Chapter 7
Countable Models of Peano Arithmetic

By GÖDEL'S COMPLETENESS THEOREM 5.5 we know that every consistent theory T has a model, and if T has an infinite model, then it has also arbitrarily large models. So, if we assume that Peano Arithmetic PA is consistent—what seems sensible—then there exists a model of PA, and because this model is infinite, PA must have arbitrarily large models as well.

In this chapter we provide a few models of PA. First, we construct the so-called *standard model*, then we extend this model to countable *non-standard models*, and finally we construct uncountable models of PA.

The Standard Model

For the sake of completeness, let us first recall the language and the seven axioms of Peano Arithmetic PA:

PA: The language PA is $\mathscr{L}_{\mathsf{PA}} = \{\mathsf{0}, \mathtt{s}, +, \cdot\}$, where "$\mathsf{0}$" is a constant symbol, "$\mathtt{s}$" is a unary function symbol, and "$+$" & "$\cdot$" are binary function symbols.

PA_0: $\neg\exists x(\mathtt{s}x = \mathsf{0})$

PA_1: $\forall x \forall y(\mathtt{s}x = \mathtt{s}y \rightarrow x = y)$,

PA_2: $\forall x(x + \mathsf{0} = x)$

PA_3: $\forall x \forall y(x + \mathtt{s}y = \mathtt{s}(x + y))$

PA_4: $\forall x(x \cdot \mathsf{0} = \mathsf{0})$

PA_5: $\forall x \forall y(x \cdot \mathtt{s}y = (x \cdot y) + x)$

If φ is any $\mathscr{L}_{\mathsf{PA}}$-formula with $x \in \text{free}(\varphi)$, then:

PA_6: $\big(\varphi(\mathsf{0}) \wedge \forall x(\varphi(x) \rightarrow \varphi(\mathtt{s}(x)))\big) \rightarrow \forall x \varphi(x)$

The domain $\mathbb{N}$ of our standard model consists of the elements in the list of natural numbers introduced in Chapter 0. So, each natural number in the set $\mathbb{N}$ is either 0

or of the form $\mathbf{s} \cdots \mathbf{s}0$ for some F I N I T E string $\mathbf{s} \cdots \mathbf{s}$. Notice the difference between $\mathtt{s}$ (which is an unary function symbol) and $\mathbf{s}$ (which is a symbol we use to build the elements of the set $\mathbb{N}$, *i.e.*, the objects in the domain of our standard model of Peano Arithmetic). In order to write this more formally, we extend the signature $\mathscr{L}_{\mathsf{PA}}$ by the unary relation symbol $\mathtt{N}$ and add the following statement as a kind of meta-axiom to PA:

$$\Phi \equiv \forall x \Big(\big\{ \mathtt{N}(0), \forall \mathtt{z} \big(\mathtt{N}(\mathtt{z}) \to \mathtt{N}(\mathtt{sz}) \big) \big\} \vdash \mathtt{N}(\mathtt{x}) \Big)$$

Notice that this statement is *not* a statement in first-order logic since it involves the symbol "$\vdash$", which incorporates implicitly the metamathematical notion of F I N I T E N E S S. However, the statement Φ makes sure that every model of $\mathsf{PA}+\Phi$ is isomorphic to the standard-model.

Now, we are going to define the standard model of PA with domain $\mathbb{N}$. For this, we have to define first an $\mathscr{L}_{\mathsf{PA}}$-structure $\mathbb{N}$. Let If σ and τ are both (possibly empty) finite strings of the form $\mathbf{s} \cdots \mathbf{s}$. Then we can interpret the non-logical symbols in $\mathscr{L}_{\mathsf{PA}}$ as follows:

$$0^{\mathbb{N}} := \mathbf{0}$$

$$\begin{aligned} \mathtt{s}^{\mathbb{N}} :\ & \mathbb{N} \to \mathbb{N} \\ & \sigma\mathbf{0} \mapsto \mathbf{s}\sigma\mathbf{0} \end{aligned}$$

$$\begin{aligned} +^{\mathbb{N}} :\quad & \mathbb{N} \times \mathbb{N} \ \to\ \mathbb{N} \\ & \langle \sigma\mathbf{0}, \tau\mathbf{0} \rangle \mapsto \sigma\tau\mathbf{0} \end{aligned}$$

$$\begin{aligned} \cdot^{\mathbb{N}} :\quad & \mathbb{N} \times \mathbb{N} \ \to \mathbb{N} \\ & \langle \sigma\mathbf{0}, \tau\mathbf{0} \rangle \mapsto \underbrace{\begin{matrix} \sigma\sigma \ \cdots \ \sigma \\ \uparrow\uparrow \ \cdots \ \uparrow \\ \mathbf{s}\,\mathbf{s} \ \cdots \ \mathbf{s} \end{matrix}}_{\tau}\mathbf{0} \end{aligned}$$

