mathbf{M}, j)$. Instead of saying "$\mathbf{M}$ is a model of $\mathbf{\Phi}$" we just write $\mathbf{M} \vDash \mathbf{\Phi}$. If φ fails in $\mathbf{M}$, then we write $\mathbf{M} \nvDash \varphi$, which is equivalent to $\mathbf{M} \vDash \neg\varphi$, because for any $\mathscr{L}$-formula φ we have *either* $\mathbf{M} \vDash \varphi$ *or* $\mathbf{M} \vDash \neg\varphi$.

Example 3.1. Example

As an immediate consequence of the definition of models we get:

FACT 3.2. *If φ is an $\mathscr{L}$-formula, ν a variable, and $\mathbf{M}$ a model, then $\mathbf{M} \vDash \varphi$ if and only if $\mathbf{M} \vDash \forall\nu\varphi$.*

This leads to the following definition: Let $\langle \nu_1, \dots, \nu_n \rangle$ be the sequence of variables which appear free in the $\mathscr{L}$-formula φ, where the variables appear in the sequence in the same order as they appear the first time in φ if one reads φ from left to right. Then the **universal closure** of φ, denoted $\overline{\varphi}$, is defined by stipulating

$$\overline{\varphi} :\equiv \forall\nu_1 \cdots \forall\nu_n \, \varphi \, .$$

As a generalisation of FACT 3.2 we get:

FACT 3.3. *If φ is an $\mathscr{L}$-formula and $\mathbf{M}$ a model, then*

$$\mathbf{M} \vDash \varphi \quad \Longleftrightarrow \quad \mathbf{M} \vDash \overline{\varphi}\,.$$

The following notation will be used later to simplify the arguments when we shall investigate the truth-value of sentences in some model $\mathbf{M}$: Suppose that $\mathbf{M}$ is a model with domain A. Let $\varphi(\nu_1,\dots,\nu_n)$ be an $\mathscr{L}$-formula whose free variables are $\nu_1,\dots,\nu_n$ and let $a_1,\dots,a_n \in A$. Then we write

$$\mathbf{M} \vDash \varphi(a_1,\dots,a_n)$$

to denote that for every assignment j in $\mathbf{M}$,

$$(\mathbf{M}, j\tfrac{a_1}{\nu_1}\dots\tfrac{a_n}{\nu_n}) \vDash \varphi.$$

Basic Notions of Model Theory

Let $\mathscr{L}$ be a signature, *i.e.*, a possibly empty set of constant symbols c, n-ary function symbols F, and n-ary relation symbols R. Two $\mathscr{L}$-structures $\mathbf{M}$ and $\mathbf{N}$ with domains A and B are **isomorphic**, denoted $\mathbf{M} \cong \mathbf{N}$, if there is a bijection $f : A \to B$ such that

$$f\big(c^{\mathbf{M}}\big) \;=\; c^{\mathbf{N}} \quad \text{(for all } c \in \mathscr{L})$$

and for all $a_1,\dots,a_n \in A$:

$$f\big(F^{\mathbf{M}}(a_1,\dots,a_n)\big) \;=\; F^{\mathbf{N}}\big(f(a_1),\dots,f(a_n)\big) \qquad \text{(for all } F \in \mathscr{L})$$

$$\langle a_1,\dots,a_n\rangle \in R^{\mathbf{M}} \;\Leftrightarrow\; \big\langle f(a_1),\dots,f(a_n)\big\rangle \in R^{\mathbf{N}} \quad \text{(for all } R \in \mathscr{L})$$

FACT 3.4. (a) *If $\mathbf{M}$ and $\mathbf{N}$ are isomorphic $\mathscr{L}$-structures and σ is an $\mathscr{L}$-sentence, then:*

$$\mathbf{M} \vDash \sigma \quad \Longleftrightarrow \quad \mathbf{N} \vDash \sigma$$

(b) *If $\mathbf{M}$ and $\mathbf{N}$ are isomorphic models of some given set of $\mathscr{L}$-formulae and φ is an $\mathscr{L}$-formula, then:*

$$\mathbf{M} \vDash \varphi \quad \Longleftrightarrow \quad \mathbf{N} \vDash \varphi$$

