Chapter 2

First-Order Logic in a Nutshell

Mathematicians devised signs, not separate from matter except in essence, yet distant from it. These were points, lines, planes, solids, numbers, and countless other characters, which are depicted on paper with certain colours, and they used these in place of the things symbolised.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

First-Order Logic is the system of Symbolic Logic concerned not only with representing the logical relations between sentences or propositions as wholes (like *Propositional Logic*), but also with their internal structure in terms of subject and predicate. First-Order Logic can be considered as a kind of language which is distinguished from higher-order languages in that it does not allow quantification over subsets of the domain of discourse or other objects of higher type. Nevertheless, First-Order Logic is strong enough to formalise all of Set Theory and thereby virtually all of Mathematics. In other words, First-Order Logic is an abstract language that in one particular case might be the language of Group Theory, and in another case might be the language of Set Theory.

The goal of this brief introduction to First-Order Logic is to illustrate and summarise some of the basic concepts of this language and to show how it is applied to fields like Group Theory and Peano Arithmetic (two theories which will accompany us for a while).

Syntax: The Grammar of Symbols

Like any other written language, First-Order Logic is based on an *alphabet*, which consists of the following *symbols*:

(a) **Variables** such as $v_0, v_1, x, y, \ldots$, which are place holders for objects of the *domain* under consideration (which can, for example, be the elements of a group, natural numbers, or sets).

(b) **Logical operators** which are "$\neg$" (*not*), "$\wedge$" (*and*), "$\vee$" (*or*), "$\rightarrow$" (*implies*), and "$\leftrightarrow$" (*if and only if*, abbreviated *iff*).

(c) **Logical quantifiers** which are the *existential quantifier* "$\exists$" (*there is* or *there exists*) and the *universal quantifier* "$\forall$" (*for all* or *for each*), where quantification is restricted to objects only and not to formulae or sets of objects (but the objects themselves may be sets).

(d) **Equality symbol** "$=$", which stands for the particular binary *equality relation*.

(e) **Constant symbols** like the number $\mathtt{0}$ in Peano Arithmetic, or the neutral element $\mathtt{e}$ in Group Theory. Constant symbols stand for fixed individual objects in the domain.

(f) **Function symbols** such as $\circ$ (the operation in Group Theory), or $+, \cdot, \mathtt{s}$ (the operations in Peano Arithmetic). Function symbols stand for fixed functions taking objects as arguments and returning objects as values. With each function symbol we associate a positive natural number, its co-called "arity" (*e.g.*, "$\circ$" is a 2-ary or binary function, and the successor operation "$\mathtt{s}$" is a 1-ary or unary function).

(g) **Relation symbols** or **predicate constants** (such as $\in$ in Set Theory) stand for fixed relations between (or properties of) objects in the domain. Again we associate an "arity" with each relation symbol (*e.g.*, "$\in$" is a binary relation).

The symbols in (a)–(d) form the core of the alphabet and are called **logical symbols**. The symbols in (e)–(g) depend on the specific topic we are investigating and are called **non-logical symbols**. The set of non-logical symbols which are used in order to formalise a certain mathematical theory is called the **signature** or **language** of this theory, denoted by $\mathscr{L}$, and *formulae* which are formulated in a language $\mathscr{L}$ are usually called $\mathscr{L}$-formulae. For example, if we investigate groups, then the only non-logical symbols we use are "$\mathtt{e}$" and "$\circ$", thus, $\mathscr{L} = \{\mathtt{e}, \circ\}$ is the signature of Group Theory.

A first step towards a proper language is to build names (*i.e.*, *terms*) with these symbols.

Terms:

(T0) Each variable is a term.

(T1) Each constant symbol is a term.

(T2) If $\tau_1, \ldots, \tau_n$ are terms and F is an n-ary function symbol, then $F\tau_1 \cdots \tau_n$ is a term.

It is convenient to use auxiliary symbols like brackets in order to make terms, relations, and other expressions easier to read. For example, we usually write $F(\tau_1, \dots, \tau_n)$ rather than $F\tau_1 \cdots \tau_n$.

To some extent, terms correspond to certain names, since they denote objects of the domain under consideration. Like real names, they are not statements and cannot express or describe possible relations between objects. So, the next step is to build more complex expressions with these terms.

Formulae:

(F0) If τ_1 and τ_2 are terms, then $\tau_1 = \tau_2$ is a formula.

(F1) If $\tau_1, \dots, \tau_n$ are terms and R is an n-ary relation symbol, then $R\tau_1 \cdots \tau_n$ is a formula.

(F2) If φ is a formula, then $\neg\varphi$ is a formula.

(F3) If φ and ψ are formulae, then $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are formulae. (To avoid the use of brackets one could write these formulae for example in *Polish notation*, *i.e.*, $\wedge\varphi\psi$, $\vee\varphi\psi$, *et cetera*. However, fully parenthesised formulae have the benefit of giving immediately obvious unique readability.)

(F4) If φ is a formula and ν a variable, then $\exists\nu\varphi$ and $\forall\nu\varphi$ are formulae.

Formulae of the form (F0) or (F1) are the most basic expressions we have, and since every formula is a logical connection or a quantification of these formulae, they are called **atomic formulae**.

For binary relations R it is convenient to write xRy instead of $R(x, y)$. For example we write $x \in y$ instead of $\in(x, y)$, and we write $x \notin y$ rather than $\neg(x \in y)$.

If a formula φ is of the form $\exists x\psi$ or of the form $\forall x\psi$ (for some formula ψ) and x occurs in ψ, then we say that x is in the *range* of a logical quantifier. Every occurrence of a variable x in a formula φ is said to be **bound** by the innermost quantifier in whose range it occurs. If an occurrence of x is not in the range of a quantifier, it is said to be **free**. Notice that it is possible for a variable to occur bound in one part of a given formula and free in another. For example, in the formula $\exists z(x = z) \wedge \forall x(x = y)$, the variable x occurs bound and free, whereas z occurs just bound and y occurs just free. However, one can always rename the variables occurring in a given formula such that no variable occurs both bound and free. For a formula φ, the set of variables occurring free in φ is denoted by $\mathrm{free}(\varphi)$. A formula φ is a **sentence** (or a **closed formula**) if it contains no free variables (*i.e.*, $\mathrm{free}(\varphi) = \emptyset$). For example, $\forall x(x = x)$ is a sentence but $(x = x)$ is just a formula.

Sometimes it is useful to indicate explicitly which variables occur free in a given formula φ, and we usually write $\varphi(x_1, \dots, x_n)$ to indicate $\{x_1, \dots, x_n\} \subseteq \mathrm{free}(\varphi)$.

If φ is a formula, ν a variable, and τ a term, then $\varphi(\nu/\tau)$ is the formula we get after replacing all *free* instances of the variable ν by τ. The process to obtain the formula $\varphi(\nu/\tau)$ is called **substitution**. Now, a substitution is **admissible** *iff* no free occurrence of ν in φ is in the range of a quantifier that binds any variable which appears in τ (*i.e.*, for each variable $\tilde{\nu}$ appearing in τ, no place where ν occurs free in φ is in the range of "$\exists\tilde{\nu}$" or "$\forall\tilde{\nu}$").

For example, if $x \notin \text{free}(\varphi)$, then $\varphi(x/\tau)$ is admissible for any term τ. In this case, the formulae φ and $\varphi(x/\tau)$ are identical, which we express by $\varphi \equiv \varphi(x/\tau)$.

So far we have letters, and we can build names and sentences. However, these sentences are just strings of symbols without any inherent meaning. Later we shall interpret formulae in the intuitively natural way by giving the symbols the intended meaning (*e.g.*, "$\wedge$" meaning "and", "$\forall x$" meaning "for all x", *et cetera*). But before we shall do so, let us stay a little bit longer on the syntactical side.

