Chapter 15
Models and Ultraproducts

The goal of this chapter is to show that every consistent $\mathscr{L}$-theory has a model, no matter whether the signature $\mathscr{L}$ is countable or uncountable. In addition, we will show that if a consistent $\mathscr{L}$-theory T has an infinite model, then, on the one hand, T has arbitrarily large models, and on the other hand, T has a model of size at most $\max\{\aleph_0, |\mathscr{L}|\}$.

In order to prove these results, we shall work within a model of ZFC, in particular, we shall make use of the Axiom of Choice. So, in contrast to the proofs of the corresponding results in Part II, the proofs below are in general not constructive. As a matter of fact we would like to mention that even though the proofs are carried out in a model of ZFC, in general, they cannot be carried out in ZFC. In fact, we do not work with ZFC as a formal system, but we just take a model of ZFC and use it as a framework in which we carry out the proofs.

Filters and Ultrafilters

In this sections, we introduce the notions of *filters* and *ultrafilters*.

Let S be an arbitrary non-empty set and let $\mathscr{P}(S)$ be the *power-set* of S, *i.e.*, the set of all subsets of S. A set $\mathscr{F} \subseteq \mathscr{P}(S)$ is called a **filter** over S, if $\mathscr{F}$ has the following properties:

- $S \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$
- $(x \in \mathscr{F} \wedge y \in \mathscr{F}) \to (x \cap y) \in \mathscr{F}$
- $(x \in \mathscr{F} \vee y \in \mathscr{F}) \to (x \cup y) \in \mathscr{F}$

In particular, if $x \in \mathscr{F}$ and $x \subseteq y$, then $y \in \mathscr{F}$. So, a filter over S is a set of subsets of S which does not contain the empty set and which is closed under intersections and supersets. For example the set $\{S\}$ is a filter over S. A more interesting example of a filter over S is the set

$$\mathscr{F} := \{x \subseteq S : S \setminus x \text{ is finite}\},$$

which is the so-called *Fréchet-filter*. Now, a set $\mathscr{U} \subseteq \mathscr{P}(S)$ is called an **ultrafilter** over S, if $\mathscr{U}$ is a filter over S and for each $x \in \mathscr{P}(S)$, either $x \in \mathscr{U}$ or $(S \setminus x) \in \mathscr{U}$. In other words, a filter $\mathscr{U}$ is an ultrafilter if $\mathscr{U}$ is not properly contained in any filter. For example, for each $a \in S$, the set

$$\mathscr{U}_a := \{x \subseteq S : a \in x\}$$

is an ultrafilter over S, a so-called *trivial ultrafilter*. In particular, every ultrafilter over a finite set is trivial. It is natural to ask whether there exist also non-trivial ultrafiters, for example, ultrafilters which contain the Fréchet-filter. Or in general, we can ask whether every filter can be extended to an ultrafiter. This is what the *Ultrafilter Theorem* states:

> Ultrafilter Theorem: If $\mathscr{F}$ is a filter over a set S, then $\mathscr{F}$ can be extended to an ultrafilter.

Surprisingly, we cannot prove the Ultrafilter Theorem without assuming some form of the Axiom of Choice. However, to prove the Ultrafilter Theorem within ZFC is not so hard (see EXERCISE 15.0).

Ultraproducts and Ultrapowers

Let $\mathscr{L}$ be an arbitrary but fixed signature, let I be an non-empty set, and for each $\iota \in I$, let $\mathbf{M}_\iota$ be an $\mathscr{L}$-structure with domain A_ι. Furthermore, let $A := \bigtimes_{\iota \in I} A_\iota$ be the Cartesian product of the sets A_ι. Below, we shall identify the elements of A with function $f : I \to \bigcup_{\iota \in I} A_\iota$, where for each $\iota \in I$, $f(\iota) \in A_\iota$. Finally, let $\mathscr{U} \subseteq \mathscr{P}(I)$ be an ultrafilter over I. With respect to $\mathscr{U}$, we define a binary relation "$\sim$" on A by stipulating

$$f \sim g \;:\Longleftrightarrow\; \{\iota \in I : f(\iota) = g(\iota)\} \in \mathscr{U}\,.$$

FACT 15.1. *The relation "$\sim$" is an equivalence relation.*

Proof. We have to show that "$\sim$" is reflexive, symmetric, and transitive.

