Chapter 14
Martin's Axiom

In this chapter, we shall introduce a set-theoretic axiom, known as Martin's Axiom, which is independent of ZFC. In the previous chapter we have compared forcing extensions with group extensions. Similarly, we could also compare forcing extensions with field extensions. Now, if we start, for example, with the field of rational numbers $\mathbb{Q}$ and extend $\mathbb{Q}$ step by step with algebraic extensions, we finally obtain an algebraic closure $\mathbb{F}$ of $\mathbb{Q}$. Since $\mathbb{F}$ is algebraically closed, we cannot extend $\mathbb{F}$ with an algebraic extension. With respect to forcing extensions, we have a somewhat similar situation: If we start, for example, with Gödel's model $\mathbf{L}$, which is a model of $\mathsf{ZFC} + \mathsf{CH}$, and extend $\mathbf{L}$ step by step with forcing notions of a certain type, we finally obtain a model of ZFC which cannot be extended by a forcing notion of that type. The model we obtain in this way is a model in which Martin's Axiom holds. In other words, models in which Martin's Axiom holds are closed under certain forcing extensions, like algebraically closed fields are closed under algebraic extensions.

As a matter of fact we would like to mention that in the presence of the Continuum Hypothesis, Martin's Axiom is vacuously true. However, if the Continuum Hypothesis fails, then Martin's Axiom becomes an interesting combinatorial statement as well as an important tool in Combinatorics which has many applications in Topology, but also in areas like Analysis and Algebra (see RELATED RESULTS 83 & 84).

Filters on Partially Ordered Sets

Below, we introduce the standard terminology of partially ordered sets, as it is used in forcing constructions.

In the context of forcing, a binary relation "$\leq$" on a set P is a **partial ordering** of P if it is transitive (*i.e.*, $p \leq q$ and $q \leq r$ implies $p \leq r$) and reflexive (*i.e.*, $p \leq p$ for every $p \in P$). Notice that we do not require that "$\leq$" is anti-symmetric (*i.e.*, $p \leq q$ and $q \leq p$ implies $p = q$), as we have done in Chapter 6. If "$\leq$" is a partial ordering on P, then $(P, \leq)$ is called a **partially ordered set**. If $\mathbb{P} = (P, \leq)$ is a partially ordered set, then the elements of P are usually called **conditions**, since in the context of forcing, elements of partially ordered sets are conditions for sentences to be true in generic extensions. Two conditions p_1 and p_2 of P are called **compatible**, denoted $p_1 \mid p_2$, if there exists a $q \in P$ such that $p_1 \leq q \geq p_2$; otherwise they are called **incompatible**, denoted $p_1 \perp p_2$.

A typical example of a partially ordered set is the set of finite partial functions with inclusion as their partial ordering: Let I and J be arbitrary sets. Then $\mathrm{Fn}(I, J)$ is the set of all functions p such that

- $\mathrm{dom}(p) \in \mathrm{fin}(I)$, *i.e.*, $\mathrm{dom}(p)$ is a finite subset of I, and
- $\mathrm{ran}(p) \subseteq J$.

For $p, q \in \mathrm{Fn}(I, J)$ define

$$p \leq q \quad \iff \quad \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge q|_{\mathrm{dom}(p)} = p.$$

If we consider functions as sets of ordered pairs, as we usually do, then $p \leq q$ is just $p \subseteq q$. We leave it as an exercise to the reader to verify that $(\mathrm{Fn}(I, J), \subseteq)$ is indeed a partially ordered set.

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and for the moment let $C \subseteq P$.

- C is called **directed** if for any $p_1, p_2 \in C$ there is a $q \in C$ such that $p_1 \leq q \geq p_2$.
- C is called **open** (or **upwards closed**), if $p \in C$ and $q \geq p$ implies $q \in C$. For example with respect to $(\mathrm{Fn}(I, J), \subseteq)$, for every $x \in I$ the set $\{p \in \mathrm{Fn}(I, J) : x \in \mathrm{dom}(p)\}$ is open.
- C is called **downwards closed** if $p \in C$ and $q \leq p$ implies $q \in C$.
- C is called **dense** if for every condition $p \in P$ there is a $q \in C$ such that $q \geq p$. For example with respect to $(\mathrm{Fn}(I, J), \subseteq)$, for every $x \in I$ the set $\{p \in \mathrm{Fn}(I, J) : x \in \mathrm{dom}(p)\}$ is dense.
- A non-empty set $F \subseteq P$ is a **filter** (on P) if it is directed and downwards closed. Notice that this definition of "filter" reverses the ordering from the definition given in Chapter 6.
- Let $\mathscr{D} \subseteq \mathscr{P}(P)$ be a set of open dense subsets of P. A filter $G \subseteq P$ is a **$\mathscr{D}$-generic filter** on P if $G \cap D \neq \emptyset$ for every set $D \in \mathscr{D}$. As an example, consider again $(\mathrm{Fn}(I, J), \subseteq)$. If F is a filter on $\mathrm{Fn}(I, J)$, then $\bigcup F : X \to J$ is a function, where X is some (possibly infinite) subset of I.

PROPOSITION 14.1. *If $(P, \leq)$ is a partially ordered set and $\mathscr{D}$ is a countable set of open dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P. Moreover, for every $p \in P$ there exists a $\mathscr{D}$-generic filter G on P which contains p.*

Proof. For $\mathscr{D} = \{D_n : n \in \omega\}$ and $p_{-1} := p$, choose for each $n \in \omega$ a $p_n \in D_n$ such that $p_n \geq p_{n-1}$, which is possible since D_n is dense. Then the set

$$G = \{q \in P : \exists n \in \omega (q \leq p_n)\}$$

is a $\mathscr{D}$-generic filter on P and $p \in G$. ⊣

It is natural to ask whether the restriction on the size of $\mathscr{D}$ can be weakened. In particular, one can ask whether $\mathscr{D}$-generic filters also exist in the case when $\mathscr{D}$ is uncountable, or whether it is at least consistent that such fiters exist. In order to get a positive answer to these questions, we have to put a restriction on the partial ordering: Let $(P, \leq)$ be a partially ordered set. A subset $A \subseteq P$ is an **anti-chain** in P if any two distinct elements of A are incompatible. As mentioned in Chapter 6, this definition of "anti-chain" is different from the one used in Order Theory. A partially ordered set $\mathbb{P} = (P, \leq)$ satisfies the **countable chain condition**, denoted *ccc*, if every anti-chain in P is at most countable (*i.e.*, finite or countably infinite).

Now we are ready to formulate Martin's Axiom in its general form.

Martin's Axiom (MA). If $\mathbb{P} = (P, \leq)$ is a partially ordered set which satisfies *ccc*, and $\mathscr{D}$ is a set of less than $\mathfrak{c}$ open dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P. In other words, MA(κ) holds for each cardinal $\kappa < \mathfrak{c}$.

If we assume CH, then $\kappa < \mathfrak{c}$ is the same as saying $\kappa \leq \omega$, thus, by PROPOSITION 14.1, CH implies MA. On the other hand, MA can replace the Continuum Hypothesis in many proofs that use CH. Furthermore, MA is consistent with ZFC + ¬CH, as we shall see in Chapter 19.

