Chapter 1
The Setting

For one cannot order or compose anything, or understand the nature of the composite, unless one knows first the things that must be ordered or combined, their nature, and their cause.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

What is Infinitary Combinatorics?

Combinatorics with all its various aspects is a broad field of mathematics which has many applications in areas like Topology, Group Theory and even Analysis. A reason for its wide range of applications might be that Combinatorics is rather a way of thinking than a homogeneous theory, and consequently Combinatorics is quite difficult to define. Nevertheless, let us start with a definition of Combinatorics, and in particular of infinitary Combinatorics, which will be suitable for our purpose:

> *Combinatorics is the branch of mathematics which studies collections of objects that satisfy certain criteria, and is in particular concerned with deciding how large or how small such collections might be. If the collections being considered are infinite, we speak of infinitary Combinatorics.*

Below we give a few examples which should illustrate some aspects of infinitary Combinatorics. At the same time, we shall present the main topics of this book, which are the *Axiom of Choice*, *cardinal characteristics of the continuum*, the *forcing technique*, and *Ramsey Theory*, but first, let us say a few words about "infinity": We shall never assume something like a "standard universe of sets" in which notions like "finiteness", "infinity", or "natural numbers" are defined semantically (*i.e.*, according to their meaning). For example, when we say that a certain set is "infinite", we mean that there is no bijection between this set and some natural number, where a natural number is an element of the set ω, which will be defined formally in Chapter 3. In particular, the elements of ω define the notion of "finiteness", and a set is infinite if and only if it is not finite. Later we shall see that there is a subtle difference between the set ω and the set $\mathbb{N}$ of the so-called standard natural numbers, but until we have defined ω, we will work with $\mathbb{N}$.

The Axiom of Choice

Let us start with an example from Graph Theory: A *graph* is a set of *vertices*, where some pairs of vertices are connected by an *edge*. Connected pairs of vertices are called *neighbours*. A graph is *infinite* if it has a infinitly many vertices. A *tree* is a *cycle-free* (*i.e.*, one cannot walk in proper cycles along edges), *connected* (*i.e.*, any two vertices are connected by a path of edges) graph, where one of its vertices is designated as the *root*. A tree is *finitely branching* if every vertex has only a finite number of neighbours. Furthermore, a *branch* through a tree is a maximal edge-path beginning at the root, in which no edge appears twice.

Now we are ready to state König's Lemma, which is often used implicitly in fields like Combinatorics, Topology, and many other branches of mathematics.

König's Lemma. Every infinite, finitely branching tree contains an infinite branch.

At first glance, this result looks straightforward and one would construct an infinite branch as follows: Let v_0 be the root. Since the tree is infinite but finitely branching, there must be a neighbour of v_0 from which we can reach infinitely many vertices without going back to v_0. Let v_1 be such a neighbour of v_0. Again, since we reach infinitely many vertices from v_1 (without going back to v_1) and the tree is finitely branching, there must be a neighbour of v_1, say v_2, from which we reach infinitely many vertices without going back to v_2. Proceeding in this way, we finally get the infinite branch $(v_0, v_1, v_2, \ldots)$.

Let us now have a closer look at this proof: Firstly, in order to prove that the set of neighbours of v_0 from which we reach infinitely many vertices without going back to v_0 is not empty, we need an infinite version of the so-called Pigeon-Hole Principle. The Pigeon-Hole Principle can be seen as the fundamental principle of Combinatorics.

Pigeon-Hole Principle. If $n+1$ pigeons roost in n holes, then at least two pigeons must share a hole. More prosaically: If m objects are coloured with n colours and $m > n$, then at least two objects have the same colour.

An infinite version of the Pigeon-Hole Principle reads as follows:

Infinite Pigeon-Hole Principle. If infinitely many objects are coloured with finitely many colours, then infinitely many objects have the same colour.

Using the Infinite Pigeon-Hole Principle we are now sure that the set of neighbours of v_0 from which we reach infinitely many vertices without going back to v_0 is not empty. However, the next problem we face is which element we should choose from that non-empty set. If the vertices are ordered in some way, then we can choose the first element with respect to that order, but otherwise, we would need some kind of choice function which selects *infinitely often* (and this is the crucial point!) one vertex from a given non-empty set of vertices. Such a choice function is guaranteed by the Axiom of Choice, denoted AC, which is introduced in Chapter 3 and discussed in great detail in Chapter 6.