Below we shall label certain formulae or types of formulae as **axioms**, which are used in connection with *inference rules* in order to derive further formulae. From a semantical point of view we can think of axioms as "true" statements from which we deduce or prove further results. We distinguish two types of axioms, namely *logical axioms* and *non-logical axioms* (which will be discussed later). A **logical axiom** is a sentence or formula φ which is universally valid (*i.e.*, φ is true in any possible universe, no matter how the variables, constants, *et cetera*, occurring in φ are interpreted). Usually one takes as logical axioms some minimal set of formulae that is sufficient for deriving all universally valid formulae (such a set is given below).

If a symbol is involved in an axiom which stands for an arbitrary relation, function, or even for a first-order formula, then we usually consider the statement as an **axiom schema** rather than a single axiom, since each instance of the symbol represents a single axiom. The following list of axiom schemata is a system of logical axioms.

Let φ, φ_1, φ_2, and ψ be arbitrary first-order formulae:

L_0: $\varphi \vee \neg\varphi$

L_1: $\varphi \to (\psi \to \varphi)$

L_2: $(\psi \to (\varphi_1 \to \varphi_2)) \to ((\psi \to \varphi_1) \to (\psi \to \varphi_2))$

L_3: $(\varphi \wedge \psi) \to \varphi$

L_4: $(\varphi \wedge \psi) \to \psi$

L_5: $\varphi \to (\psi \to (\psi \wedge \varphi))$

L_6: $\varphi \to (\varphi \vee \psi)$

L_7: $\psi \to (\varphi \vee \psi)$

L_8: $(\varphi_1 \to \varphi_3) \to ((\varphi_2 \to \varphi_3) \to ((\varphi_1 \vee \varphi_2) \to \varphi_3))$

L_9: $\neg\varphi \to (\varphi \to \psi)$

If τ is a term, ν a variable, and the substitution which leads to $\varphi(\nu/\tau)$ is admissible, then:

L_{10}: $\forall\nu\varphi(\nu) \to \varphi(\nu/\tau)$

L_{11}: $\varphi(\nu/\tau) \to \exists\nu\varphi(\nu)$

If ψ is a formula and ν a variable such that $\nu \notin \text{free}(\psi)$ then:

L_{12}: $\forall\nu(\psi \to \varphi(\nu)) \to (\psi \to \forall\nu\varphi(\nu))$

L_{13}: $\forall\nu(\varphi(\nu) \to \psi) \to (\exists\nu\varphi(\nu) \to \psi)$

What is not covered yet is the symbol "$=$", so, let us have a closer look at the binary equality relation. The defining properties of equality can already be found in Book VII, Chapter 1 of Aristotle's *Topics* [2], where one of the rules to decide whether two things are the same is as follows: . . . *you should look at every possible predicate of each of the two terms and at the things of which they are predicated and see whether there is any discrepancy anywhere. For anything which is predicated of the one ought also to be predicated of the other, and of anything of which the one is a predicate the other also ought to be a predicate.*

In our formal system, the binary equality relation is defined by the following three axioms.

If $\tau, \tau_1, \dots, \tau_n, \tau'_1, \dots, \tau'_n$ are any terms, R an n-ary relation symbol (*e.g.*, the binary relation symbol "$=$"), and F an n-ary function symbol, then:

L_{14}: $\tau = \tau$

L_{15}: $(\tau_1 = \tau'_1 \wedge \dots \wedge \tau_n = \tau'_n) \to (R(\tau_1, \dots, \tau_n) \to R(\tau'_1, \dots, \tau'_n))$

L_{16}: $(\tau_1 = \tau'_1 \wedge \dots \wedge \tau_n = \tau'_n) \to (F(\tau_1, \dots, \tau_n) = F(\tau'_1, \dots, \tau'_n))$

Finally, we define the logical operator "$\leftrightarrow$" and the binary relation "$\neq$" by stipulating

$$\varphi \leftrightarrow \psi :\iff (\varphi \to \psi) \wedge (\psi \to \varphi)$$

$$\tau_1 \neq \tau_2 :\iff \neg(\tau_1 = \tau_2)$$

i.e., $\varphi \leftrightarrow \psi$ and $\tau_1 \neq \tau_2$ are just abbreviations for $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg(\tau_1 = \tau_2)$, respectively.

This completes the list of our logical axioms. In addition to these axioms, we now add arbitrarily many theory-specific assumptions, so-called **non-logical axioms**. Such axioms are, for example, the three axioms of *Group Theory*, denoted GT, or the axioms of *Peano Arithmetic*, denoted PA.

GT: The language of Group Theory is $\mathscr{L}_{\mathsf{GT}} = \{\mathsf{e}, \circ\}$, where "e" is a constant symbol and "$\circ$" is a binary function symbol.

GT_0: $\forall x \forall y \forall z (x \circ (y \circ z) = (x \circ y) \circ z)$ (*i.e.*, "$\circ$" is *associative*)

GT_1: $\forall x(\mathtt{e} \circ x = x)$ (*i.e.*, "e" is a *left-neutral* element)

GT_2: $\forall x \exists y(y \circ x = \mathtt{e})$ (*i.e.*, every element has a *left-inverse*)

PA: The language of Peano Arithmetic is $\mathscr{L}_{\mathsf{PA}} = \{\mathtt{0}, \mathtt{s}, +, \cdot\}$, where "0" is a constant symbol, "s" is a unary function symbol, and "$+$" and "$\cdot$" are binary function symbols.

PA_0: $\neg\exists x(\mathtt{s}x = \mathtt{0})$

PA_1: $\forall x \forall y(\mathtt{s}x = \mathtt{s}y \to x = y)$

PA_2: $\forall x(x + \mathtt{0} = x)$

PA_3: $\forall x \forall y(x + \mathtt{s}y = \mathtt{s}(x + y))$

PA_4: $\forall x(x \cdot \mathtt{0} = \mathtt{0})$

PA_5: $\forall x \forall y(x \cdot \mathtt{s}y = (x \cdot y) + x)$

If φ is any $\mathscr{L}_{\mathsf{PA}}$-formula with $x \in \text{free}(\varphi)$, then:

PA_6: $\Big(\varphi(x/\mathtt{0}) \wedge \forall x\big(\varphi(x) \to \varphi(x/\mathtt{s}x)\big)\Big) \to \forall x \varphi(x)$

Notice that PA_6 is an axiom schema, known as the **induction schema**, and not just a single axiom like PA_0–PA_5.

It is often convenient to add certain *defined symbols* to a given signature so that the expressions get shorter or at least are easier to read. For example, in Peano Arithmetic—which is an axiomatic system for the natural numbers—we usually replace the expression $\mathtt{s}(\mathtt{0})$ with $\mathtt{1}$ and consequently $\mathtt{s}(x)$ by $x + \mathtt{1}$. Probably, we would like to introduce an ordering "$<$" on the natural numbers. We can do this by stipulating

$$\mathtt{1} := \mathtt{s}(\mathtt{0}) \quad \textit{and} \quad x < y \ :\Longleftrightarrow\ \exists z((x + z) + \mathtt{1} = y)\,.$$

We usually use ":=" to define constants or functions, and "$\Longleftrightarrow$" to define relations. Obviously, all that can be expressed in the language $\mathscr{L}_{\mathsf{PA}} \cup \{\mathtt{1}, <\}$ can also be expressed in $\mathscr{L}_{\mathsf{PA}}$.

So far we have a set of logical and non-logical axioms in a certain language and can define, if we wish, as many new constants, functions, and relations as we like. However, we are still not able to deduce anything from the given axioms, since we have neither *inference rules* nor the notion of *formal proof*.

