Chapter 6
Language Extensions by Definitions

Sometimes it is convenient to extend a given signature $\mathscr{L}$ by adding new non-logical symbols which have to be defined properly within the language $\mathscr{L}$ or with respect to a given $\mathscr{L}$-theory T. Let the extended signature be $\mathscr{L}^*$ and let the corresponding extended $\mathscr{L}^*$-theory be T^*. Since T is an $\mathscr{L}$-theory, we can just prove $\mathscr{L}$-sentences from T but no $\mathscr{L}^*$-sentences which contain symbols from $\mathscr{L}^* \setminus \mathscr{L}$. However, this does not imply that we can prove substantially more from T^* than from T: It might be that for each $\mathscr{L}^*$-sentence σ^* which is provable from T^* there is an $\mathscr{L}$-sentence $\tilde{\sigma}$, such that $\mathsf{T}^* \vdash \sigma^* \leftrightarrow \tilde{\sigma}$ and $\mathsf{T} \vdash \tilde{\sigma}$; which is indeed the case as we shall see below.

Defining new Relation Symbols

Let us first consider an example from Peano Arithmetic: Extend the signature $\mathscr{L}_{\mathsf{PA}}$ of Peano Arithmetic by adding the binary relation symbol "$<$" and denote the extended signature by $\mathscr{L}^*_{\mathsf{PA}} := \mathscr{L}_{\mathsf{PA}} \cup \{<\}$. In order to define the binary relation "$<$", we give an $\mathscr{L}_{\mathsf{PA}}$-formula $\psi_<$ with two free variables (*e.g.*, x and y) and say that the relation $x < y$ holds if and only if $\psi_<(x, y)$ holds. In our case, $\psi_<(x, y) \equiv \exists z(x + \mathtt{s}z = y)$. So, we would define "$<$" by stipulating:

$$x < y :\Longleftrightarrow \exists z(x + \mathtt{s}z = y)$$

The problem is now to find for each $\mathscr{L}^*_{\mathsf{PA}}$-sentence σ^* an $\mathscr{L}_{\mathsf{PA}}$-sentence $\tilde{\sigma}$ and an extension PA^* of PA, such that $\mathsf{PA}^* \vdash \sigma^* \leftrightarrow \tilde{\sigma}$ and whenever $\mathsf{PA}^* \vdash \sigma^*$, then $\mathsf{PA} \vdash \tilde{\sigma}$.

The following result provides an algorithm which transforms sentences σ^* in the extended language into equivalent sentences $\tilde{\sigma}$ in the original language:

THEOREM 6.1. *Let $\mathscr{L}$ be a signature, let R be an n-ary relation symbol which does not belong to $\mathscr{L}$, and let $\mathscr{L}^* := \mathscr{L} \cup \{R\}$. Furthermore, let $\psi_R(v_1, \ldots, v_n)$ be an $\mathscr{L}$-formula with* $\mathrm{free}(\psi_R) = \{v_1, \ldots, v_n\}$ *and let*

$$\vartheta_R \equiv \forall v_1 \cdots \forall v_n \big(R v_1 \cdots v_n \leftrightarrow \psi_R(v_1, \ldots, v_n) \big) \,.$$

Finally, let T *be a consistent* $\mathscr{L}$*-theory and let* $\mathsf{T}^* := \mathsf{T} + \vartheta_R$*.*

Then there exists an effective algorithm which transforms each $\mathscr{L}^*$*-formula* φ^* *into an* $\mathscr{L}$*-formula* $\tilde{\varphi}$*, such that:*

(a) *If* R *does not appear in* φ^**, then* $\tilde{\varphi} \equiv \varphi^*$*.*

(b) $\widetilde{\neg\varphi} \equiv \neg\tilde{\varphi}$ *(for* $\varphi^* \equiv \neg\varphi$*)*

(c) $\widetilde{\wedge\varphi_1\varphi_2} \equiv \wedge\tilde{\varphi}_1\tilde{\varphi}_2$ *(for* $\varphi^* \equiv \wedge\varphi_1\varphi_2$*)*

(d) $\widetilde{\exists\nu\varphi} \equiv \exists\nu\tilde{\varphi}$ *(for* $\varphi^* \equiv \exists\nu\varphi$*)*

