Chapter 11
Happy Families and Their Relatives

A cadence is a certain simultaneous progression of all the voices in a composition accompanying a repose in the harmony or the completion of a meaningful segment of the text.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

In this chapter we shall investigate combinatorial properties of certain families of infinite subsets of ω. In order to do so, we shall use many of the combinatorial tools developed in the preceding chapters. The families we investigate—particularly P-families and Ramsey families—will play a key role in understanding the combinatorial properties of Silver and Mathias forcing notions (see Chapter 24 and Chapter 26, respectively).

Happy Families

The P-families and Ramsey families mentioned above are relatives to the so-called *happy families*. The name "happy families" comes from a children's card game, where the idea of the game is to collect the members of fictional families. The connection to families in Set Theory is that a family $\mathscr{E} \subseteq [\omega]^\omega$ is happy if for every countable decreasing sequence $y_0 \supseteq y_1 \supseteq \cdots$ of elements of $\mathscr{E}$ there is a member of $\mathscr{E}$ which selects certain elements from the sets y_i (*cf.* PROPOSITION 11.6 (b)). This explains why happy families are also called *selective co-ideals*—which is more sober but less amusing.

Firstly recall that a family $\mathscr{F} \subseteq [\omega]^\omega$ is a filter if it is closed under supersets and finite intersections, and that the Fréchet filter is the filter consisting of all co-finite subsets of ω (*i.e.*, all $x \in [\omega]^\omega$ such that $\omega \setminus x$ is finite). To keep the notation short, for $x \subseteq \omega$ define $x^{\mathsf{c}} := \omega \setminus x$. For a filter $\mathscr{F} \subseteq [\omega]^\omega$, $\mathscr{F}^+$ denotes the collection of all sets $x \subseteq \omega$ such that x^{c} does not belong to $\mathscr{F}$, *i.e.*,

$$\mathscr{F}^+ = \{x \subseteq \omega : x^{\mathsf{c}} \notin \mathscr{F}\}.$$

Notice that since $\mathscr{F}$ is a filter, we have $\mathscr{F} \subseteq \mathscr{F}^+$. Another definition of $\mathscr{F}^+$ is given by the following

FACT 11.1. *For any filter $\mathscr{F} \subseteq [\omega]^\omega$, $x \in \mathscr{F}^+$ if and only if $x \cap z$ is non-empty whenever $z \in \mathscr{F}$.*

Proof. On the one hand, if, for some $z \in \mathscr{F}$, $x \cap z = \emptyset$, then $x^{\mathsf{c}} \supseteq z$, which implies that $x^{\mathsf{c}} \in \mathscr{F}$ and therefore $x \notin \mathscr{F}^+$. On the other hand, if $x \subseteq \omega$ is such that $x \notin \mathscr{F}^+$, then $x^{\mathsf{c}} \in \mathscr{F}$. So, for $z := x^{\mathsf{c}}$ we have $z \in \mathscr{F}$ and $x \cap z = \emptyset$. ⊣

By FACT 11.1 and the Ultrafilter Theorem we obtain the following alternative definition of $\mathscr{F}^+$:

$$\mathscr{F}^+ = \bigcup \{\mathscr{U} \subseteq \mathscr{P}(\omega) : \mathscr{U} \text{ is an ultrafilter containing } \mathscr{F}\}.$$

In particular, for every ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ we have $\mathscr{U}^+ = \mathscr{U}$.

If $\mathscr{U}$ is an ultrafilter and $x \cup y \in \mathscr{U}$, then at least one of x and y belongs to $\mathscr{U}$. In general, this is not the case for filters $\mathscr{F}$, but it holds for $\mathscr{F}^+$.

LEMMA 11.2. *Let $\mathscr{F} \subseteq [\omega]^\omega$ be a filter. If $\mathscr{F}^+$ contains $x \cup y$, then it contains at least one of x and y.*

Proof. If neither x nor y belongs to $\mathscr{F}^+$, then $x^{\mathsf{c}}, y^{\mathsf{c}} \in \mathscr{F}$. Hence, $(x \cup y)^{\mathsf{c}} = x^{\mathsf{c}} \cap y^{\mathsf{c}} \in \mathscr{F}$, and therefore $x \cup y \notin \mathscr{F}^+$. ⊣

Now, a filter $\mathscr{F} \subseteq [\omega]^\omega$ is called a **free filter** if it contains the Fréchet filter. In particular, every ultrafilter on $[\omega]^\omega$ is free. Notice that for a free filter $\mathscr{F}$, $\mathscr{F}^+ = \{x \subseteq \omega : \forall z \in \mathscr{F}(|x \cap z| = \omega)\}$, and that a filter $\mathscr{U} \subseteq [\omega]^\omega$ is an ultrafilter *iff* $\mathscr{U} = \mathscr{U}^+$. Finally, a family $\mathscr{E}$ of subsets of ω is called a **free family** if there is a free filter $\mathscr{F} \subseteq [\omega]^\omega$ such that $\mathscr{E} = \mathscr{F}^+$. In particular, $[\omega]^\omega$ and all ultrafilters on $[\omega]^\omega$ are free families. Notice that a free family does not contain any finite sets and is closed under supersets. Moreover, a free family $\mathscr{E}$ is closed under finite intersections *iff* $\mathscr{E}$ is an ultrafilter on $[\omega]^\omega$.

Recall that $\mathrm{fin}(\omega)$ denotes the set of all finite subsets of ω. To keep the notation short, for $s \in \mathrm{fin}(\omega)$ let $\bar{s} := \bigcup s$, and for $n \in \omega$ let $n^+ := n + 1$ (in other words, n^+ is the successor cardinal of n). In particular, for non-empty sets $s \in \mathrm{fin}(\omega)$ we have $\bar{s} = \max(s)$ and $\bar{s}^+ = \max(s) + 1$.

A set $x \subseteq \omega$ is said to **diagonalise** the set $\{x_s : s \in \mathrm{fin}(\omega)\} \subseteq [\omega]^\omega$ if the following conditions are satisfied:

- $x \subseteq x_\emptyset$;
- for all $s \in \mathrm{fin}(\omega)$, if $\bar{s} \in x$ then $x \setminus \bar{s}^+ \subseteq x_s$.

For $\mathscr{A} \subseteq [\omega]^\omega$ we write $\mathrm{fil}(\mathscr{A})$ for the filter generated by the members of $\mathscr{A}$, *i.e.*, $\mathrm{fil}(\mathscr{A})$ consists of all subsets of ω which are supersets of intersections of finitely many members of $\mathscr{A}$.

Now, a set $\mathscr{E} \subseteq [\omega]^\omega$ is a **happy family** if $\mathscr{E}$ is a free family and whenever $\mathrm{fil}(\{x_s : s \in \mathrm{fin}(\omega)\}) \subseteq \mathscr{E}$, there is an $x \in \mathscr{E}$ which diagonalises the set $\{x_s : s \in \mathrm{fin}(\omega)\}$.

Below, we give two examples of happy families; in the first the family is as large as possible, and in the second the family is of medium size—in the next section we shall see examples of happy families which are as small as possible.

FACT 11.3. *The family $[\omega]^\omega$ is happy.*

Proof. Let $\{x_s : s \in \mathrm{fin}(\omega)\} \subseteq [\omega]^\omega$ be a subfamily of $[\omega]^\omega$ and assume that $\mathrm{fil}(\{x_s : s \in \mathrm{fin}(\omega)\}) \subseteq [\omega]^\omega$, *i.e.*, the intersection of finitely many elements of $\{x_s : s \in \mathrm{fin}(\omega)\}$ is infinite. Let $n_0 := \bigcap x_\emptyset$ and for $k \in \omega$ choose $n_{k+1} > n_k$ such that

$$n_{k+1} \in \bigcap \bigl\{x_s : \bar{s}^+ \leq n_k + 1\bigr\}.$$

By our assumption, those choices are possible. Let $x = \{n_k : k \in \omega\}$; then $x \subseteq x_\emptyset$, and whenever $\bar{s} = n_k$ (*i.e.*, $\bar{s}^+ \leq n_k + 1$), we get

$$x \setminus \bar{s}^+ \subseteq \bigcap \bigl\{x_s : \bar{s}^+ \leq n_k + 1\bigr\}.$$

In particular, $x \setminus \bar{s}^+ \subseteq x_s$, as required. ⊣

In order to construct non-trivial examples of happy families, we first have to introduce the following notion: For a *mad* family $\mathscr{A} \subseteq [\omega]^\omega$, let $\mathscr{F}_{\mathscr{A}}$ be the collection of all subsets of ω which almost contain the complement of a finite union of members of $\mathscr{A}$.

The goal is to show that $\mathscr{F}^+_{\mathscr{A}}$ is a happy family whenever $\mathscr{A} \subseteq [\omega]^\omega$ is a *mad* family, but for this we first have to prove that $\mathscr{F}_{\mathscr{A}}$ is a free filter.

PROPOSITION 11.4. *If $\mathscr{A} \subseteq [\omega]^\omega$ is a mad family, then $\mathscr{F}_{\mathscr{A}}$ is a free filter but not an ultrafilter.*

Proof. Let $\mathscr{A} \subseteq [\omega]^\omega$ be a *mad* family and let

$$\mathscr{F}_{\mathscr{A}} = \bigl\{y \in [\omega]^\omega : \exists x_0 \dots x_n \in \mathscr{A}\bigl((x_0 \cup \dots \cup x_n)^{\mathsf{c}} \subseteq^* y\bigr)\bigr\}.$$

Firstly, $\mathscr{F}_{\mathscr{A}}$ is a free filter: By definition, $\mathscr{F}_{\mathscr{A}}$ is closed under supersets and contains all co-finite sets, and since $\mathscr{A}$ is *mad*, no co-finite set is the union of finitely many members of $\mathscr{A}$, hence, $\mathscr{F}_{\mathscr{A}}$ does not contain any finite set. Further, for any $y, y' \in \mathscr{F}_{\mathscr{A}}$ there are $x_0, \ldots, x_n$ and $x'_0, \ldots, x'_m$ in $\mathscr{A}$ such that

$$\Big(\bigcup_{i\in n} x_i\Big)^{\mathsf{c}} \subseteq^* y \quad \text{and} \quad \Big(\bigcup_{j\in m} x'_j\Big)^{\mathsf{c}} \subseteq^* y',$$

which shows that

$$\Big(\bigcup_{i\in n} x_i \cup \bigcup_{j\in m} x'_j\Big)^{\mathsf{c}} \subseteq^* y \cap y' \in \mathscr{F}_{\mathscr{A}}.$$

Secondly, $\mathscr{F}_{\mathscr{A}}$ is not an ultrafilter: We have to find a set $z \in [\omega]^\omega$ such that neither z nor z^{c} belongs to $\mathscr{F}_{\mathscr{A}}$. Let $\{x_i : i \in \omega\}$ be distinct elements of $\mathscr{A}$. Notice that it is enough to construct a set $z \in [\omega]^\omega$ such that both z and z^{c} have infinite intersection with each x_i. To construct such a set z, take a strictly increasing sequence $n_0 < \ldots < n_k < \ldots$ of natural numbers such that for each $k \in \omega$, if $k = 2^l(2m+1)$, then both n_{2k} and n_{2k+1} are in x_m and put $z = \{n_{2k} : k \in \omega\}$. ⊣

Now we are ready to give non-trivial examples of happy families. Even though the proof of the following proposition becomes considerably easier if we use the characterisation of happy families given by PROPOSITION 11.6 (b), we think it makes sense to have some non-trivial examples of happy families—and to work with the original definition—before giving an equivalent definition of happy families.

PROPOSITION 11.5. *Let* $\mathscr{A} \subseteq [\omega]^\omega$ *be a mad family. Then* $\mathscr{F}^+_{\mathscr{A}}$ *is a happy family.*

Proof. Given any family $\{y_t : t \in \text{fin}(\omega)\}$ with $\text{fil}(\{y_t : t \in \text{fin}(\omega)\}) \subseteq \mathscr{F}^+_{\mathscr{A}}$. For $s \in \text{fin}(\omega)$, let $x_s = \bigcap\{y_t : \bar{t} \leq \bar{s}\}$. Then for any $n \in \omega$, $x_{\{n\}} = x_s$ whenever $n = \bar{s}$. We shall construct an $x \in \mathscr{F}^+_{\mathscr{A}}$ which diagonalises $\{y_t : t \in \text{fin}(\omega)\}$ by showing that for all $n \in \omega$, $x \setminus n^+ \subseteq x_{\{n\}}$. For this, let x^0—constructed as in the proof of FACT 11.3—diagonalise $\{x_s : s \in \text{fin}(\omega)\}$. We may not assume that x^0 belongs to $\mathscr{F}^+_{\mathscr{A}}$, *i.e.*, there might be a $z \in \mathscr{F}$ such that $x^0 \cap z$ is finite. However, since $\mathscr{A}$ is *mad*, there is a $y^0 \in \mathscr{A}$ such that $x^0 \cap y^0$ is infinite. For each $s \in \text{fin}(\omega)$ define $x^1_s := x_s \setminus y^0$. Notice that all x^1_s are infinite and that $\text{fil}(\{x^1_s : s \in \text{fin}(\omega)\}) \subseteq \mathscr{F}^+_{\mathscr{A}}$, as $y^0 \in \mathscr{A}$. Let x^1 diagonalise $\{x^1_s : s \in \text{fin}(\omega)\}$ and let $y^1 \in \mathscr{A}$ be such that $x^1 \cap y^1$ is infinite. Since $x^1 \subseteq x^1_\emptyset \subseteq \omega \setminus y^0$ we get $y^1 \neq y^0$. Further, notice that x^1 also diagonalises $\{x_s : s \in \text{fin}(\omega)\}$. Now, for each $s \in \text{fin}(\omega)$ define $x^2_s := x_s \setminus (y^0 \cup y^1)$ and proceed as before. After countably many steps we have constructed two sequences of infinite sets, $\langle x^i : i \in \omega\rangle$ and $\langle y^i : i \in \omega\rangle$, such that each y^i belongs to $\mathscr{A}$, $y^i \neq y^j$ whenever $i \neq j$, $x^i \cap y^i$ is infinite (for all $i \in \omega$), and x^i diagonalises $\{x_s : s \in \text{fin}(\omega)\}$. Construct a strictly increasing sequence $n_0 < \ldots < n_k < \ldots$ of natural numbers such that $n_0 \in x_\emptyset$ and for each $k \in \omega$, if $k = 2^i(2m+1)$, then

$$n_k \in y^i \cap x^i \cap x_{\{n_{k-1}\}}.$$

Such a sequence of natural numbers exists because all sufficiently large numbers in x^i belong to $x_{\{n_{k-1}\}}$ and since $y^i \cap x^i$ is infinite. Finally, let $x = \{n_k : k \in \omega\}$. Then x diagonalises $\{x_s : s \in \text{fin}(\omega)\}$ and it remains to show that $x \in \mathscr{F}^+_{\mathscr{A}}$, *i.e.*, x has infinite intersection with each member of $\mathscr{F}_{\mathscr{A}}$. By construction, for each $i \in \omega$, $x \cap y^i$ is infinite, and since $\mathscr{A}$ is *mad*, $x \setminus y^i$ is infinite as well. Thus, x has infinite intersection with the complement of any finite union of elements in $\mathscr{A}$, hence, $x \in \mathscr{F}^+_{\mathscr{A}}$. ⊣

After having seen that there are non-trivial happy families, let us now give another characterisation of happy families, which will be used later in this chapter.

PROPOSITION 11.6. *For a free family $\mathscr{E}$, the following statements are equivalent:*

(a) *$\mathscr{E}$ is happy.*

(b) *If $y_0 \supseteq y_1 \supseteq \cdots \supseteq y_i \supseteq \cdots$ is a countable decreasing sequence of elements of $\mathscr{E}$, then there is a function $f \in {}^\omega\omega$ such that $f[\omega] \in \mathscr{E}$, $f(0) \in y_0$, and for all $n \in \omega$ we have $f(n+1) \in y_{f(n)}$.*

Proof. (a) ⇒ (b) Assume that $\mathscr{E}$ is happy and let $\{y_i : i \in \omega\} \subseteq \mathscr{E}$ be such that for all $i \in \omega$, $y_{i+1} \subseteq y_i$. For each $s \in \text{fin}(\omega)$ define

$$x_s = \bigcap\{y_i : i \le \bar{s}\}.$$

Notice that $\text{fil}(\{x_s : s \in \text{fin}(\omega)\}) \subseteq \mathscr{E}$. Since $\mathscr{E}$ is assumed to be happy, there is an x which diagonalises the family $\{x_s : s \in \text{fin}(\omega)\}$. Let $f = f_x$—recall that $f_x \in {}^\omega\omega$ is defined as the unique strictly increasing bijection between ω and x (defined in Chapter 9). For an arbitrary $n \in \omega$ let $s := x \cap (f(n)+1)$. Then $\bar{s}^+ = f(n)+1$ and $\bar{s} \in x$. As $f(n+1) \in x \setminus \bar{s}^+$ and $x \setminus \bar{s}^+ \subseteq x_s \subseteq y_{f(n)}$, we have $f(n+1) \in y_{f(n)}$, and since n was arbitrary, f has the required properties.

