Chapter 5
The Completeness Theorem

As in the previous chapter, we require that all formulae are written in Polish notation and that the variables are among $v_0, v_1, v_2, \ldots$ Furthermore, let $\mathscr{L}$ be a countable signature, let T be a consistent $\mathscr{L}$-theory, and let σ_0 be an $\mathscr{L}$-sentence which is not provable from T. Finally, let $\overline{\mathsf{T}}$ be the maximally consistent extension of $\mathsf{T} + \neg\sigma_0$ obtained with LINDENBAUM'S LEMMA 4.5.

We shall construct a model of $\overline{\mathsf{T}}$ as follows: In a first step, we extend the signature $\mathscr{L}$ to a signature $\mathscr{L}_c$ by adding countably many new constant symbols, so-called *special constants*. In a second the step, we extend the $\mathscr{L}$-theory $\overline{\mathsf{T}}$ to an $\mathscr{L}_c$-theory $\overline{\mathsf{T}}_c$ by adding so-called *witnesses* to existential sentences in $\in \overline{\mathsf{T}}$. In particular, for each sentence $\exists x\sigma(x) \in \overline{\mathsf{T}}$ we add an $\mathscr{L}_c$-sentence $\sigma(c)$, where c is some special constant. In a third step, we extend the $\mathscr{L}_c$-theory $\overline{\mathsf{T}}_c$ to a maximally consistent $\mathscr{L}_c$-theory $\tilde{\mathsf{T}}$, and in a last step, we build the domain of the model of $\tilde{\mathsf{T}}$ as a list of lists of closed $\mathscr{L}_c$-terms.

Extending the Language

A string of symbols is a **term-constant**, if it results from applying FINITELY many times the following rules:

(C0) Each closed (*i.e.*, variable-free) $\mathscr{L}$-term is a term-constant.
(C1) If $\tau_0, \ldots, \tau_{n-1}$ are any term-constants which we have already built and F is an n-ary function symbol, then $F\tau_0 \cdots \tau_{n-1}$ is a term-constant.
(C2) For any natural numbers i, n, if $\tau_0, \ldots, \tau_{n-1}$ are any term-constants which we have already built, then $(i, \tau_0, \ldots, \tau_{n-1}, n)$ is a term-constant.

The strings $(i, \tau_0, \ldots, \tau_{n-1}, n)$ which are built with rule (C2) are called **special constants**. Notice that for $n = 0$, $(i, \tau_0, \ldots, \tau_{n-1}, n)$ becomes $(i, 0)$.

Let $\mathscr{L}_c$ be the signature $\mathscr{L}$ extended with the countably many special constants. In order to write the special constants in a list, we first encode them and then define an ordering on the codes.

First we encode closed $\mathscr{L}$-terms as above with strings of $\mathsf{0}$'s and $\mathsf{2}$'s. Now, let $c \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ be a special constant, where the codes of $\tau_0, \ldots, \tau_{n-1}$ are already defined. Then we encode c as follows:

$$
\begin{array}{cccccccccccccc}
c & \equiv & (& \boldsymbol{i} & , & \boldsymbol{\tau_0} & , & \ldots & , & \boldsymbol{\tau_{n-1}} & , & \boldsymbol{n} &) \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\#c & \equiv & \mathsf{6} & \underbrace{\mathsf{1}\ldots\mathsf{1}}_{i\text{-times } \mathsf{1}} & \mathsf{8} & \#\tau_0 & \mathsf{8} & \ldots & \mathsf{8} & \#\tau_{n-1} & \mathsf{8} & \underbrace{\mathsf{1}\ldots\mathsf{1}}_{n\text{-times } \mathsf{1}} & \mathsf{9}
\end{array}
$$

The codes of special constants are ordered by their length and lexicographically, where $\mathsf{0} < \mathsf{1} < \mathsf{2} < \mathsf{6} < \mathsf{8} < \mathsf{9}$.

Finally, let $\Lambda_c = [c_0, c_1, \ldots]$ be the potentially infinite list of all special constants, ordered with respect to the ordering of their codes.

Extending the Theory

In this section we shall add witnesses for certain existential $\mathscr{L}_c$-sentences σ_i in the list $\overline{\mathsf{T}} = [\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots]$, where an $\mathscr{L}_c$-sentence is existential if it is of the form $\exists\nu\varphi$. The witnesses we choose from the list Λ_c of special constants. In order to make sure that we have a witness for each existential $\mathscr{L}_c$-sentence (and not just for $\mathscr{L}$-sentences), and also to make sure that the choice of witnesses do not lead to a contradiction, we have to choose the witnesses carefully.

Let $\sigma_i \in \overline{\mathsf{T}}$ and let $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$ be a special constant. Then we say that **c_j witnesses σ_i** or that **c_j is a witness for σ_i**, if:

- $i \geq 1$ and σ_i is in special Prenex Normal Form sPNF (see Chapter 3),
- "$\exists v_n$" appears in σ_i,
- for all $m < n$: if "$\exists v_m$" appears in σ_i, then $t_m \equiv (i, t_0, \ldots, t_{m-1}, m)$.