Surprisingly, just two **inference rules** are sufficient, namely:

$$\text{Modus Ponens: } \frac{\varphi \to \psi,\ \varphi}{\psi} \quad \text{and} \quad \text{Generalisation: } \frac{\varphi}{\forall \nu \varphi}\,,$$

where ν is a variable which does not occur free in any non-logical axiom.

In the former case we say that ψ is obtained from $\varphi \to \psi$ and φ by Modus Ponens, and in the latter case we say that $\forall\nu\varphi$ (where ν can be any variable) is obtained from φ by Generalisation.

Using these two inference rules, we are able to define the notion of **formal proof**:

> Let T be a possibly empty set of non-logical axioms (usually sentences), formulated in a certain language $\mathscr{L}$. An $\mathscr{L}$-formula ψ is **provable** from T (or provable in T), denoted $\mathsf{T} \vdash \psi$, if there is a *finite* sequence $\varphi_0, \ldots, \varphi_n$ of $\mathscr{L}$-formulae such that the formulae φ_n and ψ are identical, and for all i with $0 \leq i \leq n$ we have:
>
> - φ_i is a logical axiom, or
> - $\varphi_i \in \mathsf{T}$, or
> - there are $j, k < i$ such that φ_j is equal to the formula $\varphi_k \to \varphi_i$, or
> - there is a $j < i$ such that φ_i is equal to the formula $\forall x\, \varphi_j$.

If a formula ψ is not provable in T, *i.e.*, if there is no formal proof for ψ which uses just formulae from T, then we write $\mathsf{T} \nvdash \psi$.

Formal proofs, even of very simple statements, can get quite long and tricky. So, before we give an example of a formal proof, let us state a theorem which allows us to simplify formal proofs:

THEOREM 2.1 (DEDUCTION THEOREM). *If* $\{\psi_0, \ldots, \psi_n\} \cup \{\varphi_0, \ldots, \varphi_k\} \vdash \varphi$, *then*

$$\{\psi_0, \ldots, \psi_n\} \vdash (\varphi_0 \wedge \ldots \wedge \varphi_k) \to \varphi.$$

Now, as an example of a formal proof let us show that the equality relation is symmetric. We first work with $\mathsf{T}_{x=y}$, consisting only of the formula $x = y$, and show that $\mathsf{T}_{x=y} \vdash y = x$, in other words we show that $\{x = y\} \vdash y = x$:

φ_0: $(x = y \wedge x = x) \to (x = x \to y = x)$	instance of L_{15}
φ_1: $(x = y \wedge x = x) \to x = x$	instance of L_4
φ_2: $\varphi_0 \to (\varphi_1 \to ((x = y \wedge x = x) \to y = x))$	instance of L_2
φ_3: $\varphi_1 \to ((x = y \wedge x = x) \to y = x)$	from φ_2 and φ_0 by Modus Ponens
φ_4: $(x = y \wedge x = x) \to y = x$	from φ_3 and φ_1 by Modus Ponens
φ_5: $x = x$	instance of L_{14}
φ_6: $x = y$	$(x = y) \in \mathsf{T}_{x=y}$
φ_7: $x = x \to (x = y \to (x = y \wedge x = x))$	instance of L_5
φ_8: $x = y \to (x = y \wedge x = x)$	from φ_7 and φ_5 by Modus Ponens
φ_9: $x = y \wedge x = x$	from φ_8 and φ_6 by Modus Ponens
φ_{10}: $y = x$	from φ_4 and φ_9 by Modus Ponens

Thus, we have $\{x = y\} \vdash y = x$, and by the Deduction Theorem 2.1 we see that $\vdash x = y \to y = x$, and finally, by Generalisation we get

$$\vdash \forall x \forall y(x = y \to y = x).$$

We leave it as an exercise for the reader to show that the equality relation is also transitive. Therefore, since the equality relation is reflexive (by L_{14}), it is a so-called *equivalence relation* (defined in the next chapter).

Furthermore, we say that two formulae φ and ψ are **equivalent**, denoted $\varphi \Leftrightarrow \psi$, if $\vdash \varphi \leftrightarrow \psi$. In other words, if $\varphi \Leftrightarrow \psi$, then—from a logical point of view—φ and ψ state exactly the same thing, and therefore we could call φ and ψ a tautology, which means *saying the same thing twice*. However, in logic, a formula φ is a **tautology** if $\vdash \varphi$. Thus, the formulae φ and ψ are equivalent if and only if $\varphi \leftrightarrow \psi$ is a tautology.

A few examples:

- $\varphi \vee \psi \Leftrightarrow \psi \vee \varphi$, $\varphi \wedge \psi \Leftrightarrow \psi \wedge \varphi$, which shows that "$\vee$" and "$\wedge$" are commutative (up to equivalence). Moreover, "$\vee$" and "$\wedge$" are (up to equivalence) also associative—a fact which we tacitly used already.
- $\neg\neg\varphi \Leftrightarrow \varphi$, $(\varphi \vee \psi) \Leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$, which shows for example how "$\vee$" can be replaced with "$\neg$" and "$\wedge$".
- $(\varphi \to \psi) \Leftrightarrow (\neg\varphi \vee \psi)$, which shows how the logical operator "$\to$" can be replaced with "$\neg$" and "$\vee$".
- $\forall x\varphi \Leftrightarrow \neg\exists x\neg\varphi$, which shows how "$\forall$" can be replaced with "$\neg$" and "$\exists$".

Thus, some of the logical operators are redundant and we could work for example with just "$\neg$", "$\wedge$", and "$\exists$". However, it is more convenient to use all of them.

Let T be a set of $\mathscr{L}$-formulae. We say that T is **consistent**, denoted $\mathrm{Con}(\mathsf{T})$, if there is *no* $\mathscr{L}$-formula φ such that $\mathsf{T} \vdash (\varphi \wedge \neg\varphi)$, otherwise T is called **inconsistent**, denoted $\neg\,\mathrm{Con}(\mathsf{T})$.

PROPOSITION 2.2. *Let T be a set of $\mathscr{L}$-formulae.*

(a) *If $\neg\,\mathrm{Con}(\mathsf{T})$, then for all $\mathscr{L}$-formulae ψ we have $\mathsf{T} \vdash \psi$.*

(b) *If $\mathrm{Con}(\mathsf{T})$ and $\mathsf{T} \vdash \varphi$ for some $\mathscr{L}$-formula φ, then $\mathsf{T} \nvdash \neg\varphi$.*

Proof. (a) Let ψ be any $\mathscr{L}$-formula and assume that $\mathsf{T} \vdash (\varphi \wedge \neg\varphi)$ for some $\mathscr{L}$-formula φ. Then $\mathsf{T} \vdash \psi$:

φ_0:	$\varphi \wedge \neg\varphi$	provable from T by assumption
φ_1:	$(\varphi \wedge \neg\varphi) \to \varphi$	instance of L_3
φ_2:	φ	from φ_1 and φ_0 by Modus Ponens
φ_3:	$(\varphi \wedge \neg\varphi) \to \neg\varphi$	instance of L_4
φ_4:	$\neg\varphi$	from φ_3 and φ_0 by Modus Ponens
φ_5:	$\neg\varphi \to (\varphi \to \psi)$	instance of L_9
φ_6:	$\varphi \to \psi$	from φ_5 and φ_4 by Modus Ponens
φ_7:	ψ	from φ_6 and φ_2 by Modus Ponens

(b) Assume that $\mathsf{T} \vdash \varphi$ and $\mathsf{T} \vdash \neg\varphi$. Then $\mathsf{T} \vdash (\varphi \wedge \neg\varphi)$, *i.e.*, $\neg\,\mathrm{Con}(\mathsf{T})$:

φ_0:	φ	provable from T by assumption
φ_1:	$\neg\varphi$	provable from T by assumption
φ_2:	$\varphi \to (\neg\varphi \to (\varphi \wedge \neg\varphi))$	instance of L_5
φ_3:	$\neg\varphi \to (\varphi \wedge \neg\varphi)$	from φ_2 and φ_0 by Modus Ponens
φ_4:	$\varphi \wedge \neg\varphi$	from φ_3 and φ_1 by Modus Ponens

$\dashv$

Notice that PROPOSITION 2.2 (a) implies that from an inconsistent set of axioms T one can prove everything and T would be completely useless. So, if we design a set of axioms T, we have to make sure that T is consistent. However, as we shall see later, in many cases this task is impossible.