Instead of requiring that $|\mathscr{D}| < \mathfrak{c}$, we can require that $|\mathscr{D}| \leq \kappa$ for some cardinal κ.

MA(κ). If $\mathbb{P} = (P, \leq)$ is a partially ordered set which satisfies *ccc*, and $\mathscr{D}$ is a set of at most κ open dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P.

On the one hand, MA(ω) is just PROPOSITION 14.1, and therefore, MA(ω) is provable in ZFC. On the other hand, MA($\mathfrak{c}$) is just false as we will see in FACT 14.5. So, we cannot generalise MA by omitting $|\mathscr{D}| < \mathfrak{c}$. Another attempt to generalise MA would be to omit *ccc*. However, this attempt also fails.

FACT 14.2. *There exists a (non ccc) partially ordered set $\mathbb{P} = (P, \leq)$ and a set $\mathscr{D}$ of cardinality ω_1 of open dense subsets of P such that no filter on P is $\mathscr{D}$-generic.*

Proof. Consider the partially ordered set $(\mathrm{Fn}(\omega, \omega_1), \subseteq)$. Notice first that, for example, $\{\langle 0, \alpha \rangle : \alpha \in \omega_1\}$ is an uncountable anti-chain, and hence, $(\mathrm{Fn}(\omega, \omega_1), \subseteq)$ does not satisfy *ccc*. Now, for each $\alpha \in \omega_1$, the set

$$D_\alpha = \{p \in \mathrm{Fn}(\omega, \omega_1) : \alpha \in \mathrm{ran}(p)\}$$

is an open dense subset of $\mathrm{Fn}(\omega, \omega_1)$: Obviously, D_α is open. To see that D_α is also dense, take any $p \in \mathrm{Fn}(\omega, \omega_1)$. If $\alpha \in \mathrm{ran}(p)$, then $p \in D_\alpha$ and we are done. Otherwise, let $n \in \omega$ be such that $n \notin \mathrm{dom}(p)$ (notice that such an n exists since $\mathrm{dom}(p)$ is finite). Now, let $q := p \cup \{\langle n, \alpha \rangle\}$; then $q \in D_\alpha$ and $q \geq p$. Similarly, for each $n \in \omega$, the set $E_n = \{p \in \mathrm{Fn}(\omega, \omega_1) : n \in \mathrm{dom}(p)\}$ is open dense.

Let $\mathscr{D} = \{D_\alpha : \alpha \in \omega_1\} \cup \{E_n : n \in \omega\}$; then $|\mathscr{D}| = \omega_1$. Assume that $G \subseteq \mathrm{Fn}(\omega, \omega_1)$ is a $\mathscr{D}$-generic filter on $\mathrm{Fn}(\omega, \omega_1)$. Since for each $n \in \omega$, $G \cap E_n \neq \emptyset$, $f_G = \bigcup G$ is a function from ω to ω_1. Now, since for each $\alpha \in \omega_1$, $G \cap D_\alpha \neq \emptyset$, the function $f_G : \omega \to \omega_1$ is even surjective, which contradicts the definition of ω_1. $\dashv$

In order to show that *ccc* cannot be omitted in MA, we first show that for countable sets J, $\mathrm{Fn}(I, J)$ satisfies *ccc*, and for this, we first prove the following powerful combinatorial result.

LEMMA 14.3 (Δ-SYSTEM LEMMA). *Let $\mathscr{E}$ be an uncountable family of finite sets. Then there exists an uncountable family $\mathscr{C} \subseteq \mathscr{E}$ and a finite set Δ such that for any pair of distinct elements $x, y \in \mathscr{C}$: $x \cap y = \Delta$.*

Proof. Notice first that either

> *there exists an uncountable $\mathscr{E}' \subseteq \mathscr{E}$ such that for every $a \in \bigcup \mathscr{E}'$, $\{x \in \mathscr{E}' : a \in x\}$ is countable,*

or

> *for every uncountable $\mathscr{E}' \subseteq \mathscr{E}$ there exists an $a \in \bigcup \mathscr{E}'$ such that $\{x \in \mathscr{E}' : a \in x\}$ is uncountable.*

If we are in the first case, let $\mathscr{E}'$ be an uncountable subset of $\mathscr{E}$ such that for each $a \in \bigcup \mathscr{E}'$, $\{x \in \mathscr{E}' : a \in x\}$ is countable. Recall that the set of finite subsets of a countable set is always countable. So, since $\mathscr{E}'$ is an uncountable family of finite sets, $\bigcup \mathscr{E}'$ is uncountable, and notice that for any countable set $C \subseteq \bigcup \mathscr{E}'$, the set $\{x \in \mathscr{E}' : x \cap C = \emptyset\}$ must also be uncountable. By transfinite induction we construct an uncountable family $\{x_\alpha : \alpha \in \omega_1\} \subseteq \mathscr{E}'$ of pairwise disjoint sets as follows: Let x_0 be any member of $\mathscr{E}'$. If we have already constructed a set $C_\alpha = \{x_\xi : \xi \in \alpha \in \omega_1\} \subseteq \mathscr{E}'$ of pairwise disjoint sets, let $x_\alpha \in \mathscr{E}'$ be such that $x_\alpha \cap \bigcup C_\alpha = \emptyset$. Then $\mathscr{C} = \{x_\alpha : \alpha \in \omega_1\}$ and $\Delta = \emptyset$ are as required.

If we are in the second case, consider the function $\nu : \mathscr{E} \to \omega$, where for all $x \in \mathscr{E}$, $\nu(x) := |x|$. Since $\mathscr{E}$ is an uncountable family of finite sets, there is an $n \in \omega$ and an uncountable set $\mathscr{E}' \subseteq \mathscr{E}$ such that $\nu|_{\mathscr{E}'}$ is constant, say $\nu(x) = n$ for all $x \in \mathscr{E}'$. Notice that $n \geq 2$, since otherwise, there is no $a \in \bigcup \mathscr{E}'$ such that $\{x \in \mathscr{E}' : a \in x\}$ is uncountable.

The proof is now by induction on $n \geq 2$: If $n = 2$, then there is an $a \in \bigcup \mathscr{E}'$ such that $\{x \in \mathscr{E}' : a \in x\}$ is uncountable. Let $\mathscr{C} := \{x \in \mathscr{E}' : a \in x\}$ and $\Delta = \{a\}$. Then $\mathscr{C}$ is uncountable and for all $x, y \in \mathscr{C}$ we have $x \cap y = \Delta$.