(b) ⇒ (a) Assume now that $\mathscr{E}$ has property (b) and let $\{x_s : s \in \text{fin}(\omega)\} \subseteq \mathscr{E}$ be such that $\text{fil}(\{x_s : s \in \text{fin}(\omega)\}) \subseteq \mathscr{E}$. We have to find an $x \in \mathscr{E}$ which diagonalises $\{x_s : s \in \text{fin}(\omega)\}$. For each $i \in \omega$ define

$$y_i = \bigcap\{x_s : \bar{s} \le i\}.$$

Obviously, for each $i \in \omega$ we have $y_i \in \mathscr{E}$ and $y_{i+1} \subseteq y_i$. By (b) there is a function $f \in {}^\omega\omega$ such that $f[\omega] \in \mathscr{E}$ and for all $n \in \omega$ we have $f(n+1) \in y_{f(n)}$. Let $x := f[\omega]$ and let $s \in \text{fin}(\omega)$ be such that $\bar{s} \in x$. Then there exists an $n \in \omega$ such that $f(n) = \bar{s}$, and for every $k \in x \setminus \bar{s}^+$ we have $k = f(m)$ for some $m > n$, hence, $k \in y_{f(n)}$. Now, $\bar{s}^+ = f(n)+1$, and since $y_{f(n)} \subseteq x_s$ we get $k \in x_s$. Hence, for all $s \in \text{fin}(\omega)$ with $\bar{s} \in x$ we have $x \setminus \bar{s}^+ \subseteq x_s$, which shows that x diagonalises $\{x_s : s \in \text{fin}(\omega)\}$. ⊣

We leave it as an exercise to the reader to find an easier proof of PROPOSITION 11.5 by using the characterisation of happy families given by PROPOSITION 11.6 (b).

Ramsey Ultrafilters

So far we have seen two examples of happy families. In the first example (FACT 11.3), the happy family was as large as possible, and in the second example (PROPOSITION 11.5), the happy families were of medium size. Below, we consider happy families which are as small as possible, *i.e.*, happy families which are ultrafilters.

A free ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ is a **Ramsey ultrafilter** if for every colouring $\pi : [\omega]^2 \to 2$ there exists an $x \in \mathscr{U}$ which is homogeneous for π, *i.e.*, $\pi|_{[x]^2}$ is constant.

The following result gives two alternative characterisations of Ramsey ultrafilter. The first characterisation of Ramsey ultrafilters is related to P-points and Q-points (introduced below), and the second characterisation shows that a Ramsey ultrafilter is an ultrafilter that is also a happy family.

PROPOSITION 11.7. *For every free ultrafilter $\mathscr{U}$, the following conditions are equivalent:*

(a) *$\mathscr{U}$ is a Ramsey ultrafilter.*

(b) *Let $\{u_i \subseteq \omega : i \in \omega\}$ be a infinite partition of ω, i.e., $\bigcup\{u_i : i \in \omega\} = \omega$ and for any distinct $i, j \in \omega$ we have $u_i \cap u_j = \emptyset$. Then either $u_i \in \mathscr{U}$ for a (unique) $i \in \omega$, or there exists an $x \in \mathscr{U}$ such that for each $i \in \omega$, $|x \cap u_i| \leq 1$.*

(c) *$\mathscr{U}$ is happy.*

Proof. (a) $\Rightarrow$ (b) Let $\{u_i : i \in \omega\}$ be an infinite partition of ω. With respect to $\{u_i : i \in \omega\}$ define the colouring $\pi : [\omega]^2 \to 2$ by

$$\pi(\{n,m\}) = \begin{cases} 0 & \text{if there is an } i \in \omega \text{ such that } \{n,m\} \subseteq u_i, \\ 1 & \text{otherwise.} \end{cases}$$

By (a) there is an $x \in \mathscr{U}$ such that $\pi|_{[x]^2}$ is constant. Now, if $\pi|_{[x]^2}$ is constantly zero, then there exists an $i \in \omega$ such that $x \subseteq u_i$, hence, $u_i \in \mathscr{U}$. On the other hand, if $\pi|_{[x]^2}$ is constantly one, then for any distinct $n, m \in x$ and any $i \in \omega$ we find that $\{n,m\} \cap u_i$ has at most one element, hence, for each $i \in \omega$, $x \cap u_i$ has at most one element.

(b) $\Rightarrow$ (c) By PROPOSITION 11.6 it is enough to show that for every countable decreasing sequence $y_0 \supseteq y_1 \supseteq \ldots \supseteq y_n \supseteq \ldots$ of elements of $\mathscr{U}$ there is a function $f \in {}^\omega\omega$ such that $f[\omega] \in \mathscr{U}$, $f(0) \in y_0$, and for all $k \in \omega$ we have

$f(k+1) \in y_{f(k)}$. If $y = \bigcap_{n\in\omega} y_n \in \mathscr{U}$, then the function $f_y \in {}^\omega\omega$ has the required properties. So, let us assume that $\bigcap_{n\in\omega} y_n \notin \mathscr{U}$ and without loss of generality let us further assume that for all $n \in \omega$, $y_n \setminus y_{n+1} \neq \emptyset$. Consider the partition $\{y_0^{\mathsf{c}} \cup \bigcap_{n\in\omega} y_n\} \cup \{y_n \setminus y_{n+1} : n \in \omega\}$ and notice that none of the pieces are in $\mathscr{U}$. By (b), there exists a set $x = \{a_n : n \in \omega\} \in \mathscr{U}$ such that for all $n \in \omega$, $x \cap (y_n \setminus y_{n+1}) = \{a_n\}$, in particular, $x \cap \bigcap_{n\in\omega} y_n = \emptyset$. Let $g \in {}^\omega\omega$ be a strictly increasing function such that $g(0) > 0$, $g[\omega] \subseteq x$, and for all $n \in \omega$, $x \setminus g(n) \subseteq y_n$. For $k \in \omega$ let $g^{k+1}(0) := g(g^k(0))$, where $g^0(0) := 0$. Now, since $\mathscr{U}$ is an ultrafilter, either

$$u_0 = \bigcup_{k\in\omega} \big[g^{2k}(0), g^{2k+1}(0)\big) \quad \text{or} \quad u_1 = \omega \setminus u_0$$

belongs to $\mathscr{U}$—recall that $[a,b) = \{i \in \omega : a \leq i < b\}$. Without loss of generality we may assume that $u_0 \in \mathscr{U}$, and consequently $x \cap u_0 \in \mathscr{U}$. By (b) and since $\mathscr{U}$ is an ultrafilter, there exists a set $z = \{c_k : k \in \omega\} \subseteq x$ such that $z \in \mathscr{U}$ and for all $k \in \omega$,

$$z \cap \big[g^{2k}(0), g^{2k+1}(0)\big) = \{c_k\}\,.$$

By construction, for each $k \in \omega$ we have $c_{k+1} > g(c_k)$. To see this, notice that

$$c_{k+1} \in \big[g^{2k+2}(0), g^{2k+3}(0)\big),$$

which implies $c_{k+1} \geq g^{2k+2}(0)$. On the other hand,

$$c_k \in \big[g^{2k}(0), g^{2k+1}(0)\big),$$

which implies $g^{2k+1}(0) > c_k$, and because the function g is strictly increasing, we get $g^{2k+2}(0) > g(c_k)$; hence, $c_{k+1} > g(c_k)$. Finally, by the definition of g we have $x \setminus g(c_k) \subseteq y_{c_k}$, and since $c_{k+1} > g(c_k)$ and $c_{k+1} \in x$, for all $k \in \omega$ we have

$$c_{k+1} \in y_{c_k}\,.$$

Thus, if we define the function $f \in {}^\omega\omega$ by stipulating $f(k) := c_k$, then f has the required properties.

(c) ⇒ (a) Let $\mathscr{U}$ be an ultrafilter that is also a happy family, and further let $\pi : [\omega]^2 \to 2$ be an arbitrary but fixed colouring. We have to find a $y \in \mathscr{U}$ such that $\pi|_{[y]^2}$ is constant. The proof is similar to the proof of PROPOSITION 4.2. First we construct a family $\{x_s : s \in \text{fin}(\omega)\} \subseteq \mathscr{U}$. Let $x_\emptyset = \omega$, and let $x_{\{0\}} \in \mathscr{U}$ be such that $x_{\{0\}} \subseteq \omega \setminus \{0\}$ and for all $k, k' \in x_{\{0\}}$ we have $\pi(\{0,k\}) = \pi(\{0,k'\})$. Notice that since $\mathscr{U}$ is an ultrafilter, $x_{\{0\}}$ exists. In general, if x_s is defined and $n > \bar{s}$, then let $x_{s\cup\{n\}} \in \mathscr{U}$ be such that $x_{s\cup\{n\}} \subseteq x_s \setminus n^+$ and for all $k, k' \in x_{s\cup\{n\}}$ we have $\pi(\{n,k\}) = \pi(\{n,k'\})$. Since $\mathscr{U}$ is happy, there is a $y \in \mathscr{U}$ which diagonalises the family $\{x_s : s \in \text{fin}(\omega)\}$. By construction, for each $n \in y$ and for all $k, k' \in y \setminus n^+$ we have $\pi(\{n,k\}) = \pi(\{n,k'\})$ and we can define the colouring $\tau : x \to 2$ by stipulating

$$\tau(n) = \begin{cases} 0 & \text{if there is a } k \in x \setminus n^+ \text{ such that } \pi(\{n,k\}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathscr{U}$ is an ultrafilter, there exists a $x \in \mathscr{U}$ such that $x \subseteq y$ and $\tau|_x$ is constant, hence, $\pi|_{[x]^2}$ is constant. ⊣

At first glance, condition (a) is just related to PROPOSITION 4.2 and not to RAMSEY'S THEOREM. However, the following fact shows that this is not the case. Moreover, even PROPOSITION 4.8 is related to Ramsey ultrafilters (the proofs are left to the reader).

FACT 11.8. *For every free ultrafilter $\mathscr{U}$, the following conditions are equivalent:*

(a) *$\mathscr{U}$ is a Ramsey ultrafilter, i.e., for every colouring $\pi : [\omega]^2 \to 2$ there exists an $x \in \mathscr{U}$ which is homogeneous for π.*
(b) *For any $n \in \omega$, for any positive integer $r \in \omega$, and for every colouring $\pi : [\omega]^n \to r$, there exists an $x \in \mathscr{U}$ which is homogeneous for π.*
(c) *Let $\{r_k : k \in \omega\}$ and $\{n_k : k \in \omega\}$ be two (possibly finite) sets of positive integers, and for each $k \in \omega$ let $\pi_k : [\omega]^{n_k} \to r_k$ be a colouring. Then there exists an $x \in \mathscr{U}$ which is almost homogeneous for each π_k.*

It is time now to address the problem of the existence of Ramsey ultrafilters. On the one hand, it can be shown that there are models of ZFC in which no Ramsey ultrafilters exist (see PROPOSITION 26.23). Thus, the existence of Ramsey ultrafilters is not provable in ZFC. On the other hand, if we assume, for example, CH (or just $\mathfrak{p} = \mathfrak{c}$), then we can easily construct a Ramsey ultrafilter.

PROPOSITION 11.9. *If $\mathfrak{p} = \mathfrak{c}$, then there exists a Ramsey ultrafilter.*

Proof. Let $\{\pi_\alpha : \alpha \in \mathfrak{c}\}$ be an enumeration of the set of all 2-colourings of $[\omega]^2$, *i.e.*, for every colouring $\pi : [\omega]^2 \to 2$ there exists an $\alpha \in \mathfrak{c}$ such that $\pi = \pi_\alpha$. By transfinite induction we first construct a sequence $\langle x_\alpha : \alpha \in \mathfrak{c}\rangle \subseteq [\omega]^\omega$ such that $\{x_\alpha : \alpha \in \mathfrak{c}\}$ has the finite intersection property and for all $\alpha \in \mathfrak{c}$, $\pi_\alpha|_{[x_{\alpha+1}]^2}$ is constant. Let $x_0 := \omega$ and assume that for some $\alpha \in \mathfrak{c}$ we have already constructed x_β ($\beta \in \alpha$) such that $\{x_\beta : \beta \in \alpha\}$ has the finite intersection property and for all $\gamma + 1 \in \alpha$ we have $\pi_\gamma|_{[x_{\gamma+1}]^2}$ is constant. If α is a successor ordinal, say $\alpha = \beta_0 + 1$, then let $x_\alpha \in [x_{\beta_0}]^\omega$ be such that $\pi_{\beta_0}|_{[x_\alpha]^2}$ is constant (notice that by RAMSEY'S THEOREM 4.1, $x_{\alpha+1}$ exists). If α is a limit ordinal, then let x_α be a pseudo-intersection of $\{x_\beta : \beta \in \alpha\}$ (notice that since $|\alpha| < \mathfrak{p}$, $x_{\alpha+1}$ exists). In either case, the family $\{x_\beta : \beta \in \alpha\}$ has the required properties. In particular, the family $\mathscr{E} = \{x_\alpha : \alpha \in \mathfrak{c}\}$ has the finite intersection property and for each colouring $\pi : [\omega]^2 \to 2$ there is an $x \in \mathscr{E}$ such that $\pi|_{[x]^2}$ is constant. Finally, extend the family $\mathscr{E}$ to an ultrafilter $\mathscr{U}$. Then $\mathscr{U}$ is a Ramsey ultrafilter. ⊣

P-points and Q-points

Below, we consider ultrafilters which are weaker than Ramsey ultrafilters, but which share with them some combinatorial properties.

A free ultrafilter $\mathscr{U}$ is a **P-point** if for each partition $\{u_n \subseteq \omega : n \in \omega\}$ of ω, either $u_n \in \mathscr{U}$ for a (unique) $n \in \omega$, or there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap u_n$ is finite.

Furthermore, a free ultrafilter $\mathscr{U}$ is a **Q-point** if for each partition of ω into finite pieces $\{u_n \subseteq \omega : n \in \omega\}$ (*i.e.*, for each $n \in \omega$, u_n is finite), there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap u_n$ has at most one element.

An alternative definition of Q-points is in terms of so-called *interval partitions*: A partition $P = \{u_n \subseteq \omega : n \in \omega\}$ of ω is an **interval partition** if each $u_n \in P$ is of the form $[a, b]$ (for some $a, b \in \omega$), where $[a, b] := \{n \in \omega : a \leq n \leq b\}$.

FACT 11.10. *An ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ is a Q-point if and only if for each interval partition $\{I_n \subseteq \omega : n \in \omega\}$ there is an $x \in \mathscr{U}$, such that for each $n \in \omega$, $|x \cap I_n| \leq 1$.*

Proof. ($\Rightarrow$) Because every interval partition is a partition of ω into finite pieces, this direction follows immediately from the definition of Q-points.

($\Leftarrow$) Let $P = \{u_n \subseteq \omega : n \in \omega\}$ be a partition of ω into finite pieces. Let $a_0 := \max(u_0)$ and for $m \in \omega$, let

$$a_{m+1} := \max\left\{ \max(u_n) : u_n \in P \ \wedge \ u_n \cap [0, a_m + 1] \neq \emptyset \right\}.$$

Furthermore, let $I_0 := [0, a_0]$ and, for $m \in \omega$, let $I_{m+1} := (a_m, a_{m+1}]$, where $(a_m, a_{m+1}] := \{n \in \omega : a_m < n \leq a_{m+1}\}$. Since $\{I_n : n \in \omega\}$ is an interval partition, by our assumption there is an $x \in \mathscr{U}$ such that for each $n \in \omega$, $|x \cap I_n| \leq 1$. Notice that by construction, if, for some $n, m \in \omega$, $u_n \cap I_m \neq \emptyset$, then for all $k \in \omega \setminus \{m-1, m, m+1\}$ we have $u_n \cap I_k = \emptyset$. This shows that for each $n \in \omega$, $|x \cap u_n| \leq 2$, and if $|x \cap u_n| = 2$ for some $n \in \omega$, then there is an $m \in \omega$ such that $|x \cap u_n \cap I_m| = 1$ and $|x \cap u_n \cap I_{m+1}| = 1$. Now, let

$$x_0 := \{x \cap I_{2m} : m \in \omega\} \quad \text{and} \quad x_1 := \{x \cap I_{2m+1} : m \in \omega\}.$$

Then for each $n \in \omega$ we have that $x_0 \cap u_n$ as well as $x_1 \cap u_n$ contains at most one element, and since one of x_0 or x_1 belongs to $\mathscr{U}$, there is a set in $\mathscr{U}$ with the required properties. ⊣

Comparing the definitions of P-points and Q-points with PROPOSITION 11.7 (b), it is evident that a Ramsey ultrafilter is both a P-point as well as a Q-point; but the converse is also true:

FACT 11.11. *$\mathscr{U}$ is a Ramsey ultrafilter if and only if $\mathscr{U}$ is a P-point and a Q-point.*

Proof. (⇒) This follows immediately from PROPOSITION 11.7 (b) and the definitions of P-points and Q-points.