- For all $f \in A$ we obviously have $f \sim f$.
- For all $f, g \in A$ we obviously have $f \sim g \leftrightarrow g \sim f$.
- Let $f, g, h \in A$ and assume that $f \sim g$ and $g \sim h$. Furthermore, let $x := \{\iota \in I : f(\iota) = g(\iota)\}$ and $y := \{\iota \in I : g(\iota) = h(\iota)\}$. Then $x, y \in \mathscr{U}$, and since $\mathscr{U}$ is a filter, $x \cap y \in \mathscr{U}$. Thus,

$$\{\iota \in I : f(\iota) = h(\iota)\} \subseteq x \cap y \in \mathscr{U}\,,$$

which shows that $f \sim h$.

$\dashv$

For each $f \in A$, let
$$[f] := \{g \in A : g \sim f\}$$
and let
$$A^* := \bigtimes_{\iota \in I} A_\iota \,/\!\sim \quad \text{or equivalently} \quad A^* := \big\{[f] : f \in A\big\}.$$

We now construct the $\mathscr{L}$-structure $\mathbf{M}^*$ with domain A^* as follows:

- For every constant symbol $c \in \mathscr{L}$, let $f_c \in A$ be defined by stipulating
$$f_c(\iota) := c^{\mathbf{M}_\iota} \quad \text{for all } \iota \in I,$$
and let
$$c^{\mathbf{M}^*} := [f_c].$$

- For every n-ary function symbol $F \in \mathscr{L}$, let $F^{\mathbf{M}^*} : (A^*)^n \to A^*$ be such that
$$F^{\mathbf{M}^*}\big([f_0], \dots, [f_{n-1}]\big) = [f] \iff \Big\{\iota \in I : F^{\mathbf{M}_\iota}\big(f_0(\iota), \dots, f_{n-1}(\iota)\big) = f(\iota)\Big\} \in \mathscr{U}.$$

- For every n-ary relation symbol $R \in \mathscr{L}$, let $R^{\mathbf{M}^*} \subseteq (A^*)^n$ be such that
$$\big\langle [f_0], \dots, [f_{n-1}] \big\rangle \in R^{\mathbf{M}^*} \iff \Big\{\iota \in I : \big\langle f_0(\iota), \dots, f_{n-1}(\iota)\big\rangle \in R^{\mathbf{M}_\iota}\Big\} \in \mathscr{U}.$$

FACT 15.2. *The constants $c^{\mathbf{M}^*}$, the functions $F^{\mathbf{M}^*}$, and the relations $R^{\mathbf{M}^*}$ are well-defined.*

Proof. We just show that the function $F^{\mathbf{M}^*} : (A^*)^n \to A$ is well-defined and leave the proofs for $c^{\mathbf{M}^*}$ and $R^{\mathbf{M}^*}$ as an exercise (see EXERCISE 15.1). Let $F \in \mathscr{L}$ be an n-ary function symbol and let $\langle f_0, \dots, f_{n-1}\rangle, \langle g_0, \dots, g_{n-1}\rangle \in A^n$ be such that for each $0 \leq i < n$ we have
$$f_i \sim g_i \quad \text{or equivalently} \quad [f_i] = [g_i]\,.$$
Furthermore, define $f, g \in A$ by stipulating
$$f(\iota) := F^{\mathbf{M}_\iota}\big(f_0(\iota), \dots, f_{n-1}(\iota)\big) \quad \text{and} \quad g(\iota) := F^{\mathbf{M}_\iota}\big(g_0(\iota), \dots, g_{n-1}(\iota)\big)\,.$$
By definition of "$\sim$" and since $\mathscr{U}$ is an ultrafilter over I, we have
$$\big\{\iota \in I : f_0(\iota) = g_0(\iota) \,\wedge\, \cdots \,\wedge\, f_{n-1}(\iota) = g_{n-1}(\iota)\big\} \in \mathscr{U}\,,$$

and consequently we obtain

$$\Big\{\iota \in I : F^{\mathbf{M}_\iota}\big(f_0(\iota), \dots, f_{n-1}(\iota)\big) = F^{\mathbf{M}_\iota}\big(g_0(\iota), \dots, g_{n-1}(\iota)\big)\Big\} \in \mathscr{U}.$$