Now, let us assume that the lemma holds for n and that for each $x \in \mathscr{E}'$, $\nu(x) = n + 1$. Since there is an $a_0 \in \bigcup \mathscr{E}'$ such that $\{x \in \mathscr{E}' : a_0 \in x\}$ is uncountable, we can apply the induction hypothesis to the uncountable family $\mathscr{E}'_n := \{x \setminus \{a_0\} : x \in \mathscr{E}' \wedge a_0 \in x\}$ and obtain an uncountable family $\mathscr{C}_n \subseteq \mathscr{E}'_n$ and a finite set Δ_n, such that for any distinct elements $x, y \in \mathscr{C}_n$ we have $x \cap y = \Delta_n$. Then $\mathscr{C} := \big\{x \cup \{a_0\} : x \in \mathscr{C}_n\big\}$ and $\Delta := \Delta_n \cup \{a_0\}$ are as required. ⊣

COROLLARY 14.4. *If I is arbitrary and J is countable, then* $\mathrm{Fn}(I, J)$ *satisfies the countable chain condition.*

Proof. Let $\mathscr{F} \subseteq \mathrm{Fn}(I, J)$ be an uncountable family of partial functions. We have to show that $\mathscr{F}$ is not an anti-chain, *i.e.*, we have to find at least two distinct conditions in $\mathscr{F}$ which are compatible. Let $\mathscr{E} := \{\mathrm{dom}(p) : p \in \mathscr{F}\}$. Then $\mathscr{E}$ is obviously a family of finite sets. Now, since J is assumed to be countable, for every finite set $K \in \mathrm{fin}(I)$ the set $\{p \in \mathscr{E} : \mathrm{dom}(p) = K\}$ is countable, and therefore, since $\mathscr{F}$ is uncountable, $\mathscr{E}$ is uncountable as well.

Applying the Δ-SYSTEM LEMMA 14.3 to the family $\mathscr{E}$ yields an uncountable family $\mathscr{C} \subseteq \mathscr{F}$ and a finite set $\Delta \subseteq I$, such that for all distinct $p, q \in \mathscr{C}$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \Delta$.

Since J is countable and Δ is finite, uncountably many conditions of $\mathscr{C}$ must agree on Δ, *i.e.*, for some $p_0 \in \mathrm{Fn}(I, J)$ with $\mathrm{dom}(p_0) = \Delta$, the set $\mathscr{C}' = \{q \in \mathscr{C} : q|_\Delta = p_0\}$ is uncountable. So, $\mathscr{C}'$ is an uncountable subset of $\mathscr{F}$ consisting of pairwise compatible conditions, hence, $\mathscr{F}$ is not an anti-chain. ⊣

Now we show that *ccc* cannot be omitted in MA.

FACT 14.5. MA($\mathfrak{c}$) *is false.*

Proof. Consider the partially ordered set $(\mathrm{Fn}(\omega, 2), \subseteq)$. Then, by COROLLARY 14.4, $\mathrm{Fn}(\omega, 2)$ satisfies *ccc*. For each $g \in {}^\omega 2$, the set

$$D_g = \big\{p \in \mathrm{Fn}(\omega, 2) : \exists n \in \omega\big(p(n) = 1 - g(n)\big)\big\}$$

is an open dense subset of $\mathrm{Fn}(\omega, 2)$: Obviously, D_g is open, and for $p \notin D_g$ let $q := p \cup \{\langle n, 1 - g(n)\rangle\}$ where $n \notin \mathrm{dom}(p)$. Then $q \in D_g$ and $q \geq p$. Similarly, for each $n \in \omega$, the set $D_n = \{p \in \mathrm{Fn}(\omega, 2) : n \in \mathrm{dom}(p)\}$ is open dense.

Let $\mathscr{D} = \{D_g : g \in {}^\omega 2\} \cup \{D_n : n \in \omega\}$. Then $|\mathscr{D}| = |{}^\omega 2| = \mathfrak{c}$. Assume that $G \subseteq \mathrm{Fn}(\omega, 2)$ is a $\mathscr{D}$-generic filter on $\mathrm{Fn}(\omega, 2)$. Since for each $n \in \omega$, $G \cap D_n \neq \emptyset$, $f_G = \bigcup G$ is a function from ω to 2. Now, since for each $g \in {}^\omega 2$, $G \cap D_g \neq \emptyset$, $f_G \neq g$. Thus, f_G would be a function from ω to 2 which differs from every function $g \in {}^\omega 2$, which is impossible. ⊣

Weaker Forms of MA

Below, we introduce a few forms of Martin's Axiom which are in fact proper weakenings of MA (*cf.* RELATED RESULT 82).

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set. $\mathbb{P}$ is said to be **countable** if the set P is countable. Furthermore, a set $Q \subseteq P$ is called **centred** if every finite subset of Q has an upper bound (*i.e.*, for any finite set $\{q_0, \ldots, q_{n-1}\} \subseteq Q$ there is a $p \in P$ such that for each $i \in n$, $p \geq q_i$). Notice that the upper bound does not necessarily belong to Q. Finally, $\mathbb{P}$ is said to be **σ-centred** if P is the union of countably many centred sets.

Let $\mathcal{P}$ be any property of partially ordered sets, *e.g.*, $\mathcal{P} = \sigma$-centred, $\mathcal{P} = ccc$, or $\mathcal{P} =$ countable. Then MA($\mathcal{P}$) is the following statement.

MA($\mathcal{P}$). If $\mathbb{P} = (P, \leq)$ is a partially ordered set having the property $\mathcal{P}$, and $\mathscr{D}$ is a set of less than $\mathfrak{c}$ open dense subsets of P, then there exists a $\mathscr{D}$-generic filter on P.

Since every countable partially ordered set is σ-centred, and every σ-centred partially ordered set satisfies ccc, we obviously get

$$\mathsf{MA} \quad \Rightarrow \quad \mathsf{MA}(\sigma\text{-centred}) \quad \Rightarrow \quad \mathsf{MA}(\text{countable}).$$

Below, we present some consequences of Martin's Axiom for countable and σ-centred partially ordered sets.

Some Consequences of MA(σ-centred)

We now investigate the pseudo-intersection number $\mathfrak{p}$, which was introduced in Chapter 9. Recall that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

THEOREM 14.6. MA*(σ-centred) implies* $\mathfrak{p} = \mathfrak{c}$.

Proof. Let $\kappa < \mathfrak{c}$ be an infinite cardinal and let $\mathscr{F} = \{x_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ be a family with the strong finite intersection property (*i.e.*, intersections of finitely many members of $\mathscr{F}$ are infinite) of cardinality κ. Under the assumption of MA(σ-centred) we construct an infinite pseudo-intersection of $\mathscr{F}$.