Axiom of Choice. For every family $\mathscr{F}$ of non-empty sets, there is a function f, called a *choice function*, which selects one element from each member of $\mathscr{F}$ (*i.e.*, for each $x \in \mathscr{F}$, $f(x) \in x$); or equivalently, every Cartesian product of non-empty sets is non-empty.

Except in Chapter 5, where we shall see how combinatorics can, to some extent, replace the Axiom of Choice, we always work in Zermelo–Fraenkel Set Theory *with* the Axiom of Choice—even in the case as in Chapters 8 & 17 when we construct models of Set Theory in which AC fails.

Now, let us turn back to König's Lemma. In order to prove König's Lemma we do not need full AC, since it would be enough if every family of non-empty *finite* sets had a choice function—the family would consist of all subsets of neighbours of vertices. However, as we will see later, even this weaker form of AC is a proper axiom and is independent of the other axioms of Set Theory (*cf.* PROPOSITION 8.7). Thus, depending on the axioms of Set Theory we start with, AC—as well as some weakened forms of it—may fail, and consequently, König's Lemma may become unprovable. On the other hand, as we will see in Chapter 6, König's Lemma may be used as a non-trivial choice principle.

Thus, this first example shows that—with respect to our definition of Combinatorics given above—some "objects satisfying certain criteria," may, but need not, exist.

Cardinal Characteristics

The next example can be seen as a problem in infinitary *Extremal Combinatorics*. The word "extremal" describes the nature of problems dealt with in this field and refers to the second part of our definition of Combinatorics, namely "how large or how small collections satisfying certain criteria might be."

If the objects considered are infinite, then the answer, how large or how small certain sets are, depends again on the underlying axioms of Set Theory, as the next example shows.

Reaping Families. A family $\mathscr{R}$ of infinite subsets of the natural numbers $\mathbb{N}$ is said to be reaping if for every colouring of $\mathbb{N}$ with two colours there exists a monochromatic set in the family $\mathscr{R}$.

For example, the set of all infinite subsets of $\mathbb{N}$ is such a family. The *reaping number* $\mathfrak{r}$, which is a so-called *cardinal characteristic of the continuum*, is the smallest cardinality (*i.e.*, size) of a reaping family. In general, a *cardinal characteristic of the continuum* is typically defined as the smallest cardinality of a subset of a given set S which has certain combinatorial properties, where S is of the same cardinality as the continuum $\mathbb{R}$.

Consider the cardinal characteristic $\mathfrak{r}$ (*i.e.*, the size of the smallest reaping family). Since $\mathfrak{r}$ is a well-defined cardinality we can ask: How large is $\mathfrak{r}$? Can it be countable? Is it always equal to the cardinality of the continuum?

Let us show that a reaping family can never be countable: Let $\mathscr{A} = \{A_i : i \in \mathbb{N}\}$ be any countable family of infinite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, pick n_i and m_i from the set A_i in such a way that, at the end, for all i we have $n_i < m_i < n_{i+1}$. Now we colour all n_i's blue and all m_i's red. For this colouring, there is no monochromatic set in $\mathscr{A}$, and hence, $\mathscr{A}$ cannot be a reaping family. The Continuum Hypothesis, denoted CH, states that every subset of the continuum $\mathbb{R}$ is either countable or of cardinality $\mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$. Thus, if we assume CH, then any reaping family is of cardinality $\mathfrak{c}$. The same holds if we assume Martin's Axiom, which shall be introduced in Chapter 14. The question is now, can we say anything about $\mathfrak{r}$ and $\mathfrak{c}$ if we assume neither CH nor MA?

The Forcing Technique

With the *forcing technique*—invented by Paul Cohen in the early 1960s—one can show that the axioms of Set Theory do not decide CH. In other words, there are models of Set Theory in which CH holds, and other models of Set Theory in which CH fails. Moreover, we can force models of Set Theory in which CH fails and in which $\mathfrak{r} < \mathfrak{c}$. The forcing technique to construct such models is introduced in Part III and a model in which $\mathfrak{r} < \mathfrak{c}$ is given in Chapter 18.