Hence, $\big\{\iota \in I : f(\iota) = g(\iota)\big\} \in \mathscr{U}$, which shows that $[f] = [g]$ and implies that

$$F^{\mathbf{M}^*}\big([f_0], \dots, [f_{n-1}]\big) \;=\; F^{\mathbf{M}^*}\big([g_0], \dots, [g_{n-1}]\big).$$

Therefore, the value of the function $F^{\mathbf{M}^*}$ does not depend on the particular representatives we choose from the equivalence classes $[f_i]$. $\dashv$

The $\mathscr{L}$-structure $\mathbf{M}^*$ with domain A^* is called the **ultraproduct** of the $\mathscr{L}$-structures $\mathbf{M}_\iota$ ($\iota \in I$) with respect to the ultrafilter $\mathscr{U}$ over I. If for all $\iota \in I$ we have $\mathbf{M}_\iota = \mathbf{M}$ for some $\mathscr{L}$-structure $\mathbf{M}$, then $\mathbf{M}^*$ is called the **ultrapower** of $\mathbf{M}$ with respect to $\mathscr{U}$.

In the next section we show that if each $\mathscr{L}$-structure $\mathbf{M}_\iota$ is a model of some $\mathscr{L}$-theory T, then also the ultraproduct $\mathbf{M}^*$ is a model of T.

Łoś's Theorem

As above, let $\mathscr{L}$ be an arbitrary signature, let I be an non-empty set, and for each $\iota \in I$, let $\mathbf{M}_\iota$ be an $\mathscr{L}$-structure with domain A_ι. Finally, let $\mathscr{U}$ be an ultrafilter over I and let $\mathbf{M}^*$ be the ultraproduct of the $\mathscr{L}$-structures $\mathbf{M}_\iota$ ($\iota \in I$) with respect to $\mathscr{U}$.

The following result allows us to decide whether a given $\mathscr{L}$-sentence is valid in $\mathbf{M}^*$.

ŁOŠ'S THEOREM 15.3. *For each $\mathscr{L}$-sentence σ we have*

$$\mathbf{M}^* \vDash \sigma \quad\Longleftrightarrow\quad \big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma\big\} \in \mathscr{U}.$$

Proof. By THEOREM 1.2, for every $\mathscr{L}$-sentence σ there is an equivalent $\mathscr{L}$-sentence σ' which contains only "$\neg$" and "$\wedge$" as logical operators and "$\exists$" as quantifier. So, it is enough to prove ŁOŚ'S THEOREM for $\mathscr{L}$-sentence σ'. The proof is by induction on the number of the symbols "$\neg$", "$\wedge$", and "$\exists$" which appear in the $\mathscr{L}$-sentence σ'.

By construction of $\mathbf{M}^*$, ŁOŚ'S THEOREM holds for atomic $\mathscr{L}$-sentences σ'. Now, assume that $\sigma' \equiv \neg\sigma_0$ and that ŁOŚ'S THEOREM holds for σ_0. Then we have:

$$\begin{aligned}
\mathbf{M}^* \vDash \neg\sigma_0 \quad &\Longleftrightarrow \quad \mathbf{M}^* \nvDash \sigma_0 \\
&\Longleftrightarrow \quad \big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_0\big\} \notin \mathscr{U} \\
&\Longleftrightarrow \quad I \setminus \big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_0\big\} \in \mathscr{U} \\
&\Longleftrightarrow \quad \big\{\iota \in I : \mathbf{M}_\iota \vDash \neg\sigma_0\big\} \in \mathscr{U}
\end{aligned}$$

Now, assume that $\sigma' \equiv \sigma_1 \wedge \sigma_2$ and that ŁOŚ'S THEOREM holds for σ_1 and σ_2. Then we have:

$$\begin{aligned}
\mathbf{M}^* \vDash \sigma_1 \wedge \sigma_2 \quad &\Longleftrightarrow \quad \mathbf{M}^* \vDash \sigma_1 \quad \text{AND} \quad \mathbf{M}^* \vDash \sigma_2 \\
&\Longleftrightarrow \quad \underbrace{\big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_1\big\}}_{=:x_1} \in \mathscr{U} \ \text{AND} \ \underbrace{\big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_2\big\}}_{=:x_2} \in \mathscr{U} \\
&\Longleftrightarrow \quad x_1 \cap x_2 \in \mathscr{U} \\
&\Longleftrightarrow \quad \big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_1 \wedge \sigma_2\big\} \in \mathscr{U}
\end{aligned}$$

