# The hidden twin of Morley's Five Circles Theorem 

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#### Abstract

We give an algebraic proof of a slightly extended version of Morley's Five Circles Theorem. The theorem holds in all Miquelian Möbius planes obtained from a separable quadratic field extension, in particular in the classical real Möbius plane. Moreover, the calculations bring to light a hidden twin of the Five Circles Theorem that seems to have been overlooked until now.


## 1 Introduction

The classical Five Circles Theorem is due to Frank Morley [8], [9, p. 265]. We quote it here in the version of Tobias Dantzig who provided a proof based on elementary properties of the Euclidean plane in [3].

Theorem 1. If a ring of five circles be formed, the center of each upon a fixed circle and each circle of the ring intersecting the next on this fixed circle, the five other intersections when joined in succession will form a pentacle whose vertices lie one upon each of the five circles (see Figure 1).

This theorem should not be confounded with similar incidence results like Miquel's Pentagon Theorem [7, Théorème III] (see [6] for a computer assisted algebraic proof, and the gray box on page 246 in [10] for a comment) or the five circle incidence theorem in [5]. The aim of this article is to set up a simple algebraic proof of a slightly extended version of Morley's Five Circles Theorem which can be carried out by hand and which works for all Miquelian Möbius planes obtained from a separable quadratic field extension. This shows in particular that the theorem rests on a lesser axiomatic foundation than Euclidean geometry. Moreover, the careful analysis will bring to light a hidden twin of the Five Circles Theorem that seems to have been overlooked until now.

Let us restate Theorem 1 in a more formal way: A circle $K$ carries the five centers $Z_{1}, \ldots, Z_{5}$ of five circles $K_{1}, \ldots, K_{5} . K_{i-2}$ and $K_{i+2}$ intersect on $K$ in the point $P_{i}$ (indices read cyclically) and in $Q_{i}$. The line $l_{i}$ passes through $Q_{i-2}$ and $Q_{i+2}$. Then Theorem 1 claims that the intersection $R_{i}$ of $l_{i-1}$ and $l_{i+1}$ lies on $K_{i}$. Figure 1 illustrates the situation.

It is not necessary, that the centers $Z_{1}, \ldots, Z_{5}$ sit in cyclic order on $K$, as Figure 2 suggests.


Figure 1: The classical Five Circles Theorem


Figure 2: Theorem 1 with centers $Z_{1}, \ldots, Z_{5}$ not in cyclic order

This looks quite convincing, however, in Figure 3 the theorem seems to fail even though combinatorially the conditions are satisfied. So, why does the theorem work in one case but not in the other? Or more precisely: What is the exact formulation of the conditions so that the vertices $R_{i}$ of the pentagon lie on the circles $K_{i}$ ? We first analyze this question


Figure 3: Why does Theorem 1 not hold here? We will see in Theorem 5 that in this case other incidences apply instead.
in Section 2 in the classical model of the Möbius plane. In Section 3 we will generalize the results to Miquelian Möbius planes obtained from a separable quadratic field extension.

## 2 The Five Circles Theorem in the classical Möbius plane

The classical model of the Möbius plane is the Riemann sphere, which we interpret conveniently as $\mathbb{C} \cup\{\infty\}$. We use the standard notions of Möbius geometry: Circles (or blocks of the first type) are the complex solutions $z$ of the equation

$$
\begin{equation*}
\mathcal{B}_{c, r}^{1}:(z-c)(\bar{z}-\bar{c})=r \tag{1}
\end{equation*}
$$

for $c \in \mathbb{C}$ and $0<r \in \mathbb{R}$. The center of the circle is $c$, and $\sqrt{r}$ is its radius. Lines (or blocks of the second type) are the complex solutions $z$ of the equation

$$
\begin{equation*}
\mathcal{B}_{c, r}^{2}: \bar{c} z+c \bar{z}=r \tag{2}
\end{equation*}
$$

for $c \in \mathbb{C} \backslash\{0\}$ and $r \in \mathbb{R}$, together with $\infty$. In the sequel we will use $\mathbb{P}=\mathbb{C} \cup\{\infty\}$ for the set of points, and $\mathbb{B}$ for the set of blocks.

