A unique Q-point and infinitely many near-coherence classes of ultrafilters

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Abstract. We show that in the model obtained by iteratively pseudo-intersecting a Ramsey ultrafilter via a length-ω_2 countable support iteration of restricted Mathias forcing over a ground model satisfying CH, there is a unique Q-point up to isomorphism. In particular, it is consistent that there is only one Q-point while there are $2^{\mathfrak{c}}$-many near-coherence classes of ultrafilters.

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1 Introduction

Throughout this paper, read *ultrafilter* as *non-principal ultrafilter on* ω. For $x \subseteq \omega$, we denote by $[x]^\omega$ the set of infinite subsets of x and by $\text{fin}(x)$ the set of finite subsets of x.

Recall that an ultrafilter E is a *Q-point* if and only if for every interval partition $\{[k_i, k_{i+1}) : i \in \omega\}$ of ω, there exists some $x \in E$ such that $\forall i \in \omega : |x \cap [k_i, k_{i+1})| \leq 1$. Furthermore, an ultrafilter $\mathcal{U}$ is a *Ramsey ultrafilter* if and only if the *Maiden* has no winning strategy in the *ultrafilter game for $\mathcal{U}$*, played between the Maiden and *Death*:

Definition 1.1. *Let $\mathcal{U}$ be an ultrafilter. The ultrafilter game for $\mathcal{U}$ proceeds as follows: The Maiden opens the game and plays some $y_0 \in \mathcal{U}$. Death responds by playing some $n_0 \in y_0$. In the $(k+1)$-th move, the Maiden having played $y_0 \supseteq y_1 \supseteq \ldots \supseteq y_k$, and Death*

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having played $n_0 < n_1 < ... < n_k$, the Maiden plays some $y_{k+1} \in [y_k]^\omega \cap \mathcal{U}$, and Death responds by playing some $n_{k+1} \in y_{k+1}$, $n_{k+1} > n_k$.

Death wins if and only if $\{n_i : i \in \omega\} \in \mathcal{U}$.

It is well-known that every Ramsey ultrafilter is a Q-point. Canjar [5] showed that the existence of $2^\mathfrak{c}$-many Ramsey ultrafilters follows from the assumption $cov(\mathcal{M}) = \mathfrak{c}$. The weaker assumption $cov(\mathcal{M}) = \mathfrak{d}$ implies the existence of $2^\mathfrak{c}$ Q-points, as was shown by Millán [6]. It is well-known that in the Mathias model – the model obtained by a length-ω_2 countable support iteration of unrestricted Mathias forcing over a ground model satisfying CH – there are no Q-points (see [1, Proposition 26.23]). In fact, the Mathias model contains no *rapid* ultrafilters, where an ultrafilter E is rapid iff for every $f \in {}^\omega\omega$ there exists some $x \in E$ such that $\forall n \in \omega : |x \cap f(n)| \leq n$ (note that every Q-point is rapid). It follows that both the Mathias model and the model considered in this paper satisfy $cov(\mathcal{M}) = \omega_1 < \mathfrak{d} = \mathfrak{c} = \omega_2$.

In contrast to the Mathias model, our model contains $2^\mathfrak{c}$-many rapid ultrafilters: It follows from an observation of Millán [6, page 222] that the existence of a single rapid ultrafilter E implies the existence of $2^\mathfrak{c}$ of them, by considering the products $U \otimes E$ for different ultrafilters U.[2]

While the consistency of the non-existence of Q-points is a well-established fact with a variety of witnesses apart from the Mathias model[3], the construction of models containing only 'few' Q-points seems to have received less attention. However, such models do arise naturally as models containing only few near-coherence classes of ultrafilters[4]: Indeed, Mildenberger [10] has constructed models with exactly two and exactly three near-coherence classes, and it is easy to see that these contain exactly one and exactly two Q-points, respectively: In her model with exactly two near-coherence classes, one class contains a Ramsey ultrafilter, while the other class contains an ultrafilter that is ω_1-generated. Hence, this latter class cannot contain a Q-point, since her models satisfy $\mathfrak{d} = \mathfrak{c} = \omega_2$ and such a Q-point would thus have to be $<\mathfrak{d}$-generated, which is impossible. Analogously, in Mildenberger's model with exactly three near-coherence classes, two classes are represented by Ramsey ultrafilters, while the third contains an ω_1-generated ultrafilter – giving exactly two Q-points in total.

The construction of models with exactly n near-coherence classes of ultrafilters for various finite $n \geq 4$ would similarly yield the consistency of exactly m Q-points for some $m < n$.[5]

[2] $U \otimes E$ is an ultrafilter on $\omega \times \omega$ defined by $U \otimes E = \{x \subseteq \omega \times \omega : \{n \in \omega : (x)_n \in E\} \in U\}$, where $(x)_n = \{m \in \omega : \langle n, m \rangle \in x\}$.

