1.4 Real numbers: filling the gaps

We consider the set of **real numbers**, denoted by $\mathbb{R}$, as the points on an infinite straight line – called the **real line** – on which we have fixed two different points, say "0" and "1". We consider the distance between 0 and 1 as one unit, and therefore, the points 0 and 1 define a measure on the real line. This measure has also a direction, namely from 0 to 1, denoted $\vec{01}$, thus, the distance between two points can also be negative. Further, we identify a point r on the real line with its distance to 0 (with respect to the measure $\vec{01}$) and this distance is called a real number.

For any rational number $\frac{p}{q} \in \mathbb{Q}$ there is a point on the line which represents $\frac{p}{q}$, so, $\mathbb{Q} \subseteq \mathbb{R}$ (each rational number is also a real number).

A real number which is not a rational number is called **irrational**.

Are there irrational numbers?

The answer is YES, and in fact, there are even "much more" irrational numbers than rational numbers. Moreover, if we would be able to pick any number from the real line, then the likelihood of the number we picked beeing rational is 0. Of course, we will not prove this, but we will see that irrational numbers exist, *e.g.*, $\sqrt{2}$:

First we have to show that $\sqrt{2}$ is a real number: Take a right-angled triangle with two sides each of length 1, then by Pythagoras' theorem, the third side is of length $\sqrt{2}$. So, $\sqrt{2}$ is the length of a line, and hence, by definition, $\sqrt{2}$ is a real number.

Now, assume towards a contradiction that $\sqrt{2}$ is rational, so, assume that $\sqrt{2} = \frac{p}{q}$ for some positive $p, q \in \mathbb{N}$.

We may assume that not both numbers p and q are even. (Why? If p and q are both even, we can divide p as well as q by 2. Of course, $\frac{(\frac{p}{2})}{(\frac{q}{2})} = \frac{p}{q} = \sqrt{2}$. If $\frac{p}{2}$ and $\frac{q}{2}$ are both still even, we divide $\frac{p}{2}$ as well as $\frac{q}{2}$ by 2. Doing this again and again, we end up with a rational number $\frac{p'}{q'} = \sqrt{2}$, where not both numbers p' and q' are even.)

First, let us assume that p is even, and therefore, q is odd. Since p is even, $p = 2m$ (for some $m \in \mathbb{N}$). Since $\frac{p}{q} = \sqrt{2}$, we get

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} = \frac{(2m)^2}{q^2} = \frac{4m^2}{q^2}\,.$$

Hence, we get $2q^2 = 4m^2$, which implies $q^2 = 2m^2$, and therefore, *q^2 is even*. On the other hand, since q is odd and "odd"·"odd"="odd", also *q^2 is odd*, which is a contradiction.

Now, if p is odd, then also p^2 *is odd*. Again, assuming $\sqrt{2} = \frac{p}{q}$ gives us $\frac{p^2}{q^2} = 2$, and therefore $p^2 = 2q^2$, thus, p^2 *is even*, which is a contraction.

So, since there are irrational points on the real line, there are some gaps between the rational numbers and one could ask how large these gaps are. In fact, it is quite easy to see that between any two real numbers, say r and s where $r < s$, there are (infinitely) many rational numbers. (To see this, let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \frac{s-r}{4}$ and consider the set of rational numbers $\{0, \pm\frac{1}{n}, \pm\frac{2}{n}, \pm\frac{3}{n}, \ldots\}$. Then at least 4 points of this set lie between r and s. Thus, since 4 can be replaced by any number and r, s were arbitrary, there are infinitely many rational numbers between any two real numbers.) Having this picture in mind, one says that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Powers and roots

For positive natural numbers $n, m \in \mathbb{N}$ and $x \in R$ we have:

(a) $x^n \cdot y^m = x^{(n+m)}$

(b) $\big(x^n\big)^m = x^{(n\cdot m)}$

Now, we apply these two rules to define $x^{\frac{p}{q}}$ for any positive $x \in \mathbb{R}$ and any rational number $\frac{p}{q} \in \mathbb{Q}$.

First we define x^0: $x^n = x^{(n+0)} \underset{\substack{\uparrow \\ \text{by (a)}}}{=} x^n \cdot x^0$, and so, dividing both sides by x^n we get

$$1 = x^0 .$$

In the following, let $\frac{p}{q} \in \mathbb{Q}$ be a rational number and let $x \in \mathbb{R}$ be a non-negative real number. Let us define first $x^{\frac{1}{q}}$:

$$\big(x^{\frac{1}{q}}\big)^q \underset{\substack{\uparrow \\ \text{by (b)}}}{=} x^{q\frac{1}{q}} = x^1 = x\,,$$

which says that $\underbrace{x^{\frac{1}{q}} \cdot x^{\frac{1}{q}} \cdot \ldots \cdot x^{\frac{1}{q}}}_{q\text{-times}} = x$, and therefore, $x^{\frac{1}{q}} = \sqrt[q]{x}$. Hence, in general we get

$$x^{\frac{p}{q}} = \sqrt[q]{x^p} = \Big(\sqrt[q]{x}\Big)^p .$$

Let us also define $x^{-\frac{p}{q}}$:

$$x^{-\frac{p}{q}} \cdot x^{\frac{p}{q}} \underset{\substack{\uparrow \\ \text{by (a)}}}{=} x^{-\frac{p}{q}+\frac{p}{q}} = x^0 = 1\,,$$

thus,

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}\,.$$

Until now we have defined x^a for any non-negative real number $x \in \mathbb{R}$ and any rational number $a \in \mathbb{Q}$, and later we will define it also for irrational real numbers $a \in \mathbb{R}$. (We could also do it now by using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that for $0 < a < b$ we have $x^a < x^b$ (for $x > 1$) and $x^a > x^b$ (for $0 < x < 1$), but later, with the exponential function, we will get it for free.)