Let $K \in \mathbb{B}$ be a circle, and let $Z_{1}, \ldots, Z_{5} \in \mathbb{P}$ be five different points on $K$. There exists a Möbius transformation $z \mapsto a z+b$ that maps the circle $K$ to the unit circle $\mathcal{B}_{0,1}^{1}$ with
the equation $z \bar{z}=1$. This transformation maps circles to circles and lines to lines. In particular, the image of the center of a circle is the center of the image of the circle. Therefore, we may assume without loss of generality that $K$ is the unit circle. Our first goal is to identify every family consisting of five circles $K_{1}, \ldots, K_{5}$ with the property that $K_{i-2} \cap K_{i+2}=\left\{P_{i}, Q_{i}\right\}$ for $P_{i} \in K$ and $Q_{i} \in \mathbb{P}$, and such that $Z_{i}$ is the center of $K_{i}$ for each $i \in\{1, \ldots, 5\}$. To achieve this, we introduce the anti-Möbius transformation

$$
\varphi_{Z}: \mathbb{P} \rightarrow \mathbb{P}, \quad z \mapsto \varphi_{Z}(z):=Z^{2} \bar{z},
$$

for $Z \in K$. It is easy to see that $\varphi_{Z}$ has the following properties:

- If $z \in K$, then $\varphi_{Z}(z) \in K$.
- $\varphi_{Z}$ is an involution.
- $Z,-Z$ and 0 are fixed points of $\varphi_{Z}$.

Hence, $\varphi_{Z}$ is a reflection with respect to the line containing the points $Z,-Z$ and 0 .
Lemma 2. Let $Z_{1}, \ldots, Z_{5} \in K=\mathcal{B}_{0,1}^{1}$ be five different points. Then there are two quintuples (I) and (II) of circles $K_{1}, \ldots, K_{5}$ having the following properties: For each $i \in\{1, \ldots, 5\}$ the circle $K_{i}$ has center $Z_{i}$, and $K_{i-2}$ and $K_{i+2}$ intersect at $P_{i} \in K$ and $Q_{i}$. The points $P_{i}$ for the two quintuples are

$$
\begin{equation*}
\text { (I) } P_{i}=-\frac{Z_{i} Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}} \quad \text { and } \quad \text { (II) } P_{i}=\frac{Z_{i} Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}} \tag{3}
\end{equation*}
$$

respectively.

Proof. The idea is to use the map $\varphi_{Z_{i}}$ which maps the point $P_{i+2}$ to $P_{i-2}$. This corresponds to the system of equations

$$
\begin{align*}
P_{4} & =\varphi_{Z_{1}}\left(P_{3}\right)=Z_{1}^{2} \bar{P}_{3}  \tag{4}\\
P_{5} & =\varphi_{Z_{2}}\left(P_{4}\right)=Z_{2}^{2} \bar{P}_{4}=\overline{Z_{1}^{2}} Z_{2}^{2} P_{3}  \tag{5}\\
P_{1} & =\varphi_{Z_{3}}\left(P_{5}\right)=Z_{3}^{2} \bar{P}_{5}=Z_{1}^{2} \overline{Z_{2}^{2}} Z_{3}^{2} \bar{P}_{3}  \tag{6}\\
P_{2} & =\varphi_{Z_{4}}\left(P_{1}\right)=Z_{4}^{2} \bar{P}_{1}=\overline{Z_{1}^{2}} Z_{2}^{2} \overline{Z_{3}^{2}} Z_{4}^{2} P_{3}  \tag{7}\\
P_{3} & =\varphi_{Z_{5}}\left(P_{2}\right)=Z_{5}^{2} \bar{P}_{2}=Z_{1}^{2} \overline{Z_{2}^{2}} Z_{3}^{2} \overline{Z_{4}^{2}} Z_{5}^{2} \bar{P}_{3} . \tag{8}
\end{align*}
$$

We multiply (8) by $P_{3}$ and obtain

$$
P_{3}^{2}=Z_{1}^{2} \overline{Z_{2}^{2}} Z_{3}^{2} \overline{Z_{4}^{2}} Z_{5}^{2}
$$

and hence

$$
P_{3}=Z_{1} \bar{Z}_{2} Z_{3} \bar{Z}_{4} Z_{5} \quad \text { or } \quad P_{3}=-Z_{1} \bar{Z}_{2} Z_{3} \bar{Z}_{4} Z_{5} .
$$