Note that if either σ or τ is the empty string, then $\sigma\mathbf{0} \cdot^{\mathbb{N}} \tau\mathbf{0}$ is $\mathbf{0}$. The main feature of the $\mathscr{L}_{\mathsf{PA}}$-structure $\mathbb{N}$ is that every element of $\mathbb{N}$ corresponds to a certain $\mathscr{L}_{\mathsf{PA}}$-term. In order to prove this, we introduce the following notion: To each finite string $\sigma \equiv \mathbf{s} \cdots \mathbf{s}$ we assign a F I N I T E string $\underline{\sigma} \equiv \mathtt{s} \cdots \mathtt{s}$ such that $\underline{\sigma}$ is obtained from σ by replacing each occurrence of $\mathbf{s}$ with $\mathtt{s}$. As a consequence of this definition we get the following

FACT 7.1. *For all* F I N I T E *strings σ and τ of the form $\mathbf{s} \cdots \mathbf{s}$ we have:*

(a) *If σ is not the empty string, then* $\mathsf{PA} \vdash \underline{\sigma}0 \neq 0$

(b) $\mathsf{PA} \vdash \underline{\sigma}\mathsf{0} = \underline{\tau}\mathsf{0} \quad \Longleftrightarrow \quad \sigma\mathbf{0} \equiv \tau\mathbf{0}$.

Proof. (a) follows from PA_0, and (b) follows from PA_1 and L_{14}. ⊣

LEMMA 7.2. *Every element of* $\mathbb{N}$ *corresponds to a unique* FINITE *application of the function* s *to* $\mathsf{0}$*, or in other words, every element of* $\mathbb{N}$ *is equal to a unique* FINITE *application of the function* $\mathsf{s}^{\mathbb{N}}$ *to* $\mathsf{0}^{\mathbb{N}}$*. More formally, for every element* $\sigma\mathbf{0}$ *of* $\mathbb{N}$ *there is a unique* $\mathscr{L}_{\mathsf{PA}}$*-term* $\underline{\sigma}\mathsf{0}$ *such that*

$$(\underline{\sigma}\mathsf{0})^{\mathbb{N}} \quad \text{IS THE SAME OBJECT AS} \quad \sigma\mathbf{0}\,,$$

or equivalently,

$$(\underline{\sigma}\mathsf{0})^{\mathbb{N}} \equiv \sigma\mathbf{0}\,.$$

Proof. By definition of $\mathsf{s}^{\mathbb{N}}$, for every FINITE string $\tau \equiv \mathbf{s}\cdots\mathbf{s}$ we get that $\mathsf{s}^{\mathbb{N}}(\tau\mathbf{0})$ is the same element of $\mathbb{N}$ as $\mathbf{s}\tau\mathbf{0}$, and after applying this fact FINITELY many times we get:

$$\overbrace{\begin{matrix} \mathsf{s}^{\mathbb{N}} & \mathsf{s}^{\mathbb{N}} & \cdots & \mathsf{s}^{\mathbb{N}} & \mathsf{0}^{\mathbb{N}} \\ \updownarrow & \updownarrow & \cdots & \updownarrow & \updownarrow \\ \mathbf{s} & \mathbf{s} & \cdots & \mathbf{s} & \mathbf{0} \end{matrix}}^{(\underline{\sigma}\mathsf{0})^{\mathbb{N}}}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\sigma\mathbf{0}}$$

The uniqueness of $\underline{\sigma}\mathsf{0}$ follows from FACT 7.1. ⊣

Now, we are ready to prove that the $\mathscr{L}_{\mathsf{PA}}$-structure $\mathbb{N}$, which is called the **standard model** of Peano Arithmetic, is indeed a model of PA.

THEOREM 7.3. $\mathbb{N} \vDash \mathsf{PA}$.