(⇐) Let $\mathscr{U}$ be a P-point and a Q-point and let $\{u_n \subseteq \omega : n \in \omega\}$ be a partition of ω. We have to show that either $u_n \in \mathscr{U}$ for a (unique) $n \in \omega$, or there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap u_n$ has at most one element. If there is a $u_n \in \mathscr{U}$, then we are done (notice that since $\mathscr{U}$ is an ultrafilter, u_n is unique). So, assume that for all $n \in \omega$, $u_n \notin \mathscr{U}$. Since $\mathscr{U}$ is a P-point, there exists a $y_0 \in \mathscr{U}$ such that for each $n \in \omega$, $y_0 \cap u_n$ is finite. For $n \in \omega$ let $I_{2n} := y_0 \cap u_n$. Further, let $\{a_i : i \in \omega\} = \omega \setminus \bigcup_{n\in\omega}\{I_{2n} : n \in \omega\}$ and for $n \in \omega$ let $I_{2n+1} := \{a_n\}$. Then $\{I_m : m \in \omega\}$ is a partition of ω into finite pieces. Since $\mathscr{U}$ is a Q-point, there exists a $y_1 \in \mathscr{U}$ such that for each $m \in \omega$, $y_1 \cap I_m$ has at most one element. Now, let $x = y_0 \cap y_1$. Then $x \in \mathscr{U}$ and for each $n \in \omega$, $x \cap u_n$ has at most one element, hence, by PROPOSITION 11.7 (b), $\mathscr{U}$ is a Ramsey ultrafilter. ⊣

Below, we give two other characterisations of P-points. The proofs are straightforward and are left to the reader.

FACT 11.12. *For every free ultrafilter $\mathscr{U}$, the following conditions are equivalent:*

(a) *$\mathscr{U}$ is a P-point.*

(b) *For every family $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ there is an $x \in \mathscr{U}$ such that for all $n \in \omega$, $x \subseteq^* x_n$ (i.e., $x \setminus x_n$ is finite).*

(c) *For every family $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ there is a function $f \in {}^\omega\omega$ and a set $x \in \mathscr{U}$ such that for all $n \in \omega$, $x \setminus f(n) \subseteq x_n$.*

There are also characterisations of P-points which are not so obvious. As an example we give a characterisation of P-points which is seemingly stronger than the characterisation given in FACT 11.12.(b).

PROPOSITION 11.13. *For a free ultrafilter $\mathscr{U}$, the following conditions are equivalent:*

(a) *$\mathscr{U}$ is a P-point.*

(b) *For every family $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ there is an $x \in \mathscr{U}$ such that for infinitely many $n \in \omega$, $x \setminus n \subseteq x_n$.*

Proof. (b) ⇒ (a) Let $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ be an arbitrary countable subset of some ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$. For each $n \in \omega$, define

$$x'_n := \bigcap_{k\le n} x_k \,.$$

Then $\{x'_n : n \in \omega\} \subseteq \mathscr{U}$ and for all $m, n \in \omega$ with $m < n$ we have $x'_m \supseteq x'_n$. Now, by assumption there is an $x \in \mathscr{U}$ such that for infinitely many $n \in \omega$, $x \setminus n \subseteq x'_n$, then for all $n \in \omega$, $x \subseteq^* x_n$. Hence, by FACT 11.12 (b) and since $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ was arbitrary, we get that $\mathscr{U}$ is a P-point.

(a) $\Rightarrow$ (b) Since $\mathscr{U}$ is a P-point, by FACT 11.12 (c) there exists a function $f \in {}^\omega\omega$ and a set $y \in \mathscr{U}$ such that for all $n \in \omega$, $y \setminus f(n) \in x_n$. Hence, there also exists a function $g \in {}^\omega\omega$ such that $g(0) = 0$ and for all $k \in \omega$ we have $y \setminus g(k+1) \subseteq x_{g(k)}$. Since $\mathscr{U}$ is an ultrafilter, either $y_0 = \bigcup_{k\in\omega}[g(2k+1), g(2k+2))$ or $y_1 = \bigcup_{k\in\omega}[g(2k), g(2k+1))$ belongs to $\mathscr{U}$. Let $x = y \cap y_\varepsilon$, where $\varepsilon \in \{0,1\}$ is such that $y_\varepsilon \in \mathscr{U}$. Then for every $k \in \omega$ we have $x \setminus g(2k+\varepsilon) = x \setminus g(2k+\varepsilon+1) \subseteq x_{2k+\varepsilon}$. $\dashv$

P-points and Q-points, and consequently Ramsey ultrafilters, can also be characterised in terms of functions, but first we have to introduce the notion of finite-to-one functions: A function $f \in {}^\omega\omega$ is **finite-to-one** if for every $k \in \omega$, the set $\{n \in \omega : f(n) = k\}$ is finite.

PROPOSITION 11.14. *Let $\mathscr{U}$ be a free ultrafilter.*

(a) *$\mathscr{U}$ is a P-point if and only if for every function $f \in {}^\omega\omega$ there exists an $x \in \mathscr{U}$ such that $f|_x$ is constant or finite-to-one.*

(b) *$\mathscr{U}$ is a Q-point if and only if for every finite-to-one function $f \in {}^\omega\omega$ there exists an $x \in \mathscr{U}$ such that $f|_x$ is one-to-one.*

(c) *$\mathscr{U}$ is a Ramsey ultrafilter if and only if for every function $f \in {}^\omega\omega$ there exists an $x \in \mathscr{U}$ such that $f|_x$ is constant or one-to-one.*

Proof. Let $f \in {}^\omega\omega$ be an arbitrary but fixed function. For each $k \in \omega$ let $u_k := \{n \in \omega : f(n) = k\}$. Then $\{u_k : k \in \omega\} \setminus \{\emptyset\}$ is a partition of ω. The proof now follows from FACT 11.11 and the following observations (the details are left to the reader):

- For any $x \in [\omega]^\omega$, $f|_x$ is constant *iff* there is a $k \in \omega$ such that $x \subseteq u_k$.
- For any $x \in [\omega]^\omega$, $f|_x$ is finite-to-one *iff* for all $k \in \omega$ we have $x \cap u_k$ is finite.
- The function f is finite-to-one *iff* each u_k is finite.
- For any $x \in [\omega]^\omega$, $f|_x$ is one-to-one *iff* for all $k \in \omega$, $x \cap u_k$ has at most one element. $\dashv$

The next result shows that ultrafilters, and especially Q-points, must contain quite "sparse" sets.

PROPOSITION 11.15. *For free families $\mathscr{U} \subseteq [\omega]^\omega$ we have*

(a) *If $\mathscr{U}$ is a free ultrafilter, then the family $\{f_x \in {}^\omega\omega : x \in \mathscr{U}\}$ is unbounded.*

(b) *If $\mathscr{U}$ is a Q-point, then the family $\{f_x \in {}^\omega\omega : x \in \mathscr{U}\}$ is dominating.*

Proof. (a) Let $f \in {}^\omega\omega$ be arbitrary. Define $g(0) = \max\{f(0), 1\}$ and for $k \in \omega$ define $g(k+1) := g(k) + f(g(k))$. Further, let $x_0 = [0, g(0))$, and in general, for $n \in \omega$, let $x_n = [g(2n), g(2n+1))$ and $y_n = [g(2n+1), g(2n+2))$. Finally, let $x = \bigcup_{n\in\omega} x_n$ and $y = \bigcup_{n\in\omega} y_n$. We leave it as an exercise to the reader to verify that $f_x \not\leq^* f$ and $f_y \not\leq^* f$. Hence, f dominates neither f_x nor f_y. Now, since $\mathscr{U}$ is an ultrafilter, either x or y belongs to $\mathscr{U}$. Hence, f does not dominate the family $\mathscr{B} = \{f_x \in {}^\omega\omega : x \in \mathscr{U}\}$, and since f was arbitrary, $\mathscr{B}$ is unbounded.

(b) Let $g \in {}^\omega\omega$ be arbitrary. Without loss of generality we may assume that g is strictly increasing. For $n \in \omega$, let $I_n = [g(2n), g(2n+2))$. Then $\{I_n : n \in \omega\}$ is a partition of ω into finite pieces. Since $\mathscr{U}$ is a Q-point, there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap I_n$ has at most one element which implies that $g <^* f_x$. Hence, f_x dominates g, and since g was arbitrary, the family $\{f_x \in {}^\omega\omega : x \in \mathscr{U}\}$ is dominating. ⊣

As we have seen above (PROPOSITION 11.9), $\mathfrak{p} = \mathfrak{c}$ implies the existence of a Ramsey ultrafilter. On the other hand, one can show that $\mathfrak{d} = \mathfrak{c}$ is not sufficient to prove the existence of Ramsey ultrafilters (see PROPOSITION 26.23). However, as a consequence of the next result, we see that $\mathfrak{d} = \mathfrak{c}$ is sufficient to prove the existence of P-points—which shows that P-points are easier to get than Ramsey ultrafilters (*cf.* RELATED RESULTS 66 & 67).

THEOREM 11.16. *$\mathfrak{d} = \mathfrak{c}$ if and only if every free filter over a countable set which is generated by less than $\mathfrak{c}$ sets can be extended to a P-point. In particular, $\mathfrak{d} = \mathfrak{c}$ implies the existence of P-points.*

Proof. ($\Leftarrow$) Suppose that $\mathscr{E} \subseteq {}^\omega\omega$ is a family of cardinality less than $\mathfrak{c}$. For $f \in \mathscr{E}$ and $n \in \omega$ define

$$x_f = \big\{\langle n,k\rangle \in \omega\times\omega : f(n) < k\big\} \quad \text{and} \quad x_n = \big\{\langle m,k\rangle \in \omega\times\omega : n \leq m\big\},$$

and let

$$\mathscr{C} = \{x_f : f \in \mathscr{E}\} \cup \{x_n : n \in \omega\} \cup \big\{z \subseteq \omega\times\omega : (\omega\times\omega)\setminus z \text{ is finite}\big\}.$$

Notice that $|\mathscr{C}| < \mathfrak{c}$ and that each set in $\mathscr{C}$ is an infinite subset of the countable set $\omega\times\omega$. Moreover, for any finitely many members $y_0,\dots,y_n \in \mathscr{C}$ we have $y_0\cap\ldots\cap y_n$ is infinite. Now, the family $\mathscr{C}$ generates a free filter over $\omega\times\omega$, which, by assumption, can be extended to a P-point $\mathscr{U} \subseteq [\omega\times\omega]^\omega$. Consider the partition $\{u_n : n \in \omega\}$ of $\omega\times\omega$, where for $n \in \omega$, $u_n := \{n\}\times\omega$. Notice that no u_n (for $n \in \omega$) belongs to $\mathscr{U}$. Since $\mathscr{U}$ is a P-point, there exists a $y \in \mathscr{U}$ such that for all $n \in \omega$, $y \cap u_n$ is finite. Let us define the function $g \in {}^\omega\omega$ by stipulating $g(n) = \bigcup\{k \in \omega : \langle n,k\rangle \in y \cap u_n\}$. Since $y \in \mathscr{U}$, for all $f \in \mathscr{E}$ we have $y \cap x_f$ is infinite. Hence, for every $f \in \mathscr{E}$ there are infinitely many $n \in \omega$ such that

$g(n) > f(n)$. In other words, g is not dominated by any function $f \in \mathscr{E}$, which shows that no family of cardinality less than $\mathfrak{c}$ is dominating.

($\Rightarrow$) The proof is by induction using the following

CLAIM. *Suppose that the free filter $\mathscr{F} \subseteq [\omega]^\omega$ is generated by less than $\mathfrak{d}$ sets and let $\{x_n : n \in \omega\} \subseteq \mathscr{F}$. Then there exists an $x \in [\omega]^\omega$ such that for all $n \in \omega$, $x \subseteq^* x_n$, and for all $y \in \mathscr{F}$, $x \cap y$ is infinite.*

Proof of Claim. Without loss of generality we may assume that for all $n \in \omega$, $x_{n+1} \subseteq x_n$. For $y \in \mathscr{F}$ define $g_y \in {}^\omega\omega$ by stipulating $g_y(n) = \bigcap(y \cap x_n)$. Notice that the set $y \cap x_n$ is non-empty, and that if $y \subseteq y'$, then for all $n \in \omega$, $g_{y'}(n) \leq g_y(n)$. Now, since $\mathscr{F}$ is generated by less than $\mathfrak{d}$ sets, and since every free ultrafilter generated by less than $\mathfrak{d}$ sets has a basis of less than $\mathfrak{d}$ sets, there exists a function $f \in {}^\omega\omega$ such that for all $y \in \mathscr{F}$ we have $f \not\leq^* g_y$. Finally, let

$$x = \bigcup_{n\in\omega} \big(x_n \cap f(n)\big).$$

We leave it to the reader to verify that x has the required properties. $\dashv_{\text{Claim}}$

By the claim and the assumption that $\mathfrak{d} = \mathfrak{c}$ we inductively construct a P-point as follows: Let $\{X_\alpha \subseteq [\omega]^\omega : |X_\alpha| \leq \omega \wedge \alpha \in \mathfrak{c}\}$ be an enumeration of all countable subsets of $[\omega]^\omega$. Let $\mathscr{F}_0$ be any free filter which is generated by less than $\mathfrak{d}$ sets and assume that we have already constructed $\mathscr{F}_\alpha$ for some $\alpha \in \mathfrak{c}$. If $X_\alpha \cup \mathscr{F}_\alpha$ has the finite intersection property, then we use the claim to obtain a set $x_{\alpha+1}$ such that $\{x_{\alpha+1}\} \cup \mathscr{F}_\alpha$ has the finite intersection property and $x_{\alpha+1}$ is a pseudo-intersection of X_α; and let $\mathscr{F}_{\alpha+1}$ be the filter generated by $\mathscr{F}_\alpha$ and $x_{\alpha+1}$. If $X_\alpha \cup \mathscr{F}_\alpha$ does not have the finite intersection property, then let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Further, if $\alpha \in \mathfrak{c}$ is a limit ordinal and for all $\beta \in \alpha$ we have already constructed $\mathscr{F}_\beta$, then let $\mathscr{F}_\alpha = \bigcup_{\beta\in\alpha} \mathscr{F}_\beta$. Finally, let $\mathscr{F} = \bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_\alpha$. Then $\mathscr{F}$ is a P-point: Firstly, by construction, $\mathscr{F}$ is a filter, and since the free filter $\mathscr{F}_0$ is contained in $\mathscr{F}$, $\mathscr{F}$ is even a free filter. Secondly, for any $x \in [\omega]^\omega$ there exists a $\beta \in \mathfrak{c}$ such that $X_\beta = \{x\}$. Thus, either $x \in \mathscr{F}_{\beta+1}$ or there is a $y \in \mathscr{F}_\beta$ such that $x \cap y$ is finite, which implies that $x^{\mathsf{c}} \in \mathscr{F}_\beta$. Hence, $\mathscr{F}$ is a free ultrafilter. Finally, for every set $\{x_n : n \in \omega\} \subseteq \mathscr{F}$ there exists a $\gamma \in \mathfrak{c}$ such that $X_\gamma = \{x_n : n \in \omega\}$. Since $X_\gamma \cup \mathscr{F}_\gamma$ has the finite intersection property, there is an $x_{\gamma+1} \in \mathscr{F}_{\gamma+1}$ such that for all $n \in \omega$, $x_{\gamma+1} \subseteq^* x_n$. $\dashv$

Ramsey Families and P-families

Below, we give characterisations of Ramsey ultrafilters and P-points in terms of games, which lead to so-called Ramsey families and P-families respectively.