It may happen that although two $\mathscr{L}$-structures $\mathbf{M}$ and $\mathbf{N}$ are not isomorphic there is no $\mathscr{L}$-sentence that can distinguish between them. In this case we say that $\mathbf{M}$ and $\mathbf{N}$ are elementarily equivalent. More formally, we say that $\mathbf{M}$ is **elementarily equivalent** to $\mathbf{N}$, denoted $\mathbf{M} \equiv_e \mathbf{N}$, if each $\mathscr{L}$-sentence σ true in $\mathbf{M}$ is also true in $\mathbf{N}$. The following lemma shows that "$\equiv_e$" is symmetric:

LEMMA 3.5. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *are* $\mathscr{L}$*-structures and* $\mathbf{M} \equiv_e \mathbf{N}$*, then for each* $\mathscr{L}$*-sentence* σ *we have:*

$$\mathbf{M} \vDash \sigma \iff \mathbf{N} \vDash \sigma$$

Proof. One direction is immediate from the definition. For the other direction, assume that σ is not true in $\mathbf{M}$, *i.e.*, $\mathbf{M} \nvDash \sigma$. Then $\mathbf{M} \vDash \neg\sigma$, which implies $\mathbf{N} \vDash \neg\sigma$, and hence, σ is not true in $\mathbf{N}$. ⊣

As a consequence of FACT 3.3 we get:

FACT 3.6. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *are elementarily equivalent models of some given set of* $\mathscr{L}$*-formulae and* φ *is an* $\mathscr{L}$*-formula, then*

$$\mathbf{M} \vDash \varphi \iff \mathbf{N} \vDash \varphi$$

Below we investigate the relationship between syntax and semantic. In particular, we investigate the relationship between a formal proof of a formula φ from a set of formulae $\mathbf{\Phi}$ and the truth-value of φ in a model of $\mathbf{\Phi}$. In this context, two questions arise naturally:

- Is each formula φ, which is provable from some set of formulae $\mathbf{\Phi}$, valid in every model $\mathbf{M}$ of $\mathbf{\Phi}$?
- Is every formula φ, which is valid in each model $\mathbf{M}$ of $\mathbf{\Phi}$, provable from $\mathbf{\Phi}$?

Soundness Theorem

In this section we give an answer to the former question; the answer to the latter is postponed to Part II.

A logical calculus is called *sound*, if all what we can prove is valid (*i.e.*, true), which implies that we cannot derive a contradiction. The following theorem shows that First-Order Logic is sound.

THEOREM 3.7 (SOUNDNESS THEOREM). *Let* $\mathbf{\Phi}$ *be a set of* $\mathscr{L}$*-formulae and* $\mathbf{M}$ *a model of* $\mathbf{\Phi}$*. Then for every* $\mathscr{L}$*-formula* φ_0 *we have:*

$$\mathbf{\Phi} \vdash \varphi_0 \implies \mathbf{M} \vDash \varphi_0$$

Somewhat shorter we could say:

$$\forall \varphi_0 : \ \mathbf{\Phi} \vdash \varphi_0 \implies \forall \mathbf{M}\big(\mathbf{M} \vDash \mathbf{\Phi} \implies \mathbf{M} \vDash \varphi_0\big)$$

Proof. First we show that all logical axioms are valid in $\mathbf{M}$. For this we have to define truth-values of composite statements in the metalanguage.
In the previous chapter we defined for example:

$$\underbrace{\mathbf{M} \vDash \varphi \wedge \psi}_{\Theta} \iff \underbrace{\mathbf{M} \vDash \varphi}_{\Phi} \text{ AND } \underbrace{\mathbf{M} \vDash \psi}_{\Psi}$$

Thus, in the metalanguage the statement "Θ" is true if and only if the statement "Φ AND Ψ" is true. So, the truth-value of "Θ" depends on the truth-values of "Φ" and "Ψ". In order to determine truth-values of composite statement like "Φ AND Ψ" or "IF Φ THEN Ψ", where the latter statement will get the same truth-value as "NOT Φ OR Ψ", we introduce so called *truth-tables*, in which "**1**" *stands for* "**true**" and "**0**" *stands for* "**false**":

Φ	Ψ	NOT Φ	Φ AND Ψ	Φ OR Ψ	IF Φ THEN Ψ
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	1