[3] such as the Laver and Miller models (see [7] and [11], respectively).

[4] Two ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ are *nearly-coherent* iff there is some finite-to-one $f \in {}^\omega\omega$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$, where $f(\mathcal{U}_i) := \{X \subseteq \omega : f^{-1}[X] \in \mathcal{U}_i\}$. Note that two Q-points are nearly-coherent iff they are isomorphic.

[5] The inequality is strict since such a model must satisfy $\mathfrak{u} < \mathfrak{d}$, a result due to Banakh and Blass [9]. Hence, one of the n near-coherence classes contains a $<\mathfrak{d}$-generated ultrafilter and thus no Q-point.

The model considered in this paper is of a different nature, however: It contains only one Q-point while its number of near-coherence classes is $2^{\mathfrak{c}}$, i.e., the model's lack of Q-points is not the consequence of a lack of near-coherence classes. This follows from the fact that dominating reals are added at each of the ω_2-many stages of the iteration, which gives $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$ in the final extension. Since $\mathfrak{b} \leq \mathfrak{u}$ (see Solomon [8]), we have $\mathfrak{u} = \mathfrak{d} = \omega_2$ in our model, and hence there are $2^{\mathfrak{c}}$-many near-coherence classes of ultrafilters by Banakh and Blass [9].

DEFINITION 1.2. *Let $\mathcal{U}$ be a Ramsey ultrafilter. Mathias forcing restricted to $\mathcal{U}$, written $\mathbb{M}_{\mathcal{U}}$, consists of conditions $\langle s, x\rangle \in fin(\omega) \times \mathcal{U}$ with $\max s < \min x$, ordered by*

$$\langle s, x\rangle \leq_{\mathbb{M}_{\mathcal{U}}} \langle t, y\rangle :\iff s \subseteq t \wedge x \supseteq y \wedge t \setminus s \subseteq x.$$

Note that we use the convention that stronger forcing conditions are larger. The forcing notion $\mathbb{M}_{\mathcal{U}}$ clearly satisfies the c.c.c. and is therefore proper. We will need the following additional facts.

FACT 1.3 (e.g., see [1, Theorem 26.3]). *Let $\mathcal{U}$ be a Ramsey ultrafilter. The forcing notion $\mathbb{M}_{\mathcal{U}}$ has the pure decision property, i.e., for any sentence φ in the forcing language and any $\mathbb{M}_{\mathcal{U}}$-condition $\langle s, x\rangle$, there exists $y \in [x]^{\omega} \cap \mathcal{U}$ such that either $\langle s, y\rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \varphi$ or $\langle s, y\rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \neg\varphi$.*

DEFINITION 1.4. *Recall that a forcing notion $\mathbb{P}$ has the Laver property iff for every $\mathbb{P}$-name $\underset{\sim}{g}$ for an element of ${}^{\omega}\omega$ such that there exists $f \in {}^{\omega}\omega \cap \mathbf{V}$ with*

$$\mathbb{P} \Vdash \forall n \in \omega : \underset{\sim}{g}(n) \leq f(n),$$

we have that $\mathbb{P}$ forces that there exists $c : \omega \to fin(\omega)$ in $\mathbf{V}$ with

$$\forall n \in \omega : |c(n)| \leq 2^n \text{ and } \underset{\sim}{g}(n) \in c(n).$$

FACT 1.5 (e.g., see [1, Corollary 26.8]). *Let $\mathcal{U}$ be a Ramsey ultrafilter. The forcing notion $\mathbb{M}_{\mathcal{U}}$ has the Laver property.*

FACT 1.6 (e.g., see [2, Ch. VI, 2.10D]). *The Laver property is preserved under countable support iterations of proper forcing notions.*

2 Result

MAIN THEOREM. *It is consistent that there is a unique Q-point while there are* $2^{\mathfrak{c}}$*-many near-coherence classes of ultrafilters.*

Proof. Assume that the ground model $\mathbf{V}$ satisfies CH. By induction, we define:

(i) A countable support iteration $\mathbb{P}_{\omega_2} := \langle \mathbb{P}_\xi, , \underset{\sim}{Q}_\xi : \xi \in \omega_2 \rangle$ of c.c.c. forcing notions,

(ii) A sequence $\langle \underset{\sim}{\mathcal{U}}_\xi : \xi \in \omega_2 \rangle$, such that

$$\forall \xi \in \omega_2 : \mathbb{P}_\xi \Vdash \text{“}\underset{\sim}{\mathcal{U}}_\xi \text{ is a Ramsey ultrafilter extending } \bigcup_{\iota \in \xi} \underset{\sim}{\mathcal{U}}_\iota\text{”}$$

and $\underset{\sim}{Q}_\xi$ is a $\mathbb{P}_\xi$-name for Mathias forcing restricted to $\underset{\sim}{\mathcal{U}}_\xi$,

Assume that we are in step $\xi \in \omega_2$. Let G_ξ be $\mathbb{P}_\xi$-generic over $\mathbf{V}$ and work in $\mathbf{V}[G_\xi]$. Note that since $\mathbb{P}_\xi$ is a countable support iteration of proper forcing notions that are forced to be of size $\leq \omega_1$, we have $\mathbf{V}[G_\xi] \models \mathsf{CH}$ (e.g., see [3, Theorem 2.12]). For each $\iota \in \xi$, let η_ι be the Mathias real added at stage ι.