Finally, assume that $\sigma' \equiv \exists\nu\sigma_0$ (for some variable ν) and that for any $[g] \in A^*$, ŁOŚ'S THEOREM holds for $\sigma_0\big(\nu/[g]\big)$. Then

$$\begin{aligned}
\mathbf{M}^* \vDash \exists\nu\sigma_0 \quad &\Longleftrightarrow \quad \text{IT EXISTS } [g_0] \text{ IN } A^* : \quad \mathbf{M}^* \vDash \sigma_0\big(\nu/[g_0]\big) \\
&\Longleftrightarrow \quad \underbrace{\big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_0\big(\nu/g_0(\iota)\big)\big\}}_{=:x} \in \mathscr{U} .
\end{aligned}$$

Because $x \subseteq \big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\}$, we have

$$\big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} \in \mathscr{U},$$

which shows that

$$\mathbf{M}^* \vDash \exists\nu\sigma_0 \quad \Longrightarrow \quad \big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} \in \mathscr{U}.$$

In order to show the converse implication, we have to make use of the Axiom of Choice. If, for $\iota \in I$, $\mathbf{M}_\iota \vDash \exists\nu\sigma_0$, then let $a_\iota \in A_\iota$ be such that $\mathbf{M}_\iota \vDash \sigma_0(\nu/a_\iota)$, otherwise, let a_ι be an arbitrary element of A_ι. Now, for the function

$$\begin{aligned} g_0 : I &\to \bigcup A \\ \iota &\mapsto \ a_\iota \end{aligned}$$

we have

$$\big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} = \big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_0\big(\nu/g_0(\iota)\big)\big\}.$$

In particular, if $\big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} \in \mathscr{U}$, then also

$$\big\{\iota \in I : \mathbf{M}_\iota \vDash \sigma_0\big(\nu/g_0(\iota)\big)\big\} \in \mathscr{U},$$

which shows that

$$\big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} \in \mathscr{U} \quad \Longrightarrow \quad \mathbf{M}^* \vDash \exists\nu\sigma_0,$$

and consequently we obtain $\mathbf{M}^* \vDash \exists\nu\sigma_0 \Longleftrightarrow \big\{\iota \in I : \mathbf{M}_\iota \vDash \exists\nu\sigma_0\big\} \in \mathscr{U}$. $\dashv$

The Completeness Theorem for Uncountable Signatures

In Chapter 5 we have proven GÖDEL'S COMPLETENESS THEOREM 5.5 (*i.e.*, the COMPLETENESS THEOREM for countable signatures). The proof we have given was based on potentially infinite lists and the metamathematical assumptions we made were very mild. In fact, our proof for GÖDEL'S COMPLETENESS THEOREM 5.5 can be carried out effectively in a kind of *algorithmic* way. In contrast to the proof for countable signatures, the proof of the COMPLETENESS THEOREM for uncountable signatures — which will follow from the semantic form of the COMPACTNESS THEOREM 2.12 — is much more formal. In particular, it makes use of ŁOŚ'S THEOREM 15.3, which is based on the existence of ultrafilters and choice functions, and is carried out in a model of ZFC — but not in ZFC itself.

THEOREM 15.4 (Semantic Form of the Compactness Theorem). *Let* T *be an* $\mathscr{L}$*-theory such that for every finite subset* $\Phi \subseteq \mathsf{T}$ *there is an* $\mathscr{L}$*-structure* $\mathbf{M}_\Phi$ *such that* $\mathbf{M}_\Phi \vDash \Phi$*. Then* T *has a model.*