Proof. By definition of $\mathsf{s}^{\mathbb{N}}$ we get $\mathbb{N} \vDash \mathsf{PA}_0$ and by FACT 7.1 we also have $\mathbb{N} \vDash \mathsf{PA}_1$. Further, by definition of $+^{\mathbb{N}}$ and $\cdot^{\mathbb{N}}$ we get $\mathbb{N} \vDash \mathsf{PA}_2$ and $\mathbb{N} \vDash \mathsf{PA}_4$ respectively. For PA_3 let σ and τ be (possibly empty) finite strings of the form $\mathbf{s}\cdots\mathbf{s}$. Then

$$\sigma\mathbf{0} +^{\mathbb{N}} \mathsf{s}^{\mathbb{N}}\tau\mathbf{0} \equiv \sigma\mathbf{s}\tau\mathbf{0} \equiv \mathbf{s}\sigma\tau\mathbf{0} \equiv \mathsf{s}^{\mathbb{N}}(\sigma\mathbf{0} +^{\mathbb{N}} \tau\mathbf{0}).$$

Similarly, we can show $\mathbb{N} \vDash \mathsf{PA}_5$ (see EXERCISE 7.0). In order to show that $\mathbb{N} \vDash \mathsf{PA}_6$, let $\varphi(x)$ be an $\mathscr{L}_{\mathsf{PA}}$-formula and let us assume that

$$\mathbb{N} \vDash \varphi(\mathsf{0}) \wedge \forall x\big(\varphi(x) \rightarrow \varphi(\mathsf{s}x)\big)\,. \qquad (*)$$

We have to show that $\mathbb{N} \vDash \forall x\varphi(x)$. By definition of models we get that $\varphi(\mathbf{0})$ holds in $\mathbb{N}$ and for all $n \in \mathbb{N}$: if $\varphi(n)$ holds in $\mathbb{N}$, then also $\varphi(\mathsf{s}^{\mathbb{N}}n)$ holds in $\mathbb{N}$. Let $\sigma\mathbf{0}$ be an arbitrary element of $\mathbb{N}$. Since σ is a FINITE string, by $(*)$, the logical axiom L_{10}, and by applying FINITELY many times MODUS PONENS, we get $\mathbb{N} \vDash \varphi(\sigma\mathbf{0})$. Hence, since $\sigma\mathbf{0}$ was arbitrary, for every string $n \in \mathbb{N}$, $\varphi(n)$ holds in $\mathbb{N}$, and therefore, $\mathbb{N} \vDash \forall x\varphi(x)$. ⊣

As a matter of fact we would like to mention that from a metamathematical point of view, every model of PA must contain an isomorphic copy of the standard model $\mathbb{N}$. So, it would also make sense to call $\mathbb{N}$ the **minimal model** of Peano Arithmetic.

One might be tempted to think that $\mathbb{N}$ is essentially the only model of PA, but this is not the case, as we shall see now.

Countable Non-Standard Models

The previous section shows that every natural number in the standard model $\mathbb{N}$ corresponds to a unique $\mathcal{L}_{\mathsf{PA}}$-term; more precisely, every element $\sigma\mathbf{0}$ of $\mathbb{N}$ is the same object as the term $\underline{\sigma}0$. In order to simplify notations, we will from now on use variables such as $n, m, \cdots$ to denote elements of $\mathbb{N}$ and $\underline{n}, \underline{m}, \cdots$ their counterpart in the formal language $\mathcal{L}_{\mathsf{PA}}$, *i.e.*, if n stands for $\sigma\mathbf{0}$, then $\underline{n}$ denotes $\underline{\sigma}0$.

Since every model $\mathbf{M}$ of PA contains $\underline{n}^{\mathbf{M}}$, the **standard natural numbers**, for every $n \in \mathbb{N}$, it is clear that $\mathbf{M}$ contains a copy of the standard model. However, $\mathbf{M}$ can also have **non-standard natural numbers**, *i.e.*, elements which are not the interpretations of terms of the form $\underline{n}$. In the following, we present the simplest way to construct such non-standard models.

Let $\mathscr{L}_{\mathsf{PA}^+}$ be the language $\mathscr{L}_{\mathsf{PA}}$ augmented with an additional constant symbol c, which is different from 0. Note that by setting

$$x < y :\Longleftrightarrow \exists r(x + \mathsf{s}r = y)$$

one can introduce an ordering in PA, which in the standard model corresponds to the usual ordering of natural numbers (for further details see Chapters 8 and 9). Let PA^+ be the theory whose axioms are PA_0–PA_6 together with the axioms

$$\begin{aligned} &\mathsf{c} > 0 \\ &\mathsf{c} > \mathsf{s}0 \\ &\mathsf{c} > \mathsf{ss}0 \\ &\mathsf{c} > \mathsf{sss}0 \\ &\quad\vdots \end{aligned}$$

Hence, PA^+ is $\mathsf{PA} \cup \{\mathsf{c} > \underline{n} : n \in \mathbb{N}\}$.