Replacing $P_{3}$ in (4)-(7) by these expressions we obtain the formulas (3) if we use that $\bar{Z}_{i}=1 / Z_{i}$. It is obvious that these points satisfy $P_{i} \bar{P}_{i}=1$, and hence lie on $K$. Let $K_{i}$
be the circle with center $Z_{i}$ through $P_{i-2}$. It remains to verify that $P_{i+2}$ also belongs to $K_{i}$. Indeed we have

$$
\begin{aligned}
\left(P_{i-2}-Z_{i}\right)\left(\bar{P}_{i-2}-\bar{Z}_{i}\right) & =\left(Z_{i}^{2} \bar{P}_{i+2}-Z_{i}\right)\left(\overline{Z_{i}^{2}} P_{i+2}-\bar{Z}_{i}\right) \\
& =\left(Z_{i} \bar{P}_{i+2}-1\right)\left(\bar{Z}_{i} P_{i+2}-1\right) \\
& =\left(P_{i+2}-Z_{i}\right)\left(\bar{P}_{i+2}-\bar{Z}_{i}\right),
\end{aligned}
$$

and the claim follows.

Now we want to express the points $Q_{i}$ in terms of the centers $Z_{i}$.
Lemma 3. Let $K_{1}, \ldots, K_{5}$ be a quintuple of circles with centers $Z_{1}, \ldots, Z_{5} \in K=\mathcal{B}_{0,1}^{1}$. Suppose that $K_{i-2} \cap K_{i+2}=\left\{P_{i}, Q_{i}\right\}$ with $P_{i} \in K$ for each $i \in\{1, \ldots, 5\}$. Then we have

$$
\begin{equation*}
Q_{i}=Z_{i-2}+Z_{i+2}-\bar{P}_{i} Z_{i-2} Z_{i+2} \tag{9}
\end{equation*}
$$

Proof. The claim can be checked by showing that the point $Q_{i}$ given by (9) satisfies the equations of both circles $K_{i-2}$ and $K_{i+2}$. Indeed, for $K_{i-2}$, we have

$$
\begin{aligned}
\left(Q_{i}-Z_{i-2}\right)\left(\bar{Q}_{i}-\bar{Z}_{i-2}\right) & =\left(Z_{i+2}-\bar{P}_{i} Z_{i-2} Z_{i+2}\right)\left(\bar{Z}_{i+2}-P_{i} \bar{Z}_{i-2} \bar{Z}_{i+2}\right) \\
& =\left(1-\bar{P}_{i} Z_{i-2}\right)\left(1-P_{i} \bar{Z}_{i-2}\right) \\
& =\left(P_{i}-Z_{i-2}\right)\left(\bar{P}_{i}-\bar{Z}_{i-2}\right)
\end{aligned}
$$

A similar calculation shows that $Q_{i}$ lies on $K_{i+2}$.

With these preparations we are now ready to investigate the incidence relations in the two quintuples (I) and (II) of circles in Lemma 2.

### 2.1 The quintuple (I) and the classical Five Circles Theorem

Let us consider the five circles $K_{1}, \ldots, K_{5}$ having centers $Z_{1}, \ldots, Z_{5}$ on $K=\mathcal{B}_{0,1}^{1}$, and $K_{i-2} \cap K_{i+2}=\left\{Q_{i}, P_{i}\right\}$ with

$$
P_{i}=-\frac{Z_{i} Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}}, \quad Q_{i}=Z_{i-2}+Z_{i+2}-\bar{P}_{i} Z_{i-2} Z_{i+2}
$$

Inserting the expression for $P_{i}$ in the expression for $Q_{i}$ yields that

$$
Q_{i}=Z_{i-2}+Z_{i+2}+\frac{Z_{i+1} Z_{i-1}}{Z_{i}}
$$

Let $l_{i}$ denote the line through the points $Q_{i-2}, Q_{i+2}$ and $\infty$. Moreover, we will consider the lines $h_{i}$ through the points $Z_{i}, P_{i}$ and $\infty$. These lines are

$$
\begin{align*}
l_{i} & =\left\{z:\left(Q_{i+2}-z\right)\left(\bar{Q}_{i-2}-\bar{z}\right)=\left(\bar{Q}_{i+2}-\bar{z}\right)\left(Q_{i-2}-z\right)\right\} \cup\{\infty\},  \tag{10}\\
h_{i} & =\left\{z:\left(Z_{i}-z\right)\left(\bar{P}_{i}-\bar{z}\right)=\left(\bar{Z}_{i}-\bar{z}\right)\left(P_{i}-z\right)\right\} \cup\{\infty\} . \tag{11}
\end{align*}
$$