On the one hand, we have only withnesses for c_j for $\mathscr{L}$-senteces σ_i with $i \geq 1$. On the other hand, notice that since $\neg\sigma_0$ is not in sPNF, by construction of $\overline{\mathsf{T}}$ there exists an $i \geq 1$ such that σ_i and $\neg\sigma_0$ are semantically equivalent, which will be sufficient for our purposes.

If an $\mathscr{L}$-sentence $\sigma_i \in \overline{\mathsf{T}}$ is in sPNF and either "$\exists v_n$" or "$\forall v_n$" appears in σ_i, then

$$\sigma_i \equiv \exists\!\!\!\forall_0 v_0 \exists\!\!\!\forall_1 v_1 \cdots \exists\!\!\!\forall_n v_n \sigma_{i,n}(v_0, \ldots, v_n)$$

where $\sigma_{i,n}(v_0, \ldots, v_n)$ is an $\mathscr{L}$-formula in which each variable among $v_0, \ldots, v_n$ appears free. In particular, if $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$ witnesses σ_i, then

$$\sigma_i \equiv \exists\!\!\!\forall_0 v_0 \exists\!\!\!\forall_1 v_1 \cdots \exists\!\!\!\forall_{n-1} v_{n-1} \exists v_n \sigma_{i,n}(v_0, \ldots, v_n),$$

i.e., $\exists v_n$ appears in σ_i. Furthermore, if $\sigma_i \in \overline{\mathsf{T}}$ is in sPNF, $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$ is a special constant, and c_j witnesses σ_i, then let

$$\sigma_{i,n}[c_j] :\equiv \sigma_{i,n}(v_0/t_0, \dots, v_{n-1}/t_{n-1}, v_n/c_j)\,.$$

Now, we go through the list $\Lambda_c = [c_0, c_1, \dots]$ of special constants and extend step by step the list $\overline{\mathsf{T}} = [\sigma_0, \sigma_1, \dots]$. For this, we first stipulate $T_0 := \overline{\mathsf{T}}$. Assume that T_j is already defined and that $c_j \equiv (i, t_0, \dots, t_{n-1}, n)$ for some natural numbers i, n and terms $t_0, \dots, t_{n-1}$. We have the following two cases:

Case 1. The special constant c_j does not witness the $\mathscr{L}$-sentence $\sigma_i \in \overline{\mathsf{T}}$. In this case we set $T_{j+1} := T_j$.

Case 2. The special constant c_j witnesses $\sigma_i \in \overline{\mathsf{T}}$. In this case we insert the $\mathscr{L}_c$-sentence $\sigma_{i,n}[c_j]$ into the list T_j on the place which corresponds to the code $\#\sigma_{i,n}[c_j]$. The extended list is then T_{j+1}.

Finally, let $\overline{\mathsf{T}}_c$ be the resulting list, *i.e.*, $\overline{\mathsf{T}}_c$ is the union of all the T_j's.

LEMMA 5.1. $\overline{\mathsf{T}}_c$ *is consistent.*

Proof. By construction of $\overline{\mathsf{T}}$ we have $\mathrm{Con}(\overline{\mathsf{T}})$ with respect to the signature $\mathscr{L}$. We first show that $\overline{\mathsf{T}}$ is also consistent with respect to the signatur $\mathscr{L}_c$: Assume toward a contradiction that with respect to the signature $\mathscr{L}_c$, $\overline{\mathsf{T}} \vdash \boxtimes$. In that proof we replace each special constant c with a variable ν_c which does not occour in any of the finitely many formulae of the proof, such that if c and c' are distinct special constants, then ν_c and $\nu_{c'}$ are distinct variables. Notice that every logical axiom becomes a logical axiom of the same type and that $\mathscr{L}$-sentences of $\overline{\mathsf{T}}$ remain unchanged since they do not contain special constants. Furthermore, each application of MODUS PONENS or GENERALISATION becomes a new application of the same inference rule. To see this, notice that we do not apply GENERALISATION to any of the ν_c's, since otherwise, we would have applied GENERALISATION to a special constant c, but c is a term-constant and not a variable. Since the proof we obtain does not contain any special constant, we get $\overline{\mathsf{T}} \to \boxtimes$ (with respect to $\mathscr{L}$), which contradicts the fact that $\overline{\mathsf{T}}$ is consistent (with respect to $\mathscr{L}$). So, we have $\mathrm{Con}(\overline{\mathsf{T}})$ with respect to $\mathscr{L}_c$.

Now, assume towards a contradiction that $\overline{\mathsf{T}}_c$ is inconsistent, *i.e.*, $\neg\,\mathrm{Con}(\overline{\mathsf{T}}_c)$. Then, by the COMPACTNESS THEOREM 2.12, we find finitely many, pairwise distinct $\mathscr{L}_c$-sentences $\sigma_{i,n}[c_j]$ in $\overline{\mathsf{T}}_c$ such that

$$\neg\,\mathrm{Con}\left(\overline{\mathsf{T}} + \left\{\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]\right\}\right).$$

Notice that since the $\mathscr{L}_c$-sentences $\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]$ are pairwise distinct, also the special constants $c_{j_1}, \dots, c_{j_k}$ are pairwise distinct. Without loss of generality we may assume that $\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]$ are such that the sum $n_1 + \ldots + n_k + k$ is minimal.