(e) $\mathsf{T}^* \vdash \varphi^* \leftrightarrow \tilde{\varphi}$

(f) *If* $\mathsf{T}^* \vdash \varphi^*$*, then* $\mathsf{T} \vdash \tilde{\varphi}$*.*

Proof. Let φ^* be an arbitrary $\mathscr{L}^*$-formula. In φ^* we replace each occurrence of $R(v_1/\tau_1, \ldots, v_n/\tau_n)$ (where $\tau_1, \ldots, \tau_n$ are $\mathscr{L}$-terms) with a particular $\mathscr{L}^*$-formula $\psi'_R(v_1/\tau_1, \ldots, v_n/\tau_n)$ such that

$$\psi'_R(v_1, \ldots, v_n) \Leftrightarrow \psi_R(v_1, \ldots, v_n)$$

and none of the bound variables in ψ'_R is among $v_1, \ldots, v_n$ or appears in one of the $\mathscr{L}$-terms $\tau_1, \ldots, \tau_n$. In fact, to obtain ψ'_R we just have to rename the bound variables in ψ_R. For the resulting $\mathscr{L}$-formula $\tilde{\varphi}$, (a)–(d) are obviously satisfied.

We prove (e) and (f) on the semantic level: For this, we first show how we can extend a model $\mathbf{M} \vDash \mathsf{T}$ to a model $\mathbf{M}^* \vDash \mathsf{T}^*$. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A such that for each assignment j we have $(\mathbf{M}, j) \vDash \mathsf{T}$ (*i.e.*, $\mathbf{M} \vDash \mathsf{T}$). We extend $\mathbf{M}$ to an $\mathscr{L}^*$-structure $\mathbf{M}^*$ with the same domain A by stipulating $\mathbf{M}^*|_{\mathscr{L}} := \mathbf{M}$, and for any $a_1, \ldots, a_n \in A$:

$$R^{\mathbf{M}^*}(a_1, \ldots, a_n) \;:\Longleftrightarrow\; \big(\mathbf{M}, j\tfrac{a_1}{v_1} \cdots \tfrac{a_n}{v_n}\big) \vDash \psi_R(v_1, \ldots, v_n) \,.$$

Then $\mathbf{M}^*$ is an $\mathscr{L}^*$-structure and for every assignment j we have

$$(\mathbf{M}^*, j) \vDash \mathsf{T} \quad \text{and} \quad (\mathbf{M}^*, j) \vDash \vartheta_R \,,$$

and therefore we obtain

$$\mathbf{M}^* \vDash \mathsf{T}^* \,.$$

To prove (e), by the GÖDEL-HENKIN COMPLETENESS THEOREM it is enough to show that $\varphi^* \leftrightarrow \tilde{\varphi}$ holds in every model $\mathbf{M}^*$ of T^*. So, let $\mathbf{M}^*$ be an arbitrary model of T^*. In particular, $\mathbf{M}^* \vDash \vartheta_R$. If φ^* does not contain R, then we are done. Otherwise, if φ^* is atomic, then $\varphi^* \equiv R t_1 \cdots t_n$ for some $\mathscr{L}$-terms $t_1, \ldots, t_n$. Since $\mathbf{M}^* \vDash \vartheta_R$, we get

$$\mathbf{M}^* \vDash R t_1 \cdots t_n \leftrightarrow \psi'_R(t_1, \ldots, t_n) \,.$$

This shows $\mathbf{M}^* \vDash \varphi^* \leftrightarrow \tilde{\varphi}$ for atomic formulas and by (b)–(d) we get the result for arbitrary formulas.