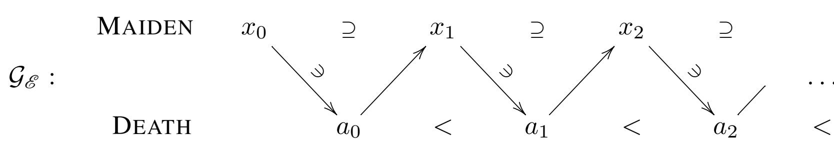
The two games we shall consider are played between two players, which we shall call DEATH and the MAIDEN. We have chosen these two players, because DEATH and the MAIDEN is a well-known motif of *Danse Macabre* or *Dance of Death*, which has received numerous treatments, especially in the second movement of Schubert's string quartet no. 14, called *Der Tod und das Mädchen*. Moreover, since the crucial point in the games we consider is that one of the players does not have a winning strategy, the player without a winning strategy is—for obvious reasons—always the MAIDEN.

Now, a **run** of an infinite two-player game consists of an infinite sequence $\langle x_0, y_0, x_1, y_1, \dots\rangle$ which is constructed alternately by the two players. More precisely, the first player starts the game with x_0 and the second player responds with y_0, then the first player plays x_1 and the second player responds with y_1, and so on. Of course, in order to get a proper game we also have to introduce some rules defining legal moves and telling which player wins a particular run of the game.

Before we introduce some further game-theoretical notions, let us illustrate the notion of rules by the following infinite two-player game, played between DEATH and the MAIDEN.

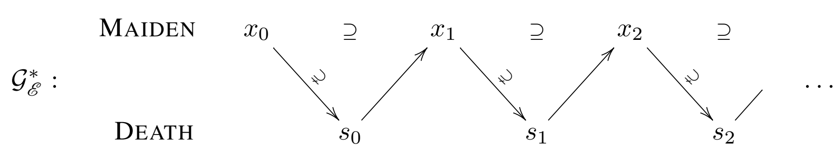
Let $\mathscr{E}$ be an arbitrary free family. Associated with $\mathscr{E}$ we define two quite similar games, denoted $\mathcal{G}_{\mathscr{E}}$ and $\mathcal{G}^*_{\mathscr{E}}$, between two players, say DEATH and the MAIDEN.

In the game $\mathcal{G}_{\mathscr{E}}$, the MAIDEN always plays members of $\mathscr{E}$ and then DEATH responds with an element of the MAIDEN's move. More precisely, the rules for $\mathcal{G}_{\mathscr{E}}$ are as follows: For each $i \in \omega$, $x_i \in \mathscr{E}$ and $a_i \in x_i$. Furthermore, we require that for each $i \in \omega$, $x_{i+1} \subseteq x_i$ and $a_i < a_{i+1}$. A run of $\mathcal{G}_{\mathscr{E}}$ is illustrated by the following diagram:



Finally, DEATH wins the game $\mathcal{G}_{\mathscr{E}}$ if and only if the set $\{a_i : i \in \omega\}$ belongs to the family $\mathscr{E}$.

In the game $\mathcal{G}^*_{\mathscr{E}}$, DEATH has slightly more freedom, since he can now play finite sets instead of singletons. A run of $\mathcal{G}^*_{\mathscr{E}}$ is illustrated by the following diagram.



Again, the sets x_i played by the MAIDEN must belong to the free family $\mathscr{E}$ and each finite set s_i played by DEATH must be a subset of the corresponding x_i. Furthermore, for each $i \in \omega$ we require that $x_{i+1} \subseteq (x_i \setminus \bigcup_{j \leq i} s_j)$. Notice that the finite sets s_i may be empty. Finally, DEATH wins the game $\mathcal{G}^*_{\mathscr{E}}$ if and only if $\bigcup\{s_i : i \in \omega\}$ belongs to the family $\mathscr{E}$.

Now we define the notion of a strategy for the MAIDEN. Roughly speaking, a **strategy** for the MAIDEN is a "rule" that tells her how to play, for each $n \in \omega$, her n^{th} move x_n, given DEATH' previous moves $m_0, \ldots, m_n$. In fact, a strategy for the MAIDEN in the game $\mathcal{G}_{\mathscr{E}}$ is a mapping σ from $\operatorname{seq}(\mathscr{E} \cup \omega)$ to $\mathscr{E}$. Intuitively, with respect to $\mathcal{G}_{\mathscr{E}}$, a strategy σ for the MAIDEN works as follows: The MAIDEN starts playing $x_0 \in \mathscr{E}$, where $x_0 = \sigma(\emptyset)$ and then DEATH responds by playing an element $a_0 \in x_0$. Then the MAIDEN plays $x_1 = \sigma(x_0, a_0)$, which—by the rules of the game—is a set in $\mathscr{E}$ and a subset of x_0, and DEATH responds with an element $a_1 \in x_1$ where $a_1 > a_0$. In general, for positive integers n, $x_n = \sigma(x_0, a_0, \ldots, x_{n-1}, a_{n-1})$, where $x_n \in \mathscr{E}$, $x_n \subseteq x_{n-1}$, $a_0, \ldots, a_{n-1}$ are the moves of DEATH, and $x_0, \ldots, x_{n-1}$ are the previous moves of the MAIDEN.

A strategy σ for the MAIDEN is a **winning strategy** if, whenever the MAIDEN follows the strategy σ, she wins the game—no matter how sophisticated DEATH plays. For example, σ is a winning strategy for the MAIDEN in the game $\mathcal{G}_{\mathscr{E}}$, if whenever $\{a_n : n \in \omega\} \subseteq \omega$ is such that $a_0 \in \sigma(\emptyset)$ and for all $n \in \omega$, $a_n < a_{n+1}$ and $a_{n+1} \in \sigma(x_0, a_0, \ldots, x_{n+1})$, then $\{a_n : n \in \omega\} \notin \mathscr{E}$.

Now, a free family $\mathscr{E}$ is called a **Ramsey family** if the MAIDEN has no winning strategy in the game $\mathcal{G}_{\mathscr{E}}$. In other words, no matter how sophisticated her strategy is, if $\mathscr{E}$ is a Ramsey family, then DEATH can win the game. Ramsey families will play an important role in the investigation of Mathias forcing notions (see Chapter 26).

Furthermore, a free family $\mathscr{E}$ is called a ***P*-family** if the MAIDEN has no winning strategy in the game $\mathcal{G}^*_{\mathscr{E}}$. P-families will play an important role in the investigation of restricted Silver forcing. In fact, in Chapter 24 it will be shown that Silver forcing restricted to a P-family (called Silver-like forcing) has the same combinatorial properties as unrestricted Silver forcing and as Grigorieff forcing, which is Silver forcing restricted to a P-point.

Obviously, the family $[\omega]^\omega$ is a Ramsey family and every Ramsey family is also a P-family. Now, the reader might guess that $[\omega]^\omega$ is not the only example and that there must be some relation between Ramsey families and Ramsey ultrafilters, as well as between P-families and P-points; this is indeed the case:

THEOREM 11.17. *Let $\mathscr{U} \subseteq [\omega]^\omega$ be a free ultrafilter. Then*

(a) *$\mathscr{U}$ is a P-point if and only if $\mathscr{U}$ is a P-family, and*

(b) *$\mathscr{U}$ is a Ramsey ultrafilter if and only if $\mathscr{U}$ is a Ramsey family.*

Proof. (a) We have to show that the MAIDEN has a winning strategy in the game $\mathcal{G}^*_{\mathscr{U}}$ if and only if $\mathscr{U}$ is not a P-point.

($\Leftarrow$) Suppose that $\mathscr{U}$ is not a P-point. Then, by FACT 11.12 (b), there exists a set $\{y_n : n \in \omega\} \subseteq \mathscr{U}$ such that whenever $y \in [\omega]^\omega$ has the property that for all $n \in \omega$, $y \setminus y_n$ is finite, then $y \notin \mathscr{U}$. Let $\sigma(\emptyset) := y_0$ (*i.e.*, $x_0 = y_0$), and for any $k \in \omega$ and $\{s_0, \ldots, s_k\} \subseteq \operatorname{fin}(\omega)$ let $\sigma(x_0, s_0, \ldots, x_k, s_k) := \bigcap_{i \leq k} y_i \setminus \bigcup_{i \leq k} s_i$. If the MAIDEN follows that strategy σ and the sequence $\langle s_k : k \in \omega \rangle$ represents the moves of DEATH, then for all $n \in \omega$ we have $(\bigcup_{k\in\omega} s_k) \setminus x_n$ is finite. Hence, $\bigcup_{k\in\omega} s_k \notin \mathscr{U}$, which shows that DEATH loses the game, or in other words, σ is a winning strategy for the MAIDEN.

($\Rightarrow$) Under the assumption that $\mathscr{U}$ is a P-point we show that no strategy for the MAIDEN is a winning strategy. Let us assume that the MAIDEN is playing according to some strategy σ. We have to show that DEATH can win. For $n \in \omega$, let X_n be the family of sets played by the MAIDEN in her first $n+1$ moves, assuming that she is following the strategy σ and DEATH plays in his first n moves only sets $s_k \subseteq n$ (for $k < n$). More formally, $x_0 = \sigma(\emptyset)$, $X_0 = \{x_0\}$, and for positive integers n, $x \in X_n$ *iff* there are $s_0, \ldots, s_k \subseteq n$ for some $k < n$, such that $x = \sigma(x_0, s_0, \ldots, x_k, s_k)$ where for all $i \leq k$, $s_i \subseteq x_i$ and $x_i = \sigma(x_0, s_0, \ldots, x_i, s_i)$. Recall that by the rules of the game, DEATH can play $\emptyset$ in any move. Clearly, for every $n \in \omega$, X_n is a finite set of elements of $\mathscr{U}$, and since $\mathscr{U}$ is an ultrafilter, $y_n := \bigcap X_n$ belongs to $\mathscr{U}$. Moreover, since $\mathscr{U}$ is a P-point, by FACT 11.12 (c) there is a set $y^* \in \mathscr{U}$ and a strictly increasing function $f \in {}^\omega\omega$ with $f(0) > 0$ such that for all $n \in \omega$, $y^* \setminus f(n) \subseteq y_n$. Let $k_0 := f(0)$, and in general, for $n \in \omega$ let $k_{n+1} := f(k_n)$. Since $\mathscr{U}$ is an ultrafilter, either

$$z_0 = \bigcup_{j\in\omega} [k_{2j}, k_{2j+1}) \quad \text{or} \quad z_1 = \omega \setminus z_0$$

belongs to $\mathscr{U}$. Without loss of generality we may assume that $z_1 \in \mathscr{U}$, which implies $z_1 \cap y^* \in \mathscr{U}$, i.e.,

$$\bigcup_{j\in\omega} \Big([k_{2j+1}, k_{2j+2}) \cap y^* \Big) \in \mathscr{U} .$$

Consider the run

$$\langle x_0, s_0^*, x_1, s_1^*, \ldots \rangle$$

of the game $\mathcal{G}^*_{\mathscr{U}}$, where the MAIDEN plays according to the strategy σ and DEATH plays

$$s_n^* = \begin{cases} [k_{2j+1}, k_{2j+2}) \cap y^* & \text{if } n = k_{2j} \text{ for some } j \in \omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that $\bigcup_{n\in\omega} s_n^* \in \mathscr{U}$. In other words, the MAIDEN loses the game if the moves of DEATH satisfy the rules of the game $\mathcal{G}^*_{\mathscr{U}}$. To see that this is indeed the case, notice first that for all positive integers j, $s^*_{k_{2j-2}} \subseteq [k_{2j-1}, k_{2j}) \subseteq k_{2j}$. Thus, if $n = k_{2j}$, then for all $k < n$ we have $s_k^* \subseteq n$. Now, if $n = k_{2j}$ for some $j \in \omega$, then $s_n^* = s^*_{k_{2j}} = [k_{2j+1}, k_{2j+2}) \cap y^*$. Further, we have

$$y^* \setminus k_{2j+1} = y^* \setminus f(k_{2j}) \subseteq y_{k_{2j}} = \bigcap X_{k_{2j}} \subseteq \bigcap \{x_0, \ldots, x_{k_{2j}}\} \subseteq x_{k_{2j}} ,$$

where $x_0, \ldots, x_{k_{2j}}$ are the moves played by the MAIDEN when DEATH plays $s_0^*, \ldots, s^*_{k_{2j-2}}$. Now, by the definition of $s^*_{k_{2j}}$ we get $s^*_{k_{2j}} \subseteq y^* \setminus k_{2j+1}$, and since

$y^* \setminus k_{2j+1} \subseteq x_{k_{2j}}$, we finally have

$$s^*_{k_{2j}} \subseteq x_{k_{2j}} .$$

So, for $n = k_{2j}$ we get $s^*_n \subseteq x_n$, which shows that for all $n \in \omega$, $s^*_n \subseteq x_n$, as required.

(b) We show that the MAIDEN has a winning strategy in the game $\mathcal{G}_{\mathscr{U}}$ if and only if the free ultrafilter $\mathscr{U}$ is not a Ramsey ultrafilter.

($\Leftarrow$) Under the assumption that the free ultrafilter $\mathscr{U}$ is not Ramsey we construct a winning strategy for the MAIDEN in the game $\mathcal{G}_{\mathscr{U}}$. If $\mathscr{U}$ is not a Ramsey ultrafilter, then, by PROPOSITION 11.6, there exists a set $\{x_n : n \in \omega\} \subseteq \mathscr{U}$ such that for each function $f \in {}^{\omega}\omega$ with $f(0) \in x$ and $f(n+1) \in x_{f(n)}$ we have $f[\omega] \notin \mathscr{U}$. Let $\sigma(\emptyset) := x_0$, and for $n \in \omega$ let $\sigma(x_0, a_0, \dots, x_n, a_n) := x_{a_n}$. By the rules of $\mathcal{G}_{\mathscr{U}}$, $a_{n+1} \in x_{a_n}$. Define $f \in {}^{\omega}\omega$ by stipulating $f(n) = a_n$. Then $f(0) \in x_0$ and for all $n \in \omega$ we have $f(n+1) \in x_{f(n)}$, and therefore $\{f(n) : n \in \omega\} \notin \mathscr{U}$. Thus, $\{a_n : n \in \omega\} \notin \mathscr{U}$, which shows that DEATH loses the game (*i.e.*, σ is a winning strategy for the MAIDEN), and consequently, $\mathscr{U}$ is not a Ramsey family.

($\Rightarrow$) Under the assumption that the free ultrafilter $\mathscr{U}$ is Ramsey we show that no strategy for the MAIDEN is a winning strategy. We will do this by following the corresponding proof of (a) after we have modified the game $\mathcal{G}_{\mathscr{U}}$: If the MAIDEN plays x_n at some stage, then we allow DEATH to respond either with a singleton $\{a_n\} \subseteq x_n$ or with the empty set. In other words, DEATH may respond to the move x_n of the MAIDEN by playing t_n, where $t_n = \{a_n\}$ (for some $a_n \in x_n$) or $t_n = \emptyset$. DEATH wins the game if and only if $\bigcup_{n\in\omega} t_n \in \mathscr{U}$. It is obvious that every winning strategy σ of the MAIDEN in the game $\mathcal{G}_{\mathscr{U}}$ corresponds to a winning strategy of the modified game: If $t_n = \emptyset$ (for some $n \in \omega$), then the MAIDEN responds with $x_{n+1} := x_n$, and if $t_n = \{a_n\}$, then she responds according to the strategy σ, where she assumes that DEATH has played a_n. Notice that if DEATH plays $t_n = \emptyset$, the MAIDEN could also respond by playing some $x_{n+1} \in \mathscr{U}$ with $x_{n+1} \subsetneq x_n$, which shows that it makes it harder to win for DEATH if he plays $\emptyset$ at some point.

Let now σ be any strategy for the MAIDEN in the modified game, which can be identified with the game $\mathcal{G}_{\mathscr{U}}$. We have to show that DEATH can win. As above, for $n \in \omega$, let X_n be the family of sets played by the MAIDEN in her first $n+1$ moves, assuming that she is following the strategy σ and DEATH plays in his first n moves only sets $t_k \subseteq n$ (for $k < n$). More formally, $x_0 = \sigma(\emptyset)$, $X_0 = \{x_0\}$, and for positive integers n, $x \in X_n$ *iff* there are $t_0, \dots, t_k \subseteq n$ for some $k < n$, such that $x = \sigma(x_0, t_0, \dots, x_k, t_k)$ where for all $i \le k$, $t_i \subseteq x_i$ and $x_i = \sigma(x_0, t_0, \dots, x_i, t_i)$. Recall that by the modified rules of the game, DEATH can always play $\emptyset$. Clearly, for every $n \in \omega$, X_n is finite, and since $\mathscr{U}$ is an ultrafilter, $y_n := \bigcap X_n$ belongs to $\mathscr{U}$. Moreover, since $\mathscr{U}$ is a Ramsey ultrafilter, and since every Ramsey ultrafilter is a P-point, by FACT 11.12 (c) there is a set $y^* \in \mathscr{U}$ and a strictly increasing function $f \in {}^{\omega}\omega$ with $f(0) > 0$ such that for all $n \in \omega$, $y^* \setminus f(n) \subseteq y_n$. Let $k_0 := f(0)$, and in general, for $n \in \omega$, let $k_{n+1} := f(k_n)$.

