Dedekind, functions on natural numbers, and large cardinals (Dedekind and the problem of categoricity)

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Richard Dedekind 1831—1916

“Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.”

In 1893 Dedekind gave an axiomatisation of the structure of natural numbers, namely of the structure:

$$\mathbb{N} = \{0, 1, 2, \ldots\}, s(n) = n + 1$$

Dedekind's axiomatization

- $s(n) \neq 0$
- $s(n) = s(m) \rightarrow n = m$
- Induction Principle: If A is a set on natural numbers such that A contains 0 and is closed under the function s, then $A = \mathbb{N}$.

Arithmetic from successor

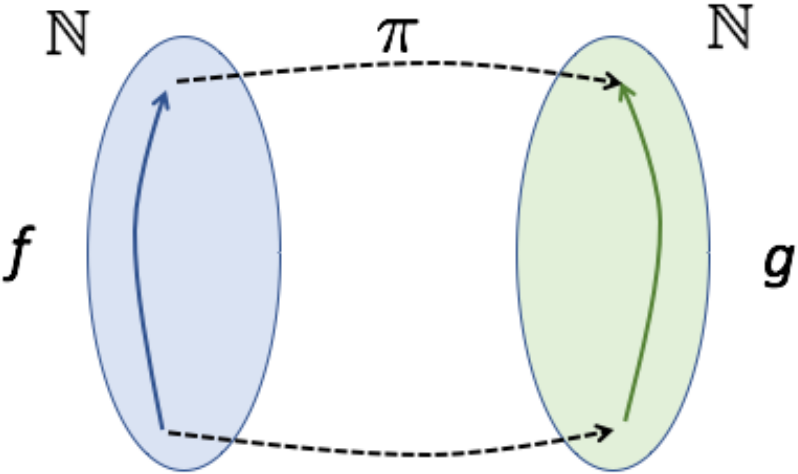
- $n + 0 = n$
- $n + s(m) = s(n + m)$
- $n \cdot 0 = 0$
- $n \cdot s(m) = n \cdot m + n$

Categorical axiomatisation

- Natural numbers together with some functions is called a structure.
- $(\mathbb{N}, s, 0)$, where $s(n) = n + 1$
- $(\mathbb{N}, +, \cdot)$
- $(\mathbb{N}, f)$ where f is a function on $\mathbb{N}$
- There are also other kinds of structures.

- Structures $(\mathbb{N}, f)$ and $(\mathbb{N}, g)$ are isomorphic[1] ("have the same form") if there is bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(f(n)) = g(\pi(n))$ for all n.
- $(\mathbb{N}, f, a)$ and $(\mathbb{N}, g, b)$: Add $\pi(a) = b$.
- An axiomatization is categorical if all structures satisfying the axioms are isomorphic.
- Dedekind's axiomatization is categorical.

[1] From the Ancient Greek: isos "equal", and morphe "form" or "shape".



ℕ
π
ℕ
f
g

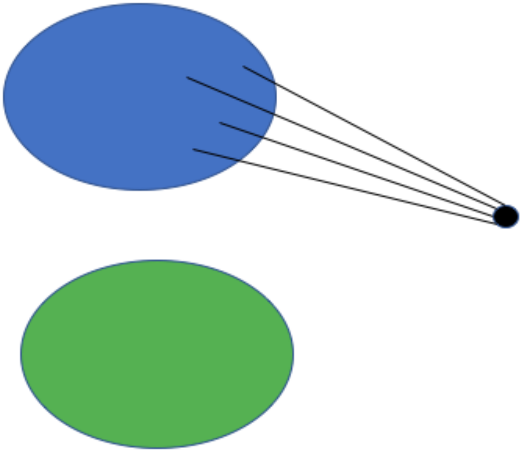
Dedekind's axiomatization is categorical

- Suppose $(\mathbb{N}, s^*, 0^*)$ is another structure that satisfies the axioms.
- Let inductively $(n+1)^*$ be $s^*(n^*)$.
- Let A be the set of $a \in \mathbb{N}$ such that $a = n^*$ for some n.
- Now $0^* \in A$, and A is closed under the function s^*.
- By the Induction Axiom, $A = \mathbb{N}$.
- Hence $n \mapsto n^*$ is an isomorphism between $(\mathbb{N}, s, 0)$ and $(\mathbb{N}, s^*, 0^*)$.