Let P be the set of all ordered pairs $\langle s, E\rangle$ such that $s \in [\omega]^{<\omega}$ and $E \in \operatorname{fin}(\kappa)$; and for $\langle s, E\rangle, \langle t, F\rangle \in P$ define

$$\langle s, E\rangle \leq \langle t, F\rangle \quad \iff \quad s \subseteq t \wedge E \subseteq F \wedge (t \setminus s) \subseteq \bigcap\{x_\alpha \in \mathscr{F} : \alpha \in E\}.$$

For $s \in [\omega]^{<\omega}$ let $P_s := \{\langle s, E\rangle \in P : E \in \operatorname{fin}(\kappa)\}$. Then any finite set $\langle s, E_1\rangle, \ldots, \langle s, E_n\rangle \in P_s$ has an upper bound, namely $\langle s, \bigcup_{i=1}^n E_i\rangle$, and since $[\omega]^{<\omega}$ is countable and $P = \bigcup\{P_s : s \in [\omega]^{<\omega}\}$, the partially ordered set $(P, \leq)$ is σ-centred. For each $\alpha \in \kappa$ and $n \in \omega$, the set

$$D_{\alpha,n} = \big\{\langle s, E\rangle \in P : \alpha \in E \wedge |s| > n\big\}$$

is an open dense subset of P. Let $\mathscr{D} = \{D_{\alpha,n} : \alpha \in \kappa \wedge n \in \omega\}$. Then $|\mathscr{D}| = \kappa$, in particular, $|\mathscr{D}| < \mathfrak{c}$. So, by MA(σ-centred) there exists a $\mathscr{D}$-generic filter G on P. Let $x_G := \bigcup\{s \in [\omega]^{<\omega} : \exists E \in \operatorname{fin}(\kappa)(\langle s, E\rangle \in G)\}$. Then, by construction, x_G is infinite. Moreover, since G intersects every $D_{\alpha,n} \in \mathscr{D}$, for every $\alpha \in \kappa$ there is a condition $\langle s, E\rangle \in G$ such that $\alpha \in E$, which implies that $x_G \setminus s \subseteq x_\alpha$. Hence, for each $\alpha \in \kappa$ we have $x_G \subseteq^* x_\alpha$, and therefore, x_G is an infinite pseudo-intersection of $\mathscr{F}$. ⊣

The key idea in the next proof, in which we will see that MA(σ-centred) $\implies 2^\kappa = \mathfrak{c}$ for all infinite cardinals $\kappa < \mathfrak{c}$, is to encode subsets of an almost disjoint family of cardinality $\kappa < \mathfrak{c}$ by subsets of ω. We will construct these codes in the following lemma, in which we will use PROPOSITION 9.6, which asserts that there is always an almost disjoint family of cardinality $\mathfrak{c}$, and therefore of any cardinality $\kappa \leq \mathfrak{c}$.

LEMMA 14.7. *Let $\kappa < \mathfrak{c}$ be an infinite cardinal and let $\mathscr{A} = \{x_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ be an almost disjoint family of cardinality κ. Furthermore, let $\mathscr{B} \subseteq \mathscr{A}$ be any subfamily of $\mathscr{A}$ and let $\mathscr{C} = \mathscr{A} \setminus \mathscr{B}$. Without loss of generality we assume that neither $\mathscr{B}$ nor $\mathscr{C}$ is empty. If we assume MA(σ-centred), then there exists a set $c \subseteq \omega$ such that for all $x \in \mathscr{A}$:*

$$|c \cap x| = \omega \iff x \in \mathscr{B}$$

Proof. Similarly to the proof of THEOREM 14.6, let P be the set of all ordered pairs $\langle s, E\rangle$ such that $s \in [\omega]^{<\omega}$ and $E \in \operatorname{fin}(\mathscr{C})$; and for $\langle s, E\rangle, \langle t, F\rangle \in P$ define

$$\langle s, E\rangle \leq \langle t, F\rangle \iff s \subseteq t \wedge E \subseteq F \wedge (t \setminus s) \cap \bigcup E = \emptyset.$$

Similarly, one shows that the partially ordered set $(P, \leq)$ is σ-centred.

Now, for each $x_\gamma \in \mathscr{C}$, the set

$$D_{x_\gamma} = \big\{\langle s, E\rangle \in P : x_\gamma \in E\big\}$$

is an open dense subset of P; and for each $x_\beta \in \mathscr{B}$ and each $k \in \omega$, the set

$$D_{x_\beta,k} = \big\{\langle s, E\rangle \in P : |s \cap x_\beta| \geq k\big\}$$

is also an open dense subset of P. Finally, let $\mathscr{D} = \{D_{x_\gamma} : x_\gamma \in \mathscr{C}\} \cup \{D_{x_\beta,k} : x_\beta \in \mathscr{B} \wedge k \in \omega\}$. Then, since $|\mathscr{B} \cup \mathscr{C}| = \kappa$, $|\mathscr{D}| = \kappa$, and since $\kappa < \mathfrak{c}$ we get $|\mathscr{D}| < \mathfrak{c}$. So, by MA(σ-centred) there exists a $\mathscr{D}$-generic filter G on P. Let $c =$

$\bigcup\{s \in [\omega]^{<\omega} : \exists E \in \text{fin}(\mathscr{C})(\langle s, E\rangle \in G)\}$. Then for any $x_\beta \in \mathscr{B}$, $|c \cap x_\beta| = \omega$; and, as in the proof of THEOREM 14.6, for any $x_\gamma \in \mathscr{C}$, $|c \cap x_\gamma| < \omega$. Thus, the set $c \subseteq \omega$ has the required properties. ⊣

Now we are ready to prove the following consequences of MA(σ-centred):

THEOREM 14.8. *If we assume* MA*(σ-centred), then for all infinite cardinals $\kappa < \mathfrak{c}$ we have $2^\kappa = \mathfrak{c}$, and as a consequence we see that $\mathfrak{c}$ is regular.*

Proof. Let $\kappa < \mathfrak{c}$ be an infinite cardinal. We have to show that $2^\kappa = \mathfrak{c}$. For this, fix an almost disjoint family $\mathscr{A} = \{x_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ of cardinality κ (which exists by PROPOSITION 9.6), and for each $u \in \mathscr{P}(\kappa)$ let $\mathscr{B}_u := \{x_\alpha \in \mathscr{A} : \alpha \in u\}$. Then, by LEMMA 14.7, there is a set $c_u \subseteq \omega$ such that for each $x \in \mathscr{A}$ we have $|c_u \cap x| = \omega \iff x \in \mathscr{B}_u$. Notice that for any distinct $u, v \in \mathscr{P}(\kappa)$ we have $c_u \neq c_v$. Indeed, if $u, v \in \mathscr{P}(\kappa)$ are distinct, then without loss of generality we may assume that there exists an $\alpha \in \kappa$ such that $\alpha \in u \setminus v$. So, $c_u \cap x_\alpha$ is infinite, whereas $c_v \cap x_\alpha$ is finite, and hence, $c_u \neq c_v$. Thus, the mapping

$$\begin{aligned} \mathscr{P}(\kappa) &\to \mathscr{P}(\omega) \\ u &\mapsto c_u \end{aligned}$$

is one-to-one, which implies that $2^\kappa \leq \mathfrak{c}$. Now, since $\omega \leq \kappa$, and consequently $\mathfrak{c} \leq 2^\kappa$, we finally get $2^\kappa = \mathfrak{c}$.

To see that $\mathfrak{c}$ is regular assume towards a contradiction that $\kappa = \text{cf}(\mathfrak{c}) < \mathfrak{c}$. Then, by COROLLARY 3.30, $\mathfrak{c} < \mathfrak{c}^\kappa$, but since $\mathfrak{c} = 2^\kappa$ we find that $\mathfrak{c}^\kappa = (2^\kappa)^\kappa = 2^\kappa = \mathfrak{c}$, a contradiction. ⊣

We conclude this section by showing that MA(σ-centred) implies that there are P-points which are not Q-points.