If $\xi = \xi' + 1$, $\eta_{\xi'}$ pseudo-intersects $\underset{\sim}{\mathcal{U}}_{\xi'}[G_\xi]$ and we may construct a Ramsey ultrafilter on $\eta_{\xi'}$ using CH (and extend it to ω to obtain $\mathcal{U}_\xi$). Similarly, if ξ is a limit ordinal and $\mathrm{cf}(\xi) = \omega$, we can build $\mathcal{U}_\xi$ on a pseudo-intersection of the tower $\langle \eta_\iota : \iota \in \xi \rangle$. Finally, if $\mathrm{cf}(\xi) = \omega_1$, then $\bigcup_{\iota \in \xi} \underset{\sim}{\mathcal{U}}_\iota[G_\xi]$ is already a Ramsey ultrafilter, since no new reals are added at stage ξ. For the same reason we also have that $\mathcal{U}_{\omega_2} := \bigcup_{\xi \in \omega_2} \underset{\sim}{\mathcal{U}}_\xi[G]$ is a Ramsey ultrafilter in $\mathbf{V}[G]$, where G is $\mathbb{P}_{\omega_2}$-generic over $\mathbf{V}$.

FACT 2.1 (e.g., see [3, Theorem 2.10]). $\mathbb{P}_{\omega_2}$ *is proper and satisfies the* ω_2*-c.c..*

We need to show that $\mathcal{U}_{\omega_2}$ is the only Q-point in $\mathbf{V}[G]$. To see this, assume by contradiction that $\mathbf{V}[G] \models$ “E is a Q-point and not isomorphic to $\mathcal{U}_{\omega_2}$”.

LEMMA 2.2. *There exists* $\delta \in \omega_2$ *such that* $E \cap \mathbf{V}[G_\delta] \in \mathbf{V}[G_\delta]$ *and* $\mathbf{V}[G_\delta] \models$ “$E \cap \mathbf{V}[G_\delta]$ *is a Q-point and not isomorphic to* $\mathcal{U}_\delta$”.

Proof. Fix $\xi \in \omega_2$ and consider names $\underset{\sim}{e}_\xi$, $\underset{\sim}{i}_\xi$, $\underset{\sim}{s}_\xi$, $\underset{\sim}{b}_\xi$ and $\underset{\sim}{f}_\xi$ such that $\mathbb{P}_{\omega_2}$ forces that

(i) “$\underset{\sim}{e}_\xi$ is an enumeration (in ω_1) of $\underset{\sim}{E} \cap \mathbf{V}[\dot{G}_\xi]$”. For each $\alpha \in \omega_1$ and $n \in \omega$ let $\mathcal{E}_{\xi,\alpha,n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding “$n \in \underset{\sim}{e}_\xi(\alpha)$”.

(ii) “$\underset{\sim}{i}_\xi$ is an enumeration (in ω_1) of the set of interval partitions of ω in $\mathbf{V}[\dot{G}_\xi]$”. Note that we may assume that $\underset{\sim}{i}_\xi$ is a $\mathbb{P}_\xi$-name.

(iii) “For all $\alpha \in \omega_1$, $\underset{\sim}{s}_\xi(\alpha)$ is an element of $\underset{\sim}{E}$ that intersects each interval in the interval partition $\underset{\sim}{i}_\xi(\alpha)$ in at most one point”. Let $\mathcal{S}_{\xi,\alpha,n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding “$n \in \underset{\sim}{s}_\xi(\alpha)$”.

(iv) "$\underset{\sim}{b}_\xi$ is an enumeration (in ω_1) of all permutations of ω in $\mathbf{V}[\dot{G}_\xi]$". We may again assume that $\underset{\sim}{b}_\xi$ is a $\mathbb{P}_\xi$-name.

(v) "For all $\alpha \in \omega_1$, $\underset{\sim}{f}_\xi(\alpha)$ is a pair $\mathrm{op}(\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\alpha)$ such that $\underset{\sim}{x}_\alpha$ is in $\underset{\sim}{E}$, $\underset{\sim}{y}_\alpha$ is in $\underset{\sim}{\mathcal{U}}_{\omega_2}$ and $\underset{\sim}{b}_\xi(\alpha)[\underset{\sim}{x}_\alpha]$ is disjoint from $\underset{\sim}{y}_\alpha$". Let $\mathcal{X}_{\xi,\alpha,n} \subseteq \mathbb{P}_{\omega_2}$ be a maximal antichain deciding "$n \in \underset{\sim}{x}_\alpha$", and define $\mathcal{Y}_{\xi,\alpha,n}$ analogously.

