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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 1, 2002

The pdf-file you may download from
`http://www.math.berkeley.edu/~halbeis/4students/gt.html`

Please hand in your solutions (stapled together with your full name on the first page) at the lecture on Monday, 4 February 2002.

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1. Let $\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$, where $\mathbb{Q}$ denotes the set of all rational numbers.
   (a) Show that $(\mathbb{Q}^*, \cdot)$ has exactly one neutral element.
   (b) Show that in $(\mathbb{Q}^*, \cdot)$, every element has exactly one inverse.

2. Let "$\circ$" be a binary associative operation on the set S, so, for any $x, y, z \in S$ we have $x \circ (y \circ z) = (x \circ y) \circ z$.

   Show that for any $a, b, c, d \in S$ we have:

   $$(a \circ b) \circ (c \circ d) = \big(a \circ (b \circ c)\big) \circ d$$

3. Show that the binary operation

   $$\begin{array}{rccc} \sharp: & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & (x, y) & \longmapsto & x \cdot (y+1) \end{array}$$

   is neither commutative nor associative.

4. Let $A = \{1, 2, 3, 4, 6, 12\}$ and let

   $$\begin{array}{rccc} \star: & A \times A & \longrightarrow & A \\ & (a, b) & \longmapsto & \gcd(a, b) \end{array}$$

   where $\gcd(a, b)$ denotes the greatest common divisor of a and b.
   (a) Show that the operation "$\star$" is commutative and associative.
   (b) Show that $(A, \star)$ has a neutral element.
   (c) Why is $(A, \star)$ not a group?

5. Let "$\bullet$" be the binary operation on $\mathbb{Q}^*$ defined as follows:

   $$\begin{array}{rccc} \bullet: & \mathbb{Q}^* \times \mathbb{Q}^* & \longrightarrow & \mathbb{Q}^* \\ & (p, q) & \longmapsto & p \cdot (q+q) \end{array}$$

   Show that $(\mathbb{Q}^*, \bullet)$ is an abelian group.