With these truth-tables one can show that all logical axioms are valid in $\mathbf{M}$. As an example we show that every instance of L_1 is valid in $\mathbf{M}$: For this, let φ_1 be an instance of L_1, *i.e.*, $\varphi_1 \equiv \varphi \to (\psi \to \varphi)$ for some $\mathscr{L}$-formulae φ and ψ. Then $\mathbf{M} \vDash \varphi_1$ *iff* $\mathbf{M} \vDash \varphi \to (\psi \to \varphi)$:

$$\underbrace{\mathbf{M} \vDash \varphi \to (\psi \to \varphi)}_{\Theta} \iff \text{IF } \underbrace{\mathbf{M} \vDash \varphi}_{\Phi} \text{ THEN } \underbrace{\mathbf{M} \vDash \psi \to \varphi}$$
$$\iff \text{IF } \Phi \text{ THEN IF } \underbrace{\mathbf{M} \vDash \psi}_{\Psi} \text{ THEN } \underbrace{\mathbf{M} \vDash \varphi}_{\Phi}$$

This shows that

$$\Theta \iff \text{IF } \Phi \text{ THEN } (\text{IF } \Psi \text{ THEN } \Phi).$$

Writing the truth-table of "Θ", we see that the statement "Θ" is always true (*i.e.*, φ_1 is valid in $\mathbf{M}$):

Therefore, $\mathbf{M} \vDash \varphi_1$, and since φ_1 was an arbitrary instance of L_1, every instance of L_1 is valid in $\mathbf{M}$.

In order to show that also the logical axioms L_{10}–L_{16} are valid in $\mathbf{M}$, we need somewhat more than just truth-tables:

Let A be the domain of $\mathbf{M}$, let j be an arbitrary assignment, and let $\mathbf{I} = (\mathbf{M}, j)$ be the corresponding $\mathscr{L}$-interpretation.

Now, we show that every instance of L_{10} is valid in $\mathbf{M}$. For this, let φ_{10} be an instance of L_{10}, *i.e.*, $\varphi_{10} \equiv \forall x \varphi(x) \to \varphi(t)$ for some $\mathscr{L}$-formula φ, where x is a

Φ	Ψ	IF Ψ THEN Φ	IF Φ THEN (IF Ψ THEN Φ)
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

variable, t a term, and the substitution $\varphi(x/t)$ is admissible. We work with $\mathbf{I}$ and show that $\mathbf{I} \vDash \varphi_{10}$.
By definition we have:

$$\mathbf{I} \vDash \forall x\varphi(x) \rightarrow \varphi(t) \quad \Longleftrightarrow \quad \text{IF } \mathbf{I} \vDash \forall x\varphi(x) \text{ THEN } \mathbf{I} \vDash \varphi(t)$$

Again by definition we have:

$$\mathbf{I} \vDash \forall x\varphi(x) \quad \Longleftrightarrow \quad \text{FOR ALL } a \text{ IN } A: \ \mathbf{I}\frac{a}{\nu} \vDash \varphi$$

In particular we get:

$$\mathbf{I} \vDash \forall x\varphi(x) \quad \Longrightarrow \quad \mathbf{I}\frac{I(t)}{\nu} \vDash \varphi$$

Furthermore, by FACT 3.1.(a) we get:

$$\mathbf{I} \vDash \varphi(t) \quad \Longleftrightarrow \quad \mathbf{I}\frac{I(t)}{\nu} \vDash \varphi(x)$$

Hence, we get

$$\text{IF } \mathbf{I} \vDash \forall x\varphi(x) \text{ THEN } \mathbf{I} \vDash \varphi(t)$$

which shows that

$$(\mathbf{M}, j) \vDash \forall x\varphi(x) \rightarrow \varphi(t)$$

and since the assignment j was arbitrary, we finally get:

$$\mathbf{M} \vDash \forall x\varphi(x) \rightarrow \varphi(t)$$

Therefore, $\mathbf{M} \vDash \varphi_{10}$, and since φ_{10} was an arbitrary instance of L_{10}, every instance of L_{10} is valid in $\mathbf{M}$.

