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Group Theory (Module 210PMA208)

Department of Pure Mathematics

Week 3, 2002

You may download the ps-file from
`http://www.math.berkeley.edu/~halbeis/4students/gt.html`

Please hand in your solutions (stapled together with your full name on the first page) at the lecture on Monday, 18 February 2002.

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11. Let $(G, *_G)$ and $(H, *_H)$ be groups and let the operation "$\circ$" on $G \times H$ be defined as follows:
$$\langle g_1, h_1\rangle \circ \langle g_2, h_2\rangle := \langle g_1 *_G g_2, h_1 *_H h_2\rangle$$
Show that $(G \times H, \circ)$ is a group and that it is abelian iff G and H are abelian.

12. Let H_1 and H_2 be subgroups of G. Show that $H_1 \cap H_2$ is a subgroup of G.

13. Show that every subgroup H of $(\mathbb{Z}, +)$ is of the form $H = m\mathbb{Z}$ (for some $m \in \mathbb{N}$).

14. The additive groups $4\mathbb{Z}$, $12\mathbb{Z}$ and $27\mathbb{Z}$ are subgroups of $(\mathbb{Z}, +)$, and so are (by Question 12) the groups $4\mathbb{Z} \cap 12\mathbb{Z}$, $4\mathbb{Z} \cap 27\mathbb{Z}$ and $12\mathbb{Z} \cap 27\mathbb{Z}$. By Question 13 we know that for some $m_1, m_2, m_3 \in \mathbb{N}$
$$\begin{aligned} 4\mathbb{Z} \cap 12\mathbb{Z} &= m_1\mathbb{Z}, \\ 4\mathbb{Z} \cap 27\mathbb{Z} &= m_2\mathbb{Z}, \\ 12\mathbb{Z} \cap 27\mathbb{Z} &= m_3\mathbb{Z}. \end{aligned}$$
(a) Compute m_1, m_2 and m_3.
(b) Prove that for any $a, b \in \mathbb{N}$, $a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a, b)\mathbb{Z}$, where $\operatorname{lcm}(a, b)$ is the least common multiple of a and b.
(c) Show that $4\mathbb{Z} \cup 27\mathbb{Z}$ is not a subgroup of $(\mathbb{Z}, +)$.

15. Let T, C and D be the *tetrahedron-group*, the *cube-group* and the *dodecahedron-group* respectively.
(a) Show that $T < C$.
(b) Show that $T < D$.
(c) Show that $C \not< D$.

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Office hours: Monday 3 pm–4 pm, Friday 1 pm–2 pm