Since $\mathscr{U}$ is an ultrafilter, either

$$z_0 = \bigcup_{j\in\omega}[k_{2j}, k_{2j+1}) \quad \text{or} \quad z_1 = \omega \setminus z_0$$

belongs to $\mathscr{U}$. Without loss of generality we may assume that $z_1 \in \mathscr{U}$, which implies $z_1 \cap y^* \in \mathscr{U}$, *i.e.*,

$$\bigcup_{j\in\omega}\Big([k_{2j+1}, k_{2j+2}) \cap y^*\Big) \in \mathscr{U} .$$

Now, since $\mathscr{U}$ is a Ramsey ultrafilter, by PROPOSITION 11.7 (b) there exists a set $\{a_{k_{2j}} : j \in \omega\} \subseteq \omega$ such that

$$\forall j \in \omega \left(a_{k_{2j}} \in [k_{2j+1}, k_{2j+2})\right) \quad \text{and} \quad \{a_{k_{2j}} : j \in \omega\} \in \mathscr{U} .$$

Consider the run $\langle x_0, t_0^*, x_1, t_1^*, \ldots\rangle$ of the game $\mathcal{G}_{\mathscr{U}}$, where the MAIDEN plays according to her strategy σ and DEATH plays

$$t_n^* = \begin{cases} \{a_{k_{2j}}\} & \text{if } n = k_{2j} \text{ and } a_{k_{2j}} \in y^*, \\ \emptyset & \text{otherwise.} \end{cases}$$

Because $\{a_{k_{2j}} : j \in \omega\} \in \mathscr{U}$, we get $\{a_{k_{2j}} : j \in \omega\} \cap y^* \in \mathscr{U}$ which shows that $\bigcup_{n\in\omega} t_n^* \in \mathscr{U}$. In other words, the MAIDEN loses the game if the moves of DEATH satisfy the rules of the game $\mathcal{G}_{\mathscr{U}}$. For this, notice first that for all positive integers j, $a_{k_{2j-2}} \in [k_{2j-1}, k_{2j}) \subseteq k_{2j}$, in particular we get $a_{k_{2j-2}} < k_{2j}$. Thus, if $n = k_{2j}$, then for all $k < n$ we have $t_k^* \subseteq n$ where t_k^* is either $\{a_k\}$ or $\emptyset$. Now, if $n = k_{2j}$ for some $j \in \omega$, then $a_n = a_{k_{2j}} \in [k_{2j+1}, k_{2j+2})$, and hence $t_{k_{2j}}^* \subseteq [k_{2j+1}, k_{2j+2}) \cap y^*$. Further, we have

$$y^* \setminus k_{2j+1} = y^* \setminus f(k_{2j}) \subseteq y_{k_{2j}} = \bigcap X_{k_{2j}} \subseteq \bigcap\{x_0, \ldots, x_{k_{2j}}\} \subseteq x_{k_{2j}} ,$$

where $x_0, \ldots, x_{k_{2j}}$ are the moves played by the MAIDEN when DEATH plays $t_0, \ldots, t_{k_{2j-2}}$. Now, by the definition of $t_{k_{2j}}^*$ we get $t_{k_{2j}}^* \subseteq y^* \setminus k_{2j+1}$, and since $y^* \setminus k_{2j+1} \subseteq x_{k_{2j}}$, we finally have

$$t_{k_{2j}}^* \subseteq x_{k_{2j}} .$$

So, for each $n \in \omega$ we have either $t_n^* = \{a_n\}$ and $a_n \in x_n$ or $t_n^* = \emptyset$, and in both cases we have $t_n^* \subseteq x_n$. This shows that if the MAIDEN plays according to the strategy σ, DEATH can win. Hence, σ is not a winning strategy for the MAIDEN. ⊣

Roughly speaking, Ramsey families are a kind of generalised Ramsey ultrafilter and P-families are a kind of generalised P-point.

Let us turn back to happy families and let us compare them with Ramsey families. At first glance, happy families and Ramsey families look very similar. However, it turns out that the conditions for a Ramsey family are slightly stronger than for a happy family. This is because in the definition of happy families we require that they contain sets which diagonalise certain subfamilies having the finite intersection property. On the other hand, a strategy of the MAIDEN in the game $\mathcal{G}_{\mathscr{E}}$ can be quite arbitrary: Even though the sets played by her in a run of $\mathcal{G}_{\mathscr{E}}$ form a decreasing sequence, the family of possible moves of the MAIDEN does not necessarily have the finite intersection property. Of course, by restricting the set of strategies the MAIDEN can choose from, we could make sure that all happy families are Ramsey. In fact we just have to require that all the moves of the MAIDEN—no matter what DEATH is playing—belong to some family which has the finite intersection property. However, the definition of Ramsey families given above has the advantage that the MAIDEN is able—by a winning strategy—to defeat DEATH in the game $\mathcal{G}_{\mathscr{E}}$ even in some cases when $\mathscr{E}$ is happy (see PROPOSITION 11.19).

Below, we show first that every Ramsey family is happy, and then we show that there are happy families which are not even P-families. Thus, Ramsey families are smaller "clans" (*i.e.*, families who originate from the same family and have the same name) than happy families.

FACT 11.18. *Every Ramsey family is happy.*

Proof. Let $\mathscr{E}$ be a free family which is not happy. Thus, there is exists a set $\mathscr{C} = \{y_s : s \in \text{fin}(\omega)\} \subseteq \mathscr{E}$ such that $\text{fil}(\mathscr{C}) \subseteq \mathscr{E}$ but no $y \in \mathscr{E}$ diagonalises $\mathscr{A}$. Let $\sigma(\emptyset) := x_\emptyset$ and for $n \in \omega$ and $s = \{a_0, \ldots, a_n\} \in \text{fin}(\omega)$ let $\sigma(x_0, a_0, \ldots, x_n, a_n) := \bigcap_{s' \subseteq s} y_s$. It is not hard to verify that in the game $\mathcal{G}_{\mathscr{E}}$, σ is a winning strategy for the MAIDEN. $\dashv$

By PROPOSITION 11.5 we know that every *mad* family induces a happy family. This type of happy family provides examples of happy families which are not Ramsey families, in fact, which are not even P-families.

PROPOSITION 11.19. *Not every happy family is Ramsey; moreover, not every happy family is a P-family.*

Proof. It is enough to construct a happy family which is not a P-family: Let $\{t_k : k \in \omega\}$ be an enumeration of $\bigcup_{n\in\omega} {}^n\omega$ such that for all $i, j \in \omega$, $t_i \subseteq t_j$ implies $i \leq j$, in particular, $t_0 = \emptyset$. For functions $f \in {}^\omega\omega$ define the set $x_f \in [\omega]^\omega$ by stipulating

$$x_f := \big\{k \in \omega : \exists n, i, j \in \omega \, (f|_n = t_i \wedge f|_{n+1} = t_j \wedge i \leq k < j \wedge t_i \subseteq t_k)\big\}.$$

Obviously, for any distinct functions $f, g \in {}^\omega\omega$, $x_f \cap x_g$ is finite (compare with the sets constructed in the proof of PROPOSITION 9.6). Now, let $\mathscr{A}_0 := \{x_f : f \in {}^\omega\omega\}$. Then $\mathscr{A}_0 \subseteq [\omega]^\omega$ is a set of pairwise almost disjoint sets which can be extended to a *mad* family, say $\mathscr{A}$. Recall that by PROPOSITION 11.5, $\mathscr{F}^+_{\mathscr{A}}$ is a happy family.

We show that $\mathscr{F}^+_{\mathscr{A}}$ is not a P-family: Let $k_0 := 0$ and let $x_0 := \omega$ be the first move of the MAIDEN, and let s_0 be DEATH's response. In general, if s_n is DEATH's n^{th} move, then the MAIDEN chooses k_{n+1} such that $k_{n+1} \geq \max(s_n)$, $|t_{k_{n+1}}| = n+1$, and $t_{k_n} \subseteq t_{k_{n+1}}$, and then she plays

$$x_{n+1} = \{i \in \omega : t_{k_{n+1}} \subseteq s_i\}.$$

Obviously, for every $n \in \omega$ we have $x_{n+1} \subsetneq x_n$. Moreover, all moves of the MAIDEN are legal:

CLAIM. *For every* $n \in \omega$, $x_n \in \mathscr{F}^+_{\mathscr{A}}$.

PoC Firstly, for every $n \in \omega$, x_n has infinite intersection with infinitely many members of $\mathscr{A}_0$. Indeed, $x_n \cap x_f$ is infinite whenever $f|_n = t_{k_n}$. Secondly, for every $z \in \mathscr{F}_{\mathscr{A}}$ there are finitely many $y_0, \ldots, y_k \in \mathscr{A}$ such that $(y_0 \cup \ldots \cup y_k)^{\mathsf{c}} \subseteq^* z$. Now, for x_n let $x_f \in \mathscr{A}_0 \setminus \{y_0, \ldots, y_k\}$ such that $x_f \cap x_n$ is infinite. Then, since $x_f \cap (y_0 \cup \ldots \cup y_k)$ is finite, $x_f \subseteq^* z$. Hence, $x_n \cap z$ is infinite which shows that $x_n \in \mathscr{F}^+_{\mathscr{A}}$. $\dashv_{\text{Claim}}$

By the MAIDEN's strategy, $\bigcup_{n\in\omega} t_{k_n} = f$ for some particular function $f \in {}^\omega\omega$. Moreover, $\bigcup_{n\in\omega} s_n \subseteq x_f \in \mathscr{A}_0$, and since subsets of members of $\mathscr{A}_0$ do not belong to $\mathscr{F}^+_{\mathscr{A}}$, $\bigcup_{n\in\omega} s_n \notin \mathscr{F}^+_{\mathscr{A}}$. Hence, DEATH loses the game, no matter what he is playing, which shows that the MAIDEN has a winning strategy in the game $\mathcal{G}^*_{\mathscr{F}^+_{\mathscr{A}}}$. In other words, the happy family $\mathscr{F}^+_{\mathscr{A}}$ is not a P-family. $\dashv$

The Rudin–Keisler Ordering of Ultrafilters over ω

In this section, we introduce an ordering on the set of all ultrafilters over ω. For this, we first define the image of an ultrafilter under a function $f : \omega \to \omega$.

For $f \in {}^\omega\omega$ and an ultrafilter $\mathscr{V} \subseteq \mathscr{P}(\omega)$, let

$$f(\mathscr{V}) := \big\{x \subseteq \omega : \exists y \in \mathscr{V}\big(f[y] \subseteq x\big)\big\}.$$

We leave it as an exercise to the reader to show that

$$f(\mathscr{V}) = \big\{x \subseteq \omega : f^{-1}[x] \in \mathscr{V}\big\},$$

where $f^{-1}[x] := \{n \in \omega : f(n) \in x\}$.

FACT 11.20. *If* $\mathscr{V} \subseteq \mathscr{P}(\omega)$ *is an ultrafilter over* ω *and* $\mathscr{U} = f(\mathscr{V})$, *then* $\mathscr{U}$ *is also an ultrafilter over* ω.

Proof. Since $f^{-1}[\omega] = \omega$, we get $\omega \in \mathscr{U}$, and since $f^{-1}[\emptyset] = \emptyset$, we get $\emptyset \notin \mathscr{U}$.

If $x \subseteq x'$ and $x \in f(\mathscr{V})$ (i.e., $x \in \mathscr{U}$), then $f[y_0] \subseteq x$ for some $y_0 \in \mathscr{V}$, and therefore $f[y_0] \subseteq x'$, which shows that $x' \in f(\mathscr{V})$ (i.e., $x' \in \mathscr{U}$).

If $x, x' \in f(\mathscr{V})$ (i.e., $x, x' \in \mathscr{U}$), then $f^{-1}[x], f^{-1}[x'] \in \mathscr{V}$, and since $\mathscr{V}$ is an ultrafilter, $\big(f^{-1}[x] \cap f^{-1}[x']\big) \in \mathscr{V}$. Now, since $f^{-1}[x] \cap f^{-1}[x'] = f^{-1}[x \cap x']$, we get $x \cap x' \in f(\mathscr{V})$ (i.e., $x \cap x' \in \mathscr{U}$). ⊣

The so-called **Rudin–Keisler ordering** "$\leq_{RK}$" on the set of ultrafilters over ω is now defined as follows:

$$\mathscr{U} \leq_{RK} \mathscr{V} :\iff \exists f \in {}^{\omega}\omega\big(\mathscr{U} = f(\mathscr{V})\big)$$

Furthermore, for ultrafilters $\mathscr{U}, \mathscr{V} \subseteq \mathscr{P}(\omega)$ we define

$$\mathscr{U} \equiv_{RK} \mathscr{V} :\iff \mathscr{U} = f(\mathscr{V}) \quad \text{for some bijection } f \in {}^{\omega}\omega.$$

FACT 11.21. (a) *The relation "$\leq_{RK}$" is reflexive and transitive.*

(b) *The relation "$\equiv_{RK}$" is an equivalence relation on the set of ultrafilters over ω.*

Proof. (a) For the identity function $\iota : \omega \to \omega$ we obviously have $\iota(\mathscr{U}) = \mathscr{U}$, hence, $\mathscr{U} \leq_{RK} \mathscr{U}$. Furthermore, if $f(\mathscr{W}) = \mathscr{V}$ and $g(\mathscr{V}) = \mathscr{U}$ for some functions $f, g \in {}^{\omega}\omega$, then $g \circ f(\mathscr{W}) = \mathscr{U}$, hence, $\mathscr{U} \leq_{RK} \mathscr{V}$ and $\mathscr{V} \leq_{RK} \mathscr{W}$ implies $\mathscr{U} \leq_{RK} \mathscr{W}$.

(b) Notice that if $f, g \in {}^{\omega}\omega$ are bijections, then f^{-1}, g^{-1}, and $f \circ g$ are also bijections. From this observation it follows easily that the relation "$\equiv_{RK}$" is reflexive, symmetric, and transitive (e.g., if $f(\mathscr{U}) = \mathscr{V}$, where f is a bijection, then $f^{-1}(\mathscr{V}) = \mathscr{U}$). ⊣

The following lemma will be crucial in the proof of THEOREM 11.23.

LEMMA 11.22. *For any ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ and any function $f \in {}^{\omega}\omega$ we have*

$$f(\mathscr{U}) = \mathscr{U} \longrightarrow \{n \in \omega : f(n) = n\} \in \mathscr{U}.$$

Proof. Let $f \in {}^{\omega}\omega$ be an arbitrary but fixed function and let $\mathscr{U} \subseteq \mathscr{P}(\omega)$ be an ultrafilter such that $f(\mathscr{U}) = \mathscr{U}$. We consider the following three sets:

$$\begin{aligned} D &:= \{n \in \omega : f(n) < n\} \quad \text{(decreasing)} \\ E &:= \{n \in \omega : f(n) = n\} \quad \text{(equal)} \\ I &:= \{n \in \omega : f(n) > n\} \quad \text{(increasing)} \end{aligned}$$

Since $\mathscr{U}$ is an ultrafilter, exactly one of the sets D, E, I belongs to $\mathscr{U}$. If $E \in \mathscr{U}$, then we are done. So, we have to show that neither D nor I belongs to $\mathscr{U}$.

Assume towards a contradiction that $D \in \mathscr{U}$. Then for every $n \in D$ we consider the sequence $\langle f^k(n) : k \in \omega\rangle$ where $f^0(n) := n$ and $f^{k+1}(n) := f\big(f^k(n)\big)$. By the definition of D, for every $n \in D$ there is a least $k_n \in \omega$ such that $f^{k_n}(n) \notin D$. Then D is the disjoint union of the sets $D' := \{n \in D : k_n \text{ is odd}\}$ and $D'' := \{n \in D : k_n \text{ is even}\}$, and since $\mathscr{U}$ is an ultrafilter and by assumption $D \in \mathscr{U}$, exactly one of these two sets belongs to $\mathscr{U}$. Now, since $f(D') = D''$ and $f(D'') = D'$, this is a contradiction to $f(\mathscr{U}) = \mathscr{U}$, which shows that $D \notin \mathscr{U}$.