With similar arguments one can show that also every instance of L_{11}, L_{12}, or L_{13} is valid in $\mathbf{M}$ (see EXERCISES 3.6.(a)–(c)). Furthermore, one can also show that also L_{14}, L_{15}, and L_{16} are valid in $\mathbf{M}$ (see EXERCISES 3.6.(d)–(f)).

Let $\mathbf{\Phi}$ be a set of formulae, let $\mathbf{M}$ be a model of $\mathbf{\Phi}$, and assume that $\mathbf{\Phi} \vdash \varphi_0$. We shall show that $\mathbf{M} \vDash \varphi_0$. For this, we notice first the following facts:

- As we have seen above, each instance of a logical axiom is valid in $\mathbf{M}$.
- Since $\mathbf{M} \vDash \mathbf{\Phi}$, each formula of $\mathbf{\Phi}$ is valid in $\mathbf{M}$.
- By the truth-tables we get

$$\text{IF } \big(\mathbf{M} \vDash \varphi \to \psi \text{ AND } \mathbf{M} \vDash \varphi\big) \text{ THEN } \mathbf{M} \vDash \psi$$

and therefore, every application of MODUS PONENS in the proof of φ_0 from $\mathbf{\Phi}$ yields a valid formula (if the premisses are valid).

- Since, by FACT 3.2,

$$\mathbf{M} \vDash \varphi \iff \mathbf{M} \vDash \forall \nu \varphi(\nu)$$

every application of the GENERALISATION in the proof of φ_0 from $\mathbf{\Phi}$ yields a valid formula.

From these facts it follows immediately that *each* formula in the proof of φ_0 from $\mathbf{\Phi}$ is valid in $\mathbf{M}$. In particular we get

$$\mathbf{M} \vDash \varphi_0$$

which completes the proof. ⊣

The following fact summarises a few consequences of the SOUNDNESS THEOREM.

FACT 3.8.

(a) *Every tautology is valid in each model:*

$$\forall \varphi : \vdash \varphi \implies \forall \mathbf{M} : \mathbf{M} \vDash \varphi$$

(b) *If a set of formulae* $\mathbf{\Phi}$ *has a model, then* $\mathbf{\Phi}$ *is consistent:*

$$\exists \mathbf{M} : \mathbf{M} \vDash \mathbf{\Phi} \implies \text{Con}(\mathbf{\Phi})$$

(c) *The logical axioms are consistent:*

$$\text{Con}(\mathsf{L}_0\text{-}\mathsf{L}_{16})$$

(d) *If a sentence* σ *is not valid in* $\mathbf{M}$*, where* $\mathbf{M}$ *is a model of* $\mathbf{\Phi}$*, then* σ *is not provable from* $\mathbf{\Phi}$*:*

$$\text{IF } \big(\mathbf{M} \nvDash \sigma \text{ AND } \mathbf{M} \vDash \mathbf{\Phi}\big) \text{ THEN } \mathbf{\Phi} \nvdash \sigma$$

Substitution of Variables

In Part II & III we shall encode formulae by strings of certain symbols and by natural numbers, respectively. In order to do so, we have to make sure that the variables are among a well-defined set of symbols, namely among $v_0, v_1, \ldots$ where the index n of v_n is a natural number, *i.e.*, a member of $\mathbb{N}$. Before we prove the next result, we introduce the following notion.

We say that two $\mathscr{L}$-formulae φ and ψ are **semantically equivalent** if for all $\mathscr{L}$-structures $\mathbf{M}$ and every assignment j we have:

$$(\mathbf{M}, j) \vDash \varphi \quad \Longleftrightarrow \quad (\mathbf{M}, j) \vDash \psi$$

THEOREM 3.9 (VARIABLE SUBSTITUTION THEOREM). *For every sentence* σ *there is a semantically equivalent sentence* $\tilde{\sigma}$ *which contains just variables among* $v_0, v_1, \ldots$, *where for any* $n \in \mathbb{N}$ *we have* v_n *appears in* σ', *then also* v_m *appears in* $\tilde{\sigma}$ *for all* $m < n$.