By the ω_2-c.c. of $\mathbb{P}_{\omega_2}$, there exists for each $\xi \in \omega_2$ some $\gamma_\xi \in \omega_2$ greater than ξ such that all the above antichains consist of $\mathbb{P}_{\gamma_\xi}$-conditions. Recursively define $\lambda(0) = 0$, $\lambda(\xi+1) = \gamma_{\lambda(\xi)}$ and for limit ordinals $\xi : \lambda(\xi) = \bigcup_{\iota\in\xi} \lambda(\iota)$, for $\xi \leq \omega_1$. Set $\delta := \lambda(\omega_1)$ and consider the extension $\mathbf{V}[G_\delta]$. Since $\mathrm{cf}(\delta) = \omega_1$, we have that $E \cap \mathbf{V}[G_\delta] = \bigcup_{\iota\in\omega_1} E \cap \mathbf{V}[G_{\lambda(\iota)}]$, and since each $E \cap \mathbf{V}[G_{\lambda(\iota)}]$ is an element of $\mathbf{V}[G_\delta]$ by (i), $E \cap \mathbf{V}[G_\delta]$ is an element of $\mathbf{V}[G_\delta]$ (and an ultrafilter). Furthermore, any interval partition of ω in $\mathbf{V}[G_\delta]$ already appears in some $\mathbf{V}[G_{\lambda(\iota)}]$, $\iota \in \omega_1$, where it equals $\underset{\sim}{i}_{\lambda(\iota)}[G_{\lambda(\iota)}](\alpha)$ for some $\alpha \in \omega_1$. Since $\underset{\sim}{s}_{\lambda(\iota)}[G_\delta](\alpha) \in E \cap \mathbf{V}[G_\delta]$, we obtain that $E \cap \mathbf{V}[G_\delta]$ is a Q-point. Finally and analogously, any permutation of ω in $\mathbf{V}[G_\delta]$ already appears in $\mathbf{V}[G_{\lambda(\iota)}]$ for some $\iota \in \omega_1$ and hence there are witnesses $\underset{\sim}{x}_\alpha[G_\delta] \in E \cap \mathbf{V}[G_\delta]$ and $\underset{\sim}{y}_\alpha[G_\delta] \in \mathcal{U}_{\omega_2} \cap \mathbf{V}[G_\delta] = \mathcal{U}_\delta$ witnessing that $E \cap \mathbf{V}[G_\delta]$ and $\mathcal{U}_\delta$ are not isomorphic. ⊣

We now designate $\mathbf{V}[G_\delta]$ as the new ground model and rename the Q-point $E \cap \mathbf{V}[G_\delta]$ to E and the Ramsey ultrafilter $\mathcal{U}_\delta$ to $\mathcal{U}$. Note that by the Factor-Lemma (e.g., see [4, Theorem 4.6]), the quotient $\mathbb{P}_{\omega_2}/G_\delta$ is again isomorphic to a countable support iteration of restricted Mathias forcings. In particular, by Facts 1.5 and 1.6, $\mathbb{P}_{\omega_2}/G_\delta$ is isomorphic to the two-step iteration $\mathbb{M}_\mathcal{U} * \underset{\sim}{R}$, where $\mathbb{M}_\mathcal{U} \Vdash$ "$\underset{\sim}{R}$ has the Laver property".

It remains to show the following.

PROPOSITION 2.3. *Let E be a Q-point and $\mathcal{U}$ a Ramsey ultrafilter such that E and $\mathcal{U}$ are not isomorphic. Let $\mathbb{M}_\mathcal{U}$ be Mathias forcing restricted to $\mathcal{U}$ and let $\underset{\sim}{R}$ be a $\mathbb{M}_\mathcal{U}$-name such that $\mathbb{M}_\mathcal{U} \Vdash$ "$\underset{\sim}{R}$ has the Laver property". Then $\mathbb{M}_\mathcal{U} * \underset{\sim}{R} \Vdash$ "E cannot be extended to a Q-point".*