PROPOSITION 14.9. *If we assume* MA*(σ-centred), then there exists an ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ which is a P-point but not a Q-point.*

Proof. We shall construct a P-point $\mathscr{U} \subseteq [\omega]^\omega$ which is not a Q-point. For this, we first define a partition $\mathscr{I} = \{I_n \subseteq \omega : n \in \omega\}$ of ω into finite blocks I_n: Let $I_0 := [0, 1)$ (in fact, $I_0 = \{0\}$), and for $n \in \omega$ let $I_{n+1} := [2^n, 2^{n+1})$. Furthermore, let

$$\mathscr{A} := \{a_\alpha \in [\omega]^\omega : \alpha \in \mathfrak{c}\}$$

be an enumeration of all infinite subsets of ω such that for each $n \in \omega$, $|a_\alpha \cap I_n| \leq 1$. For each $\alpha \in \mathfrak{c}$, let $c_\alpha := \omega \setminus a_\alpha$ and let $\mathscr{C} := \{c_\alpha : \alpha \in \mathfrak{c}\}$. Then, since the blocks of the partition $\mathscr{I}$ are arbitrarily large, the family $\mathscr{C}$ has the strong finite intersection property, denoted *sfip*. To see this, let $C \in \text{fin}(\mathscr{C})$. Then $|C| = k$ for some $k \in \omega$, which implies that for every $n \geq k$,

$$\Big|\bigcap C \cap I_{n+1}\Big| \geq 2^n - k\,,$$

and therefore,

$$\Big|\bigcap C\Big| = \Big|\bigcup_{n\in\omega}\Big(\bigcap C \cap I_n\Big)\Big| = \omega\,.$$

Now, let $\mathscr{P} := \{P_\alpha : \alpha \in \mathfrak{c}\}$ be an enumeration of all partitions of ω, where $P_0 := \mathscr{I}$. By induction on $\mathfrak{c}$, for every $\alpha \in \mathfrak{c}$ we define a family $\mathscr{F}_\alpha \subseteq [\omega]^\omega$ of filters:

Let $\mathscr{F}_0 := \{\omega \setminus s : s \in [\omega]^{<\omega}(\omega)\}$ be the Fréchet filter. Notice that $\mathscr{F}_0 \cup \mathscr{C}$ has the *sfip* and that $|\mathscr{F}_0| < \mathfrak{c}$.

Assume now that $\alpha \in \mathfrak{c}$ is a successor ordinal, say $\alpha = \beta_0 + 1$, and assume that $\mathscr{F}_{\beta_0} \cup \mathscr{C}$ has the *sfip* and that $|\mathscr{F}_{\beta_0}| < \mathfrak{c}$. Let

$$\mathscr{E} := \Big\{\bigcap X : X \in \operatorname{fin}\big(\mathscr{F}_{\beta_0} \cup \mathscr{C}\big)\Big\}$$

be the collection of all intersections of finitely many members of $\mathscr{F}_{\beta_0} \cup \mathscr{C}$. Since $\mathscr{F}_{\beta_0} \cup \mathscr{C}$ has the *sfip*, we get that $\mathscr{E} \subseteq [\omega]^\omega$. Now, we consider the following three cases:

Case 1. There exists a $Y_0 \in P_{\beta_0}$ such that for each $E \in \mathscr{E}$, $|E \cap Y_0| = \omega$. Then $\{Y_0 \cap c_{\beta_0}\} \cup \mathscr{F}_{\beta_0} \cup \mathscr{C}$ has the *sfip* and we define

$$\mathscr{F}_\alpha := \mathscr{F}_{\beta_0} \cup \{Y_0 \cap c_{\beta_0}\}\,.$$

Notice that by construction, $\mathscr{F}_\alpha \cup \mathscr{C}$ has the *sfip* and that $|\mathscr{F}_\alpha| < \mathfrak{c}$.

Case 2. There exists an $E_0 \in \mathscr{E}$ such that for each $Y \in P_{\beta_0}$, $|E_0 \cap Y| < \omega$. Then we define

$$\mathscr{F}_\alpha := \mathscr{F}_{\beta_0} \cup \{E_0 \cap c_{\beta_0}\}\,.$$

Notice that $\mathscr{F}_\alpha \cup \mathscr{C}$ has the *sfip* and that $|\mathscr{F}_\alpha| < \mathfrak{c}$.

Case 3. If we are neither in Case 1 nor in Case 2, then

$$\textit{for each } Y_0 \in P_{\beta_0} \textit{ there is an } E \in \mathscr{E} \textit{ such that } |E \cap Y_0| < \omega, \tag{1}$$

and

$$\textit{for each } E_0 \in \mathscr{E} \textit{ there is a } Y \in P_{\beta_0} \textit{ such that } |E_0 \cap Y| = \omega. \tag{2}$$

In order to construct $\mathscr{F}_\alpha$, we shall define a σ-centred partially ordered set and apply MA(σ-centred), but first we have to prove the following two claims:

CLAIM 1. *For each $E \in \mathscr{E}$ there are infinitely many parts $Y \in P_{\beta_0}$ such that $|E \cap Y| = \omega$.*

Proof of Claim 1. Assume towards a contradiction that there exists an $E_0 \in \mathscr{E}$ such that $\mathscr{Y} := \big\{Y \in P_{\beta_0} : |E_0 \cap Y| = \omega\big\}$ is finite. Let $E \in \mathscr{E}$ be arbitrary. Then

$E \cap E_0 \in \mathscr{E}$ and by (2) there exists a $Y_0 \in P_{\beta_0}$ with $|(E \cap E_0) \cap Y_0| = \omega$, which implies that $Y_0 \in \mathscr{Y}$. Since $E \in \mathscr{E}$ was arbitrary, there is a $Y \in \mathscr{Y}$ such that for all $E \in \mathscr{E}$, $|E \cap Y| = \omega$, which contradicts (1). Hence, for each $E \in \mathscr{E}$, the set

$$\mathscr{Y}_{\mathscr{E}} := \big\{Y \in P_{\beta_0} : |E \cap Y| = \omega\big\}$$

is infinite. $\dashv_{\text{Claim 1}}$

CLAIM 2. *For each* $\tilde{\mathscr{Y}} \in \operatorname{fin}(P_{\beta_0})$, *each* $E \in \mathscr{E}$, *and each* $k \in \omega$, *there is an* $n \in \omega$ *such that*

$$\Big|(E \cap I_n) \setminus \bigcup \tilde{\mathscr{Y}}\Big| > k\,.$$

Proof of Claim 2. Assume towards a contradiction that there exists a finite set $\tilde{\mathscr{Y}}_0 \in \operatorname{fin}(P_{\beta_0})$, a set $E_0 \in \mathscr{E}$, and some $k_0 \in \omega$, such that for each $n \in \omega$ we have

$$\Big|(E_0 \cap I_n) \setminus \bigcup \tilde{\mathscr{Y}}_0\Big| \leq k_0\,.$$

Then there is a finite set $\{a_{\alpha_i} : i \in k_0\} \subseteq \mathscr{A}$ such that for all $n \in \omega$,

$$(E_0 \cap I_n) \setminus \bigcup \tilde{\mathscr{Y}}_0 \subseteq \bigcup_{i \in k_0} a_{\alpha_i}\,.$$