Proof. Let $\mathscr{L}$ be a signature and let σ be an $\mathscr{L}$-sentence written in Polish notation. Let $\exists\!\!\!\forall_0$ be the left most quantifier which appears in σ, *i.e.*, $\exists\!\!\!\forall_0$ is either "$\exists$" or "$\forall$". $\exists\!\!\!\forall_0$ is followed by a variable, say ν_0, and a formula, say φ_0. Let us assume that the quantifier $\exists\!\!\!\forall_0$ is "$\exists$" (the case when $\exists\!\!\!\forall_0$ is "$\forall$" is similar). Let now $\mathbf{M}$ be an arbitrary $\mathscr{L}$-structure with domain A and let j be an arbitrary assignment. If necessary, we extend j such that j is defined on each of the variables $v_0, v_1, \ldots$, *i.e.*, for each $n \in \mathbb{N}$, $j(v_n)$ is an element of A. Finally, let $m \in \mathbb{N}$ be the least natural number such that for all $k \in \mathbb{N}$, the variable v_{m+k} does not occur in σ. Then we have:

$$\begin{aligned}
(\mathbf{M}, j) \vDash \exists \nu_0 \varphi_0 \quad &\Longleftrightarrow \quad \text{IT EXISTS } a_0 \text{ IN } A: \quad (\mathbf{M}, j\tfrac{a_0}{\nu_0}) \vDash \varphi_0 \\
&\Longleftrightarrow \quad \text{IT EXISTS } a_0 \text{ IN } A: \quad (\mathbf{M}, j\tfrac{a_0}{v_m}) \vDash \varphi_0(\nu_0/v_m) \\
&\Longleftrightarrow \quad (\mathbf{M}, j) \vDash \exists v_m \varphi_0(\nu_0/v_m)
\end{aligned}$$

In particular, we have

$$\mathbf{M} \vDash \exists \nu_0 \varphi_0 \quad \Longleftrightarrow \quad \mathbf{M} \vDash \exists v_m \varphi_0(\nu_0/v_m)\,.$$

Proceeding this way, we can replace all the variables variables $\nu_0, \nu_1, \ldots$ appearing in σ with the variables $v_m, v_{m+1}, \ldots$ and obtain a sentence σ' with

$$\mathbf{M} \vDash \sigma \iff \mathbf{M} \vDash \sigma'\,.$$

With the same arguments we can replace the variables $v_m, v_{m+1}, \ldots$ in σ' with the variables $v_0, v_1, \ldots$ and obtain a sentence $\tilde{\sigma}$ with the required properties. $\dashv$

Completion of Theories

A set of $\mathscr{L}$-sentences is called an **$\mathscr{L}$-theory**., and an $\mathscr{L}$-theory T is called **complete**, if for every $\mathscr{L}$-sentence σ we have *either* $\mathsf{T} \vdash \sigma$ *or* $\mathsf{T} \vdash \neg\sigma$. Furthermore, for an $\mathscr{L}$-theory T let $\mathbf{Th}(\mathsf{T})$ be the set of all $\mathscr{L}$-sentences σ, such that $\mathsf{T} \vdash \sigma$. By these definitions we get that a consistent $\mathscr{L}$-theory T is complete *iff* for every $\mathscr{L}$-sentence σ we have *either* $\sigma \in \mathbf{Th}(\mathsf{T})$ *or* $\neg\sigma \in \mathbf{Th}(\mathsf{T})$.

PROPOSITION 3.10. *If* T *is an* $\mathscr{L}$*-theory wich has a model, then there exists a complete* $\mathscr{L}$*-theory* $\bar{\mathsf{T}}$ *which contains* T*. In particular, every* $\mathscr{L}$*-theory which has a model can be completed, i.e., can be extended to a complete theory.*

Proof. Let $\mathbf{M}$ be a model of some $\mathscr{L}$-theory T and let $\bar{\mathsf{T}}$ be the set of $\mathscr{L}$-sentences σ, such that $\mathbf{M} \vDash \sigma$. Since for each $\mathscr{L}$-sentence σ_0 we have either $\mathbf{M} \vDash \sigma_0$ or $\mathbf{M} \vDash \neg\sigma_0$, we get either $\sigma_0 \in \bar{\mathsf{T}}$ or $\neg\sigma_0 \in \bar{\mathsf{T}}$, which shows that $\bar{\mathsf{T}}$ is complete, and since $\mathbf{M} \vDash \mathsf{T}$, we get that $\bar{\mathsf{T}}$ contains T. $\dashv$

It is natural to ask whether also the converse of PROPOSITION 3.10 holds, *i.e.*, whether every $\mathscr{L}$-theory which can be completed has a model. Notice that if an $\mathscr{L}$-theory T can be completed, then T must be consistent. So, one may ask whether every consistent theory has a model. An affirmative answer to this question together with FACT 3.8 (b) would imply that an $\mathscr{L}$-theory T is consistent if and only if T has a model—which is indeed the case, as we shall see.