Thus, the second example shows that—depending on the additional axioms of Set Theory we start with—we can get different answers when we try to "decide how large or how small certain collections might be."

Many more cardinal characteristics like $\mathfrak{hom}$ and $\mathfrak{par}$ (see below) are introduced in Chapter 9. Possible (*i.e.*, consistent) relations between these cardinals are investigated in Part III and more systematically in Part IV—where the cardinal characteristics are also used to distinguish the combinatorial features of certain forcing notions.

Ramsey Theory

Another field of Combinatorics is the so-called Ramsey Theory, and since many results in this work rely on Ramsey-type theorems, let us give a brief description of Ramsey Theory.

Loosely speaking, *Ramsey Theory* (which can be seen as a part of extremal Combinatorics) is the branch of Combinatorics which deals with structures preserved under partitions, or colourings. Typically, one looks at the following kind of question: If a particular object (*e.g.*, algebraic, geometric or combinatorial) is arbitrarily coloured with finitely many colours, what kinds of monochromatic structure can we find?

For example, VAN DER WAERDEN'S THEOREM, which will be proved in Chapter 12, tells us that *for any positive integers r and n, there is a positive integer N such that for every r-colouring of the set $\{0, 1, \ldots, N\}$ we can always find a monochromatic (non-constant) arithmetic progression of length n.*

Even though VAN DER WAERDEN'S THEOREM is one of the earliest results in Ramsey Theory, the most famous result in Ramsey Theory is surely RAMSEY'S THEOREM (which will be discussed in detail in Chapter 4):

RAMSEY'S THEOREM. *Let n be any positive integer. If we colour all n-element subsets of $\mathbb{N}$ with finitely many colours, then there exists an infinite subset of $\mathbb{N}$ all of whose n-element subsets have the same colour.*

There is also a finite version of RAMSEY'S THEOREM which gives an answer to problems like the following:

How many people must be invited to a party in order to make sure that three of them mutually shook hands on a previous occasion or three of them mutually did not shake hands on a previous occasion?

It is quite easy to show that at least six people must be invited. On the other hand, if we ask how many people must get invited such that there are five people who all mutually shook hands or did not shake hands on a previous occasion, then the precise number is not known—but it is conjectured that it is sufficient to invite 43 people.

As we shall see later, RAMSEY'S THEOREM has many—sometimes unexpected—applications. For example, if we work in Set Theory without AC, then RAMSEY'S THEOREM can help to construct a choice function, as we will see in Chapter 5. Sometimes we get Ramsey-type (or anti-Ramsey-type) results even for partitions into infinitely many classes (*i.e.*, using infinitely many colours). For example, one can show that there is a colouring of the points in the Euclidean plane with countably many colours, such that no two points of any "copy of the rationals" have the same colour. This result can be seen as an anti-Ramsey-type theorem (since we are far away from "monochromatic structures"), and it shows that Ramsey-type theorems cannot be generalised arbitrarily. However, concerning RAMSEY'S THEOREM, we can ask for a "nice" family $\mathscr{F}$ of infinite subsets of $\mathbb{N}$, such that for every colouring of the n-element subsets of $\mathbb{N}$ with finitely many colours, there exists a homogeneous set in the family $\mathscr{F}$, where an infinite set $x \subseteq \mathbb{N}$ is called *homogeneous* if all n-element subsets of x have the same colour. Now, "nice" could mean "as small as possible" but also "is an ultrafilter." In the former case, this leads to the *homogeneous number* $\mathfrak{hom}$, which is the smallest cardinality of a family $\mathscr{F}$ which contains a homogeneous set for every 2-colouring of the 2-element subsets of $\mathbb{N}$. One can show that $\mathfrak{hom}$ is uncountable and—like for the reaping number—that the axioms of Set Theory do not decide whether or not $\mathfrak{hom}$ is equal to $\mathfrak{c}$ (see Chapter 18). The latter case, where "nice" means "is an ultrafilter," leads to so-called *Ramsey ultrafilters*. It is not difficult to show that Ramsey ultrafilters exist if one assumes CH (see Chapter 11), but on the other hand, the axioms of Set Theory alone do not imply the existence of Ramsey ultrafilters (see PROPOSITION 26.23). A somewhat anti-Ramsey-type question would be to ask how many 2-colourings of the 2-element subsets of $\mathbb{N}$ we need to make sure that no single infinite subset of $\mathbb{N}$ is almost homogeneous for all these colourings, where a set H is called *almost homogeneous* if there is a finite set K such that $H \setminus K$ is homogeneous. This question leads to the *partition number* $\mathfrak{par}$. Again, $\mathfrak{par}$ is uncountable and the axioms of Set Theory do not decide whether or not $\mathfrak{par}$ is equal to $\mathfrak{c}$ (see, for example, Chapter 18).