For term-constants τ we define the height $h(\tau)$ as follows: If τ is a closed $\mathscr{L}$-term, then $h(\tau) := 0$. If $\tau_0, \dots, \tau_{n-1}$ are term-constants and $F \in \mathscr{L}$ is an n-ary function symbol, then

$$h(F\tau_0 \cdots \tau_{n-1}) := \max\left\{h(\tau_0), \dots, h(\tau_{n-1})\right\}.$$

Finally, if $\tau \equiv (i, \tau_0, \dots, \tau_{n-1}, n)$ is a special constant, then

$$h(\tau) := 1 + \max\big\{h(\tau_0), \dots, h(\tau_{n-1})\big\} \quad \text{where } \max\emptyset := 0\,.$$

Without loss of generality we may assume that

$$h(c_{j_k}) = \max\big\{h(c_{j_1}), \dots, h(c_{j_k})\big\},$$

i.e., for each special constant c_j occurring in c_{j_k} we have $h(c_j) < h(c_{j_k})$.

Let us now consider the formula $\sigma_{i_k,n_k}[c_{j_k}]$. To simplify the notation, we write i, n, j instead of i_k, n_k, j_k respectively; in particular, $\sigma_{i_k,n_k}[c_{j_k}]$ becomes $\sigma_{i,n}[c_j]$. Furthermore, let

$$\Sigma := \big\{\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_{k-1},n_{k-1}}[c_{j_{k-1}}]\big\},$$

and let $c_j \equiv (i, t_0, \dots, t_{n-1}, n)$, *i.e.*,

$$\sigma_{i,n}[c_j] \equiv \sigma_{i,n}(v_0/t_0, \dots, v_{n-1}/t_{n-1}, v_n/c_j)\,.$$

Since c_j witnesses σ_i, "$\exists v_n$" appears in σ_i, *i.e.*,

$$\sigma_{i,n-1}(v_0, \dots, v_{n-1}) \equiv \exists v_n \sigma_{i,n}(v_0, \dots, v_{n-1}, v_n)\,.$$

To simplify the notation again, we set

$$\tilde\sigma(v_n) :\equiv \sigma_{i,n}(v_0/t_0, \dots, v_{n-1}/t_{n-1}, v_n)\,.$$

Notice that v_n is the only variable which appears free in $\tilde\sigma$.

CLAIM. $\neg\,\mathrm{Con}\big(\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j]\big) \quad \Longrightarrow \quad \neg\,\mathrm{Con}\big(\overline{\mathsf{T}} + \Sigma + \exists v_n \tilde\sigma(v_n)\big)$

Proof of Claim. If $\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j]$ is inconsistent, then $\overline{\mathsf{T}} + \Sigma + \sigma_{i,n}[c_j] \vdash$ ⧉ and with the DEDUCTION THEOREM we get

$$\overline{\mathsf{T}} + \Sigma \;\vdash\; \sigma_{i,n}[c_j] \to \text{⧉}.$$

In the latter proof we replace the special constant c_j throughout the proof with a variable ν which does not occour in $\sigma_{i,n}$ and which does not occur in any of the finitely many formulae of the former proof. Notice that every logical axiom becomes a logical axiom of the same type and that $\mathscr{L}$-sentences of $\overline{\mathsf{T}}$ are not affected (since they do not contain special constants). Furthermore, also $\mathscr{L}_c$-sentences of Σ are not affected since they do not contain the special constant c_j (recall that the special constants $c_{j_1}, \dots, c_{j_k}$ are pairwise distinct). Finally, each application of MODUS PONENS or GENERALISATION becomes a new application of the same inference rule (notice that we do not apply GENERALISATION to ν, since otherwise, we would have applied GENERALISATION to c_j, but c_j is a term-constant). Now, we construct a proof of $\exists v_n \tilde\sigma(v_n) \to$ ⧉ from $\overline{\mathsf{T}} + \Sigma$ as follows:

$\overline{\mathsf{T}} + \Sigma \vdash \tilde{\sigma}(\nu) \to \boxtimes$	by assumption
$\overline{\mathsf{T}} + \Sigma \vdash \forall\nu\big(\tilde{\sigma}(\nu) \to \boxtimes\big)$	by GENERALISATION
$\overline{\mathsf{T}} + \Sigma \vdash \forall\nu\big(\tilde{\sigma}(\nu) \to \boxtimes\big) \to \big(\exists\nu\tilde{\sigma}(\nu) \to \boxtimes\big)$	L_{13}
$\overline{\mathsf{T}} + \Sigma \vdash \exists\nu\tilde{\sigma}(\nu) \to \boxtimes$	by MODUS PONENS
$\overline{\mathsf{T}} + \Sigma \vdash \exists v_n\tilde{\sigma}(v_n) \to \boxtimes$	TAUTOLOGY **??**