Semantics: Making Sense of the Symbols

Let T be any set of $\mathscr{L}$-formulae (for some signature $\mathscr{L}$). There are two different ways to approach T, namely the *syntactical* and the *semantical* approaches. The above presented syntactical approach considers the set T just as a set of well-formed formulae—regardless of their intended sense or meaning—from which we can prove some other formulae.

On the other hand, we can consider T also from a semantical point of view by interpreting the symbols of the language $\mathscr{L}$ in a reasonable way, and then seeking for a *model* in which all formulae of T are true. To be more precise, we first have to define how models are built and what "true" means.

In order to define models, we have to assume some notions of *Set Theory* like *subset*, *cartesian product*, or *relation*, which shall be defined properly in the next chapter. Furthermore, we also make use of the the binary *membership relation* "$\in$", which is the only non-logical symbol of Set Theory.

Let $\mathscr{L}$ be an arbitrary but fixed language. An **$\mathscr{L}$-structure $\mathbf{M}$** consists of a non-empty set A, called the **domain** of $\mathbf{M}$, together with a mapping which assigns to each constant symbol $c \in \mathscr{L}$ an element $c^{\mathbf{M}} \in A$, to each n-ary relation symbol $R \in \mathscr{L}$ a set of n-tuples $R^{\mathbf{M}}$ of elements of A, and to each n-ary function symbol

$F \in \mathscr{L}$ a function $F^{\mathbf{M}}$ from n-tuples of A to A. In other words, the constant symbols become elements of A, n-ary relation symbols become subsets of A^n (*i.e.*, subsets of the n-fold cartesian product of A), and n-ary functions symbols become n-ary functions from A^n to A.

The interpretation of variables is given by a so-called assignment: An **assignment** in an $\mathscr{L}$-structure $\mathbf{M}$ is a mapping j which assigns to each variable an element of the domain A.

Finally, an **$\mathscr{L}$-interpretation $\mathbf{I}$** is a pair $(\mathbf{M}, j)$ consisting of an $\mathscr{L}$-structure $\mathbf{M}$ and an assignment j in $\mathbf{M}$. For a variable ν, an element $a \in A$, and an assignment j in $\mathbf{M}$ we define the assignment $j\frac{a}{\nu}$ by stipulating

$$j\tfrac{a}{\nu}(\nu') = \begin{cases} a & \text{if } \nu' = \nu, \\ j(\nu') & \text{otherwise.} \end{cases}$$

For an interpretation $\mathbf{I} = (\mathbf{M}, j)$ and an element $a \in A$, let

$$\mathbf{I}\tfrac{a}{\nu} := (\mathbf{M}, j\tfrac{a}{\nu})\,.$$

We associate with every interpretation $\mathbf{I} = (\mathbf{M}, j)$ and every $\mathscr{L}$-term τ an element $\mathbf{I}(t) \in A$ as follows:

- For a variable ν, let $\mathbf{I}(\nu) := j(\nu)$.
- For a constant symbol $c \in \mathscr{L}$, let $\mathbf{I}(c) := c^{\mathbf{M}}$.
- For an n-ary function symbol $F \in \mathscr{L}$ and terms $\tau_1, \dots, \tau_n$, let

$$\mathbf{I}\big(F(\tau_1, \dots, \tau_n)\big) := F^{\mathbf{M}}\big(\mathbf{I}(\tau_1), \dots, \mathbf{I}(\tau_n)\big).$$

Now, we are able to define precisely when a formula φ becomes *true* under an interpretation $\mathbf{I} = (\mathbf{M}, j)$; in which case we write $\mathbf{I} \vDash \varphi$ and say that φ is **true** in $\mathbf{I}$ (or that φ **holds** in $\mathbf{I}$). The definition is by induction on the complexity of the formula φ. By the rules (F0)–(F4), φ must be of the form $\tau_1 = \tau_2$, $R(\tau_1, \dots, \tau_n)$, $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \rightarrow \psi_2$, $\exists\nu\psi$, or $\forall\nu\psi$:

$$\begin{aligned}
\mathbf{I} \vDash \tau_1 = \tau_2 \quad &:\Longleftrightarrow \quad \mathbf{I}(\tau_1) \text{ IS THE SAME OBJECT AS } \mathbf{I}(\tau_2) \\
\mathbf{I} \vDash R(\tau_1, \dots, \tau_n) \quad &:\Longleftrightarrow \quad \big\langle \mathbf{I}(\tau_1), \dots, \mathbf{I}(\tau_n) \big\rangle \text{ BELONGS TO } R^{\mathbf{M}} \\
\mathbf{I} \vDash \neg\psi \quad &:\Longleftrightarrow \quad \text{NOT } \mathbf{I} \vDash \psi \\
\mathbf{I} \vDash \psi_1 \wedge \psi_2 \quad &:\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \text{ AND } \mathbf{I} \vDash \psi_2 \\
\mathbf{I} \vDash \psi_1 \vee \psi_2 \quad &:\Longleftrightarrow \quad \mathbf{I} \vDash \psi_1 \text{ OR } \mathbf{I} \vDash \psi_2 \\
\mathbf{I} \vDash \psi_1 \rightarrow \psi_2 \quad &:\Longleftrightarrow \quad \text{NOT } \mathbf{I} \vDash \psi_1 \text{ OR } \mathbf{I} \vDash \psi_2
\end{aligned}$$

$$\mathbf{I} \vDash \exists\nu\psi \quad :\Longleftrightarrow \quad \text{IT EXISTS } a \text{ IN } A: \ \mathbf{I}\tfrac{a}{\nu} \vDash \psi$$

$$\mathbf{I} \vDash \forall\nu\psi \quad :\Longleftrightarrow \quad \text{FOR ALL } a \text{ IN } A: \ \mathbf{I}\tfrac{a}{\nu} \vDash \psi$$

Notice that by the logical rules in our informal language, for *every* $\mathscr{L}$-formula φ we have either $\mathbf{I} \vDash \varphi$ or $\mathbf{I} \vDash \neg\varphi$. So, every $\mathscr{L}$-formula is either true or false in $\mathbf{I}$.

Let T be an arbitrary set of $\mathscr{L}$-formulae. Then an $\mathscr{L}$-structure $\mathbf{M}$ is a **model of** T if for every assignment j and for each formula $\varphi \in \mathsf{T}$ we have $(\mathbf{M}, j) \vDash \varphi$, *i.e.*, φ is true in the $\mathscr{L}$-interpretation $\mathbf{I} = (\mathbf{M}, j)$. Instead of saying "$\mathbf{M}$ is a model of T" we just write $\mathbf{M} \vDash \mathsf{T}$. If φ fails in $\mathbf{M}$, then we write $\mathbf{M} \nvDash \varphi$, which is equivalent to $\mathbf{M} \vDash \neg\varphi$, because for any $\mathscr{L}$-formula φ we have *either* $\mathbf{M} \vDash \varphi$ *or* $\mathbf{M} \vDash \neg\varphi$.