RAMSEY'S THEOREM, as well as Ramsey Theory in general, play an important role throughout this book. For example, in most chapters of Part II we shall meet—sometimes unexpectedly—RAMSEY'S THEOREM in one form or the other.

NOTES

Gioseffo Zarlino. All citations of Zarlino (1517–1590) are taken from Part Three of his book entitled *Le Istitutioni Harmoniche* (*cf.* [1]). This section of Zarlino's *Institutioni* is concerned primarily with the art of counterpoint, which is, according to Zarlino, *the concordance or agreement born of a body with diverse parts, its various melodic lines accommodated to the total composition, arranged so that voices are separated by commensurable, harmonious intervals.* The word "counterpoint" presumably originated at the beginning of the 14th century and was derived from "punctus contra punctum," *i.e.*, point against point or note against note. Zarlino himself was an Italian music theorist and composer. While he composed a number of masses, motets and madrigals, his principal claim to fame is as a music theorist: For example, Zarlino was ahead of his time in proposing that the octave should be divided into twelve equal semitones, that is to say, he advocated a practice in the 16th century which was universally adopted three centuries later. He also advocated equal temperament for keyboard instruments and just intonation for unaccompanied vocal music and strings—a system which has been successfully practised up to the present day. Furthermore, Zarlino arranged the modes in a different order of succession, beginning with the Ionian mode instead of the Dorian mode. This arrangement seems almost to have been dictated by a prophetic anticipation of the change which was to lead to the abandonment of the modes in favour of a newer tonality, for his series begins with a form which corresponds exactly with our modern major mode and ends with the prototype of the descending minor scale of modern music. (For the terminology of music theory we refer the interested reader to Benson [2].)

Zarlino's most notable student was the music theorist and composer Vincenzo Galilei, the father of Galileo Galilei.

König's Lemma and Ramsey's Theorem. A proof of König's Lemma can be found in König's book on Graph Theory [3, VI, §2, Satz 6], where he called the result *Unendlichkeitslemma*. As a first application of the *Unendlichkeitslemma* he proved the following theorem of de la Vallée Poussin: *If E is a subset of the open unit interval $(0,1)$ which is closed in $\mathbb{R}$ and I is a set of open intervals covering E, then there is a natural number n, such that if one partitions $(0,1)$ into 2^n intervals of length 2^{-n}, each of these intervals containing a point of E is contained in an interval of I.* Using the *Unendlichkeitslemma*, König also showed that VAN DER WAERDEN'S THEOREM is equivalent to the following statement: *If the positive integers are finitely coloured, then there are arbitrarily long monochromatic arithmetic progressions.* In a similar way we will use König's Lemma to derive the FINITE RAMSEY THEOREM from RAMSEY'S THEOREM (*cf.* COROLLARY 4.3).

At first glance, König's Lemma and RAMSEY'S THEOREM seem to be quite unrelated statements. In fact, König's Lemma is a proper (but rather weak) choice principle, whereas RAMSEY'S THEOREM is a very powerful combinatorial tool. However, as we shall see in Chapter 6, RAMSEY'S THEOREM can also be considered as a proper choice principle which turns out to be even stronger than König's Lemma (see THEOREM 6.14).

REFERENCES

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2. DAVID J. BENSON, ***Music: a mathematical offering***, Cambridge University Press, Cambridge, 2007.
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