EXERCISES

3.0 Let T be a set of $\mathscr{L}$-sentences (for some signature $\mathscr{L}$) and let $\mathbf{M}$ be an $\mathscr{L}$-structure such that $\mathbf{M} \vDash \mathsf{T}$. Furthermore, let $\mathscr{L}'$ be an extension of $\mathscr{L}$ (*i.e.*, $\mathscr{L}'$ is a signature which contains $\mathscr{L}$). Then there is an $\mathscr{L}'$-structure $\mathbf{M}'$ with the same domain as $\mathbf{M}$, such that $\mathbf{M} \vDash \mathsf{T}$.

Hint: Let a_0 be an arbitrary but fixed element of the domain A of $\mathbf{M}$. For each constant symbol $c \in \mathscr{L}'$ which does not belong to $\mathscr{L}$, let $c^{\mathbf{M}'} := a_0$. Similarly, for each n-ary function symbol $F \in \mathscr{L}'$ which does not belong to $\mathscr{L}$, let $F^{\mathbf{M}'} : A^n \to A$ be such that $F^{\mathbf{M}'}$ maps each element of A^n to a_0. Finally, for each n-ary relation symbol $R \in \mathscr{L}'$ which does not belong to $\mathscr{L}$, let $R^{\mathbf{M}'} := A^n$.

3.1 If an $\mathscr{L}$-theory T has, up to isomorphisms, a unique model, then T is complete.

3.2 If two structures $\mathbf{M}$ and $\mathbf{N}$ are isomorphic, then they are elementarily equivalent.

3.3 Let DLO be the theory of dense linearly ordered sets without endpoints: The signature $\mathscr{L}_{\mathsf{DLO}}$ contains just the binary relation symbol "$<$", and the non-logical axioms of DLO are the following sentences:

DLO_0 $\forall x \neg(x < x)$
DLO_1 $\forall x \forall y \forall z\big((x < y \wedge y < z) \to x < z\big)$
DLO_2 $\forall x \forall y\big(x < y \vee x = y \vee y < x\big)$
DLO_3 $\forall x \forall y \exists z\big(x < y \to (x < z \wedge z < y)\big)$
DLO_4 $\forall x \exists y \exists z\big(y < x \wedge x < z\big)$

Show that the theory DLO is complete, *i.e.*, for all $\mathscr{L}_{\mathsf{DLO}}$-sentences σ we have *either* $\mathsf{DLO} \vdash \sigma$ *or* $\mathsf{DLO} \vdash \neg\sigma$.

3.4 Show that the converse of EXERCISE 3.2 does not hold.

Hint: Let $\mathbb{Q}$ be the set of rational numbers, let $\mathbb{R}$ be the set of real numbers, and let "$<$" be the natural ordering on $\mathbb{Q}$ and $\mathbb{R}$, respectively. Then the two non-isomorphic $\mathscr{L}_{\mathsf{DLO}}$-structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are both models of DLO.

3.5 Show that every countable model of DLO is isomorphic to $(\mathbb{Q}, <)$.

Hint: Enumerate both $\mathbb{Q}$ and some model $\mathbf{M}$ of DLO, and construct an isomorphism by recursion such that in the n-th step the n-th element of M is mapped to an element of $\mathbb{Q}$ so that the order is preserved.

3.6 Let $\mathscr{L}$ be an arbitrary signature and let $\mathbf{M}$ be an arbitrary $\mathscr{L}$-structure. Then

(a) L_{11} is valid in $\mathbf{M}$,
(b) L_{12} is valid in $\mathbf{M}$,
(c) L_{13} is valid in $\mathbf{M}$,
(d) L_{14} is valid in $\mathbf{M}$,
(e) L_{15} is valid in $\mathbf{M}$,
(f) L_{16} is valid in $\mathbf{M}$.