So, assume towards a contradiction that $I \in \mathscr{U}$. Then for every $n \in I$ we consider again the sequence $\langle f^k(n) : k \in \omega\rangle$. If, for $n \in I$, there is a $k \in \omega$ such that $f^k(n) \notin I$, then let k_n be the least such number; otherwise, let $k_n := \omega$. Then I is the disjoint union of the sets $I_0 := \{n \in I : k_n \in \omega\}$ and $I_\omega := \{n \in I : k_n = \omega\}$. Since $\mathscr{U}$ is an ultrafilter and $I \in \mathscr{U}$ (by assumption), exactly one of the sets I_0 and I_ω belongs to $\mathscr{U}$. If $I_0 \in \mathscr{U}$, then exactly one of the sets $I_0' := \{n \in I_0 : k_n \text{ is odd}\}$ and $I_0'' := \{n \in I_0 : k_n \text{ is even}\}$ belongs to $\mathscr{U}$; but since $f(I_0') = I_0''$ and $f(I_0'') = I_0'$, this is a contradiction to $f(\mathscr{U}) = \mathscr{U}$. So, $I_0 \notin \mathscr{U}$, which implies that $I_\omega \in \mathscr{U}$. Now, for each $n \in I_\omega$ there exists a least number $m_n \in I_\omega$ such that there is a $k \in \omega$ with $f^k(m_n) = n$. Let $I_\omega' := \big\{n \in I_\omega : \exists k \in \omega\big(f^{2k+1}(m_n) = n\big)\big\}$ and $I_\omega'' := \big\{n \in I_\omega : \exists k \in \omega\big(f^{2k}(m_n) = n\big)\big\}$. Since the two sets I_ω' and I_ω'' are disjoint and their union is I_ω, either I_ω' or I_ω'' belongs to $\mathscr{U}$, but not both. Furthermore, we get $f(I_\omega') = I_\omega''$ and $f(I_\omega'') = I_\omega'$, which is again a contradiction to $f(\mathscr{U}) = \mathscr{U}$. So, I_ω also does not belong to $\mathscr{U}$, which shows that $I \notin \mathscr{U}$.

Since $\mathscr{U}$ is an ultrafilter and $D \cup E \cup I$ belongs to $\mathscr{U}$, but neither D nor I belongs to $\mathscr{U}$, we get that E belongs to $\mathscr{U}$, which completes the proof. $\dashv$

The following result shows that up to "$\equiv_{RK}$-equivalence", the Rudin–Keisler ordering "$\leq_{RK}$" is antisymmetric.

THEOREM 11.23. *For all ultrafilters $\mathscr{U}, \mathscr{V} \subseteq \mathscr{P}(\omega)$ we have*

$$(\mathscr{U} \leq_{RK} \mathscr{V} \wedge \mathscr{V} \leq_{RK} \mathscr{U}) \longrightarrow \mathscr{U} \equiv_{RK} \mathscr{V}.$$

Proof. Assume that $\mathscr{U} \leq_{RK} \mathscr{V}$ and $\mathscr{V} \leq_{RK} \mathscr{U}$ and let $f, g \in {}^\omega\omega$ be such that $f(\mathscr{V}) = \mathscr{U}$ and $g(\mathscr{U}) = \mathscr{V}$. Notice that $f{\circ}g(\mathscr{U}) = \mathscr{U}$. So, by LEMMA 11.22, there is an $x_0 \in \mathscr{U}$ such that for all $n \in x_0$, $f{\circ}g(n) = n$, *i.e.*, $f{\circ}g|_{x_0}$ is the identity function. Hence, $g|_{x_0}$ as well as $f|_{g[x_0]}$ is one-to-one, *i.e.*, f and g are both bijections between the sets x_0 and $g[x_0]$. Now, we show that there exists a set $x_0' \subseteq x_0$ in $\mathscr{U}$ such that $g|_{x_0'}$ can be extended to a bijection $\tilde{g} \in {}^\omega\omega$. If $|\omega \setminus x_0| = |\omega \setminus g[x_0]|$, take any bijection h between $\omega \setminus x_0$ and $\omega \setminus g[x_0]$. Then, for $x_0' := x_0$, $\tilde{g} := g \cup h$ has the required properties. Otherwise, the set x_0 must be infinite and we can split x_0 into two disjoint infinite parts x_0' and x_0'' where x_0' belongs to $\mathscr{U}$. In this case, take any bijection h between the two infinite sets $\omega \setminus x_0'$ and $\omega \setminus g[x_0']$ and let $\tilde{g} := g \cup h$.

Since $\tilde{g} \in {}^\omega\omega$ is a bijection, $x_0' \in \mathscr{U}$, $g(\mathscr{U}) = \mathscr{V}$, and $g|_{x_0'} = \tilde{g}|_{x_0'}$, we get that $\tilde{g}[x_0'] \in \mathscr{V}$. It remains to show that this implies $\tilde{g}(\mathscr{U}) = \mathscr{V}$. Since $g(\mathscr{U}) = \mathscr{V}$, we get

$$\big\{g[x] : x \in \mathscr{U}\big\} \subseteq \mathscr{V} \qquad \text{and} \qquad \big\{g^{-1}[y] : y \in \mathscr{V}\big\} \subseteq \mathscr{U}\,.$$

Furthermore, by construction of $\tilde{g}$ we have $g|_{x_0'} = \tilde{g}|_{x_0'}$. Now, for every $y \in \mathscr{V}$ let $y' := y \cap \tilde{g}[x_0']$ and let $x' := \tilde{g}^{-1}[y']$. Then $y' \in \mathscr{V}$, $x' \in \mathscr{U}$, and $\tilde{g}[x'] \subseteq y$, which shows that $\tilde{g}(\mathscr{U}) = \mathscr{V}$. ⊣

For the sake of completeness we give the following

FACT 11.24. *For any ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ and any function $f \in {}^{\omega}\omega$ we have*

$$f(\mathscr{U}) \equiv_{RK} \mathscr{U} \;\longrightarrow\; \exists x \in \mathscr{U}\big(f|_x \text{ is one-to-one}\big).$$

Proof. Assume $f(\mathscr{U}) \equiv_{RK} \mathscr{U}$, where $f \in {}^{\omega}\omega$ and $\mathscr{U} \subseteq \mathscr{P}(\omega)$ is an ultrafilter. By definition of "$\equiv_{RK}$", there exists a bijection $g \in {}^{\omega}\omega$ such that $g\circ f(\mathscr{U}) = \mathscr{U}$. Hence, by LEMMA 11.22, there is an $x_0 \in \mathscr{U}$ such that $g\circ f|_{x_0}$ is the identity function, and since $g|_{f[x_0]}$ is one-to-one, $f|_{x_0}$ is also one-to-one. ⊣

So far, we have not seen an example of an ultrafilter $\mathscr{W} \subseteq [\omega]^{\omega}$ which is neither a P-point nor a Q-point. The following result gives now such an example.

THEOREM 11.25. *For any ultrafilters $\mathscr{U}, \mathscr{V} \subseteq [\omega]^{\omega}$ there is an ultrafilter $\mathscr{W} \subseteq [\omega]^{\omega}$, which is neither a P-point nor a Q-point, such that*

$$\mathscr{U} \leq_{RK} \mathscr{W} \quad \text{and} \quad \mathscr{V} \leq_{RK} \mathscr{W}\,.$$

Proof. In a first step we construct an ultrafilter $\mathscr{W} \subseteq [\omega]^{\omega}$ which is above $\mathscr{U}$ and $\mathscr{V}$, and in a second step we show that $\mathscr{W}$ is neither a P-point nor a Q-point.

Firstly, let

$$\mathscr{W}^* = \Big\{X \subseteq \omega \times \omega : \big\{a \in \omega : \{b \in \omega : \langle a, b\rangle \in X\} \in \mathscr{V}\big\} \in \mathscr{U}\Big\}.$$

Then $\mathscr{W}^*$ is a non-principal ultrafilter over $\omega \times \omega$. To see this, notice first that $\emptyset \notin \mathscr{W}^*$, that $\omega \times \omega \in \mathscr{W}^*$, that $\mathscr{W}^* \subseteq [\omega \times \omega]^{\omega}$ (this is because $\mathscr{U}, \mathscr{V} \subseteq [\omega]^{\omega}$), and that $X \in \mathscr{W}^*$ and $X \subseteq X' \subseteq \omega \times \omega$ implies $X' \in \mathscr{W}^*$. Furthermore, let $X_0 \subseteq \omega \times \omega$ be such that $X_0 \notin \mathscr{W}^*$. Then

$$\big\{a \in \omega : \{b \in \omega : \langle a, b\rangle \in X_0\} \in \mathscr{V}\big\} \notin \mathscr{U}\,,$$

which implies, since $\mathscr{U}$ is an ultrafilter, that

$$\big\{a' \in \omega : \{b \in \omega : \langle a, b\rangle \in X_0\} \notin \mathscr{V}\big\} \in \mathscr{U}\,,$$

and consequently, since $\mathscr{V}$ is an ultrafilter, we get

$$\big\{a' \in \omega : \{b' \in \omega : \langle a', b'\rangle \notin X_0\} \in \mathscr{V}\big\} \in \mathscr{U}\,,$$

which shows that $(\omega \times \omega) \setminus X_0 \in \mathscr{W}^*$. Finally, let $j_0 : \omega \times \omega \to \omega$ be a bijection. Then $\mathscr{W} := \big\{ j_0[X] : X \in \mathscr{W}^* \big\}$ is an ultrafilter over ω. In order to show that $\mathscr{W}$ is above both ultrafilters $\mathscr{U}$ and $\mathscr{V}$, we work with $\mathscr{W}^*$ and define the projections $\pi_{\mathscr{U}}$ and $\pi_{\mathscr{V}}$ by stipulating

$$\begin{array}{rccc} \pi_{\mathscr{U}} : & \mathscr{P}(\omega \times \omega) & \longrightarrow & \mathscr{P}(\omega) \\ & X & \longmapsto & \big\{ a \in \omega : \exists b \in \omega(\langle a, b\rangle \in X) \big\} \\ \pi_{\mathscr{V}} : & \mathscr{P}(\omega \times \omega) & \longrightarrow & \mathscr{P}(\omega) \\ & X & \longmapsto & \big\{ b \in \omega : \exists a \in \omega(\langle a, b\rangle \in X) \big\} \end{array}$$

We leave it as an exercise to the reader to show that $\mathscr{U} = \pi_{\mathscr{U}}[\mathscr{W}^*]$ and that $\mathscr{V} = \pi_{\mathscr{V}}[\mathscr{W}^*]$. Now, we define $f, g \in {}^{\omega}\omega$ by stipulating

$$\begin{aligned} f : \omega &\to \omega \\ n &\mapsto \pi_{\mathscr{U}}\big(\{ {j_0}^{-1}(n) \}\big) \\ g : \omega &\to \omega \\ m &\mapsto \pi_{\mathscr{V}}\big(\{ {j_0}^{-1}(m) \}\big) \end{aligned}$$

where j_0 is as above. Then, since $\big\{ {j_0}^{-1}[z] : z \in \mathscr{W} \big\} = \mathscr{W}^*$ and $\mathscr{U} = \big\{ \pi_{\mathscr{U}}(X) : X \in \mathscr{W}^* \big\}$, for every $x_0 \in \mathscr{U}$ there are $X_0 \in \mathscr{W}^*$ and $z_0 \in \mathscr{W}$, such that $X_0 = {j_0}^{-1}[z_0]$ and $\pi_{\mathscr{U}}(X_0) = x_0$, *i.e.*, $\pi_{\mathscr{U}}\big({j_0}^{-1}[z_0]\big) = x_0$. Hence, $f[z_0] = x_0$ where $z_0 \in \mathscr{W}$, and since $x_0 \in \mathscr{U}$ was arbitrary, we get $f(\mathscr{W}) = \mathscr{U}$. This shows that $\mathscr{U} \leq_{\text{RK}} \mathscr{W}$—the relation $\mathscr{V} \leq_{\text{RK}} \mathscr{W}$ is shown similarly.

It remains to prove that $\mathscr{W}$ is neither a P-point nor a Q-point. We work again with the ultrafilter $\mathscr{W}^* \subseteq [\omega \times \omega]^\omega$ and show that $\mathscr{W}^*$ is neither a P-point nor a Q-point.

$\mathscr{W}^*$ is not a Q-point: Firstly, let

$$D := \big\{ \langle a, b\rangle \in \omega \times \omega : a \leq b \big\}.$$

Notice that D belongs to $\mathscr{W}^*$. Now, define $\pi : \omega \times \omega \twoheadrightarrow D$ by stipulating

$$\pi(\langle a, b\rangle) = \begin{cases} \langle a, b\rangle & \text{if } a \leq b, \\ \langle a, a\rangle & \text{otherwise,} \end{cases}$$

and for each $m \in \omega$, let

$$u_m := \big\{ \langle a, b \in \omega \times \omega : \pi(\langle a, b\rangle) = \langle a, m\rangle \big\}.$$

Then $\{u_m : m \in \omega\}$ is a partition of $\omega \times \omega$ where each u_m is finite—in fact, $|u_m| = 2m + 1$. Assume towards a contradiction that $\mathscr{W}^*$ is a Q-point. Then there is a $Y_Q \in \mathscr{W}^*$ such that for each $m \in \omega$, $|Y_Q \cap u_m| \leq 1$. Since $\mathscr{W}^*$ is an ultrafilter, $(Y_Q \cap D) \in \mathscr{W}^*$. Above we have seen that $\mathscr{V} = \pi_{\mathscr{V}}[\mathscr{W}^*]$, so, for $y_Q := \pi_{\mathscr{V}}(Y_Q \cap D)$

we get that $y_Q \in \mathscr{V}$. Furthermore, by definition of $\mathscr{W}^*$ and since $(Y_Q \cap D) \in \mathscr{W}^*$, for each $n_0 \in y_Q$ we get that the set

$$V_{n_0} := \{m \in \omega : \langle n_0, m\rangle \in (Y_Q \cap D)\}$$

belongs to the ultrafilter $\mathscr{V}$. Now, if n_0 and n_0' are distinct members of y_Q, then $V_{n_0} \cap V_{n_0'} \in \mathscr{V}$, in particular, $V_{n_0} \cap V_{n_0'}$ is non-empty. Let m_0 be an element of $V_{n_0} \cap V_{n_0'}$. Then $\langle n_0, m_0\rangle$ and $\langle n_0', m_0\rangle$ are two distinct elements of $Y_Q \cap D$ which both belong to u_{m_0}. So, $|Y_Q \cap u_{m_0}| \geq 2$, which contradicts our assumption and shows that $\mathscr{W}^*$ is not a Q-point.

$\mathscr{W}^*$ is not a P-point: For each $n \in \omega$, let

$$u_n := \{\langle n, m\rangle : m \in \omega\}.$$

Then $\{u_n : n \in \omega\}$ is a partition of $\omega \times \omega$. Assume towards a contradiction that there is an $X_P \in \mathscr{W}^*$ such that for each $n \in \omega$, $X_P \cap u_n$ is finite. Let $x_P := \pi_{\mathscr{U}}(X_P)$ be the projection of X_P. Then, since $X_P \in \mathscr{W}^*$, $x_P \in \mathscr{U}$. Now, since $\mathscr{V}$ contains only infinite sets and $X_P \cap u_n$ is finite for each $n \in \omega$, we get that for each $n_0 \in x_P$, $\{m \in \omega : \langle n_0, m\rangle \in X_P\}$ is finite and therefore does not belong to $\mathscr{V}$. Consequently, $X_P \notin \mathscr{W}^*$, which contradicts our assumption and shows that $\mathscr{W}^*$ is not a P-point. ⊣

The next result shows that Ramsey ultrafilters are minimal with respect to the Rudin–Keisler ordering.