By taking the complements of the a_{α_i}'s we get $\big((E_0 \cap I_n) \setminus \bigcup \tilde{\mathscr{Y}}_0\big) \cap \bigcap_{i \in k_0} c_{\alpha_i} = \emptyset$, and since $\bigcup_{n \in \omega} I_n = \omega$, we have

$$\Big(E_0 \setminus \bigcup \tilde{\mathscr{Y}}_0\Big) \cap \bigcap_{i \in k_0} c_{\alpha_i} = \emptyset\,.$$

This implies that for every $Y \in P_{\beta_0} \setminus \tilde{\mathscr{Y}}_0$ we have

$$\Big(E_0 \cap \bigcap_{i \in k_0} c_{\alpha_i}\Big) \cap Y = \emptyset\,.$$

Now, since $E_0 \cap \bigcap_{i \in k_0} c_{\alpha_i} \in \mathscr{E}$ and $\tilde{\mathscr{Y}}_0$ is finite, this contradicts CLAIM 1. $\dashv_{\text{Claim 2}}$

Now we are ready to construct the family $\mathscr{F}_\alpha$. For this, let $P_{\beta_0} = \{Y_m : m \in \omega\}$ and let $\mathbb{P} = (P, \leq)$ be defined as follows: Conditions $p \in P$ are ordered pairs of the form $p = (\langle s_{n_i} : i \in k+1\rangle, X)$, where

- $k \in \omega$ and $\{n_i : i \in k+1\} \subseteq \omega$,
- for all $i \in k$, $n_i < n_{i+1}$,
- for all $i \in k+1$, $s_{n_i} \subseteq \big(I_{n_i} \setminus \bigcup_{m \in i} Y_m\big)$ and $|s_{n_i}| = 2^i$,
- $X \in \operatorname{fin}(\mathscr{F}_{\beta_0})$.

For two conditions $p = (\langle s_{n_i} : i \in k+1\rangle, X)$ and $p' = (\langle s'_{n'_i} : i \in k'+1\rangle, X')$ we define

$$p \leq p' :\Longleftrightarrow \bigcup_{i \leq k} s_{n_i} \subseteq \bigcup_{i \leq k'} s'_{n'_i} \quad \wedge \quad X \subseteq X' \quad \wedge \bigcup_{k < i \leq k'} s'_{n'_i} \subseteq \bigcap X\,.$$

Notice that since there are just countably many finite subsets of ω, there are just countably many first coordinates, and since any two conditions with the same first coordinate are compatible, $\mathbb{P}$ is σ-centred. Now, for each $x \in \mathscr{F}_{\beta_0}$ let

$$D_x := \left\{(\langle s_{n_i} : i \in k+1\rangle, X) \in P : x \in X\right\}.$$

By CLAIM 2, for each $x \in \mathscr{F}_{\beta_0}$, the set D_x is an open dense subset of P. Hence, since $|\mathscr{F}_{\beta_0}| < \mathfrak{c}$, the set $\mathscr{D} := \{D_x \subseteq P : x \in \mathscr{F}_{\beta_0}\}$ is of cardinality less than $\mathfrak{c}$ and by MA(σ-centred), there exists a $\mathscr{D}$-generic filter G on P; let

$$x_G := \bigcup \left\{ \bigcup_{i \leq k} s_{n_i} : \exists X \in \operatorname{fin}(\mathscr{F}_{\beta_0})\big((\langle s_{n_i} : i \in k+1\rangle, X) \in G\big)\right\}.$$

By definition of the conditions $p \in P$, we get that for each $Y \in P_{\beta_0}$, $|x_G \cap Y| < \omega$. Furthermore, since the sets D_x are open dense, we get that for each $x \in \mathscr{F}_{\beta_0}$, $|x_G \cap x| = \omega$, which shows that $\mathscr{F}_{\beta_0} \cup \{x_G\}$ has the *sfip*. Moreover, by construction even the family $\mathscr{F}_{\beta_0} \cup \mathscr{C} \cup \{x_G\}$ has the *sfip*. Finally, let

$$\mathscr{F}_\alpha := \mathscr{F}_{\beta_0} \cup \{x_G \cap c_{\beta_0}\}\,.$$

This completes the construction of $\mathscr{F}_\alpha$ in the case when α is a successor ordinal. Notice that $\mathscr{F}_\alpha \cup \mathscr{C}$ has the *sfip* and that $|\mathscr{F}_\alpha| < \mathfrak{c}$.

If $\alpha \in \mathfrak{c}$ is a limit ordinal and for all $\beta \in \beta' \in \alpha$ we have $\mathscr{F}_\beta \subseteq \mathscr{F}_{\beta'}$, $\mathscr{F}_\beta \cup \mathscr{C}$ has the *sfip*, and $|\mathscr{F}_\beta| < \mathfrak{c}$, then let $\mathscr{F}_\alpha := \bigcup_{\beta \in \alpha} \mathscr{F}_\beta$. Notice that since $\alpha < \mathfrak{c}$, by the properties of $\mathscr{F}_\beta$ we get that $\mathscr{F}_\alpha \cup \mathscr{C}$ has the *sfip* and $|\mathscr{F}_\alpha| < \mathfrak{c}$.

Let now $\mathscr{F}_\mathfrak{c} := \bigcup_{\alpha \in \mathfrak{c}} \mathscr{F}_\alpha$. Then $\mathscr{F}_\mathfrak{c}$ has the *sfip*, and therefore, by the Ultrafilter Theorem, $\mathscr{F}_\mathfrak{c}$ can be extended to some ultrafilter $\mathscr{U}$. It remains to show that $\mathscr{U}$ is a P-point but not a Q-point: Firstly notice that since $\mathscr{F}_\mathfrak{c}$ contains the Fréchet filter, $\mathscr{U}$ is non-principal (*i.e.*, $\mathscr{U} \subseteq [\omega]^\omega$). Furthermore, for each partition P of ω, either there is a $Y \in P$ which belongs to $\mathscr{U}$, or there is an $x \in \mathscr{U}$ such that for each $Y \in P$, $x \cap Y$ is finite. Hence, $\mathscr{U}$ is a P-point. Finally, by construction we get that $\mathscr{C} \subseteq \mathscr{U}$, which shows that $\mathscr{A} \cap \mathscr{U} = \emptyset$. In other words, there is no $x \in \mathscr{U}$ such that for all $n \in \omega$ we have $|x \cap I_n| \leq 1$, which shows that $\mathscr{U}$ is not a Q-point. $\dashv$

MA(countable) *Implies the Existence of Ramsey Ultrafilters*

In this section we shall see that MA(countable) implies the existence of $2^{\mathfrak{c}}$ pairwise non-isomorphic Ramsey ultrafilters, where two Ramsey ultrafilters $\mathscr{U}_1$ and $\mathscr{U}_2$ are isomorphic if, up to a permutation of ω, they are equal (see FACT 11.21). By PROPOSITION 11.9, it would be enough to show that MA(countable) implies $\mathfrak{p} = \mathfrak{c}$. However, this is not the case (*cf.* RELATED RESULTS 80–82 and COROLLARY 22.11).