Therefore, we finally have $\neg\operatorname{Con}\big(\overline{\mathsf{T}} + \Sigma + \exists v_n\tilde{\sigma}(v_n)\big)$. $\dashv_{\text{Claim}}$

Let us now consider σ_i. Let $m \leq n$ be the largest natural number such that for each l with $1 \leq l \leq m$ we have that "$\forall v_{n-l}$" appears in σ_i. For example if $m = 0$, then for no $n' < n$, $\forall\!\!\!\exists_{n'}$ is the quantifier "$\forall$", and if $m = n$, then for no $n' < n$, $\forall\!\!\!\exists_{n'}$ is the quantifier "$\exists$". In general, σ_i is of the form

$$\sigma_i \equiv \underbrace{\forall\!\!\!\exists_0 v_0 \quad \cdots \quad}_{\text{"}\exists\text{" or "}\forall\text{"}} \exists v_{n-m-1} \underbrace{\forall v_{n-m} \quad \cdots \quad \forall v_{n-1}}_{\text{only "}\forall\text{"}} \exists v_n\, \sigma_{i,n}(v_0, \ldots, v_n)\,.$$

Consider now the formula

$$\tilde{\sigma}_m :\equiv \sigma_{i,n-m-1}(v_0/t_0, \ldots, v_{n-m-1}/t_{n-m-1})\,.$$

Then either $\tilde{\sigma}_m \in \overline{\mathsf{T}}$ (in case $m = n$), or "$\exists v_{n-m-1}$" appears in σ_i (in case $m < n$), and therefore, we are in one of following two cases:

$\tilde{\sigma}_m \in \overline{\mathsf{T}}$: First notice that in this case,

$$\tilde{\sigma}_m \equiv \forall v_n \cdots \forall v_{n-1} \exists v_n \sigma_{i,n}(v_0, \ldots, v_n)\,.$$

Since $\tilde{\sigma}_m \in \overline{\mathsf{T}}$ and $t_0, \ldots, t_{n-1}$ are closed terms, by L_{10} we get $\overline{\mathsf{T}} \vdash \exists v_n\tilde{\sigma}(v_n)$. Hence, by the CLAIM, $\neg\operatorname{Con}(\overline{\mathsf{T}} + \Sigma)$. This shows that we do not need $\sigma_{i_k,n_k}[c_{j_k}]$ to derive a contradiction from

$$\overline{\mathsf{T}} + \big\{\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_k,n_k}[c_{j_k}]\big\}\,,$$

which is a contradiction to the minimality of the sum $n_1 + \ldots n_k + k$.

"$\exists v_{n-m-1}$" appears in σ_i: Notice that since $c_j \equiv (i, t_0, \ldots, t_{n-1}, n)$ witnesses σ_i,

$$t_{n-m-1} \equiv (i, t_0, \ldots, t_{n-m-2}, n-m-1)$$

witnesses σ_i, too. Similar as above, with L_{10} we get

$$\overline{\mathsf{T}} + \sigma_{i,n-m-1}[t_{n-m-1}] \vdash \exists v_n\tilde{\sigma}(v_n)\,,$$

and with the DEDUCTION THEOREM we obtain

$$\overline{\mathsf{T}} \vdash \sigma_{i,n-m-1}[t_{n-m-1}] \to \exists v_n\tilde{\sigma}(v_n)\,.$$

This shows that if we derive a contradiction from

$$\overline{\mathsf{T}} + \Sigma + \exists v_n \tilde{\sigma}(v_n)\,,$$

then we also derive a contradiction from

$$\overline{\mathsf{T}} + \Sigma + \sigma_{i,n-m-1}[t_{n-m-1}]\,,$$

which is again a contradiction to the minimality of the sum $n_1 + \dots n_k + k$.

Therefore, $\overline{\mathsf{T}} + \{\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]\}$ is consistent, and since the finitely many $\mathscr{L}_c$-sentences $\sigma_{i_1,n_1}[c_{j_1}], \dots, \sigma_{i_k,n_k}[c_{j_k}]$ were arbitrary, we obtain that $\overline{\mathsf{T}}_c$ is consistent, which completes the proof. ⊣

The Completeness Theorem for Countable Signatures

In this section we shall construct a model of the $\mathscr{L}_c$-theory $\overline{\mathsf{T}}_c$, which is of course also a model of the $\mathscr{L}$-theory $\mathsf{T} + \neg\sigma_0$. However, since we extended the signature $\mathscr{L}$, we first have to extend the binary relation "=", as well as relation symbols in $\mathscr{L}$, to the new closed $\mathscr{L}_c$-terms.