Proof. It suffices to show that if $\langle p, \underset{\sim}{q}\rangle \in \mathbb{M}_\mathcal{U} * \underset{\sim}{R}$ and a $\mathbb{M}_\mathcal{U} * \underset{\sim}{R}$-name $\underset{\sim}{a}$ for a strictly increasing element of ${}^\omega\omega$ are such that

$$\langle p, \underset{\sim}{q}\rangle \Vdash_{\mathbb{M}_\mathcal{U} * \underset{\sim}{R}} \forall n \in \omega : \underset{\sim}{a}(n) \in (\dot{\eta}(n-1), \dot{\eta}(n)],$$

then there exists some $v \in E$ and some $\langle \bar{p}, \underset{\sim}{\bar{q}}\rangle$ greater than $\langle p, \underset{\sim}{q}\rangle$ such that

$$\langle \bar{p}, \underset{\sim}{\bar{q}}\rangle \Vdash_{\mathbb{M}_\mathcal{U} * \underset{\sim}{R}} |\mathrm{range}(\underset{\sim}{a}) \cap v| < \omega.$$

Recall that $\dot{\eta}$ is the canonical $\mathbb{M}_\mathcal{U}$-name for the Mathias real (assume $\mathbb{M}_\mathcal{U} \Vdash \dot{\eta}(-1) = -\infty$).

Note that $\underset{\sim}{a}$ is forced by $\mathbb{M}_{\mathcal{U}}$ to be dominated by $\dot{\eta}$. Hence, by the Laver property of $\underset{\sim}{R}$, there exists a $\mathbb{M}_{\mathcal{U}}$-name $\underset{\sim}{c}$ for a function from ω to $\mathrm{fin}(\omega)$ and some $\langle p', \underset{\sim}{q}' \rangle \geq_{\mathbb{M}_{\mathcal{U}} * \underset{\sim}{R}} \langle p, \underset{\sim}{q} \rangle$ such that

$$\langle p', \underset{\sim}{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underset{\sim}{R}} \forall n \in \omega : \underset{\sim}{a}(n) \in \underset{\sim}{c}(n) \text{ and } |\underset{\sim}{c}(n)| \leq 2^n.$$

We may assume without loss of generality that $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \forall n \in \omega : \underset{\sim}{c}(n) \subseteq (\dot{\eta}(n-1), \dot{\eta}(n)]$. Let $\underset{\sim}{C}$ be a $\mathbb{M}_{\mathcal{U}}$-name for an element of $[\omega]^\omega$ such that $p' \Vdash_{\mathbb{M}_{\mathcal{U}}} \underset{\sim}{C} = \bigcup \mathrm{range}(\underset{\sim}{c})$. Hence, we have

$$\langle p', \underset{\sim}{q}' \rangle \Vdash_{\mathbb{M}_{\mathcal{U}} * \underset{\sim}{R}} \forall n \in \omega : \underset{\sim}{a}(n) \in \underset{\sim}{C} \cap (\dot{\eta}(n-1), \dot{\eta}(n)] \text{ and } |\underset{\sim}{C} \cap (\dot{\eta}(n-1), \dot{\eta}(n)]| \leq 2^n.$$

Lemma 2.4. *Write $p' = \langle s, x_0 \rangle$. There exists $x_1 \in [x_0]^\omega \cap \mathcal{U}$ such that the $\mathbb{M}_{\mathcal{U}}$-condition $\langle s, x_1 \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_0 \rangle$ has the following property:*

For every $t \in \mathit{fin}(x_1)$, there exists $C_t \in \mathit{fin}(\omega)$ such that

$$\langle s \cup t, x_1 \setminus (\max t)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underset{\sim}{C} \cap (\max t)^+ = C_t.$$

Proof. We define a strategy for the Maiden in the ultrafilter game for $\mathcal{U}$, which will not be a winning strategy since $\mathcal{U}$ is a Ramsey ultrafilter.

Since $\mathbb{M}_{\mathcal{U}}$ has pure decision, there exists $C_\emptyset \subseteq (\max s)^+$ and $y_0 \in [x_0]^\omega \cap \mathcal{U}$ such that $\langle s, y_0 \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underset{\sim}{C} \cap (\max s)^+ = C_\emptyset$. The Maiden starts by playing y_0.

Assume $y_0 \supseteq y_1 \supseteq ... \supseteq y_k$ and $n_0 < n_1 < ... < n_k$ have been played, where $\forall i \leq k : y_i \in \mathcal{U}$ and $n_i \in y_i$. Again by pure decision, for each $t \subseteq \{n_0, n_1, ..., n_k\}$ with $\max t = n_k$, there exists $z_t \in [y_k \setminus n_k^+]^\omega \cap \mathcal{U}$ and $C_t \subseteq n_k^+$ such that $\langle s \cup t, z_t \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underset{\sim}{C} \cap (n_k)^+ = C_t$. The Maiden plays

$$y_{k+1} := \bigcap_{\substack{t \subseteq \{n_i : i \leq k\} \\ \max t = n_k}} z_t.$$

Since Death wins, we have that $x_1 := \{n_i : i \in \omega\} \in \mathcal{U}$. It is easy to check that this x_1 satisfies the lemma. ⊣

The following lemma strengthens the previous one.