For example S_7 (*i.e.*, the set of all permutations of seven different items) is a model of GT, where the interpretation of the binary operation is composition and the neutral element is interpreted as the identity permutation. In this case, the elements of the domain of S_7 can be real and can even be heard, namely when the seven items are seven bells and a peal of, for example, Stedman Triples consisting of all 5040 permutations of the seven bells is rung—which happens quite often, since Stedman Triples are very popular with change-ringers. However, the objects of models of mathematical theories usually do not belong to our physical world and are no more real than, for example, the *number zero* or the *empty set*.

The Completeness Theorem

The following two theorems, which we state without proofs, are the main connections between the syntactical and the semantical approach to first-order theories. On the one hand, the SOUNDNESS THEOREM 2.3 just tells us that our deduction system is sound, *i.e.*, if a sentence φ is provable from T then φ is true in each model of T. On the other hand, GÖDEL'S COMPLETENESS THEOREM 2.4 tells us that our deduction system is even complete, *i.e.*, every sentence which is true in all models of T is provable from T. As a consequence we find that $\mathsf{T} \vdash \varphi$ if and only if φ is true in each model of T. In particular, if T is empty, this implies that every tautology (*i.e.*, universally valid formula) is provable.

THEOREM 2.3 (SOUNDNESS THEOREM). *Let T be a set of $\mathscr{L}$-sentences and let φ be any $\mathscr{L}$-sentence. If $\mathsf{T} \vdash \varphi$, then in any model $\mathbf{M}$ such that $\mathbf{M} \vDash \mathsf{T}$ we have $\mathbf{M} \vDash \varphi$.*

THEOREM 2.4 (GÖDEL'S COMPLETENESS THEOREM). *Let T be a set of $\mathscr{L}$-sentences and let σ be any $\mathscr{L}$-sentence. If $\mathsf{T} \nvdash \sigma$, then there is a model $\mathbf{M}$ such that $\mathbf{M} \vDash \mathsf{T} \cup \{\neg\sigma\}$. In particular, every consistent set of $\mathscr{L}$-sentences has a model.*

As a matter of fact we would like to mention that if the signature $\mathscr{L}$ is uncountable, then, in general, GÖDEL'S COMPLETENESS THEOREM cannot be proved without using some form of the Axiom of Choice.

One of the main consequences of GÖDEL'S COMPLETENESS THEOREM 2.4 is that formal proofs—which are usually quite long and involved—can be replaced with informal ones: Let T be a consistent set of $\mathscr{L}$-formulae and let φ be any $\mathscr{L}$-sentence. Then, by GÖDEL'S COMPLETENESS THEOREM 2.4, in order to show that $\mathsf{T} \vdash \varphi$ it is enough to show that $\mathbf{M} \vDash \varphi$ whenever $\mathbf{M} \vDash \mathsf{T}$. In fact, we would take an arbitrary model $\mathbf{M}$ of T and show that $\mathbf{M} \vDash \varphi$.

As an example, let us show that $\mathsf{GT} \vdash (y \circ x = \mathsf{e}) \to (x \circ y = \mathsf{e})$: First, let $\mathbf{G}$ be a model of GT with domain G, and let x and y be any elements of G. By GT_2 we know that every element of G has a left-inverse. In particular, y has a left-inverse, say $\bar{y}$, and we have $\bar{y} \circ y = \mathsf{e}$. By GT_1 we have $x \circ y = (\bar{y} \circ y) \circ (x \circ y)$, and by GT_0 we get $(\bar{y} \circ y) \circ (x \circ y) = \bar{y} \circ ((y \circ x) \circ y)$. Now, if $y \circ x = \mathsf{e}$, then we have $x \circ y = \bar{y} \circ y$ and consequently we get $x \circ y = \mathsf{e}$. Notice that we tacitly used that the equality relation is symmetric and transitive.

We leave it as an exercise for the reader to find the corresponding formal proof of this basic result in Group Theory. In a similar way, one can show that every left-neutral element is also a right-neutral element (called a *neutral element*) and that there is just one neutral element in a group.

The following result, which is a consequence of GÖDEL'S COMPLETENESS THEOREM 2.4, shows that *every* consistent set of formulae has a model.

PROPOSITION 2.5. *Let T be any set of $\mathscr{L}$-sentences. Then $\operatorname{Con}(\mathsf{T})$ if and only if T has a model.*

Proof. ($\Rightarrow$) If T has no model, then, by GÖDEL'S COMPLETENESS THEOREM 2.4, for *every* $\mathscr{L}$-sentence σ we have $\mathsf{T} \vdash \sigma$ (otherwise, there would be a model of $\mathsf{T} \cup \{\neg\sigma\}$, in particular there would be a model of T). So, taking σ to be $\varphi \wedge \neg\varphi$ (for some $\mathscr{L}$-sentence φ) we get $\mathsf{T} \vdash (\varphi \wedge \neg\varphi)$, hence T is inconsistent.

($\Leftarrow$) If T is inconsistent, then, by PROPOSITION 2.2 (a), for every $\mathscr{L}$-sentence σ we have $\mathsf{T} \vdash \sigma$, in particular, $\mathsf{T} \vdash \varphi \wedge \neg\varphi$ (for some $\mathscr{L}$-sentence φ). Now, the SOUNDNESS THEOREM 2.3 implies that in all models $\mathbf{M} \vDash \mathsf{T}$ we have $\mathbf{M} \vDash \varphi \wedge \neg\varphi$; thus, there are no models of T. ⊣

A set of *sentences* T is usually called a **theory**. A consistent theory T (in a certain language $\mathscr{L}$) is said to be **complete** if for every $\mathscr{L}$-sentence σ, *either* $\mathsf{T} \vdash \sigma$ *or* $\mathsf{T} \vdash \neg\sigma$. If T is not complete, we say that T is **incomplete**.

The following result is an immediate consequence of PROPOSITION 2.5.

COROLLARY 2.6. *Every consistent theory is contained in a complete theory.*

Proof. Let T be a theory with signature $\mathscr{L}$. If T is consistent, then it has a model, say $\mathbf{M}$. Now let $\bar{\mathsf{T}}$ be the set of all $\mathscr{L}$-sentences σ such that $\mathbf{M} \vDash \sigma$. Obviously, $\bar{\mathsf{T}}$ is a complete theory which contains T. ⊣

Let T be a set of $\mathscr{L}$-sentences and let σ be any $\mathscr{L}$-sentence not contained in T. The sentence σ is said to be **consistent relative to** T (or that σ is **consistent with** T) if $\mathrm{Con}(\mathsf{T})$ implies $\mathrm{Con}(\mathsf{T} \cup \{\sigma\})$ (later we write $\mathsf{T} + \sigma$ instead of $\mathsf{T} \cup \{\sigma\}$). If both σ and $\neg\sigma$ are consistent with T, then σ is said to be **independent** of T. In other words, if $\mathrm{Con}(\mathsf{T})$, then σ is independent of T if *neither* $\mathsf{T} \vdash \sigma$ *nor* $\mathsf{T} \vdash \neg\sigma$. By GÖDEL'S COMPLETENESS THEOREM 2.4 we see that if $\mathrm{Con}(\mathsf{T})$ and σ is independent of T, then there are models $\mathbf{M}_1$ and $\mathbf{M}_2$ of T such that $\mathbf{M}_1 \vDash \sigma$ and $\mathbf{M}_2 \vDash \neg\sigma$. A typical example of a sentence which is independent of GT is $\forall x \forall y (x \circ y = y \circ x)$ (*i.e.*, the binary operation is commutative), and indeed, there are abelian as well as non-abelian groups.

In order to prove that a certain sentence σ is independent of a given (consistent) theory T, one could try to find two different models of T such that σ holds in one model and fails in the other. However, this task is quite difficult, in particular if one cannot prove that T has a model at all (as is the case for Set Theory).