Proof. Let I be the set of all finite subsets of T, *i.e.*,

$$I := \big\{\Phi \subseteq \mathsf{T} : \Phi \text{ is finite}\big\},$$

and for each $\Phi \in I$, let $\mathbf{M}_\Phi$ be an $\mathscr{L}$-structure with domain A_Φ such that $\mathbf{M}_\Phi \vDash \Phi$. Furthermore, for every $\Phi \in I$ let

$$\Delta(\Phi) := \big\{\Phi' \in I : \Phi \subseteq \Phi'\big\}.$$

In other words, $\Delta(\Phi)$ is the set of all finite supersets $\Phi' \supseteq \Phi$. In particular, for every $\Phi \in I$ we have $\Phi \in \Delta(\Phi)$. Now, for all $\Phi_1, \Phi_2 \in I$ we have $\Delta(\Phi_1) \cap \Delta(\Phi_1) = \Delta(\Phi_1 \cup \Phi_2)$, where $\Phi_1 \cup \Phi_2 \in I$. Therefore, the set

$$\mathscr{F} := \big\{\Psi \subseteq I : \exists \Phi \in I\big(\Delta(\Phi) \subseteq \Psi\big)\big\}$$

is a filter over I, which, by the Ultrafilter Theorem, can be extended to an ultrafilter $\mathscr{U}$.

Let $\mathbf{M}^*$ with domain A^* be the ultraproduct of the $\mathscr{L}$-structures $\mathbf{M}_\Phi$ ($\Phi \in I$) with respect to the ultrafilter $\mathscr{U}$ over I, and let $\sigma_0 \in \mathsf{T}$ be an arbitrary $\mathscr{L}$-sentence. Then $\{\sigma_0\} \in I$, and $\mathbf{M}_{\{\sigma_0\}} \vDash \sigma_0$. Moreover, for every $\Phi \in \Delta\big(\{\sigma_0\}\big)$ we have $\mathbf{M}_\Phi \vDash \sigma_0$. Therefore, we have

$$\Delta\big(\{\sigma_0\}\big) = \big\{\Phi \in I : \sigma_0 \in \Phi\big\} \subseteq \big\{\Phi \in I : \mathbf{M}_\Phi \vDash \sigma_0\big\}.$$

Now, since $\Delta\big(\{\sigma_0\}\big) \in \mathscr{F} \subseteq \mathscr{U}$, by ŁOŚ'S THEOREM 15.3 we obtain

$$\mathbf{M}^* \vDash \sigma_0,$$

and since $\sigma_0 \in \mathsf{T}$ was arbitrary, this shows that $\mathbf{M}^* \vDash \mathsf{T}$, hence, T has a model. $\dashv$

As a consequence of THEOREM 15.4 and GÖDEL'S COMPLETENESS THEOREM 5.5 we obtain the the COMPLETENESS THEOREM for arbitrarily large signatures.

COMPLETENESS THEOREM 15.5. *If $\mathscr{L}$ is an arbitrary signature and T is a consistent set of $\mathscr{L}$-sentences, then T has a model.*

Proof. Firstly, if T is consistent, then, by the COMPACTNESS THEOREM 2.12, every finite subset $\Phi \subseteq \mathsf{T}$ is consistent. Secondly, as in the proof of GÖDEL'S COMPLETENESS THEOREM 5.5, for every finite subset of $\Phi \subseteq \mathsf{T}$ we can construct an $\mathscr{L}'$-structure $\mathbf{M}'_\Phi$ with domain A_Φ, such that $\mathbf{M}'_\Phi \vDash \Phi$, where $\mathscr{L}'$ is the finite subset of $\mathscr{L}$ consisting of all non-logical symbols which appear in sentences of Φ. Now, we extend each $\mathscr{L}'$-structure $\mathbf{M}'_\Phi$ to an $\mathscr{L}$-structure $\mathbf{M}_\Phi$ with the same domain A_Φ, such that $\mathbf{M}_\Phi \vDash \Phi$ (see EXERCISE 3.2). Hence, for every finite subset of $\Phi \subseteq \mathsf{T}$ there is an $\mathscr{L}$-structure $\mathbf{M}_\Phi$ such that $\mathbf{M}_\Phi \vDash \Phi$, and therefore, we can apply THEOREM 15.4 in order to construct a model $\mathbf{M}^* \vDash \mathsf{T}$. ⊣

As an immediate consequences of the COMPLETENESS THEOREM 15.5 and the SOUNDNESS THEOREM **??** we obtain the following

COROLLARY 15.6. *For any signature $\mathscr{L}$, a set T of $\mathscr{L}$-sentences has a model if and only if T is consistent.*

The Upward Löwenheim-Skolem Theorem

We show now that every $\mathscr{L}$-theory which has an infinite model, has arbitrarily large models.