FACT 11.26. *If $\mathscr{U}, \mathscr{U}' \subseteq [\omega]^\omega$ are ultrafilters, where $\mathscr{U}$ is a Ramsey ultrafilter, then*

$$\mathscr{U}' \leq_{RK} \mathscr{U} \longrightarrow \mathscr{U} \equiv_{RK} \mathscr{U}'.$$

Proof. Assume that $\mathscr{U}' \leq_{RK} \mathscr{U}$, where $\mathscr{U}$ is a Ramsey ultrafilter. By definition of "$\leq_{RK}$", there exists a function $f \in {}^\omega\omega$, such that $f(\mathscr{U}) = \mathscr{U}'$, and since $\mathscr{U}$ is a Ramsey ultrafilter, by PROPOSITION 11.14 (c), there exists an $x \in \mathscr{U}$ such that $f|_x$ is constant or one-to-one. If $f|_x$ is constant, then the ultrafilter $f(\mathscr{U})$ would be principal, which contradicts the fact that $f(\mathscr{U}) = \mathscr{U}'$ and $\mathscr{U}' \subseteq [\omega]^\omega$. So, $f|_x$ is one-to-one. With similar arguments as in the proof of THEOREM 11.23 we find an $x' \subseteq x$ in $\mathscr{U}$ such that $f|_{x'}$ can be extended to a bijection $\bar{f} \in {}^\omega\omega$, such that $\bar{f}(\mathscr{U}) = \mathscr{U}'$, which shows that $\mathscr{U} \equiv_{RK} \mathscr{U}'$. ⊣

In order to state the following lemma—which will play a key role in the construction of a model of ZFC in which there are up to Rudin–Keisler equivalence just finitely many Ramsey ultrafilters (*cf.* PROPOSITION 27.6)—we first have to define a certain game: Let $\mathscr{U}, \mathscr{V} \subseteq [\omega]^\omega$ be two free families. Then the game $\mathcal{G}^{\mathscr{U}}_{\mathscr{V}}$ is the composition of the games $\mathcal{G}_{\mathscr{U}}$ and $\mathcal{G}^*_{\mathscr{V}}$, visualised by the following figure:

MAIDEN	$x_0 \in \mathscr{U}$		$y_1 \in \mathscr{V}$		$x_2 \in \mathscr{U}$		$y_3 \in \mathscr{V}$		$\dots$
DEATH		$t_0 \in [x_0]^{<2}$		$s_1 \in [y_1]^{<\omega}$		$t_2 \in [x_2]^{<2}$		$s_3 \in [y_3]^{<\omega}$	

The rules for $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$ are as follows: For each $i \in \omega$, $x_{2i} \in \mathscr{U}$, $y_{2i+1} \in \mathscr{V}$, t_{2i} is either the empty set or a singleton $\{a_{2i}\}$ with $a_{2i} \in x_{2i}$, and s_{2i+1} is a finite subset of y_{2i+1}. Finally, DEATH wins the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$ if and only if $\bigcup\{t_{2i} : i \in \omega\} \in \mathscr{U}$ *and* $\bigcup\{s_{2i+1} : i \in \omega\} \in \mathscr{V}$.

LEMMA 11.27. *Let $\mathscr{U}$ be a Ramsey ultrafilter and $\mathscr{V}$ be a P-point. Then $\mathscr{U} \leq_{RK} \mathscr{V}$ if and only if the* MAIDEN *has a winning strategy in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$.*

Proof. ($\Rightarrow$) First we show that if $\mathscr{U} \leq_{RK} \mathscr{V}$, then the MAIDEN has a winning strategy σ in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. So, assume that $\mathscr{U} \leq_{RK} \mathscr{V}$ and let $f \in {}^{\omega}\omega$ be such that $f(\mathscr{V}) = \mathscr{U}$. Since $\mathscr{V}$ is a P-point, there exists a set $y_0 \in \mathscr{V}$ such that f is finite-to-one on y_0. Let $x_0 := f[y_0]$; then $x_0 \in \mathscr{U}$ and define $\sigma(\emptyset) := x_0$. Assume now that $t_0 \in [x_0]^{<2}$ is the first move of DEATH. Since f is finite-to-one on y_0, $f^{-1}[t_0] \cap y_0$ is finite. Let

$$y_1 := y_0 \setminus \big(\max(f^{-1}[t_0] \cap y_0) + 1\big)$$

and define $\sigma(\emptyset, x_0, t_0) := y_1$. Assume that $s_1 \in [y_1]^{<\omega}$ is the second move of DEATH. Then let

$$x_2 := f[y_1] \setminus \big(\max(f[s_1]) + 1\big)$$

and define $\sigma(\emptyset, x_0, t_0, y_1, s_1) := x_2$. The next moves of the MAIDEN are

$$y_3 := y_1 \setminus \big(\max(f^{-1}[t_2] \cap y_1) + 1\big) \quad \text{and} \quad x_4 := f[y_3] \setminus \big(\max(f[s_3]) + 1\big),$$

respectively. Proceeding in this way we finally get

$$\bigcup_{i\in\omega} t_{2i} \in \mathscr{U} \quad \Longleftrightarrow \quad \bigcup_{i\in\omega} s_{2i+1} \notin \mathscr{V},$$

which shows that DEATH loses the game whenever the MAIDEN plays according to the strategy σ—no matter what he plays. Hence, σ is a winning strategy for the MAIDEN.

($\Leftarrow$) By contraposition we show that if $\mathscr{U} \not\leq_{RK} \mathscr{V}$, then no strategy σ for the MAIDEN is a winning strategy. For this we first combine the proofs of THEOREM 11.17 (a) & (b) and then use the premise that $\mathscr{U} \not\leq_{RK} \mathscr{V}$.

Let σ be any strategy for the MAIDEN in the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. We have to show that DEATH can win. Let $x_0 := \sigma(\emptyset)$ (*i.e.*, $x_0 \in \mathscr{U}$), let $X_0 := \{x_0\}$, and for positive integers n, $x \in X_n$ if and only if for some $k < n$ there are $t_0, t_2, \dots, t_{2k} \subseteq n$ and $s_1, s_3, \dots, s_{2k+1} \subseteq n$ such that $x = \sigma(x_0, t_0, y_1, \dots, s_{2k+1})$, where for all $i \leq k$ we have:

$$t_{2i} \in [x_{2i}]^{<2} \quad \text{where} \quad x_{2i} = \sigma(x_0, t_0, y_1 \dots, s_{2i-1}),$$

and

$$s_{2i+1} \in [y_{2i+1}]^{<\omega} \quad \text{where} \quad y_{2i+1} = \sigma(x_0, t_0, y_1 \dots, t_{2i}).$$

Similarly, for $n \in \omega$ we define Y_n by stipulating $y \in Y_n$ if and only if for some $k \leq n$ there are $t_0, t_2, \dots, t_{2k} \subseteq n$ and $s_1, s_3, \dots, s_{2k-1} \subseteq n$ such that $y = \sigma(x_0, t_0, y_1, \dots, t_{2k})$, where for all $i \leq k$ we have:

$$t_{2i} \in [x_{2i}]^{<2} \quad \text{where} \quad x_{2i} = \sigma(x_0, t_0, y_1 \dots, s_{2i-1}),$$

and

$$s_{2i-1} \in [y_{2i-1}]^{<\omega} \quad \text{where} \quad y_{2i-1} = \sigma(x_0, t_0, y_1 \dots, t_{2i-2}).$$

Recall that by the rules of the game, DEATH can always play $\emptyset$. Clearly, for every $n \in \omega$, both sets X_n and Y_n are finite subsets of $\mathscr{U}$ and $\mathscr{V}$, respectively. Hence, for each $n \in \omega$, $\bigcap X_n \in \mathscr{U}$ and $\bigcap Y_n \in \mathscr{V}$. Moreover, since both ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are P-points, there are sets $x^* \in \mathscr{U}$ and $y^* \in \mathscr{V}$, and a strictly increasing function $f \in {}^{\omega}\omega$ with $f(0) > 0$ such that for all $n \in \omega$,

$$x^* \setminus f(n) \subseteq \bigcap X_n \quad \text{and} \quad y^* \setminus f(n) \subseteq \bigcap Y_n .$$

Let $k_0 := f(0)$, and in general, for $m \in \omega$, let $k_{m+1} := f(k_m)$. Furthermore, for $m \in \omega$, let $u_m := [k_m, k_{m+1})$. Since $\mathscr{U}$ is a Ramsey ultrafilter, there is a set $x = \{a_m : m \in \omega\}$ in $\mathscr{U}$ such that for each $m \in \omega$, $u_m \cap x = \{a_m\}$. Define the two sets $\mathscr{S}, \mathscr{T} \subseteq [\omega]^\omega$ by stipulating

$$S \in \mathscr{S} :\iff \{a_m : m \in S\} \in \mathscr{U},$$

$$T \in \mathscr{T} :\iff \bigcup\{u_m : m \in T\} \in \mathscr{V}.$$

Notice that for any $S, S' \in \mathscr{S}$ we have $S \cap S' \in \mathscr{S}$, in particular, $S \cap S' \in [\omega]^\omega$; similarly for $T, T' \in \mathscr{T}$. In fact, since $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters, $\mathscr{S}$ and $\mathscr{T}$ are ultrafilters, too. We show now that due to the fact that $\mathscr{U} \not\leq_{RK} \mathscr{V}$, the two ultrafilters $\mathscr{S}$ and $\mathscr{T}$ can be separated. For this we prove the following two claims.

CLAIM 1. *There are $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $S \cap T = \emptyset$.*

Proof of Claim 1. If there are $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $S \cap T$ is finite, then $S' = S \setminus (S \cap T)$ is in $\mathscr{S}$ and $S' \cap T = \emptyset$. So, assume towards a contradiction that for all $S \in \mathscr{S}$ and $T \in \mathscr{T}$ we have $|S \cap T| = \omega$.

First we show that this implies that for all $S \in \mathscr{S}$ and $T \in \mathscr{T}$, $S \cap T \in \mathscr{S} \cap \mathscr{T}$, and consequently we get $\mathscr{S} = \mathscr{T}$. Indeed, if $S_0 \cap T_0 \notin \mathscr{S}$ for some $S_0 \in \mathscr{S}$ and $T_0 \in \mathscr{T}$, then $S_0' := \omega \setminus (S_0 \cap T_0)$ belongs to $\mathscr{S}$, and since $\mathscr{S}$ is a filter, $S_0' \cap S_0 \in \mathscr{S}$. Hence, $(S_0' \cap S_0) \cap T_0 = S_0' \cap (S_0 \cap T_0) = \emptyset$ and for $S := S_0' \cap S_0$ in $\mathscr{S}$ and $T = T_0$ in $\mathscr{T}$ we have $S \cap T = \emptyset$, which contradicts our assumption.

Now we show that $\mathscr{S} = \mathscr{T}$ implies $\mathscr{U} \leq_{RK} \mathscr{V}$, which contradicts the fact that $\mathscr{U} \nleq_{RK} \mathscr{V}$: Let $g \in {}^\omega\omega$ be such that for all $m \in \omega$ we have $g[u_m] := \{a_m\}$. Then for each $y \in \mathscr{V}$ we get $g[y] \in \mathscr{U}$. To see this, notice that the set $\{m \in \omega : y \cap u_m \neq \emptyset\}$ belongs to $\mathscr{T}$ and therefore, by the definition of g and since $\mathscr{S} = \mathscr{T}$, we get $g[y] \in \mathscr{U}$. So, $g(\mathscr{V}) = \mathscr{U}$, which implies that $\mathscr{U} \leq_{RK} \mathscr{V}$. $\dashv_{\text{Claim 1}}$

CLAIM 2. *There are* $S \in \mathscr{S}$ *and* $T \in \mathscr{T}$ *such that* $S \cap T = \emptyset$ *and for all distinct* $m, m' \in S \cup T$, $|m - m'| \geq 2$, *where* $|m - m'|$ *denotes the absolute value of the difference* $m - m'$.

Proof of Claim 2. By CLAIM 1 there are $\tilde{S} \in \mathscr{S}$ and $\tilde{T} \in \mathscr{T}$ such that $\tilde{S} \cap \tilde{T} = \emptyset$. Let $A := \{2k : k \in \omega\}$ and $B := \{2k+1 : k \in \omega\}$. Then either the set $\tilde{S} \cap A$ or the set $\tilde{S} \cap B$ belongs to $\mathscr{S}$; similarly, either the set $\tilde{T} \cap A$ or the set $\tilde{T} \cap B$ belongs to $\mathscr{T}$. Without loss of generality, let us assume $\tilde{S} \cap A \in \mathscr{S}$.

If $\tilde{T} \cap A \in \mathscr{T}$, let $S_0 := \tilde{S} \cap A$ and $T_0 := \tilde{T} \cap A$. Then $S_0 \in \mathscr{S}$, $T_0 \in \mathscr{T}$, and because S and T are disjoint, S_0 and T_0 are disjoint subsets of A and for all distinct $m, m' \in S_0 \cup T_0$ we have $|m - m'| \geq 2$.

If $\tilde{T} \cap A \notin \mathscr{T}$, then $\tilde{T} \cap B \in \mathscr{T}$. Now, by the definition of $\mathscr{S}$ and $\mathscr{T}$, and since $\mathscr{U}$ and $\mathscr{V}$ are filters, the sets

$$x_A := \{a_{2k} : k \in \omega\} \qquad \text{and} \qquad y_B := \bigcup\{u_{2k+1} : k \in \omega\}$$

belong to $\mathscr{U}$ and $\mathscr{V}$, respectively. Let $g_+, g_- \in {}^\omega\omega$ be functions such that for all $k \in \omega$ we have: $g_+[u_{2k+1}] := \{a_{2k+2}\}$, $g_-[u_{2k+1}] := \{a_{2k}\}$, and $g_+[u_{2k+1}] = g_-[u_{2k}] := \{0\}$. In particular, we get $g_+[y_B] = x_A \setminus \{a_0\}$ and $g_-[y_B] = x_A$, *i.e.*, both sets $g_+[y_B]$ and $g_-[y_B]$ belong to $\mathscr{U}$. On the other hand, since $\mathscr{U} \nleq_{RK} \mathscr{V}$, we have that neither $g_+(\mathscr{V}) = \mathscr{U}$ nor $g_-(\mathscr{V}) = \mathscr{U}$. Hence, there are $y_+, y_- \in [y_B]^\omega$ which belong to $\mathscr{V}$ such that neither $g_+[y_+]$ nor $g_-[y_-]$ belongs to $\mathscr{U}$. So, for $\bar{y} := y_+ \cap y_-$ we get that $\bar{y} \subseteq y_B$, $\bar{y} \in \mathscr{V}$, and

$$g_+[\bar{y}] \notin \mathscr{U} \quad \text{and} \quad g_-[\bar{y}] \notin \mathscr{U}\,.$$

Now, since $\mathscr{U}$ is an ultrafilter and $g_+[\bar{y}] \notin \mathscr{U}$, we get $(\omega \setminus g_+[\bar{y}]) \in \mathscr{U}$, which implies that $x_+ := x_A \cap (\omega \setminus g_+[\bar{y}])$ belongs to $\mathscr{U}$; similarly, we get that $x_- := x_A \cap (\omega \setminus g_-[\bar{y}])$ belongs to $\mathscr{U}$. For $\bar{x} := x_+ \cap x_-$ we get $\bar{x} \subseteq x_A$, $\bar{x} \in \mathscr{U}$, and

$$g_+[\bar{y}] \cap \bar{x} = \emptyset \quad \text{and} \quad g_-[\bar{y}] \cap \bar{x} = \emptyset\,.$$

With respect to $\bar{x}$ and $\bar{y}$, consider the two sets

$$S_0 := \big\{2k \in \omega : a_{2k} \in \bar{x}\big\} \quad \text{and} \quad T_0 := \big\{2k+1 \in \omega : \bar{y} \cap u_{2k+1} \neq \emptyset\big\}\,.$$

By definition, $S_0 \in \mathscr{S}$, $T_0 \in \mathscr{T}$, and $S_0 \cap T_0 = \emptyset$. Furthermore, if $n \in T_0$, then $n = 2k+1$ (for some $k \in \omega$) and $\bar{y} \cap u_{2k+1} \neq \emptyset$. Hence, by definition of g_+ and g_-,

$$a_{2k+2} \in g_+[\bar{y}] \quad \text{and} \quad a_{2k} \in g_-[\bar{y}]\,,$$

which implies that neither a_{2k+2} nor a_{2k} belongs to $\bar{x}$, and consequently neither $2k+2$ nor $2k$ belongs to S_0. In other words, if $n \in T_0$, then neither $n+1$ nor $n-1$ belongs to S_0, which shows that for all $m \in S_0$ and $n \in T_0$, $|m-n| \geq 2$. Furthermore, since $\bar{x} \subseteq x_A$, for any distinct $m, m' \in \bar{x}$ we have $|m-m'| \geq 2$. Similarly, since $\bar{y} \subseteq y_B$, for any distinct $m, m' \in \bar{y}$ we have $|m-m'| \geq 2$. Thus, $S_0 \cap T_0 = \emptyset$ and for all distinct $m, m' \in S_0 \cup T_0$ we have $|m-m'| \geq 2$, as required. $\dashv_{\text{Claim 2}}$