Limits of First-Order Logic

We begin this section with a useful result, called the COMPACTNESS THEOREM. On the one hand, it is just a consequence of the fact that formal proofs are finite (*i.e.*, finite sequences of formulae). On the other hand, the COMPACTNESS THEOREM is the main tool used to prove that a certain sentence (or a set of sentences) is consistent with a given theory. In particular, the COMPACTNESS THEOREM is implicitly used in every set-theoretic consistency proof which is obtained by forcing (for details, see Chapter 16).

THEOREM 2.7 (COMPACTNESS THEOREM). *Let T be a set of $\mathscr{L}$-sentences. Then T is consistent if and only if every finite subset $\mathsf{T}' \subseteq \mathsf{T}$ is consistent.*

Proof. Obviously, if T is consistent, then every finite subset $\mathsf{T}' \subseteq \mathsf{T}$ must be consistent. On the other hand, if T is inconsistent, then there is an $\mathscr{L}$-sentence σ such that $\mathsf{T} \vdash \sigma \wedge \neg\sigma$. In other words, there is a proof of $\sigma \wedge \neg\sigma$ from T. Now, since every proof is finite, there are only finitely many sentences of T involved in this proof, and if T' is this finite set of sentences, then $\mathsf{T}' \vdash \sigma \wedge \neg\sigma$, which shows that T', a finite subset of T, is inconsistent. ⊣

Recall that by GÖDEL'S COMPLETENESS THEOREM 2.4 we get that every consistent theory has a model. So, together with the COMPACTNESS THEOREM we get that a theory T has a model if and only if every finite subset $\mathsf{T}' \subseteq \mathsf{T}$ has a model. This statement also holds for Propositional Logic, where instead of models we have the corresponding notion of *satisfiability* (defined in Chapter 6):

Compactness Theorem for Propositional Logic. *If Σ is a set of formulae of Propositional Logic, then Σ is satisfiable if and only if every finite subset $\Sigma' \subseteq \Sigma$ is satisfiable.*

As a matter of fact, we would like to mention that the Compactness Theorem for Propositional Logic is equivalent to the Prime Ideal Theorem, which implies that in general, the Compactness Theorem for Propositional Logic cannot be proved without using some form of the Axiom of Choice (see THEOREM 6.7).

A simple application of the COMPACTNESS THEOREM 2.7 shows that if PA is consistent, then there is more than one model of PA (*i.e.*, beside the intended model of natural numbers with domain $\mathbb{N}$, there are also so-called *non-standard* models of PA with larger domains):

First, we extend the signature $\mathscr{L}_{\mathsf{PA}} = \{0, \mathtt{s}, +, \cdot\}$ by adding a new constant symbol $\mathtt{n}$. Secondly we extend PA by adding the formulae

$$\underbrace{\mathtt{n} \neq 0}_{\varphi_0}, \quad \underbrace{\mathtt{n} \neq \mathtt{s}(0)}_{\varphi_1}, \quad \underbrace{\mathtt{n} \neq \mathtt{s}\big(\mathtt{s}(0)\big)}_{\varphi_2}, \quad \dots$$

and let Ψ be the set of these formulae. Now, if PA has a model $\mathbf{N}$ with domain say $\mathbb{N}$, and Φ is any finite subset of Ψ, then, by interpreting $\mathtt{n}$ in a suitable way, $\mathbf{N}$ is also a model of $\mathsf{PA} \cup \Phi$, which implies that $\mathsf{PA} \cup \Phi$ is consistent. Thus, by the COMPACTNESS THEOREM 2.7, $\mathsf{PA} \cup \Psi$ is also consistent and therefore has a model, say $\mathbf{N}^*$. Now, $\mathbf{N}^* \vDash \mathsf{PA} \cup \Psi$, but since $\mathtt{n}$ is different from every standard natural number of the form $\mathtt{s}(\mathtt{s}(\dots\mathtt{s}(0)\dots))$, the domain of $\mathbf{N}^*$ must be very different from $\mathbb{N}$ (since it contains a kind of infinite number, whereas all standard natural numbers are finite).

This example shows that we cannot axiomatise Peano Arithmetic in First-Order Logic in such a way that all the models we get have essentially the same domain $\mathbb{N}$.

By PROPOSITION 2.5 we know that a set of first-order formulae T is consistent if and only if it has a model, *i.e.*, there is a model $\mathbf{M}$ such that $\mathbf{M} \vDash \mathsf{T}$. So, in order to prove, for example, that the axioms of Set Theory are consistent we only have to find a single model in which all these axioms hold. However, as a consequence of the following theorems—which we state again without proof—this turns out to be impossible (at least if one restricts oneself to methods formalisable in Set Theory).

THEOREM 2.8 (GÖDEL'S INCOMPLETENESS THEOREM). *Let T be a theory with signature $\mathscr{L}$ which consists of finitely many axioms or axiom schemata. Furthermore, assume that T is consistent and sufficiently strong to prove the axioms of Peano Arithmetic PA. Then there is always an $\mathscr{L}$-sentence σ which is independent of T, i.e., neither $\mathsf{T} \vdash \sigma$ nor $\mathsf{T} \vdash \neg\sigma$.*

As a consequence of GÖDEL'S COMPLETENESS THEOREM 2.4 we get that such theories have at least two essentially different models, one in which the sentence σ is true and one in which it fails. In particular, since PA is such a theory, we find that the very same number-theoretic sentence σ is true in some models of PA and fails in some other, which shows that the theory PA is incomplete.

On the one hand, GÖDEL'S INCOMPLETENESS THEOREM tells us that in any theory T which is sufficiently strong, there are always statements which are independent of T (*i.e.*, which can neither be proved nor disproved in T). On the other hand, statements which are independent of a given theory (*e.g.*, of Set Theory or of Peano Arithmetic) are often very interesting, since they say something unexpected, but in a language we can understand. From this point of view it is good to have GÖDEL'S INCOMPLETENESS THEOREM, which guarantees the existence of such statements in theories like Set Theory or Peano Arithmetic.

In Part III we shall present a technique with which we can prove the independence of certain set-theoretical statements from the axioms of Set Theory, which are introduced and discussed in the next chapter.

NOTES

Let us give a brief overview of the history of mathematical logic (for a comprehensive history of logic we refer the reader to Bocheński [6] and W. & M. Kneale [41]).

Aristotle's "Organon". Aristotle's logical treatises contain the earliest formal study of logic (*i.e.*, of Propositional Logic, which is concerned about logical relations between propositions as wholes) and consequently he is commonly considered the first logician. Aristotle's logical works were grouped together by the ancient commentators under the title *Organon* [1], consisting of *Categories, On Interpretation, Prior Analytics, Posterior Analytics, Topics*, and *On Sophistical Refutations*. Probably the most famous part of Aristotle's Logic is the *syllogistic*, which he defines at the beginning of the *Prior Analytics* as follows: *A syllogism is a discourse in which from certain propositions that are laid down something other than what is stated follows of necessity.* Aristotle's syllogistic can be considered as a precursor of modern *Predicate Logic*. For a modern view of Aristotle's syllogistic we refer the reader to Łukasiewicz [42] and Russinoff [48].

Chrysippus and the Stoics. In its core, Stoic Logic is a *Propositional Logic*, since the inference concerns the relations between items that have the structure of propositions. The Stoics were the first to work out in detail a theory of arguments involving the conditional and other forms of complex propositions. The theory of conditional propositions was also central in Chrysippus' Logic, who was the main representative of the Stoic school. The Stoics distinguished between *indemonstrable arguments*, called *indemonstrables*, and arguments that can be reduced to indemonstrables. To some extent, indemonstrables can be seen as logical axioms, and even though they are not rules of inference, one of the five types of indemonstrables introduced by Chrysippus is very similar to Modus Ponens. The leading premisses of Chrysippus' indemonstrables use only Chrysippus' connectives, which are "and", "if", "or" (exclusive), and the negation "not". Not only had the Stoics (and Chrysippus in particular) a very deep understanding of Propositional Logic, they even claimed that their system of Propositional Logic was complete in the sense that every valid argument could be reduced to a series of arguments of the five basic types of indemonstrables. Moreover, even the method of reduction was not left vague, but was exactly characterised by four meta-rules, of which we possess just two (a third meta-rule is uncertain). For Stoic Logic, in particular for its completeness or incompleteness, we refer the reader to Bobzien [4] and Milne [43].