UPWARD LÖWENHEIM-SKOLEM THEOREM 15.7. *Let T be an $\mathscr{L}$-theory which has an infinite model, and let C be an arbitrarily large set. Then there exists a model $\mathbf{M}^* \vDash \mathsf{T}$ with domain A^* such that $|A^*| \geq |C|$ (i.e., the cardinality of A^* is at least the cardinality of C).*

Proof. For each $\gamma \in C$ we define a constant symbol c_γ which does not belong to $\mathscr{L}$. Let $\mathscr{L}^* := \mathscr{L} \cup \{c_\gamma : \gamma \in C\}$. Furthermore, let T^* be the $\mathscr{L}^*$-theory consisting of the sentences in T together with the sentences $c_\gamma \neq c_{\gamma'}$ (for any distinct $\gamma, \gamma' \in C$). As in the proof of THEOREM 15.4, let I be the set of all finite subsets of T^*. Now, let $\mathbf{M} \vDash \mathsf{T}$ be a model with infinite domain A. For any $\Phi \in I$ we can extent the $\mathscr{L}$-structure $\mathbf{M}$ to an $\mathscr{L}^*$-structure $\mathbf{M}_\Phi$ such that

$$\mathbf{M}_\Phi \vDash \mathsf{T} + \Phi.$$

To see this, notice that the domain A of $\mathbf{M}$ is infinite and that there are just finitely many constant symbols c_γ which appear in Φ. Therefore, we can apply THEOREM 15.4 in order to construct an $\mathscr{L}^*$-structure $\mathbf{M}^*$ with domain A^* such that

$\mathbf{M}^* \vDash \mathsf{T}^*$. Finally, by definition of T^*, the elements $c_\gamma^{\mathbf{M}^*}$ in A^* (for $\gamma \in C$) are pairwise distinct, which shows that $|A^*| \geq |C|$. ⊣

As an immediate consequence of the UPWARD LÖWENHEIM–SKOLEM THEOREM 15.7 we get the following

COROLLARY 15.8. *If an $\mathscr{L}$-theory T has a countably infinite model, then T has also an uncountable model. In particular,* PA *has an uncountable model.*

As a matter of fact, we would like to mention that the proof of the UPWARD LÖWENHEIM–SKOLEM THEOREM 15.7 can be carried out neither in the formal language of ZFC (since we use an infinite set of constant symbols), nor in the language of meta-mathematics (since we use THEOREM 15.4, which is based on ŁOŚ'S THEOREM 15.3 and therefore on ultrafilters).

The Downward Löwenheim-Skolem Theorem

The last result of this chapter provides an upper bound for the minimum size of a model of a given theory.

DOWNWARD LÖWENHEIM-SKOLEM THEOREM 15.9. *If a consistent $\mathscr{L}$-theory T has an infinite model, then T has a model of size at most $\max\{\aleph_0, |\mathscr{L}|\}$.*

Proof. If the signature $\mathscr{L}$ is countable, then, by GÖDEL'S COMPLETENESS THEOREM 5.5, T has a model, which is — by construction — a countable model.

Now, assume that $|\mathscr{L}|$ (*i.e.*, the cardinality of $\mathscr{L}$) is uncountable. First notice that with the signature $\mathscr{L}$ we can build at most $\mathscr{L}$ terms. To see this, recall that a term is just a special finite string of logical and non-logical symbols, and by FACT 13.7, the cardinality of the set of such strings is $\max\{\aleph_0, |\mathscr{L}|\}$. Now, in order to build a model $\mathbf{M} \vDash \mathsf{T}$ of cardinality at most $\max\{\aleph_0, |\mathscr{L}|\}$, we can essentially follow the proof of GÖDEL'S COMPLETENESS THEOREM 5.5. However, instead of potentially infinite lists we have to work with actual infinite sequences of length at most $|\mathscr{L}|$. At the end of the construction, the domain of $\mathbf{M}$ will be a sequence of length at most $|\mathscr{L}|$ of sequences of length at most $|\mathscr{L}|$. ⊣

As an immediate consequence of the DOWNWARD LÖWENHEIM–SKOLEM THEOREM 15.9 we get the following

COROLLARY 15.10. *If T is a consistent $\mathscr{L}$-theory and the signature $\mathscr{L}$ is countable, then T has a countable model.*