PROPOSITION 14.10. MA(countable) *implies that there exist* $2^{\mathfrak{c}}$ *pairwise non-isomorphic Ramsey ultrafilters.*

Proof. Since there are just $\mathfrak{c}$ permutations of ω, in order to get $2^{\mathfrak{c}}$ pairwise *non-isomorphic* Ramsey ultrafilters it is enough to find $2^{\mathfrak{c}}$ *distinct* Ramsey ultrafilters. The $2^{\mathfrak{c}}$ pairwise distinct Ramsey ultrafilters are constructed by transfinite induction. In fact, we shall construct a binary tree of height $\mathfrak{c}$, such that every branch of the tree corresponds to a Ramsey ultrafilter. More precisely, for every $\gamma : \mathfrak{c} \to 2$ and every $\alpha \in \mathfrak{c}$ we construct a set

$$\mathscr{F}_{\gamma|_\alpha} = \big\{x_{\beta,\gamma(\beta)} : \beta \in \alpha\big\} \subseteq [\omega]^\omega$$

with the *sfip* such that the filter generated by $\bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_{\gamma|_\alpha}$ is a Ramsey ultrafilter. In addition, we make sure that for any two distinct $\gamma, \gamma' \in {}^{\mathfrak{c}}2$, the filters generated by $\bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_{\gamma|_\alpha}$ and $\bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_{\gamma'|_\alpha}$ are distinct. In order to get Ramsey ultrafilters at the end, by PROPOSITION 11.7 (b) it is enough to make sure that for every infinite partition $\{Y_n : n \in \omega\}$ of ω, either there is an $n_0 \in \omega$ such that $Y_{n_0} \in \bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_{\gamma|_\alpha}$, or there exists an $x \in \bigcup_{\alpha\in\mathfrak{c}} \mathscr{F}_{\gamma|_\alpha}$ such that for all $n \in \omega$, $|x \cap Y_n| \leq 1$.

Let $\{\mathscr{P}_\alpha : \alpha \in \mathfrak{c}\}$ be the set of all infinite partial partitions of ω. Thus, for each $\alpha \in \mathfrak{c}$, $\mathscr{P}_\alpha = \{Y_n^\alpha : n \in \omega\}$ is a set of pairwise disjoint subsets of ω such that $\bigcup \mathscr{P}_\alpha = \omega$. Furthermore, let

$$x_{0,0} := \{2n : n \in \omega\}, \quad x_{0,1} := \{2n+1 : n \in \omega\},$$

and for $\delta \in \{0,1\}$, let

$$\mathscr{F}_{\{\langle 0,\delta\rangle\}} := \{x_{0,\delta}\} \cup \big\{x \subseteq \omega : |\omega \setminus x| < \omega\big\}.$$

Obviously, both sets $\mathscr{F}_{\{\langle 0,0\rangle\}}$ and $\mathscr{F}_{\{\langle 0,1\rangle\}}$ have the *sfip*. Let $\alpha \in \mathfrak{c}$ and assume that for each $\eta \in {}^{\alpha}2$ and each $\beta \in \alpha$ we have already constructed a set

$$\mathscr{F}_{\eta|_\beta} = \big\{x_{\iota,\eta(\iota)} : \iota \in \beta\big\} \subseteq [\omega]^\omega$$

with the *sfip*, such that for any $\beta_0 \in \beta_1 \in \alpha$ we have $\mathscr{F}_{\eta|_{\beta_0}} \subseteq \mathscr{F}_{\eta|_{\beta_1}}$. In order to construct $\mathscr{F}_\eta$ we have to consider two cases:

α *limit ordinal*: If α is a limit ordinal, then let

$$\mathscr{F}_\eta = \bigcup_{\beta \in \alpha} \mathscr{F}_{\eta|_\beta}.$$

Since the sets $\mathscr{F}_{\eta|_\beta}$ are increasing and each of these sets has the *sfip*, $\mathscr{F}_\eta$ has the *sfip* as well.

α *successor ordinal*: If α is a successor ordinal, say $\alpha = \beta_0 + 1$, then we proceed as follows: Consider the partition $\mathscr{P}_{\beta_0} = \{Y_n : n \in \omega\}$ and notice that *either* there is an $n_0 \in \omega$ such that $\mathscr{F}_{\eta|_{\beta_0}} \cup \{Y_{n_0}\}$ has the *sfip*, *or* for every $n \in \omega$, Y_n belongs to the dual ideal of $\mathscr{F}_{\eta|_{\beta_0}}$, *i.e.*, is a subset of the complement of a finite intersection of members of $\mathscr{F}_{\eta|_{\beta_0}}$. We consider the two cases separately:

Case 1: Let $n_0 \in \omega$ be such that $\mathscr{F}_{\eta|_{\beta_0}} \cup \{Y_{n_0}\}$ has the *sfip*. Let $P_1 = \mathrm{Fn}(Y_{n_0}, 2)$ and, for $p, q \in P_1$, let $p \leq q \iff p \subseteq q$. Then $(P_1, \leq)$ is countable and for every finite set $E \in \mathrm{fin}(\beta_0)$, every $n \in \omega$ and each $\delta \in \{0, 1\}$, the set

$$D_{E,n,\delta} = \left\{ p \in P_1 : \left| p^{-1}(\delta) \cap \bigcap_{\iota \in E} x_{\iota, \eta(\iota)} \right| \geq n \right\}$$

is an open dense subset of P_1. Now let

$$\mathscr{D} = \big\{ D_{E,n,\delta} : E \in \mathrm{fin}(\beta_0) \wedge n \in \omega \wedge \delta \in \{0,1\} \big\}.$$

Then $|\mathscr{D}| \leq \max\{|\alpha|, \omega\} < \mathfrak{c}$ and by MA(countable) there exists a $\mathscr{D}$-generic filter G on P_1. For $\delta \in \{0, 1\}$, let

$$x_{\beta_0, \delta} := \bigcup \big\{ p^{-1}(\delta) : p \in G \big\}.$$

For $\delta \in \{0, 1\}$ we find that $x_{\beta_0,\delta} \in [Y_{n_0}]^\omega$ and that $\mathscr{F}_\eta := \mathscr{F}_{\eta|_{\beta_0}} \cup \{x_{\beta_0, \eta(\beta_0)}\}$ has the *sfip*. Finally, let $\eta, \eta' \in {}^\alpha 2$ be such that $\eta(\beta_0) = 1 - \eta'(\beta_0)$. Because $x_{\beta_0,0} \cap x_{\beta_0,1} = \emptyset$, we obviously have $\mathscr{F}_\eta \neq \mathscr{F}_{\eta'}$. Moreover, by construction we see that $\mathscr{F}_\eta \cup \mathscr{F}_{\eta'}$ lacks the *sfip*, and therefore no ultrafilter can extend both $\mathscr{F}_\eta$ and $\mathscr{F}_{\eta'}$.