Boole's "Laws of Thought". In 1854, Boole published in *An Investigation of the Laws of Thought* [8] (see also [7]) a new approach to logic by reducing it to a kind of algebra and thereby incorporated logic into Mathematics: Boole noticed that Aristotle's Logic was essentially dealing with classes of objects and he further observed that these classes can be denoted by symbols like x, y, z, subject to the ordinary rules of algebra, with the following interpretations.

(a) xy denotes the class of members of x which are also members of y.

(b) If x and y have no members in common, then $x+y$ denotes the class of objects which belong either to x or to y.

(c) $1-x$ denotes all the objects not belonging to the class x.

(d) $x=0$ means that the class x has no members.

However, Boole's Logic was still Propositional Logic, but just 25 years later this weakness was eliminated.

Frege's "Begriffsschrift". In 1879, Frege published in his *Begriffsschrift* [15], the most important advance in logic since Aristotle and Chrysippus. In this work, Frege presented for the first time what we would recognise today as a logical system with negation, implication, universal quantification, logical axioms, *et cetera*. Even though Frege's achievement in logic was a major step towards First-Order Logic, his subsequent work [16, 17] led to some antinomy, which was discovered by Russell. For Frege's presentation of logic after Russell's discovery we refer the reader to Frege's lectures on his *Begriffsschrift* between 1910 and 1913 (see [18]).

Peano's "Arithmetices Principia". Written in Latin, [46] was Peano's first attempt at an axiomatisation of Mathematics—and in particular of Arithmetic—in a symbolic language. The initial arithmetic notions are *number*, *one*, *successor*, *is equal to*, and nine axioms are stated concerning theses notions. (Today, "$=$" belongs to the underlying language of logic, and so, Peano's axioms dealing with equality become redundant; further, we start the natural numbers with *zero*, rather than *one*.) Concerning the problem of whether the natural numbers can be considered as symbols without inherent meaning, we refer the reader to the discussion between Müller [45] and Bernays [3]. For Peano's work in logic, and in particular for the development of the axioms for natural numbers, we refer the reader to Jourdain [38, pp. 270–314] (where one can also find some comments by Peano) and to Wang [50]. According to Jourdain (*cf.* [38, p. 273]), Peano [46] succeeded in writing out wholly in symbols the propositions and proofs of a complete treatise on the arithmetic of positive numbers. However, in the arithmetical demonstrations, Peano made extensive use of Grassmann's work [27], and in fundamental questions of arithmetic as well as in the theory of logical functions, he used Dedekind's work [9]. The main feature of Wang's paper [50] is the printing of a letter (mentioned by Noether on p. 490 of [10]) from Dedekind to a headmaster in Hamburg, dated 27 February, 1890. In that letter, Dedekind points out the appearance of non-standard models of axioms for natural numbers (see Kaye [39]) and explains how one could avoid such unintended models by using his *Kettentheorie* (*i.e.*, concept of chains) which he developed in [9]. He also refers to Frege's works [15, 16] and notes that Frege's method of defining a kind of "successor relation" agrees in essence with his concept of chains.

Russell's "Principia Mathematica". One of these steps was taken by Russell and Whitehead in their *Principia Mathematica* [51], which is a three-volume work on the foundations of Mathematics, published between 1910 and 1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The main inspiration and motivation for the *Principia Mathematica* was Frege's earlier work on logic, especially the contradictions discovered by Russell (as mentioned above). The questions remained whether a contradiction could also be derived from the axioms given in the *Principia Mathematica*, and whether there exists a mathematical statement which could neither be proven nor disproven in the system (for Russell's search for truth, we refer the reader to Doxiadis and Papadimitriou [12]). It took another twenty odd years until these questions were answered by Gödel's Incompleteness Theorem, but before that, the logical axioms had to be settled.

Hilbert's "Grundzüge der theoretischen Logik". In 1928, Ackermann and Hilbert published in their *Grundzüge der theoretischen Logik* [36], to some extent the final version of the logical axioms (for the development of these axioms see, for example, Hilbert [33, 34, 35]).

Gödel's Completeness Theorem. Gödel proved the COMPLETENESS THEOREM in his doctoral dissertation *Über die Vollständigkeit des Logikkalküls* [19] which was completed in 1929. In 1930, he published the same material as in the doctoral dissertation in a rewritten and shortened form in [20]. The standard proof of GÖDEL'S COMPLETENESS THEOREM is Henkin's proof, which can be found in [30] (see also [31]) as well as in most other textbooks on logic. A slightly different approach can be found, for example, in Kleene [40§72].

Gödel's Incompleteness Theorem. In 1930, Gödel announced in [21] his INCOMPLETENESS THEOREM (published later in [22]), which is probably the most famous theorem in logic. The theorem as it is stated above is Satz VI of [22]. GÖDEL'S INCOMPLETENESS THEOREM 2.4 is discussed in great detail in Hoffmann [37] and in Mostowski [44] (see also Goldstern and Judah [26, Chapter 4]); and for a different proof of GÖDEL'S INCOMPLETENESS THEOREM, not just a different version of Gödel's proof, see Putnam [47]. For more historical background—as well as for Gödel's platonism—we refer the reader to Goldstein [24].

Our approach to First-Order Logic presented in this chapter is partially taken from the first few sections of the hyper-textbook for students by Detlovs and Podnieks (these sections are an extended translation of the corresponding chapters of Detlovs [11]). For other rules of inference see, for example, Hermes [32] or Ebbinghaus, Flum, and Thomas [13, 14], and for the nature and technique of formal proofs see, for example, Hales [28], Harrison [29], and Wiedijk [52].

REFERENCES

1. ARISTOTLE, ***Organon***, Athens, *published by Andronikos of Rhodos around* 40 B.C.
2. ______, ***Topics***, Athens, *published by Andronikos of Rhodos around* 40 B.C.
3. PAUL BERNAYS, *Erwiderung auf die Note von Herrn Aloys Müller: Über Zahlen als Zeichen*, ***Mathematische Annalen***, vol. 90 (1923), 159–163.
4. SUSANNE BOBZIEN, *Stoic logic*, *in* ***The Cambridge Companion to the Stoics*** (Brad Inwood, ed.), Cambridge University Press, Cambridge, 2003, pp. 85–123.
5. JOSEPH M. BOCHEŃSKI, ***A History of Formal Logic***, 2nd ed., [translated and edited by Ivo Thomas], Chelsea Publishing Company, New York, 1970.
6. ______, ***Formale Logik***, 4th ed., Orbis Academicus: Problemgeschichten der Wissenschaft in Dokumenten und Darstellungen, Verlag Karl Alber, Freiburg, 1978 (see [5] for a translation into English).
7. GEORGE BOOLE, *The calculus of logic*, ***The Cambridge and Dublin Mathematical Journal***, vol. 3 (1848), 183–198.
8. ______, ***An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities***, Walton and Maberley, London, 1854 [reprint: Prometheus Books, New York, 2003].
9. RICHARD DEDEKIND, ***Was sind und was sollen die Zahlen***, Friedrich Vieweg & Sohn, Braunschweig, 1888 (see also [10, pp. 335–390]).