LEMMA 5.2. *The list $\overline{\mathsf{T}}_c$ can be extended to a consistent list $\tilde{\mathsf{T}}$ of $\mathscr{L}_c$-sentence, such that the new $\mathscr{L}_c$-sentences are variable-free and for each variable-free $\mathscr{L}_c$-senctence σ we have*

$$\text{either} \quad \sigma \in \tilde{\mathsf{T}} \quad \text{or} \quad \neg\sigma \in \tilde{\mathsf{T}}\,.$$

Proof. Like in the proof of LINDENBAUM'S LEMMA 4.5, we go through the list of all variable-free $\mathscr{L}_c$-sentences and successively extend the list $\overline{\mathsf{T}}_c$ to a maximally consistent list $\tilde{\mathsf{T}}$. ⊣

Now we are ready to construct the domain of a model of $\tilde{\mathsf{T}}$, which shall be a list of lists: For this, let

$$\Lambda_\tau = [t_0, t_1, \dots, t_n, \dots]$$

be the list of all term-constants (ordered with respect to their codes). We go through the list Λ_τ and construct step by step a list of lists: First, we set $A_0 := [\ ]$. Now, assume that A_n is already defined and consider the $\mathscr{L}_c$-sentences

$$t_n = t_0, \quad t_n = t_1, \quad \dots \quad t_n = t_{n-1}\,.$$

If, for some m with $0 \leq m < n$, the sentence $t_n = t_m$ belongs to $\tilde{\mathsf{T}}$, then we append t_n to that list in A_n which contains t_m; the resulting list is A_{n+1}. If none of the sentences $t_n = t_m$ (for $0 \leq m < n$) belongs to $\tilde{\mathsf{T}}$ (*e.g.*, if $n = 0$), then $A_{n+1} := A_n + \big[[t_n]\big]$. Finally, let $A = \big[[t_{n_0}, \dots], [t_{n_1}, \dots] \dots\big]$ be the resulting list. Then, A is a finite or potentially infinite list of potentially infinite lists.

The lists in the list A are the objects of the domain of our model $\mathbf{M} \vDash \tilde{\mathsf{T}}$. To simplify the notation, for term-constants τ let $\tilde{\tau}$ be the unique list of A which contains τ.

In order to get an $\mathscr{L}_c$-structure $\mathbf{M}$ with domain A, we have to define a mapping which assigns to each constant symbol $c \in \mathscr{L}_c$ an element $c^{\mathbf{M}} \in A$, to each n-ary function symbol $F \in \mathscr{L}$ a function $F^{\mathbf{M}} : A^n \to A$, and to each n-ary relation symbol $R \in \mathscr{L}$ a set $R^{\mathbf{M}} \subseteq A^n$:

- If $c \in \mathscr{L}_c$ is a constant symbol of $\mathscr{L}$ or a special constant, then let
$$c^{\mathbf{M}} := \tilde{c}\,.$$
- If $F \in \mathscr{L}$ is an n-ary function symbol and $\tilde{t}_1, \dots, \tilde{t}_n$ are elements of A, then let
$$F^{\mathbf{M}}\tilde{t}_1 \cdots \tilde{t}_n := \widetilde{Ft_1 \cdots t_n}\,.$$
- If $R \in \mathscr{L}$ is an n-ary relation symbol and $\tilde{t}_1, \dots, \tilde{t}_n$ are elements of A, then we define
$$\langle \tilde{t}_1, \dots, \tilde{t}_n \rangle \in R^{\mathbf{M}} \quad :\Longleftrightarrow \quad Rt_1 \cdots t_n \in \tilde{\mathsf{T}}\,.$$

FACT 5.3. *The definitions above, which rely on representatives of the lists in A, are well-defined.*

Proof. This follows easily by L_{14}, L_{15}, and L_{16}, and the construction of $\tilde{\mathsf{T}}$; the details are left as an exercise to the reader. ⊣

THEOREM 5.4. *The $\mathscr{L}_c$-structure $\mathbf{M}$ is a model of $\tilde{\mathsf{T}}$, and therefore also of the $\mathscr{L}$-theory $\mathsf{T} + \neg\sigma_0$.*

Proof. We have to show that for each $\mathscr{L}_c$-sentence σ, if $\sigma \in \tilde{\mathsf{T}}$ then $\mathbf{M} \vDash \sigma$. We show slightly more, namely that for each $\mathscr{L}_c$-sentence σ we have
$$\tilde{\mathsf{T}} \vdash \sigma \implies \mathbf{M} \vDash \sigma\,, \quad \text{or equivalently} \quad \mathbf{M} \nvDash \sigma \implies \tilde{\mathsf{T}} \nvdash \sigma\,.$$

First we consider the case when σ is variable-free: The proof is by induction on the number of logical operators. By LEMMA 5.2 we know that for each variable-free $\mathscr{L}_c$-sentences σ we have either $\sigma \in \tilde{\mathsf{T}}$ or $\neg\sigma \in \tilde{\mathsf{T}}$. Hence, we must show that for each variable-free $\mathscr{L}_c$-sentences σ we have $\sigma \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma$.

If σ is variable-free and does not contain logical operators, then σ is atomic. In this case, we have either $\sigma \equiv t_1 = t_2$ (for some term-constants t_1 and t_1) or $\sigma \equiv Rt_1 \cdots t_n$ (for an n-ary relation symbol $R \in \mathscr{L}$ and term-constants $t_1, \dots, t_n$), and by construction of $\mathbf{M}$, in both cases we get $\sigma \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma$.