The exponential function

Before we define the exponential function, we have to introduce some notation.

For a positive natural number $n \in \mathbb{N}$, define $n! := 1 \cdot 2 \cdot \ldots \cdot n$, called ***n* factorial**. Further define $0! := 1$. For example, $1! = 1$, $3! = 1 \cdot 2 \cdot 3 = 6$, $6! = 1 \cdot \ldots \cdot 6 = 720$, and we have $n! = (n-1)! \cdot n$.

Assume we have a sequence of real numbers, say $s_0, s_1, s_2, \ldots$, then

$$\sum_{i=k} \ell s_i = (s_k + s_{k+1} + \ldots + s_\ell)$$

where "$\sum$" is the capital Greek letter "S", called "sigma", and stands for "sum"; and

$$\sum_{i=k} \infty s_i = (s_k + s_{k+1} + \ldots)$$

where "∞" stands for "infinity".

Some examples:

1. Let $s_n := \frac{1}{2^n}$, thus, $s_0 = \frac{1}{2^0} = 1$, $s_1 = \frac{1}{2^1} = \frac{1}{2}$, $s_2 = \frac{1}{2^2} = \frac{1}{4}, \ldots, s_{10} = \frac{1}{2^{10}} = \frac{1}{1024}, \ldots$ For example, we have the following:

$$\begin{aligned} \textstyle\sum_{i=2}^4 &= (s_2 + s_3 + s_4) = \left(\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}\right) = \tfrac{7}{16} \\ \textstyle\sum_{i=0}^0 &= (s_0) = 1 \\ \textstyle\sum_{i=0}^\infty &= (s_0 + s_1 + \ldots) = \left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\right) = 2 \quad \text{(will be proved later)} \end{aligned}$$

2. Further, we have $\sum\limits_{n=0}^{\infty} \frac{1}{n+1} = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots) = \infty$ (will be proved later). So, *not* every infinite sum (also called series) where the summands tends to 0 (which means get smaller and smaller) if finite!

3. On the other, an alternating series (a series the terms of which are alternately positive and negative) where the summands tend to 0 is *always* finite.

   For example, $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \ldots\big) \approx 0.6931471\ldots$

Now, we are prepared to define the (real-valued) *function* exp as follows:

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

So, $\exp(z) = \big(\frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots\big) = \big(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots\big)$ and for any z, $\exp(z)$ is a series.

One can show that $\exp(z)$ has the same features as a power function, *e.g.*, $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$ (like $x^{(n+m)} = x^n \cdot x^m$), and so, there must be a real number e such that $\exp(z) = e^z$. What is e? Since $e = e^1 = \exp(1)$ we get

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$$

The number e is called **Euler number** and is named after Leonhard Euler (1707–1783), one of the most prolific mathematicians of all time.

What is the value of e? To get a good approximation, we consider the first few terms of the series of e^{-1}, which is an alternating series:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \underbrace{\overbrace{\underbrace{\overbrace{\underbrace{\overbrace{\underbrace{1}_{1 > e^{-1}} - 1}^{0 < e^{-1}} + \frac{1}{2}}_{\frac{1}{2} > e^{-1}} - \frac{1}{6}}^{\frac{1}{3} < e^{-1}} + \frac{1}{24}}_{\frac{3}{8} > e^{-1}} - \frac{1}{120}}^{\frac{11}{30} < e^{-1}} + \frac{1}{720}}_{\frac{53}{144} > e^{-1}} \mp \ldots$$

Thus, $\frac{11}{30} < \frac{1}{e} < \frac{53}{144}$ and therefore, $\frac{30}{11} < e < \frac{144}{53}$. In fact,

$$e \approx 2.7182818284590455348848081484902650117874145507 81\ldots$$

Now, we like to show that e is irrational, but first we show the simple fact that for any positive natural number p, $p! \cdot e^{-1} \notin \mathbb{N}$:

$$p! \cdot e^{-1} = p!\big(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \ldots\big) = \underbrace{p! - p! + \frac{p!}{2} - \ldots \pm \frac{p!}{p!}}_{\substack{\in \mathbb{N},\text{ since } \frac{p!}{n!} \in \mathbb{N} \\ \text{whenever } n \leq p}} \underbrace{\overset{\substack{\text{depending on the parity of } p \\ \downarrow}}{\mp} \frac{p!}{(p+1)!} \pm \ldots}_{\substack{\notin \mathbb{N},\text{ since it is an alter-} \\ \text{nating series and the first} \\ \text{term has absolute value } < 1}}$$

e is irrational: Assume towards a contradiction, that e is rational, so, $e = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. If $e = \frac{p}{q}$, then $\frac{1}{e} = e^{-1} = \frac{q}{p} = \frac{q \cdot (p-1)!}{p \cdot (p-1)!} = \frac{q \cdot (p-1)!}{p!}$, and therefore, $p! \cdot e^{-1} = q \cdot (p-1)! \in \mathbb{N}$, which contradicts the previous fact that $p! \cdot e^{-1} \notin \mathbb{N}$.