10. ______, ***Gesammelte mathematische Werke III***, edited by R. Fricke, E. Noether, and Ö. Ore, Friedrich Vieweg & Sohn, Braunschweig, 1932.
11. VILNIS DETLOVS, ***Matemātiskā logika***, Izdevniecība "Zvaigzne", Riga, 1974.
12. APOSTOLOS DOXIADIS AND CHRISTOS H. PAPADIMITRIOU, ***Logicomix: An Epic Search for Truth***, *(a graphic novel with art by Alecos Papadatos and Annie di Donna)*, Bloomsbury, New York, 2009.
13. HEINZ-DIETER EBBINGHAUS, JÖRG FLUM, AND WOLFGANG THOMAS, ***Mathematical Logic***, 2nd ed., *English translation of* [14], [Undergraduate Texts in Mathematics], Springer-Verlag, New York, 1994.
14. ______, ***Einführung in die mathematische Logik***, 4th ed., Spektrum Akademischer Verlag, Heidelberg · Berlin, 1996.
15. GOTTLOB FREGE, ***Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens***, L. Nebert, Halle, 1879 (see [49] for a translation into English).
16. ______, ***Die Grundlagen der Arithmetik, eine logisch-mathematische Untersuchung über den Begriff der Zahl***, Verlag von Wilhelm Koebner, Breslau, 1884. [English translation (with German original): *The foundations of arithmetic. A logo-mathematical enquiry into the concept of number*, translated by J. L. Austin, Basil Blackwell, Oxford, 1950].
17. ______, ***Die Grundgesetze der Arithmetik I & II***, Verlag von Hermann Pohle, Jena, 1893 (I), 1903 (II).
18. ______, *Vorlesungen über Begriffsschrift (nach der Mitschrift von Rudolf Carnap, mit Einleitung und Anmerkungen herausgegeben von Gottfried Gabriel)*, ***History and Philosophy of Logic***, vol. 17 (1996), iii–xvi, 1–48.
19. KURT GÖDEL, *Über die Vollständigkeit des Logikkalküls*, Dissertation (1929), University of Vienna (Austria), (reprinted and translated into English in [23]).
20. ______, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), 349–360 (see [49, 23] for a translation into English).
21. ______, *Einige metamathematische Resultate über Entscheidungsdefinitheit und Widerspruchsfreiheit*, ***Anzeiger der Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse***, vol. 67 (1930), 214–215 (see [49, 23] for a translation into English).
22. ______, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, ***Monatshefte für Mathematik und Physik***, vol. 38 (1931), 173–198 (see [49, 23] for a translation into English).
23. ______, ***Collected Works, Volume I: Publications 1929–1936***, edited by S. Feferman (Editor-in-chief), J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, J. van Heijenoort, Oxford University Press, New York, 1986.
24. REBECCA GOLDSTEIN, ***Incompleteness – The proof and paradox of Kurt Gödel***, Great Discoveries, Atlas Books / W. W. Norton & Company, New York and London, 2005 (see [25] for a translation into German).
25. ______, ***Kurt Gödel, Jahrhundertmathematiker und großer Entdecker***, [translated into German by Thorsten Schmidt], Piper, München and Zürich, 2006.

26. MARTIN GOLDSTERN AND HAIM JUDAH, ***The incompleteness phenomenon. A new course in mathematical logic***, A. K. Peters, Wellesley, 1995.
27. HERMANN GRASSMANN, ***Lehrbuch der Arithmetik für höhere Lehranstalten***, Th. Chr. Fr. Enslin, Berlin, 1861.
28. THOMAS C. HALES, *Formal proof*, ***Notices of the American Mathematical Society***, vol. 55 (2008), 1370–1380.
29. JOHN HARRISON, *Formal proof—theory and practice*, ***Notices of the American Mathematical Society***, vol. 55 (2008), 1395–1406.
30. LEON HENKIN, *The completeness of the first-order functional calculus*, ***The Journal of Symbolic Logic***, vol. 14 (1949), 159–166.
31. ______, *The discovery of my completeness proofs*, ***The Bulletin of Symbolic Logic***, vol. 2 (1996), 127–158.
32. HANS HERMES, ***Einführung in die mathematische Logik. Klassische Prädikatenlogik***, Mathematische Leitfäden, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1963.
33. DAVID HILBERT, *Neubegründung der Mathematik. Erste Mitteilung*, ***Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität***, vol. 1 (1922), 157–177.
34. ______, *Die logischen Grundlagen der Mathematik*, ***Mathematische Annalen***, vol. 88 (1923), 151–165.
35. ______, *Die Grundlagen der Mathematik. Vortrag, gehalten auf Einladung des Mathematischen Seminars im Juli 1927 in Hamburg.*, ***Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität***, vol. 6 (1928), 65–85 (see [49] for a translation into English).
36. DAVID HILBERT AND WILHELM ACKERMANN, ***Grundzüge der theoretischen Logik***, [Die Grundlehren der mathematischen Wissenschaften 27], Julius Springer, Berlin, 1928 (6. Auflage, 1972).
37. DIRK W. HOFFMANN, ***Die Gödel'schen Unvollständigkeitssätze: Eine geführte Reise durch Kurt Gödels historischen Beweis***, Springer Spektrum, Heidelberg, 2013.
38. PHILIP E. B. JOURDAIN, *The development of the theories of mathematical logic and the principles of mathematics*, ***The Quarterly Journal of Pure and Applied Mathematics***, vol. 43 (1912), 219–314.
39. RICHARD KAYE, ***Models of Peano Arithmetic***, [Oxford Logic Guides 15], The Clarendon Press Oxford University Press, New York, 1991.
40. STEPHEN COLE KLEENE, ***Introduction to Metamathematics***, [Bibliotheca Mathematica, A Series of Monographs on Pure and Applied Mathematics, Volume I], North-Holland, Amsterdam, 1959.
41. WILLIAM KNEALE AND MARTHA KNEALE, ***The Development of Logic***, Oxford University Press, Oxford, 1962.
42. JAN ŁUKASIEWICZ, ***Aristotle's Syllogistic, from the standpoint of modern formal logic***, Oxford University Press, Oxford, 1951.
43. PETER MILNE, *On the completeness of non-Philonian Stoic logic*, ***History and Philosophy of Logic***, vol. 16 (1995), 39–64.

44. ANDRZEJ MOSTOWSKI, ***Sentences undecidable in formalized arithmetic. An exposition of the theory of Kurt Gödel***, [Studies in logic and the foundations of mathematics], North-Holland, Amsterdam, 1952.
45. ALOYS MÜLLER, *Über Zahlen als Zeichen*, ***Mathematische Annalen***, vol. 90 (1923), 153–158.
46. GIUSEPPE PEANO, ***Arithmetices principia, nova methoda exposita***, Fratres Bocca, Torino, 1889 (see [49] for a translation into English).
47. HILARY PUTNAM, *Nonstandard models and Kripke's proof of the Gödel theorem*, ***Notre Dame Journal of Formal Logic***, vol. 41 (2000), 53–58.
48. I. SUSAN RUSSINOFF, *The syllogism's final solution*, ***The Bulletin of Symbolic Logic***, vol. 5 (1999), 451–469.
49. JEAN VAN HEIJENOORT, ***From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931***, [Source Books in the History of Science], Harvard University Press, Cambridge, Massachusetts, 1967.
50. HAO WANG, *The axiomatization of arithmetic*, ***The Journal of Symbolic Logic***, vol. 22 (1957), 145–158.
51. ALFRED NORTH WHITEHEAD AND BERTRAND RUSSELL, ***Principia Mathematica, Vol. I–III***, Cambridge University Press, Cambridge, 1910–1913.
52. FREEK WIEDIJK, *Formal proof—getting started*, ***Notices of the American Mathematical Society***, vol. 55 (2008), 1408–1417.