LEMMA 7.4. $\mathrm{Con}(\mathsf{PA}^+)$, *i.e., the theory* PA^+ *is consistent.*

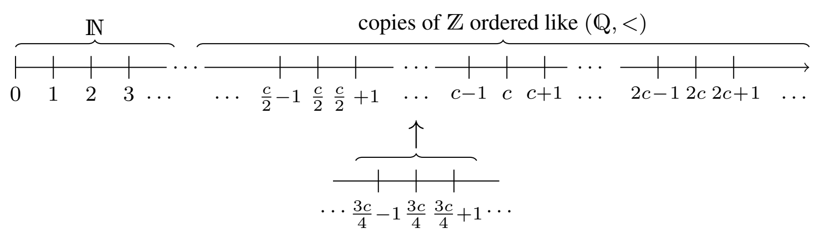
Proof. By the COMPACTNESS THEOREM it suffices to prove that every FINITE subset of PA^+ is consistent. Let T be a FINITE subset of PA^+. Now let $n \in \mathbb{N}$ be maximal such that the formula $\mathsf{c} \neq \underline{n}$ belongs to T. Notice that such n exists, since T is finite. Then we can define a model $\mathbf{M}$ of T with domain $\mathbb{N}$ by interpreting the constant and function symbols by $0^{\mathbf{M}} \equiv \mathbf{0}$, $\mathtt{s}^{\mathbf{M}} \equiv \mathbf{s}$, $+^{\mathbf{M}} \equiv +^{\mathbb{N}}$, $\cdot^{\mathbf{M}} \equiv \cdot^{\mathbb{N}}$ and $\mathtt{c}^{\mathbf{M}} \equiv \mathbf{s}n$. Since $\mathbb{N} \vDash \mathsf{PA}$ we get that $\mathbf{M} \vDash \mathsf{PA}$ and by construction $\mathbf{M} \vDash \mathtt{c} \neq \underline{m}$ for every $m \leq n$ and hence $\mathbf{M} \vDash \mathsf{PA}^+$. ⊣

Now, since $\mathscr{L}_{\mathsf{PA}^+}$ is a countable signature, by THEOREM 5.6 it follows that PA^+ has a countable model $\mathbf{M}$ which is also a **non-standard model** of PA, *i.e.*, a model which is not isomorphic to the standard model $\mathbb{N}$. What does the order structure of $\mathbf{M}$ look like?

Note that c has a successor $\mathtt{s}c = c + 1$, and $c + 1$ in turn has a successor, and so on. Furthermore, since

$$\mathsf{PA} \vdash \forall x\big(x = 0 \vee \exists y(x = \mathtt{s}y)\big)$$

(see LEMMA 8.4), c also has a predecessor, *i.e.*, there exists $c - 1$ with the property that $\mathtt{s}(c - 1) = c$, and the same argument yields that c in fact has infinitely many predessors, which are all non-standard. Hence, the order structure of c and its predecessors and successors corresponds to $(\mathbb{Z}, <)$, so there are infinitely many such $\mathbb{Z}$-chains. Moreover, each multiple of c yields a further copy $\mathbb{Z}$-chain. Now, one can easily prove in PA that every number is even or odd (see EXERCISE 8.1), and hence there is d such that $2d = c$ or $2d = c + 1$. We denote d by $\frac{c}{2}$. This shows that between the copy of the standard model and the $\mathbb{Z}$-chain given by c, there is a further $\mathbb{Z}$-chain given by $\frac{c}{2}$ and its predessors and successors. In fact, the $\mathbb{Z}$-chains are ordered like $(\mathbb{Q}, <)$ (see EXERCISE 7.6).



Note that the proof of LEMMA 7.4 implies that there are non-standard models of PA which are elementarily equivalent to $\mathbb{N}$. To see this, let $\mathbf{Th}(\mathbb{N})$ denote the theory of all $\mathscr{L}_{\mathsf{PA}}$-sentences which are true in $\mathbb{N}$. Then one could simply replace PA by $\mathbf{Th}(\mathbb{N})$ in LEMMA 7.4 and thus obtain a model of $\mathbf{Th}(\mathbb{N})$ augmented with all formulae of the form $\mathtt{c} > \underline{n}$ for $n \in \mathbb{N}$. By construction, this model is elementarily equivalent to $\mathbb{N}$. For a more general result see EXERCISE 7.2.