For (f), we first extend an arbitrary model $\mathbf{M} \vDash \mathsf{T}$ to a model $\mathbf{M}^* \vDash \mathsf{T}^*$. By (e), for each $\mathscr{L}^*$-formula φ^* we have

$$\mathbf{M}^* \vDash \varphi^* \iff \mathbf{M}^* \vDash \tilde{\varphi}\,.$$

Now, if $\mathsf{T}^* \vdash \varphi^*$, then $\mathbf{M}^* \vDash \varphi^*$, which implies that $\mathbf{M}^* \vDash \tilde{\varphi}$. Since $\tilde{\varphi}$ is an $\mathscr{L}$-formula, we get $\mathbf{M} \vDash \tilde{\varphi}$, and since the model $\mathbf{M}$ of T was arbitrary, by the GÖDEL-HENKIN COMPLETENESS THEOREM we get $\mathsf{T} \vdash \tilde{\varphi}$. ⊣

Defining new Function Symbols

The situation is slightly more subtle if we define new functions. However, there is also an algorithm which transforms sentences σ^* in the extended language into equivalent sentences $\tilde{\sigma}$ in the original language:

THEOREM 6.2. *Let $\mathscr{L}$ be a signature, let f be an n-ary function symbol which does not belong to $\mathscr{L}$, let $\mathscr{L}^* := \mathscr{L} \cup \{f\}$ and let T be a consistent $\mathscr{L}$-theory. Furthermore, let $\psi_f(v_1, \ldots, v_n, y)$ be an $\mathscr{L}$-formula with $\mathrm{free}(\psi_f) = \{v_1, \ldots, v_n, y\}$ such that*

$$\mathsf{T} \vdash \forall v_1 \cdots \forall v_n \exists! y\, \psi_f(v_1, \ldots, v_n, y)\,.$$

Finally, let

$$\vartheta_f \equiv \forall v_1 \cdots \forall v_n \forall y \big(f v_1 \cdots v_n = y \leftrightarrow \psi_f(v_1, \ldots, v_n, y) \big)$$

and let $\mathsf{T}^* := \mathsf{T} + \vartheta_f$.

Then there exists an effective algorithm which transforms each $\mathscr{L}^*$-formula φ^* into an $\mathscr{L}$-formula $\tilde{\varphi}$, such that:

(a) *If f does not appear in φ^*, then $\tilde{\varphi} \equiv \varphi^*$.*

(b) $\widetilde{\neg\varphi} \equiv \neg\tilde{\varphi}$ *(for $\varphi^* \equiv \neg\varphi$)*

(c) $\widetilde{\wedge\varphi_1\varphi_2} \equiv \wedge\tilde{\varphi}_1\tilde{\varphi}_2$ *(for $\varphi^* \equiv \wedge\varphi_1\varphi_2$)*

(d) $\widetilde{\exists\nu\varphi} \equiv \exists\nu\tilde{\varphi}$ *(for $\varphi^* \equiv \exists\nu\varphi$)*

(e) $\mathsf{T}^* \vdash \varphi^* \leftrightarrow \tilde{\varphi}$

(f) *If $\mathsf{T}^* \vdash \varphi^*$, then $\mathsf{T} \vdash \tilde{\varphi}$.*

Proof. By an *elementary f-term* we mean an $\mathscr{L}^*$-term of the form $f t_1 \cdots t_n$, where $t_1, \ldots, t_n$ are $\mathscr{L}^*$-terms which do not contain the symbol f. We first prove the theorem for atomic $\mathscr{L}^*$-formulae φ^* (i.e., for formulae which are free of quantifiers and logical operators). Let $\varphi^*(f\,|\,w)$ be the result of replacing the leftmost occurence of an elementary f-term in φ^* with a new symbol w, which stands for a new variable. Then, the formula

$$\exists w\big(\psi_f(t_1,\ldots,t_n,w)\wedge\varphi^*(f\,|\,w)\big)$$

is called the *f-transform of* φ^*. If φ^* does not contain f, then let φ^* be its own f-transform. Before we procceed, let us prove the following

CLAIM. $\mathsf{T}^*\vdash\exists w\big(\psi_f(t_1,\ldots,t_n,w)\wedge\varphi^*(f\,|\,w)\big)\leftrightarrow\varphi^*$

Proof of Claim. Let $\mathbf{M}^*$ be a model of T^* with domain A, let j be an arbitrary assignment which assigns to w an element of A, and let $\mathbf{M}^*_j := (\mathbf{M}^*, j)$ be the corresponding $\mathscr{L}^*$-interpretation.