Lemma 2.5. *Assume $\langle s, x_1 \rangle$ is as in the conclusion of the previous lemma. There exists $x_2 \in [x_1]^\omega \cap \mathcal{U}$ such that $\langle s, x_2 \rangle$ has the following property:*

For every $t \in \mathit{fin}(x_2)$, every $m \in x_2 \setminus \max t$ and all $n, n' \in x_2 \setminus m^+$, it holds that $C_{t \cup \{n\}} \cap m^+ = C_{t \cup \{n'\}} \cap m^+$.

Proof. We again prove this by playing the ultrafilter game for $\mathcal{U}$. Assume $y_0 := x_1 \supseteq y_1 \supseteq ... \supseteq y_k$ and $n_0 < n_1 < ... < n_k$ have been played. For every $t \subseteq \{n_0, n_1, ..., n_k\}$ and every $d \subseteq n_k^+$ consider the set

$$P_{t,d} := \{n \in y_k \setminus n_k^+ : C_{t \cup \{n\}} \cap n_k^+ = d\}.$$

Note that for every $t \subseteq \{n_0, n_1, ..., n_k\}$, the set $\{P_{t,d} : d \subseteq n_k^+\}$ is a partition of $y_k \setminus n_k^+$ into finitely many pieces. Hence, there exists one $d_t \subseteq n_k^+$ such that $P_{t,d_t} \in \mathcal{U}$.

The Maiden plays

$$y_{k+1} := \bigcap_{t \subseteq \{n_i : i \leq k\}} P_{t,d_t}.$$

Death will win and hence $x_2 := \{n_i : i \in \omega\} \in \mathcal{U}$. It is again not hard to check that x_2 satisfies the lemma. $\dashv$

The following fact will be needed later.

FACT 2.6. *Without loss of generality, we may assume that for all $n \in \{\max s\} \cup x_2$, if n is the j'th element of $s \cup x_2$ in increasing order, then $n > 2^{j+1}$.*

Proof. Note that the conclusion of Lemmas 2.4 and 2.5 also holds for each $\langle s', x' \rangle \geq_{\mathbb{M}_{\mathcal{U}}} \langle s, x_2 \rangle$. Hence, we simply trim x_2 such that the enumeration of $s \cup x_2$ dominates 2^{j+1} above $|s|$ and replace s with $s \cup \{\min x_2\}$ and x_2 with $x_2 \setminus \{\min x_2\}$. $\dashv$

Next, let N be a countable elementary submodel of some large enough $\mathcal{H}_\chi$ such that $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \underset{\sim}{C}, \langle s, x_2 \rangle\} \in N$. By induction, construct a sequence $N_0 \subseteq N_1 \subseteq ...$ of finite subsets of N such that

(i) $\{\mathcal{U}, \mathbb{M}_{\mathcal{U}}, \underset{\sim}{C}, \langle s, x_2 \rangle, s, x_2\} \subseteq N_0$,

(ii) $\bigcup_{i \in \omega} N_i = N$,

(iii) $\forall i \in \omega : k_i := N_i \cap \omega \in \omega$.

(iv) $\forall i \in \omega : \forall t \in \text{fin}(\omega) : t \in N_i \iff t \subseteq N_i$,

(v) If $\langle m, l, D \rangle \in (\omega \times \omega \times \text{fin}(\omega)) \cap N_i$, then $m, l, D \in N_i$ (and hence $D \subseteq N_i$ by the previous condition).

(vi) $\forall i \in \omega :$ If $\varphi(x, a_0, ..., a_l)$ is a formula of length less than 2025 with $a_0, ..., a_l \in N_i$ and $N \models \exists x \varphi(x, a_0, ..., a_l)$, then there exists $b \in N_{i+1}$ such that $N \models \varphi(b, a_0, ..., a_l)$.

LEMMA 2.7. *$\langle s, x_2 \rangle$ forces that*

$$\forall i \in \omega \setminus \{0,1\} : \underset{\sim}{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i) \neq \emptyset \implies \begin{cases} \mathit{range}(\dot{\eta}) \cap [k_{i-2}, k_{i-1}) \neq \emptyset, \ \mathit{or} \\ \mathit{range}(\dot{\eta}) \cap [k_{i-1}, k_i) \neq \emptyset, \ \mathit{or} \\ \mathit{range}(\dot{\eta}) \cap [k_i, k_{i+1}) \neq \emptyset. \end{cases}$$

Proof. Assume $\langle s \cup t, x' \rangle \geq_{\mathbb{M}_\mathcal{U}} \langle s, x_2 \rangle$, $a \in \omega \setminus (\max s)^+$ and $i \in \omega \setminus \{0, 1\}$ are such that

$$\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_\mathcal{U}} a \in \underset{\sim}{C} \setminus (\max s)^+ \cap [k_{i-1}, k_i).$$

We show that $\langle s \cup t, x' \rangle$ forces one of the three possible conclusions in the statement of the lemma.