Before we consider the case when σ is variable-free and contains logical operators, recall that for any $\mathscr{L}_c$-sentence $\tilde{\sigma}$ with $\sigma \Leftrightarrow \tilde{\sigma}$, by the SOUNDNESS THEOREM 3.7 we get $\mathbf{M} \vDash \sigma$ if and only if $\mathbf{M} \vDash \tilde{\sigma}$. So, by the 3-SYMBOLS THEOREM 1.2, we may assume that σ is either of the form $\neg\sigma'$ or of the form $\wedge\sigma_1\sigma_2$. Now, let σ be a non-atomic, variable-free $\mathscr{L}_c$-sentence, and assume that for each variable-free $\mathscr{L}_c$-sentence σ' which contains fewer logical operators than σ, we have $\sigma' \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma'$. By our former assumption, we just have to consider the following two cases:

$\sigma \equiv \neg\sigma'$: Since σ' has fewer logical operators than σ, we have $\sigma' \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma'$. This shows that

$$\neg\sigma' \notin \tilde{\mathsf{T}} \iff \mathbf{M} \nvDash \neg\sigma', \quad \text{or equivalently} \quad \sigma \in \tilde{\mathsf{T}} \iff \mathbf{M} \vDash \neg\sigma .$$

$\sigma \equiv \wedge\sigma_1\sigma_2$: Since each if σ_1 and σ_2 has fewer logical operators than $\tilde{\sigma}$, we have $\sigma_1 \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma_1$, and $\sigma_2 \in \tilde{\mathsf{T}}$ if and only if $\mathbf{M} \vDash \sigma_2$. Hence, we obtain

$$\wedge\sigma_1\sigma_2 \in \tilde{\mathsf{T}} \iff \sigma_1 \in \tilde{\mathsf{T}} \ \text{AND} \ \sigma_2 \in \tilde{\mathsf{T}} \iff \mathbf{M} \vDash \sigma_1 \ \text{AND} \ \mathbf{M} \vDash \sigma_2 \iff \mathbf{M} \vDash \wedge\sigma_1\sigma_2$$

which shows that $\sigma \in \tilde{\mathsf{T}} \iff \mathbf{M} \vDash \sigma$.

We now consider the case when the $\mathscr{L}_c$-sentence $\sigma \in \tilde{\mathsf{T}}$ contains variables: The proof is by induction on the number of different variables which appear in σ. If $\sigma \in \tilde{\mathsf{T}}$ is an $\mathscr{L}_c$-sentence which contains variables, then, by the construction of $\overline{\mathsf{T}}_c$, there is a $\tilde{\sigma} \in \overline{\mathsf{T}}_c$ in sPNF, say

$$\tilde{\sigma} \equiv \forall\!\!\!\exists_0 v_0 \cdots \forall\!\!\!\exists_n v_n \sigma_{i,n}(v_0, \dots, v_n), \quad \text{where } \sigma_{i,n} \text{ is quantifier free,}$$

such that for some natural numbers i, k, n with $k \leq n$ and some term-constants $t_0, \dots, t_{k-1}$ we have

$$\sigma \equiv \forall\!\!\!\exists_k v_k \cdots \forall\!\!\!\exists_n v_n \sigma_{i,n}(v_0/t_0, \dots, v_{k-1}/t_{k-1}, v_k, \dots, v_n).$$

Now, let σ an $\mathscr{L}_c$-sentence of the above form and assume that for each $\mathscr{L}_c$-sentence σ' which contains fewer variables than σ we have

$$\tilde{\mathsf{T}} \vdash \sigma' \implies \mathbf{M} \vDash \sigma', \quad \text{or equivalently} \quad \mathbf{M} \nvDash \sigma' \implies \tilde{\mathsf{T}} \nvdash \sigma'.$$

We are in exactly one of the following two cases:

- If $\forall\!\!\!\exists_k$ is the quantifier "$\exists$" and $\tilde{\mathsf{T}} \vdash \sigma$, then $\sigma \in \overline{\mathsf{T}}_c$ and for the special constant

  $$t_k \equiv (i, t_0, \dots, t_{k-1}, k)$$

  and the $\mathscr{L}_c$-sentence

  $$\sigma' \equiv \forall\!\!\!\exists_{k+1} v_{k+1} \cdots \forall\!\!\!\exists_n v_n \sigma_{i,n}(v_0/t_0, \dots, v_k/t_k, v_{k+1}, \dots),$$

  we have $\sigma' \in \overline{\mathsf{T}}_c$, and consequently $\sigma' \in \tilde{\mathsf{T}}$. Now, since σ' has fewer variables than σ, by our assumption we conclude that $\mathbf{M} \vDash \sigma'$, and therefore, by L_{11} and the SOUNDNESS THEOREM 3.7, we obtain $\mathbf{M} \vDash \sigma$. Hence,

  $$\tilde{\mathsf{T}} \vdash \sigma \implies \mathbf{M} \vDash \sigma .$$

- If $\forall\!\!\!\exists_k$ is the quantifier "$\forall$" and $\mathbf{M} \nvDash \sigma$, then $\mathbf{M} \vDash \neg\sigma$. Now, for the $\mathscr{L}_c$-sentence