Assume that

$$\mathbf{M}^*_j\vDash\exists w\big(\psi_f(t_1,\ldots,t_n,w)\wedge\varphi^*(f\,|\,w)\big)\,.$$

Then, since $\mathsf{T}^*\vdash\forall v_1\cdots\forall v_n\exists!y\,\psi_f(v_1,\ldots,v_n,y)$, there exists a unique $b\in A$ such that

$$\mathbf{M}^*_{j\frac{b}{w}}\vDash\psi_f(t_1,\ldots,t_n,w)\wedge\varphi^*(f\,|\,w)\,,$$

which is the same as saying that

$$\mathbf{M}^*_j\vDash\psi_f(t_1,\ldots,t_n,b)\wedge\varphi^*(f\,|\,b)\,.$$

Now, since $\mathbf{M}^*_j\vDash\vartheta_f$, b is the same object as $f^{\mathbf{M}^*_j}t_1^{\mathbf{M}^*_j}\cdots t_n^{\mathbf{M}^*_j}$. This implies

$$\mathbf{M}^*_j\vDash ft_1\cdots t_n=b\,,$$

and shows that

$$\mathbf{M}^*_j\vDash\varphi^*\,.$$

For the reverse implication assume that $\mathbf{M}^*_j\vDash\varphi^*$ and let b be the same object as $f^{\mathbf{M}^*_j}t_1^{\mathbf{M}^*_j}\cdots t_n^{\mathbf{M}^*_j}$. Then $\mathbf{M}^*_j\vDash\varphi^*(f\,|\,b)$ and, since $\mathbf{M}^*_j\vDash\vartheta_f$,

$$\mathbf{M}^*_j\vDash\psi_f(t_1,\ldots,t_n,w)\leftrightarrow ft_1\cdots t_n=w\,.$$

In particular we get

$$\mathbf{M}^*_{j\frac{b}{w}}\vDash\psi_f(t_1,\ldots,t_n,b)\leftrightarrow ft_1\cdots t_n=b\,,$$

and because $f^{\mathbf{M}^*_j}t_1^{\mathbf{M}^*_j}\cdots t_n^{\mathbf{M}^*_j}$ is the same object as b, we get $\mathbf{M}^*_j\vDash\psi_f(t_1,\ldots,t_n,b)$, and since we already know $\mathbf{M}^*_j\vDash\varphi^*(f\,|\,b)$, we have

$$\mathbf{M}^*_j\vDash\psi_f(t_1,\ldots,t_n,b)\wedge\varphi^*(f\,|\,b)\,.$$

So, there exists a b in A, such that

$$\mathbf{M}^*_{j\frac{b}{w}} \vDash \psi_f(t_1,\dots,t_n,w) \wedge \varphi^*(f|w)\,,$$

which is the same as saying that

$$\mathbf{M}^*_j \vDash \exists w\big(\psi_f(t_1,\dots,t_n,w) \wedge \varphi^*(f|w)\big)\,.$$

Since the model $\mathbf{M}^*$ of T^* was arbitrary, by the GÖDEL-HENKIN COMPLETENESS THEOREM we get $\mathsf{T}^* \vdash \exists w\big(\psi_f(t_1,\dots,t_n,w) \wedge \varphi^*(f|w)\big) \leftrightarrow \varphi^*$. $\dashv_{\text{Claim}}$

Since the f-transform $\exists w\big(\psi_f(t_1,\dots,t_n,w) \wedge \varphi^*(f|w)\big)$ of φ^* contains one less f than φ^*, if we take successive f-transforms (introducing always new variables), eventually we obtain an an atomic $\mathscr{L}$-formula $\tilde{\varphi}$ (*i.e.*, a formula which does not contain f) such that $\mathsf{T}^* \vdash \varphi^* \leftrightarrow \tilde{\varphi}$. We call $\tilde{\varphi}$ the *f-less transform of φ^**.

In order to get f-less transforms of non-atomic $\mathscr{L}^*$-formulae φ^*, we just extend the definition by letting $\widetilde{\neg\varphi}$ be $\neg\tilde{\varphi}$, $\widetilde{\wedge\varphi_1\varphi_2}$ be $\wedge\tilde{\varphi}_1\tilde{\varphi}_2$, and $\widetilde{\exists\nu\varphi}$ be $\exists\nu\tilde{\varphi}$; properties (a)–(e) are then obvious.