By possibly extending t, we may assume that t contains at least one element that is greater than a. Let $l_0 := \max(t \cap a)$ and $l^* := \min(t \setminus a)$. Furthermore, let $m^* := \max(x_2 \cap l^*)$. Hence, l_0 and l^* are consecutive elements of t and $l_0 \leq m^* < l^*$ and $l_0 < a \leq l^*$. We distinguish between two cases:

Case I. Assume $l_0 \leq m^* \leq a \leq l^*$.

If $l^* \in [k_{i-1}, k_i)$, we are done, since this means that $\langle s \cup t, x' \rangle \Vdash_{\mathbb{M}_\mathcal{U}} l^* \in \text{range}(\underset{\sim}{\eta}) \cap [k_{i-1}, k_i)$. Hence, assume $l^* \notin [k_{i-1}, k_i)$, which means that $l^* \notin N_i$, since l^* is certainly not in N_{i-1} (if it were, a would be as well by (iii)). Note that l^* witnesses that

$$N \models \exists l : l = \min(x_2 \setminus a).$$

Hence, by (v), we have that $l^* \in N_{i+1}$ and thus $l^* \in [k_i, k_{i+1})$.

Case II. Assume $l_0 < a < m^* < l^*$.

Let $t' := t \cap a$, i.e., $l_0 := \max t'$, and let $i^* \in \omega \setminus \{0\}$ be such that $l_0 \in [k_{i^*-1}, k_{i^*})$, i.e., l_0 first appears in N_{i^*}. If $i^* = i$, we are again done, hence assume that $a \notin N_{i^*}$. We will show that $i^* = i - 1$.

Let $j \in \omega$ be such that l^* is the j'th elements of $s \cup t$ in increasing order. By Lemmas 2.4 and 2.5, there is $C_{t' \cup \{l^*\}} \subseteq (l^*)^+$ such that

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \Vdash_{\mathbb{M}_\mathcal{U}} \underset{\sim}{C} \cap (l^*)^+ = C_{t' \cup \{l^*\}}.$$

Set $D^* := C_{t' \cup \{l^*\}} \cap (l_0, m^*)$. Since

$$\langle s \cup t' \cup \{l^*\}, x_2 \setminus (l^*)^+ \rangle \leq_{\mathbb{M}_\mathcal{U}} \langle s \cup t, x' \rangle,$$

and since $l_0 < a < m^*$ by assumption, we must have $a \in D^*$. Furthermore, note that $D^* \subseteq C_{t' \cup \{l^*\}} \cap (l_0, l^*]$ and thus $|D^*| =: \gamma \leq 2^j$.

Now, m^*, l^* and D^* witness that

$$N \models \exists \langle m, l, D \rangle : \begin{cases} m, l \in x_2 \setminus {l_0}^+, m < l, \text{ and} \\ D \subseteq (l_0, m), \text{ and} \\ |D| = \gamma, \text{ and} \\ \langle s \cup t' \cup \{l\}, x_2 \setminus l^+ \rangle \Vdash_{\mathbb{M}_\mathcal{U}} \underset{\sim}{C} \cap (l_0, m) = D. \end{cases}$$

Since l_0 is the $(j-1)$'th element of $s \cup t'$, we have $l_0 > 2^j$ by Fact 2.6.[6] Hence, since $l_0 \in N_{i^*}$, it follows that $\gamma \in N_{i^*}$. Thus, all the parameters in the above formula lie in N_{i^*}, which implies that there exists $\langle m^\dagger, l^\dagger, D^\dagger \rangle \in N_{i^*+1}$ satisfying the formula.

Claim. $l^\dagger \geq a$

Note that the proof of this claim will finish the proof of the Lemma, since $l^\dagger \in N_{i^*+1}$ by (v) and thus $a \in N_{i^*+1} \setminus N_{i^*}$.

Proof. Assume by contradiction that $l^\dagger < a$, i.e.,

$$l_0 < m^\dagger < l^\dagger < a < m^* < l^*.$$

By Lemma 2.5, we have that

$$C_{t' \cup \{l^\dagger\}} \cap (m^\dagger) = C_{t' \cup \{l^*\}} \cap (m^\dagger).$$

Since $\langle s \cup t' \cup \{l^\dagger\}, x_2 \setminus (l^\dagger)^+ \rangle \Vdash_{\mathbb{M}_{\mathcal{U}}} \underset{\sim}{C} \cap (l_0, m^\dagger) = D^\dagger$, it follows that $C_{t' \cup \{l^*\}} \cap (m^\dagger) = D^\dagger$ and hence $D^\dagger = D^* \cap (l_0, m^\dagger)$. However, both $D^\dagger$ and D^* have size γ and thus $D^* \subseteq (l_0, m^\dagger)$, which is a contradiction to the fact that $a \in D^*$ and $a > m^\dagger$. ■ ⊣

We now only need one final lemma to finish the proof of the proposition and thus of the main theorem.