Case 2: If for each $n \in \omega$, Y_n belongs to the dual ideal of $\mathscr{F}_{\eta|_{\beta_0}}$, then each finite intersection of members of $\mathscr{F}_{\eta|_{\beta_0}}$ meets infinitely many sets of $\mathscr{P}_{\beta_0}$. Let $P_2 \subseteq \mathrm{Fn}(\omega, 2)$ be such that $p \in P_2$ *iff* for every $Y \in \mathscr{P}_{\beta_0}$ we have

$$\max\big\{ \left|p^{-1}(0) \cap Y\right|, \left|p^{-1}(1) \cap Y\right| \big\} \leq 1,$$

and for $p, q \in P_2$ let $p \leq q \iff p \subseteq q$. As before, $(P_2, \leq)$ is countable and for every finite set $E \in \mathrm{fin}(\beta_0)$, every $n \in \omega$ and each $\delta \in \{0, 1\}$, the set

$$D_{E,n,\delta} = \left\{ p \in P_2 : \left| p^{-1}(\delta) \cap \bigcap_{\iota \in E} x_{\iota, \eta(\iota)} \right| \geq n \right\}$$

is an open dense subset of P_2. Let

$$\mathscr{D} = \{D_{E,n,\delta} : E \in \text{fin}(\beta_0) \land n \in \omega \land \delta \in \{0,1\}\}$$

and let G be a $\mathscr{D}$-generic filter on P_2. Finally, for $\delta \in \{0,1\}$, let $x_{\beta_0,\delta} := \bigcup\{p^{-1}(\delta) : p \in G\}$. Then $\mathscr{F}_\eta := \mathscr{F}_{\eta|_{\beta_0}} \cup \{x_{\beta_0,\eta(\beta_0)}\}$ has the *sfip*, and in addition, $x_{\beta_0,\eta(\beta_0)}$ is such that for all $n \in \omega$, $|x_{\beta_0,\eta(\beta_0)} \cap Y_n| \leq 1$. Furthermore, for $\eta, \eta' \in {}^{\alpha}2$ with $\eta(\beta_0) = 1 - \eta'(\beta_0)$, no ultrafilter can extend both $\mathscr{F}_\eta$ and $\mathscr{F}_{\eta'}$.

Finally, for each $\gamma \in {}^{\mathfrak{c}}2$, let $\mathscr{F}_\gamma$ be the filter generated by the set $\bigcup_{\alpha \in \mathfrak{c}} \mathscr{F}_{\gamma|_\alpha}$. By construction, for any two distinct $\gamma, \gamma' \in {}^{\mathfrak{c}}2$, $\mathscr{F}_\gamma$ and $\mathscr{F}_{\gamma'}$ are two distinct Ramsey ultrafilters, and consequently there exist $2^{\mathfrak{c}}$ pairwise non-isomorphic Ramsey ultrafilters. ⊣

NOTES

Martin's Axiom. MA was first formulated by Martin and Solovay [11]. The paper contains various equivalent formulations of MA and numerous applications (including THEOREM 14.8, see also Chapter 20). They also stress the usefulness of MA as a viable alternative to CH and point out that many of the traditional problems solved using CH can be solved using MA. Roughly speaking, this is because under MA, sets of cardinality less than $\mathfrak{c}$ usually behave like countable sets (but of course, there are exceptions).

For equivalents of MA, consequences, weaker forms, history, *et cetera*, we refer the reader to Kunen [10, Chapter II, §2–§5], Fremlin [7], Weiss [17], Rudin [13], Blass [3, Section 7], and Jech [9, Chapter 16].

MA(σ-centred) *and P-points*. The result that under MA(σ-centred) there are P-points which are not Q-points (*i.e.*, PROPOSITION 14.9) is due to Booth (see [4, Theorems 4.10 & 4.12]).

MA(countable) *and Ramsey ultrafilters*. PROPOSITION 14.10 is due to Canjar [5] (who actually proved even more), but the proof given above was communicated to me by Michael Hrušák (compare PROPOSITION 14.10 with Chapter 11 | RELATED RESULT 64).

The Δ-System Lemma. This powerful combinatorial result was first proved by Shanin [15] (see Kunen [10, Chapter II, §1] for a slightly more general result).

RELATED RESULTS

80. MA(σ-centred) $\iff$ $\mathfrak{p} = \mathfrak{c}$. As we have seen above in THEOREM 14.6, MA(σ-centred) implies $\mathfrak{p} = \mathfrak{c}$. On the other hand, the converse is also true, *i.e.*, $\mathfrak{p} = \mathfrak{c}$ implies MA(σ-centred). This somewhat surprising result was first proved by Bell [1] (see also Fremlin [7, 14C] or the proof of THEOREM 19.4).

81. MA(countable) $\iff$ $cov(\mathcal{M}) = \mathfrak{c}$. Fremlin and Shelah showed in [8] that MA(countable) is equivalent to $cov(\mathcal{M}) = \mathfrak{c}$, where $cov(\mathcal{M})$ denotes the *covering number* of the meagre ideal (defined in Chapter 22). See also Martin and Solovay [11§4], Blass [3, Theorem 7.13], and Miller [12] for some further results concerning $cov(\mathcal{M})$.

82. MA(σ-linked). A partially ordered set $(P, \leq)$ is said to be **σ-linked** if we can write $P = \bigcup_{n\in\omega} P_n$, where each set P_n consists of pairwise compatible elements.

    On the one hand, it is easily verified that

    $$\mathsf{MA} \implies \mathsf{MA}(\sigma\text{-linked}) \implies \mathsf{MA}(\sigma\text{-centred}) \implies \mathsf{MA}(\text{countable})$$

    but on the other hand, to show that none of the converse implications hold requires quite sophisticated techniques. For the corresponding references we refer the reader to Fremlin [7, Appendix B1].

83. *The existence of magic sets under* MA(σ-centred). A set of reals $M \subseteq \mathbb{R}$ is called **magic** if for any two continuous, nowhere constant real-valued functions $f, g : \mathbb{R} \to \mathbb{R}$ we have

    $$f \neq g \iff f[M] \nsubseteq g[M] \ \wedge\ g[M] \nsubseteq f[M] .$$

    In 1993, Berarducci and Dikranjan [2, Theorem 8.5] proved that under CH, magic sets exist. This result can be improved by showing that even MA(σ-centred) implies the existence of magic sets (see Schumacher [14]). On the other hand, there are also models of ZFC in which there are no magic sets (see Ciesielski and Shelah [6]).

84. *Whitehead's Problem*. One of the earliest applications of Martin's Axiom in Algebra was an answer to Whitehead's problem, which is the question of whether every Whitehead group is free.

    A Whitehead group is an abelian group G such that for every group H and any surjective group homomorphism $h : H \to G$ with kernel isomorphic to $\mathbb{Z}$, there exists a group homomorphism $g : G \to H$ such that $h \circ g$ is the identity.

One can show that every countable Whitehead group is free, but for uncountable Whitehead groups G, the question of whether G is free is undecidable in ZFC. In particular, Shelah [16] showed that if MA is true and CH is false, then there is a non-free Whitehead group (see also Fremlin [7, Section 34]).

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