- What if the successor function is replaced by some other function?
- New structures emerge.
- Do they have categorical axiomatizations?

- $(\mathbb{N}, f)$, where $f(n)$ is the n^{th} decimal of π.
- $(\mathbb{N}, f)$, where $f(n)$ is zero or one determined by tossing a coin.
- ...

- $(\mathbb{N}, E)$, where there is an E-edge between n and m according to a fair coin toss.
- Extension axiom: If V and W are finite disjoint sets then there is a vertex with an edge to everything in V and to nothing in W.
- This (with a few other axioms) is a categorical axiomatization of $(\mathbb{N}, E)$.



- A. Fraenkel asked in 1923 whether every structure on the natural numbers has a categorical axiomatization.
- A positive answer was conjectured by R. Carnap in the late 20s.

Abraham Fraenkel 1891—1965

Rudolph Carnap 1891—1970

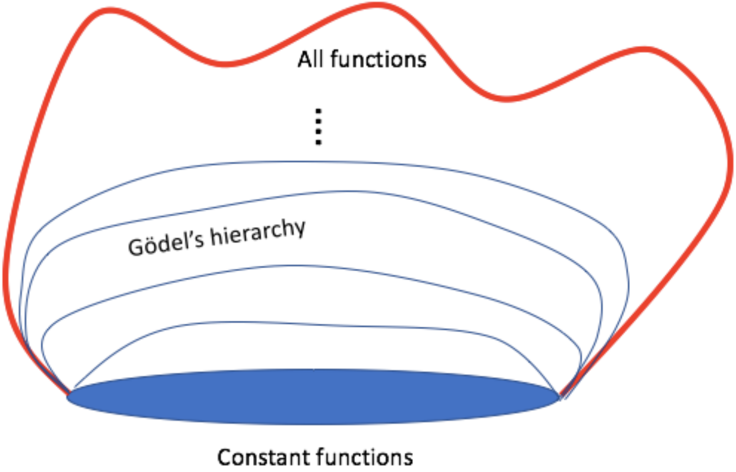
Surprise!

Ajtai proved in 1978 that the conjecture is independent of the axioms of set theory, which means more or less that the conjecture is really hard to solve, and some people think that it means there is no answer.

Miklos Ajtai 1946—

- Let us look at the set P of all functions on $\mathbb{N}$.
- In 1938 Kurt Gödel introduced a Ramified Hierarchy inside P: So-called constructible sets.
- The assumption, called Axiom of Constructibility, that constructible sets cover P (and in fact the entire universe) completely, is consistent with the axioms of set theory, i.e. no contradiction can be derived from it, unless there is a contradiction already in set theory itself.

Kurt Gödel 1906—1978

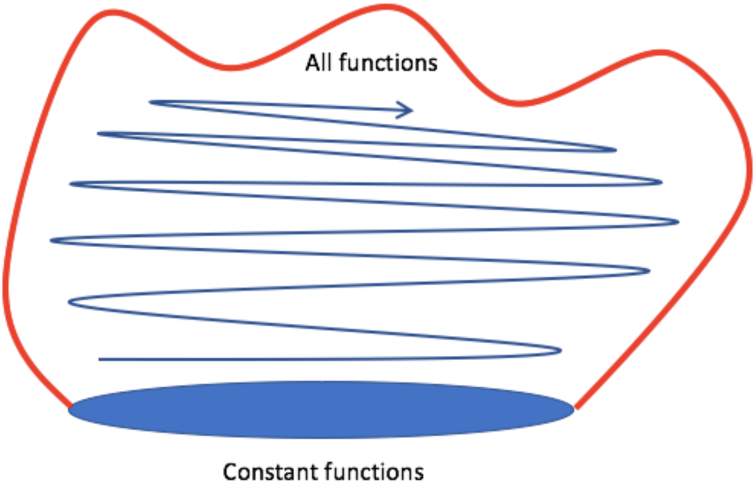

All functions
⋮
Gödel's hierarchy
Constant functions

- Assume the Axiom of Constructibility.
- There is a definable well-ordering $<_L$ of P:

First function, second function, third function, etc ...

