SATZ. *Für Ultrafilter $\mathscr{U} \subseteq [\omega]^\omega$ sind die folgenden Aussagen äquivalent:*

(a) *$\mathscr{U}$ ist Ramsey.*

(b) *Für jede absteigende Folge $y_0 \supseteq y_1 \supseteq \ldots \supseteq y_n \supseteq \ldots$ von Mengen aus $\mathscr{U}$ existiert eine Funktion $f \in {}^\omega\omega$ so dass $f[\omega] \in \mathscr{U}$, $f(0) \in y_0$, und für alle $k \in \omega$ gilt $f(k+1) \in y_{f(k)}$.*

(c) *Ist $\{u_n : n \in \omega\}$ eine Partition von ω, so gilt entweder $u_{n_0} \in \mathscr{U}$ für ein $n_0 \in \omega$, oder es existiert ein $x \in \mathscr{U}$, so dass für alle $n \in \omega$ gilt $|x \cap u_n| \leq 1$.*

(d) *$\mathscr{U}$ ist sowohl ein P-point wie auch ein Q-point.*

Proof. (c) ⇒ (b) If $y = \bigcap_{n\in\omega} y_n \in \mathscr{U}$, then the function $f_y \in {}^\omega\omega$ has the required properties. So, let us assume that $\bigcap_{n\in\omega} y_n \notin \mathscr{U}$ and without loss of generality let us further assume that for all $n \in \omega$, $y_n \setminus y_{n+1} \neq \emptyset$. Consider the partition $\{y_0^{\mathsf{c}} \cup \bigcap_{n\in\omega} y_n\} \cup \{y_n \setminus y_{n+1} : n \in \omega\}$ and notice that none of the pieces are in $\mathscr{U}$. By (b), there exists a set $x = \{a_n : n \in \omega\} \in \mathscr{U}$ such that for all $n \in \omega$, $x \cap (y_n \setminus y_{n+1}) = \{a_n\}$, in particular, $x \cap \bigcap_{n\in\omega} y_n = \emptyset$. Let $g \in {}^\omega\omega$ be a strictly increasing function such that $g(0) > 0$, $g[\omega] \subseteq x$, and for all $n \in \omega$, $x \setminus g(n) \subseteq y_n$. For $k \in \omega$ let $g^{k+1}(0) := g(g^k(0))$, where $g^0(0) := 0$. Now, since $\mathscr{U}$ is an ultrafilter, either

$$u_0 = \bigcup_{k\in\omega} \big[g^{2k}(0), g^{2k+1}(0)\big) \quad \text{or} \quad u_1 = \omega \setminus u_0$$

belongs to $\mathscr{U}$—recall that $[a,b) = \{i \in \omega : a \leq i < b\}$. Without loss of generality we may assume that $u_0 \in \mathscr{U}$, and consequently $x \cap u_0 \in \mathscr{U}$. By (b) and since $\mathscr{U}$ is an ultrafilter, there exists a set $z = \{c_k : k \in \omega\} \subseteq x$ such that $z \in \mathscr{U}$ and for all $k \in \omega$,

$$z \cap \big[g^{2k}(0), g^{2k+1}(0)\big) = \{c_k\}\,.$$

By construction, for each $k \in \omega$ we have $c_{k+1} > g(c_k)$. To see this, notice that

$$c_{k+1} \in \big[g^{2k+2}(0), g^{2k+3}(0)\big)$$

which implies $c_{k+1} \geq g^{2k+2}(0)$. On the other hand,

$$c_k \in \big[g^{2k}(0), g^{2k+1}(0)\big)$$

which implies $g^{2k+1}(0) > c_k$, and because g is strictly increasing, we get $g^{2k+2}(0) > g(c_k)$; hence, $c_{k+1} > g(c_k)$. Finally, by the definition of g we have $x \setminus g(c_k) \subseteq y_{c_k}$, and since $c_{k+1} > g(c_k)$ and $c_{k+1} \in x$, for all $k \in \omega$ we have:

$$c_{k+1} \in y_{c_k}$$

Thus, if we define the function $f \in {}^\omega\omega$ by stipulating $f(k) := c_k$, then f has the required properties. ⊣