$$\tilde{\sigma} \equiv \exists v_k \overline{\exists\!\!\!\forall}_{k+1} v_{k+1} \cdots \overline{\exists\!\!\!\forall}_n v_n \neg\sigma_{i,n}(v_0/t_0, \dots, v_{k-1}/t_{k-1}, v_k, \dots, v_n)\,,$$

where for $k < i \leq n$, the quantifier $\overline{\exists\!\!\!\forall}_i$ is "$\exists$" if $\exists\!\!\!\forall_i$ is "$\forall$", and vice versa, we have $\tilde{\sigma} \Leftrightarrow \neg\sigma$, and therefore $\mathbf{M} \vDash \tilde{\sigma}$. For the special constant

$$t_k \equiv (i, t_0, \dots, t_{k-1}, k)$$

and the $\mathscr{L}_c$-sentence

$$\bar{\sigma} \equiv \overline{\exists\!\!\!\forall}_{k+1} v_{k+1} \cdots \overline{\exists\!\!\!\forall}_n v_n \neg\sigma_{i,n}(v_0/t_0, \dots, v_k/t_k, v_{k+1}, \dots, v_n)\,,$$

we obtain $\mathbf{M} \vDash \bar{\sigma}$, *i.e.*, $\mathbf{M} \nvDash \neg\bar{\sigma}$. Now, since $\neg\bar{\sigma}$ has fewer variables than σ, by our assumption we conclude that $\tilde{\mathsf{T}} \nvdash \neg\bar{\sigma}$, *i.e.*,

$$\tilde{\mathsf{T}} \nvdash \exists\!\!\!\forall_{k+1} v_{k+1} \cdots \exists\!\!\!\forall_n v_n \sigma_{i,n}(v_0/t_0, \dots, v_k/t_k, v_{k+1}, \dots, v_n)\,,$$

and finally obtain $\tilde{\mathsf{T}} \nvdash \sigma$ (notice that by L_{10}, $\tilde{\mathsf{T}} \vdash \sigma$ implies $\tilde{\mathsf{T}} \vdash \neg\bar{\sigma}$). Hence,

$$\mathbf{M} \nvDash \sigma \implies \tilde{\mathsf{T}} \nvdash \sigma\,,$$

⊣

The following theorem just summarises what we have achieved so far:

GÖDEL'S COMPLETENESS THEOREM 5.5. *If $\mathscr{L}$ is a countable signature and T is a consistent set of $\mathscr{L}$-sentences, then T has a model. Moreover, if $\mathsf{T} \nvdash \sigma_0$ (for some $\mathscr{L}$-sentence σ_0), then $\mathsf{T} + \neg\sigma_0$ has a model.*

In our construction, it was essential that the signature $\mathscr{L}$ was countable, so that the symbols in $\mathscr{L}$ could be encoded by finite strings. However, in the more formal setting of axiomatic Set Theory, we can prove the COMPLETENESS THEOREM also for arbitrarily large signatures (see Chapter **??**).

Some Consequences and Equivalents

We conclude this chapter by discussing some consequences and equivalent formulations of GÖDEL'S COMPLETENESS THEOREM 5.5 which follow directly or in combination with the COMPACTNESS THEOREM 2.12.

Let $\mathscr{L}$ be a countable signature, T a set of $\mathscr{L}$-sentences, and σ_0 an $\mathscr{L}$-sentence.

- If $\mathsf{T} \nvdash \sigma_0$, then there is an $\mathscr{L}$-structure $\mathbf{M}$ such that $\mathbf{M} \vDash \mathsf{T} + \neg\sigma_0$:

$$\mathsf{T} \nvdash \sigma_0 \implies \exists\, \mathbf{M}\, \big(\mathbf{M} \vDash \mathsf{T} + \neg\sigma_0\big)$$

  This is just a reformulation of GÖDEL'S COMPLETENESS THEOREM 5.5.

- If T is consistent, then T has a model:

$$\mathrm{Con}(\mathsf{T}) \quad \Longrightarrow \quad \exists\, \mathbf{M}\, \big(\mathbf{M} \vDash \mathsf{T}\big)$$

  This follows from the fact that $\mathrm{Con}(\mathsf{T})$ is equivalent to the existence of an $\mathscr{L}$-sentence σ_0 such that $\mathsf{T} \nvdash \sigma_0$.

- If each model of T is also a model of σ_0, then $\mathsf{T} \vdash \sigma_0$:

$$\forall\, \mathbf{M}\, \big(\mathbf{M} \vDash \mathsf{T} \Longrightarrow \mathbf{M} \vDash \sigma_0\big) \quad \Longrightarrow \quad \mathsf{T} \vdash \sigma_0$$

  This follows by contraposition: If $\mathsf{T} \nvdash \sigma_0$, then, by GÖDEL'S COMPLETENESS THEOREM 5.5, there is a model $\mathbf{M} \vDash \mathsf{T} + \neg\sigma_0$.