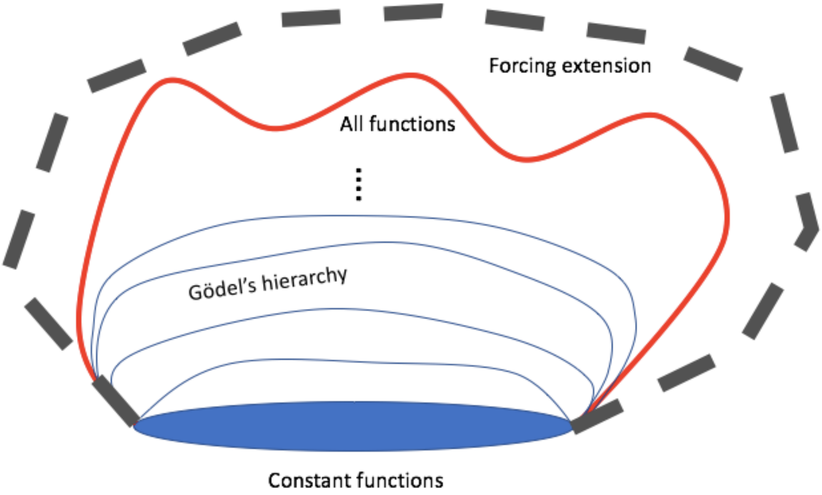
- Given $(\mathbb{N}, f)$, the axioms that characterize $(\mathbb{N}, f)$ categorically say that f is the $<_L$-smallest f^* such that $(\mathbb{N}, f^*)$ is isomorphic to $(\mathbb{N}, f)$.
- *(Sounds circular, but is not!)*

A well-ordering of P



In 1963 Paul Cohen introduced an alternative to Gödel's Axiom of Constructibility: forcing.

Paul Cohen 1934—2007.


Forcing extension
All functions
Gödel's hierarchy
Constant functions

The basic concept of forcing is a set of finite pieces of information about a potential function on $\mathbb{N}$. Technically, a piece of information is a finite set of entries from the table of values of the function, such as

$$f(1,1) = 49$$
$$f(50,0) = 1$$
$$f(2016,201) = 5599$$

- Cohen's forcing yields a function f such that any statement about f is "forced" by some finite piece of information about f.
- We call such f generic, because you can only say generic or very common things about f.
- Cannot insist $f(n+1) = f(n) + 1$ for all n.
- Can insist $f(n+1) = f(n) + 1$ for some n, and also $f(n+1) \neq f(n) + 1$ for some other n.

- Suppose f is generic - obtained by Cohen's forcing method.
- Let $\mathcal{F}$ be the (countable) set of functions g which agree with f apart from a finite number of arguments.
- Let $R(g, n, m)$ be the relation $g(n) = m$.
- The structure we cannot axiomatize categorically consists of $\mathbb{N}$, $\mathcal{F}$ and the relation R.

Large cardinals

- Gödel's method gives positive answer.
- Cohen's method gives negative answer.
- Axioms of set theory cannot solve our conjecture.
- We need new axioms.
- So-called Large Cardinal Axioms lead to a negative answer to the conjecture.

- S. Ulam 1930: The cardinality κ of a set A is measurable if there is a non-trivial two-valued κ-additive measure on the set of all subsets of A.
- Non-trivial means that singletons get measure 0 and the set A gets measure 1.
- Axiom of Choice implies there is such a measure on the subsets of $\mathbb{N}$.
- Axiom of Measurable Cardinals: There is a measurable cardinal bigger than the cardinality of $\mathbb{N}$.

Stanisław Ulam 1909—1984.

L. Keskinen 2011:

- Assume large cardinals. (We need actually cardinals bigger than measurable ones).
- In 1988 Hugh Woodin showed that the theory of $L(\mathbb{R})$ (a canonical extension of Gödel's constructible universe by inclusion of all the reals) cannot be changed by forcing.
- Our conjecture can be expressed as a property of $L(\mathbb{R})$.
- We can force the conjecture to fail with the forcing method.
- Hence the Conjecture must fail also without the forcing.
- Fraenkel's problem has been solved!

Hugh Woodin 1955—

Thus Dedekind's axiomatisation of the successor function is really particular and not something all functions on natural numbers enjoy, but to prove that this is so, one has to go deeply into foundations of mathematics.

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