Let $S_0 \in \mathscr{S}$ and $T_0 \in \mathscr{T}$ be such that $S_0 \cap T_0 = \emptyset$ and for all distinct $m, m' \in S \cup T$, $|m-m'| \geq 2$. Consider the run $\langle x_0, t_0^*, y_1, s_1^*, \ldots \rangle$ of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$, where the MAIDEN plays according to her strategy σ and DEATH plays

$$t_{2n}^* := \begin{cases} \{a_{m+1}\} & \text{if } n = k_m,\ m+1 \in S_0, \text{ and } a_{m+1} \in x^*, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$s_{2n+1}^* = \begin{cases} y^* \cap u_{m+1} & \text{if } n = k_m \text{ and } m+1 \in T_0, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear that $\bigcup_{n \in \omega} t_{2n}^* \in \mathscr{U}$ and that $\bigcup_{n \in \omega} s_{2n+1}^* \in \mathscr{V}$. In other words, the MAIDEN loses the game if the moves of DEATH satisfy the rules of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$. To see this, notice first that for any $m \in \omega$ we have

$$x^* \setminus k_{m+1} = x^* \setminus f(k_m) \subseteq \bigcap X_{k_m} \subseteq \bigcap \{x_0, \ldots, x_{2k_m}\} \subseteq x_{2k_m},$$

where $x_0, y_1 \ldots, x_{2k_m}$ are the moves played by the MAIDEN when DEATH plays $t_0^*, s_1^* \ldots, s_{2k_m-1}^*$; and

$$y^* \setminus k_{m+1} = y^* \setminus f(k_m) \subseteq \bigcap Y_{k_m} \subseteq \bigcap \{y_1, \ldots, y_{2k_m+1}\} \subseteq y_{2k_m+1},$$

where $x_0, y_1 \ldots, x_{2k_m}, y_{2k_m+1}$ are the moves played by the MAIDEN when DEATH plays $t_0^*, s_1^* \ldots, t_{2k_m}^*$. By definition, for all $m \in \omega$, $t_{2k_m}^*$ and $s_{2k_m+1}^*$ are both subsets of k_{m+2}—in fact, they are subsets of $[k_{m+1}, k_{m+2})$. Now, recall that whenever $m+1 \in S_0$ ($m+1 \in T_0$), then $m+1 \notin T_0$ ($m+1 \notin S_0$) and neither $m \in S_0$ nor $m \in T_0$. In particular, if $m' < m$ and $m'+1, m+1 \in S_0 \cup T_0$, then $m' \leq m-2$. Hence, for $n = k_m$, $m' < m$, and $m+1 \in S_0 \cup T_0$, we get that $t_{2k_{m'}}^*$ and $s_{2k_{m'}+1}^*$ are both subsets of n (*e.g.*, if $m' = m-1$, then both sets $t_{2k_{m'}}^*$ and $s_{2k_{m'}+1}^*$ are empty). This shows that the moves of DEATH satisfy the rules of the game $\mathcal{G}_{\mathscr{V}}^{\mathscr{U}}$, which completes the proof. $\dashv$

NOTES

Happy Families and Ramsey Ultrafilters. Happy families were introduced by Mathias [12] in order to investigate the Ramsey property as well as Ramsey ultrafilters. Furthermore, happy families are closely related to *Mathias forcing*—also introduced in [12]—which will be discussed in Chapter 26. FACT 11.3 and PROPOSITION 11.5 are taken from Mathias [12, p. 61 ff.]. PROPOSITION 11.6 is due to Mathias [12, Proposition 0.8] and the characterisation of Ramsey ultrafilters (*i.e.*, PROPOSITION 11.7 and FACT 11.8) is taken from Bartoszyński and Judah [1, Theorem 4.5.2] and Booth [3, Theorem 4.9] (according to Booth [3, p. 19], most of [3, Theorem 4.9] is due to Kunen).

On P-points. A point x of a topological space X is called a ***P*-point** if every intersection of countably many open sets containing x contains an open set containing x. Now, the ultrafilters we called P-points are in fact the P-points of the topological space $\beta\omega \setminus \omega$ (defined on p. 273). The existence of P-points of the space $\beta\omega \setminus \omega$ cannot be shown in ZFC (see RELATED RESULT 68). However, by THEOREM 11.16, which is due to Ketonen [10] (see also Bartoszyński and Judah [1, Theorem 4.4.5]), it follows that P-points exist if we assume CH—which was first proved by Rudin [16]. PROPOSITION 14.9 is due to Booth.

Ramsey Families and P-families. Ramsey families and P-families were first introduced and studied by Laflamme in [11], where the filters associated to a Ramsey family are called *+-Ramsey filters*, and the filters associated to a P-family are called *P+-filters*. Furthermore, PROPOSITION 11.19 is a generalisation of Halbeisen [6, Proposition 6.2].

Characterisation of Ramsey ultrafilters and P-points in terms of games. THEOREM 11.17 can be found in terms of functions, which can be interpreted as strategies for certain games, in Grigorieff's paper [5] (see Proposition 6.4 and Corollary 1.16). At about the same time (*i.e.*, around 1970), THEOREM 11.17 was also discovered by Galvin, Hechler, and McKenzie, but their paper was never published (see RELATED RESULT 71). The proof of THEOREM 11.17 presented here is taken from Bartoszyński and Judah [1, Theorems 4.5.3 & 4.4.4]. Furthermore, LEMMA 11.27 is due to Shelah [18, Lemma 5.11], who proved it in a slightly different way using finite models.

The Rudin–Keisler Ordering. The basic properties of the so-called Rudin–Keisler ordering of ultrafilters over ω were studied by Keisler [9] and Rudin [14] (see also Rudin [15, p. 355 ff.]). However, it was apparently Katětov [8] who first defined an ordering of arbitrary filters equivalent to the Rudin–Keisler ordering. The results about the Rudin–Keisler ordering presented above were discovered independently by many people and can be found for example in Booth [3]. For the Rudin–Keisler ordering of P-points we refer the reader to Blass [2] and to Comfort and Negrepontis [4§16].

RELATED RESULTS

64. *On the existence of Ramsey ultrafilters*. Mathias showed that under CH, every happy family contains a Ramsey ultrafilter (see Mathias [12, Proposition 0.11]). In particular, this shows that Ramsey ultrafilters exist if we assume CH (according to Booth [3, p. 23], this was first shown by Galvin). However, by PROPOSITION 11.9 we know that $\mathfrak{p} = \mathfrak{c}$ is sufficient for the existence of Ramsey ultrafilters. With Martin's Axiom in place of $\mathfrak{p} = \mathfrak{c}$, this result is due to Booth [3, Theorem 4.14]. Furthermore, Keisler showed that if we assume CH, then there are $2^{\mathfrak{c}}$ pairwise non-isomorphic Ramsey ultrafilters (see Blass [2, p. 148]). Finally, by combining the proofs of Keisler and Booth, Blass [2, Theorem 2] showed that $\mathfrak{t} = \mathfrak{c}$ is enough to get $2^{\mathfrak{c}}$ pairwise non-isomorphic Ramsey ultrafilters (see PROPOSITION 14.10 for a slightly more general result and for $\mathfrak{t}$ see Chapter 9 | RELATED RESULT 55). On the other hand, we shall see in Chapter 26 & 27 that the existence of Ramsey ultrafilters is independent of ZFC (see also Chapter 22 | RELATED RESULT 121).

65. *How many Ramsey ultrafilters exist?* As mentioned above, there may be $2^{\mathfrak{c}}$ pairwise non-isomorphic Ramsey ultrafilters and there are models of ZFC in which there are no Ramsey ultrafilters. Moreover, in Chapter 23 we shall construct a model in which there are exactly $\mathfrak{c}$ Ramsey ultrafilters, and in Chapter 27 we shall see that it is consistent with ZFC that there are exactly κ Ramsey ultrafilters for any cardinal κ with $0 \leq \kappa \leq \omega_2$.

66. *There may be P-points which are not Ramsey*. It is consistent with ZFC that there exists a P-point which is not a Ramsey ultrafilter (see PROPOSITION 14.9). Moreover, there is a model of ZFC in which there are just P-points but no Q-points (see Chapter 26 | RELATED RESULT 155).

67. *On the existence of Q-points*. Mathias [13, Proposition 10] showed that $\mathfrak{d} = \omega_1$ implies the existence of Q-points. Recall that by PROPOSITION 11.9, $\mathfrak{p} = \mathfrak{c}$ implies the existence of Ramsey ultrafilters; in particular the existence of P-points and Q-points. Thus, the existence of Q-points is consistent with $\mathfrak{d} > \omega_1$. However, if there are just P-points but no Q-points, then we must have $\mathfrak{d} > \omega_1$.

68. *On the existence of P-points*. P-points were studied by Rudin [16], who proved, assuming CH, that they exist and that any of them can be mapped to any other by a homeomorphism of $\beta\omega \setminus \omega$ onto itself. In particular, CH implies the existence of P-points. Of course, this follows from the fact that

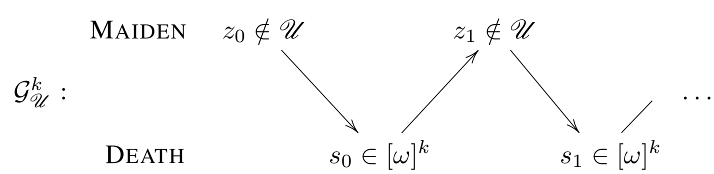
CH implies the existence of Ramsey ultrafilters, and Ramsey ultrafilters are P-points. However, as mentioned above, the converse is not true. Now, it is natural to ask whether there are also models of ZFC in which there are no P-points. This is indeed the case, as Shelah showed in [17] (see also Shelah [18, VI, §4], Wimmers [20], or Bartoszyński and Judah [1, 4.4.7]). Moreover, as for Ramsey ultrafilters, it is consistent with ZFC that, up to permutations of ω, there exists a single P-point (see Shelah [18, XVIII, §4]).

69. *Simple P_κ-points*. For any regular uncountable cardinal κ, a free ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ is called a **simple P_κ-point** if $\mathscr{U}$ is generated by an almost decreasing (*i.e.*, modulo finite) κ-sequence of infinite subsets of ω. Clearly, every simple P_κ-point is a P-point. It is conjectured that the existence of both, a simple P_{ω_1}-point and a P_{ω_2}-point, is consistent with ZFC. (For weak P-points and other points in $\beta\omega \setminus \omega$ see, for example, van Mill [19, Section 4].)

70. *Rapid and unbounded filters*. A free filter $\mathscr{F} \subseteq [\omega]^\omega$ is called a **rapid filter** if for each $f \in {}^\omega\omega$ there exists an $x \in \mathscr{F}$ such that for all $n \in \omega$, $|x \cap f(n)| \leq n$. By definition, if $\mathscr{F}$ is rapid filter, then $\{f_x : x \in \mathscr{F}\}$ is a dominating family. It is not hard to verify that all Q-points are rapid (see FACT 26.22), but the converse does not hold (see, for example, Bartoszyński and Judah [1, Lemma 4.6.3] and in particular the remark after the proof of that lemma). However, as for P-points or Q-points, the existence of a rapid filter is independent of ZFC (see PROPOSITION 26.23). A weaker notion than that of rapid filters is the notion of unbounded filters, where a free filter $\mathscr{F} \subseteq [\omega]^\omega$ is called **unbounded** if the family $\{f_x : x \in \mathscr{F}\}$ is unbounded. Since every free ultrafilter induces an unbounded family (*cf.* PROPOSITION 11.15 (a)), unbounded filters always exist. Furthermore, one can show that every unbounded filter induces a set which does not have the Ramsey property (for a slightly more general result see Judah [7, Fact 8]).

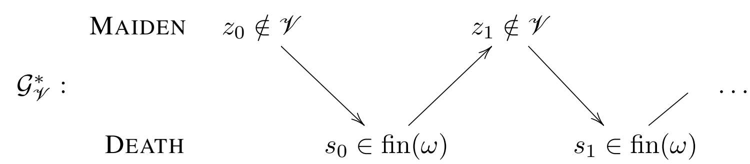
71. *Characterisations of ultrafilters by games*. In their unpublished notes, Galvin, Hechler, and McKenzie characterised different types of ultrafilters in terms of games.

    For example they characterised Ramsey ultrafilters by the following game, where $\mathscr{U} \subseteq [\omega]^\omega$ is some ultrafilter:



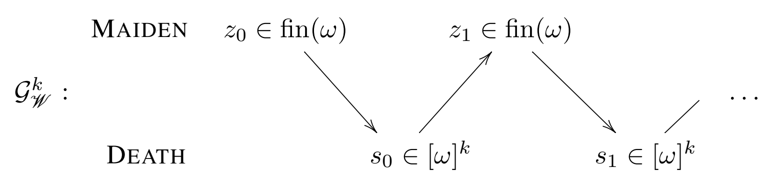
    where k is a positive integer, the sets $z_0, s_0, z_1, s_1, \ldots$ are pairwise disjoint, and the MAIDEN wins the game if and only if $\bigcup_{n\in\omega} z_n \in \mathscr{U}$. Now, the ultrafilter $\mathscr{U}$ is a Ramsey ultrafilter if and only if the MAIDEN does not have a winning strategy in the game $\mathcal{G}^k_{\mathscr{U}}$.

Furthermore, they characterised P-points by the following game, where $\mathscr{V} \subseteq [\omega]^\omega$ is some ultrafilter:

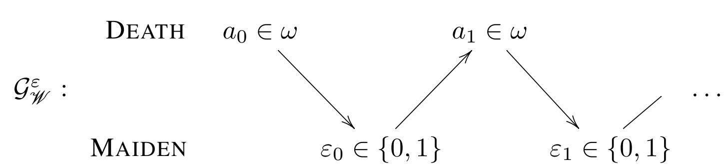


where the sets $z_0, s_0, z_1, s_1, \dots$ are pairwise disjoint and the MAIDEN wins the game if and only if $\bigcup_{n\in\omega} z_n \in \mathscr{V}$. Now, the ultrafilter $\mathscr{V}$ is a P-point if and only if the MAIDEN does not have a winning strategy in the game $\mathcal{G}^*_{\mathscr{V}}$.

Finally, they characterised Q-points by the following two games, where in both games, $\mathscr{W} \subseteq [\omega]^\omega$ is some ultrafilter:



where k is a positive integer, the sets $z_0, s_0, z_1, s_1, \dots$ are pairwise disjoint, and the MAIDEN wins the game if and only if $\bigcup_{n\in\omega} z_n \in \mathscr{W}$.



where $a_0, a_1, \dots$ are pairwise distinct and the MAIDEN wins the game if and only if the set $\{a_n : \varepsilon_n = 1\}$ is infinite and does not belong to $\mathscr{W}$.

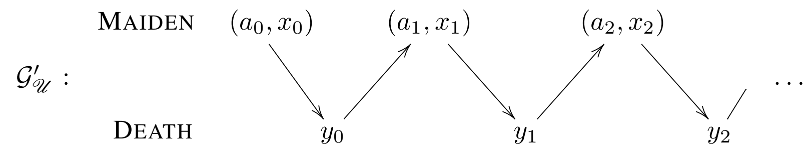
Now, the following statements are equivalent:

(a) $\mathscr{W}$ is a Q-point.

(b) The MAIDEN does not have a winning strategy in the game $\mathcal{G}^k_{\mathscr{W}}$.

(c) The MAIDEN does not have a winning strategy in the game $\mathcal{G}^\varepsilon_{\mathscr{W}}$.

72. *Another characterisation of Ramsey ultrafilters*. Let $\mathscr{U} \subseteq [\omega]^\omega$ be an ultrafilter. The game $\mathcal{G}'_{\mathscr{U}}$ is defined by



The sets y_i and x_i played by DEATH and the MAIDEN respectively must belong to the ultrafilter $\mathscr{U}$, and for each $i \in \omega$, a_{i+1} must be a member of y_i. Furthermore, for each $i \in \omega$ we require that $x_{i+1} \subseteq y_i \subseteq x_i$ and that $a_i < \min(x_i)$. Finally, the MAIDEN wins the game $\mathcal{G}'_{\mathscr{U}}$ if and only if $\{a_i : i \in \omega\}$ does *not* belong to the ultrafilter $\mathscr{U}$.

In 2002, Claude Laflamme showed me that $\mathscr{U}$ is a Ramsey ultrafilter if and only if the MAIDEN has no winning strategy in the game $\mathcal{G}'_{\mathscr{U}}$.

73. *On strongly maximal almost disjoint families**. A *mad* family $\mathscr{A}$ is called **strongly maximal almost disjoint** if given countably many members of $\mathscr{F}^+_{\mathscr{A}}$, there is a member of $\mathscr{A}$ that meets each of them in an infinite set.

For a free family $\mathscr{E}$, consider the following game: The moves of the MAIDEN are members of $\mathscr{E}$ and DEATH responds as in the game $\mathcal{G}_{\mathscr{E}}$. Furthermore, DEATH wins if and only if the set of integers played by DEATH belongs to $\mathscr{A}$, but has infinite intersection with each set played by the MAIDEN.

If $\mathscr{A}$ is a *mad* family, then obviously, in the game described above, the MAIDEN has a winning strategy if and only if $\mathscr{A}$ is not strongly maximal almost disjoint, which motivates the following question: Is it the case that for a *mad* family $\mathscr{A}$, $\mathscr{F}^+_{\mathscr{A}}$ is Ramsey if and only if $\mathscr{A}$ is strongly maximal almost disjoint?

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