It remains to prove property (f). Let $\mathbf{M}_0$ be an abitrary model of T with domain A. Then, since $\mathsf{T} \vdash \forall v_1 \cdots \forall v_n \exists! y\, \psi_f(v_1,\dots,v_n,y)$, for all $a_1,\dots,a_n$ in A there exists a unique b in A such that

$$\mathbf{M}_0 \vDash \psi_f(a_1,\dots,a_n,b)$$

and we define the n-ary function f^* on A by stipulating:

$$f^*(a_1,\dots,a_n) := b$$

With this definition, we can extend the $\mathscr{L}$-structure $\mathbf{M}_0$ to an $\mathscr{L}^*$-structure $\mathbf{M}^*_0$, where we still have $\mathbf{M}^* \vDash \mathsf{T}$. With the definition of f^* we get in addition $\mathbf{M}^*_0 \vDash \vartheta_f$, which implies $\mathbf{M}^*_0 \vDash \mathsf{T}^*$. If we have $\mathsf{T}^* \vdash \varphi^*$, for some $\mathscr{L}^*$-formula φ^*, then there exists an $\mathscr{L}$-formula $\tilde{\varphi}$, such that $\mathsf{T}^* \vdash \varphi^* \leftrightarrow \tilde{\varphi}$, *i.e.*, $\mathsf{T}^* \vdash \tilde{\varphi}$. Since $\mathsf{T}^* \vdash \tilde{\varphi}$ implies $\mathbf{M}^*_0 \vDash \tilde{\varphi}$, and because $\tilde{\varphi}$ is an $\mathscr{L}$-formula, we have $\mathbf{M}_0 \vDash \tilde{\varphi}$. Now, since the model $\mathbf{M}_0$ of T was arbitrary, by the GÖDEL-HENKIN COMPLETENESS THEOREM we get $\mathsf{T} \vdash \tilde{\varphi}$. $\dashv$

Defining new Constant Symbols

Constant symbols can be handled like 0-are function symbols:

FACT 6.3. *Let $\mathscr{L}$ be a signature, let c be constant symbol which does not belong to $\mathscr{L}$, let $\mathscr{L}^* := \mathscr{L} \cup \{c\}$ and let T be a consistent $\mathscr{L}$-theory. Furthermore, let $\psi_c(y)$ be an $\mathscr{L}$-formula with $\mathrm{free}(\psi_c) = \{y\}$ such that $\mathsf{T} \vdash \exists! y\, \psi_c(y)$. Finally, let*

$$\vartheta_c \equiv \forall y\big(c = y \leftrightarrow \psi_c(y)\big)$$

and let $\mathsf{T}^* := \mathsf{T} + \vartheta_c$.

Then there exists an effective algorithm which transforms each $\mathscr{L}^*$-formula φ^* into an $\mathscr{L}$-formula $\tilde{\varphi}$, such that:

(a) *If f does not appear in φ^*, then $\tilde{\varphi} \equiv \varphi^*$.*

(b) $\widetilde{\neg\varphi} \equiv \neg\tilde{\varphi}$ *(for $\varphi^* \equiv \neg\varphi$)*

(c) $\widetilde{\wedge\varphi_1\varphi_2} \equiv \wedge\tilde{\varphi}_1\tilde{\varphi}_2$ *(for $\varphi^* \equiv \wedge\varphi_1\varphi_2$)*

(d) $\widetilde{\exists\nu\varphi} \equiv \exists\nu\tilde{\varphi}$ *(for $\varphi^* \equiv \exists\nu\varphi$)*

(e) $\mathsf{T}^* \vdash \varphi^* \leftrightarrow \tilde{\varphi}$

(f) *If $\mathsf{T}^* \vdash \varphi^*$, then $\mathsf{T} \vdash \tilde{\varphi}$.*

Proof. The algorithm is constructed in exactly the same way as in the proof of THEOREM 6.2. ⊣

NOTES

In this chapter, we mainly followed Mendelson [25, Sec. 2.9].

EXERCISES

6.0 Show that in a signature $\mathscr{L}$, constant symbols and functions symbols are dispensable (*i.e.*, as non-logical symbols we need only relation symbols).

Hint: Notice that n-ary function symbols can be replaced with $n + 1$-ary relation symbols, and that constant symbols can be replaced with unary relation symbols.