NOTES

An early attempt at formalising arithmetic was given by Hermann Grassmann [19] in 1861, who defined addition and multiplication and proved elementary results such as the associative and commutative laws using induction. Richard Dedekind [4] also identified induction as a key principle in 1888 as well as the first two axioms of Peano Arithmetic; however, he introduced them as a definition rather than as axioms. Giuseppe Peano [36] presented his five axioms in 1889, where he only introduces zero and the successor function axiomatically, and the induction axiom is given in second-order logic in the following form: Every set of natural numbers which contains 0 and is closed under the successor function is the set of all natural numbers. The version of Peano's Axioms formalised in first-order logic, where the induction axiom is replaced by an axiom schema, and the axioms defining addition and multipliation are included, goes back to the advent of first-order logic in the 1920's. The first explicit construction of a non-standard model of arithmetic was given by Thoralf Skolem in [41]. Fur further reading on non-standard models consult [28].

EXERCISES

7.0 Prove that $\mathbb{N} \vDash \mathsf{PA}_5$.

7.1 Prove that PA_0 and PA_1 are independent of the other axioms of PA.

7.2 Show that there are uncountably many countable models of PA which are all elementarily equivalent and pairwise non-isomorphic.

Hint: Let $\mathbb{P}$ be the set of prime numbers and let c be a constant symbol which is different from $\mathsf{0}$. For any distinct prime numbers p and q let $\varphi_{p,q}$ be the formula

$$p \mid \mathsf{c} \wedge q \nmid \mathsf{c}\,.$$

For every subset $S \subseteq \mathbb{P}$, let Φ_S be the collection of all formulae $\varphi_{p,q}$ such that $p \in S$ and $q \notin S$. Now, for each $S \subseteq \mathbb{P}$, $\mathbb{N}$ is a model for every finite subset of $\mathsf{T}(\mathbb{N}) + \Phi_S$, and hence, for every $S \subseteq \mathbb{P}$, $\mathsf{T}(\mathbb{N}) + \Phi_S$ has a countable model, say $\mathbf{N}_S$. Notice for all these models $\mathbf{N}_S$ we have $\mathbf{N} \vDash \mathsf{T}(\mathbb{N})$, and that for each model $\mathbf{N}_S$, there are only countably many subsets $S \subseteq \mathbb{P}$ such that $\mathbf{N}_S \vDash \Phi_S$. Since by CANTOR'S THEOREM 13.4 the set of all subsets $S \subseteq \mathbb{P}$ is uncountable, we obtain uncountably many countable models $\mathbf{N}_S$ of PA which are pairwise non-isomorphic.

7.3 Prove the following so-called *Overspill Principle*: If $\mathbf{M}$ is a non-standard model of PA with domain M, φ is a formula with $n+1$ free variables and $b_1, \ldots, b_n \in M$, then

$$\mathbf{M} \vDash \varphi(\underline{n}, b_1, \ldots, b_n) \quad \text{for all } n \in \mathbb{N}$$

implies that there is a non-standard element $a \in M$ such that

$$\mathbf{M} \vDash \forall x(x < a \rightarrow \varphi(x, b_1, \ldots, b_n)).$$

7.4 Show that it is not possible to introduce a relation $\mathrm{standard}(x)$ by a language extension of $\mathscr{L}_{\mathsf{PA}}$ such that for every model $\mathbf{M}$ of PA with domain M and for every $a \in M$ we have $\mathbf{M} \vDash \mathrm{standard}(a)$ if and only if $a = \underline{n}^{\mathbf{M}}$ for some $n \in \mathbb{N}$.

7.5 Let $\mathbf{M}$ be a non-standard model of PA with domain M. Show that there is an $a \in M$ such that every standard prime number divides a.

7.6 Let $\mathbf{M}$ be a countable non-standard model of PA with domain M. For every non-standard element c of M, let

$$\mathbb{Z}_c :\equiv \big\{d \in M : \text{there exists } n \in \mathbb{N} \text{ such that } d + n = c \text{ or } c + n = d\big\}.$$

and let $\mathbb{Z}_c < \mathbb{Z}_d$, if $c + n < d$ for all $n \in \mathbb{N}$. Show that $\{\mathbb{Z}_c : c \in M \text{ is non-standard}\}$ is a dense linearly ordered set (see EXERCISE 3.5) and use EXERCISE 5.1 to conclude that the order structure of $\mathbf{M}$ corresponds to the disjoint union of $\mathbb{N}$ and $\mathbb{Q} \times \mathbb{Z}$.