Lemma 2.8. *Let $I := \{[k_i, k_{i+1}) : i \in \omega\}$ be any interval partition of ω and E and $\mathcal{U}$ non-isomorphic Q-points. Then there exist $v \in E$ and $u \in \mathcal{U}$ such that*

$$\forall i \in \omega \setminus \{0\} : v \cap [k_i, k_{i+1}) \neq \emptyset \implies \begin{cases} u \cap [k_{i-1}, k_i) = \emptyset, \text{ and} \\ u \cap [k_i, k_{i+1}) = \emptyset, \text{ and} \\ u \cap [k_{i+1}, k_{i+2}) = \emptyset. \end{cases}$$

Proof. Say that a Q-point element *selects* from an interval partition if it intersects each interval in exactly one point. Let $v_0 \in E$ and $u_0 \in \mathcal{U}$ be such that they select from I. Let f be an order-preserving bijection from v_0 to u_0, extended to a permutation of ω. Thus, for each $i \in \omega$, f sends the element selected by v_0 in $[k_i, k_{i+1})$ to the element selected by u_0 in $[k_i, k_{i+1})$. Since E and $\mathcal{U}$ are non-isomorphic, there exist $v_1 \in [v_0]^\omega \cap E$ and $u_1 \in [u_0]^\omega \cap \mathcal{U}$ such that $u_1 \cap f[v_1] = \emptyset$. Hence, for all $i \in \omega \setminus \{0\}$:

$$v_1 \cap [k_i, k_{i+1}) \neq \emptyset \implies u_1 \cap [k_i, k_{i+1}) = \emptyset.$$

Both E and $\mathcal{U}$ contain the set

$$y_\varepsilon := \bigcup_{\substack{i \in \omega \\ i \equiv \varepsilon \pmod 3}} [k_i, k_{i+1}),$$

[6] Note that the additional requirement in Fact 2.6 that $\max s$ is already larger than $2^{|s|}$ is needed here, since l_0 could be $\max s$.

each for exactly one $\varepsilon = \varepsilon(E), \varepsilon(\mathcal{U}) \in 3$. Let $v_2 := v_1 \cap y_{\varepsilon(E)} \in E$ and $u_2 := u_1 \cap y_{\varepsilon(\mathcal{U})} \in \mathcal{U}$. If $\varepsilon(E) = \varepsilon(\mathcal{U})$ then v_2 and u_2 satisfy the lemma, hence assume without loss of generality that $\varepsilon(E) = 0$ and $\varepsilon(\mathcal{U}) = 1$.

Let $\bar{v}_0 \in E$ and $\bar{u}_0 \in \mathcal{U}$ be elements that select from the interval partition

$$\{[k_i, k_{i+2}) : i \in \omega, i \equiv 0 \pmod 3\} \cup \{[k_i, k_{i+1}) : i \in \omega, i \equiv 2 \pmod 3\}.$$

Again, by considering a permutation of ω that maps the element selected by $\bar{v}_0$ in any interval to the element selected by $\bar{u}_0$ in the same interval, we find $\bar{v}_1 \in [\bar{v}_0]^\omega \cap E$ and $\bar{u}_1 \in [\bar{u}_0]^\omega \cap \mathcal{U}$ such that $\bar{v}_1$ and $\bar{u}_1$ never select from the same interval. Now, clearly, $v_1 \cap \bar{v}_1 \in E$ and $u_1 \cap \bar{u}_1 \in \mathcal{U}$ work. $\dashv$

We can now finish the proof of the proposition and hence of the main theorem: Let $v \in E$, $u \in \mathcal{U}$ be given by the previous lemma for the interval partition $\{[k_i, k_{i+1}) : i \in \omega\} \cup \{[0, k_0)\}$ constructed in the proof of Lemma 2.7. Let $G * H$ be any $\mathbb{M}_{\mathcal{U}} * \underset{\sim}{R}$-generic filter containing $\langle\langle s, x_2\rangle, \underset{\sim}{q}'\rangle$. By Lemma 2.7, we have that in $\mathbf{V}[G * H]$, whenever $\text{range}(\underset{\sim}{a}[G * H]) \setminus (\max s)^+$ intersects one of the intervals $[k_i, k_{i+1})$, then the Mathias real η intersects $[k_i, k_{i+1})$ or one of the adjacent intervals $[k_{i-1}, k_i)$ or $[k_{i+1}, k_{i+2})$. Since $\text{range}(\eta)$ is almost contained in u, the same is true for u in place of η above some $n \geq (\max s)^+$. Hence, $\text{range}(\underset{\sim}{a}[G * H]) \setminus n$ is disjoint from v. $\dashv$

$\dashv$

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