- In combination with the COMPACTNESS THEOREM 2.12 we obtain the following implication:

  If every finite subset T' of T has a model, then T has a model.

  If every finite subset T' of T has a model, then every finite subset T' of T is consistent, and therefore, by the COMPACTNESS THEOREM 2.12, T is consistent. Thus, T has a model.

The most important consequence of the COMPACTNESS THEOREM 2.12 and the SOUNDNESS THEOREM 3.7 is the following equivalence:

$$\underbrace{\forall\, \mathbf{M}\, \big(\mathbf{M} \vDash \mathsf{T} \Longrightarrow \mathbf{M} \vDash \sigma_0\big)}_{\text{denoted } \ \mathsf{T} \vDash \sigma_0} \quad \Longleftrightarrow \quad \mathsf{T} \vdash \sigma_0$$

This equivalence allows us to replace *formal proofs* with *mathematical proofs*: For example, instead of proving formally the uniqueness of the neutral element in groups from the axioms of Group Theory GT, we just show that in every model of GT (*i.e.*, in every group), the neutral element is unique. So, instead of $\mathsf{GT} \vdash \sigma_0$, we just show $\mathsf{GT} \vDash \sigma_0$.

As a last consequence we would like to mention the so-called *Skolem's Paradox*, which is in fact just the countable version of the DOWNWARD LÖWENHEIM–SKOLEM THEOREM **??**.

THEOREM 5.6 (Skolem's Paradox). *If $\mathscr{L}$ is a countable signature and T is a consistent set of $\mathscr{L}$-sentences, then T has a countable model.*

Proof. In the previous chapter, we began with a countable signature $\mathscr{L}$ and a consistent set of $\mathscr{L}$-sentences T, and at the end, we obtained a model of T whose domain was a finite or potentially infinite list of lists. So, the model of T we constructed is countable. $\dashv$

NOTES

The COMPLETENESS THEOREM for countable signatures was first proved by Gödel [10, 11]. Later, a modified proof was given by Henkin, LeonHenkin [15] (see also [16]). The proof given here is essentially Henkin's proof, but in contrast to Henkin's proof, our construction does not rely on the assumption that an *actually infinite* set exists.

EXERCISES

5.0 Let $\mathscr{L}$ be a countable signature and let T be a consistent set of $\mathscr{L}$-sentences. For each subset $\Phi \subseteq \mathsf{T}$ let $\mathbf{M}_\Phi$ be a model for Φ, let $\Sigma := \big\{\mathbf{M}_\Phi : \Phi \subseteq \mathsf{T} \textit{ and } \mathrm{Con}(\Phi)\big\}$, and for each $\mathscr{L}$-sentence $\varphi \in \mathsf{T}$, let $X_\varphi := \{\mathbf{M} \in \Sigma : \mathbf{M} \vDash \varphi\}$.

(a) Show that the set $\{X_\varphi : \varphi$ an $\mathscr{L}$-sentence$\}$ is a basis for a topology on Σ.

(b) Show that for each $\varphi \in \mathsf{T}$, X_φ is closed.

(c) Show with the topological compactness theorem that each open covering of Σ contains a finite sub-covering, *i.e.*, the topological space Σ is compact.

5.1 Let DLO be the — assumingly consistent — theory of dense linearly ordered sets without end-points (see EXERCISE 3.5).

(a) Show that the theory DLO is complete, (*i.e.*, for all $\mathscr{L}_{\mathsf{DLO}}$-sentences σ we have *either* $\mathsf{DLO} \vdash \sigma$ *or* $\mathsf{DLO} \vdash \neg\sigma$).

Hint: Assume towards a contradiction that there exists an $\mathscr{L}_{\mathsf{DLO}}$-sentence σ, such that $\mathsf{DLO} \nvdash \neg\sigma$ and $\mathsf{DLO} \nvdash \sigma$. Then $\mathsf{DLO} + \sigma$ and $\mathsf{DLO} + \neg\sigma$ are both consistent, and therefore, by SKOLEM'S PARADOX 5.6, there are countable models $\mathbf{M}$ and $\mathbf{N}$ such that $\mathbf{M} \vDash \mathsf{DLO} + \sigma$ and $\mathbf{N} \vDash \mathsf{DLO} + \neg\sigma$, which contradicts the fact that any two countable models of DLO are isomorphic (see EXERCISE 3.5).

(b) Show that the converse of EXERCISE 3.4 does not hold.

Hint: Let $\mathbb{Q}$ be the set of rational numbers, let $\mathbb{R}$ be the set of real numbers, and let "$<$" be the natural ordering on $\mathbb{Q}$ and $\mathbb{R}$, respectively. Then the two non-isomorphic $\mathscr{L}_{\mathsf{DLO}}$-structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are both models of DLO.

5.2 Let $\mathscr{L}$ be a countable signature and let T be a consistent set of $\mathscr{L}$-sentences sucht that T has arbitrarily large finite models.

(a) Show that T has an infinite model.

Hint: Use the COMPACTNESS THEOREM 2.12.

(b) Show that the notion of F I N I T E N E S S cannot be formalised in First-Order Logic.