Now we claim that the lines $l_{i-1}, l_{i+1}$ and $h_{i}$ meet in the point

$$
\begin{equation*}
R_{i}=Z_{i}\left(\frac{Z_{i+2}}{Z_{i+1}}+\frac{Z_{i-2}}{Z_{i-1}}+1\right) \tag{12}
\end{equation*}
$$

Let us check that $R_{i}$ belongs to $l_{i-1}$. If we use the expression (12) for $R_{i}$ in place of $z$ in (10) we obtain for the bracket factors

$$
\begin{aligned}
Q_{i+1}-R_{i} & =\frac{\left(Z_{i-1}-Z_{i}\right)\left(Z_{i-2}+Z_{i-1}\right)}{Z_{i-1}} \\
\bar{Q}_{i-3}-\bar{R}_{i} & =\frac{\left(Z_{i} Z_{i+2}-Z_{i+1} Z_{i-1}\right)\left(Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}\right)}{Z_{i} Z_{i+1} Z_{i+2} Z_{i-2} Z_{i-1}} \\
\bar{Q}_{i+1}-\bar{R}_{i} & =\frac{\left(Z_{i}-Z_{i-1}\right)\left(Z_{i-2}+Z_{i-1}\right)}{Z_{i} Z_{i-2} Z_{i-1}} \\
Q_{i-3}-R_{i} & =\frac{\left(Z_{i+1} Z_{i-1}-Z_{i} Z_{i+2}\right)\left(Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}\right)}{Z_{i+1} Z_{i+2} Z_{i-1}}
\end{aligned}
$$

Indeed the product of the first two expressions agrees with the product of the last two. Hence, $z=R_{i}$ satisfies the equation of the line $l_{i-1}$. Similar calculations show that $R_{i}$ also lies on the lines $l_{i+1}$ and $h_{i}$.

In order to prove the original version of the Five Circles Theorem, we need to show that $R_{i}$ belongs to the circle $K_{i}$ with center $Z_{i}$ through the points $P_{i+2}, P_{i-2}, Q_{i-2}, Q_{i+2}$ which is given by the equation

$$
\begin{equation*}
\left(Z_{i}-z\right)\left(\bar{Z}_{i}-\bar{z}\right)=\left(Z_{i}-P_{i+2}\right)\left(\bar{Z}_{i}-\bar{P}_{i+2}\right) \tag{13}
\end{equation*}
$$

If we use the expression (12) for $R_{i}$ in place of $z$ in (13) we obtain for the bracket factors

$$
\begin{aligned}
Z_{i}-R_{i} & =-\frac{Z_{i}\left(Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}\right)}{Z_{i+1} Z_{i-1}} \\
\bar{Z}_{i}-\bar{R}_{i} & =-\frac{Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}}{Z_{i} Z_{i+2} Z_{i-2}} \\
Z_{i}-P_{i+2} & =\frac{Z_{i}\left(Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}\right)}{Z_{i+1} Z_{i-2}} \\
\bar{Z}_{i}-\bar{P}_{i+2} & =\frac{Z_{i+1} Z_{i-2}+Z_{i+2} Z_{i-1}}{Z_{i} Z_{i+2} Z_{i-1}}
\end{aligned}
$$

Indeed the product of the first two expressions agrees with the product of the last two. Hence, $R_{i} \in K_{i}$ as claimed.

Observe that we actually proved a slightly enhanced version of the Five Circles Theorem since we showed that the lines $h_{i}$ also pass through the points $R_{i}$ (see Figure 4). In fact, there is yet another incidence to be discovered in this configuration: Let us reflect the point $Z_{i}$ at the perpendicular bisector of $Z_{i-1}$ and $Z_{i+1}$. The mirrored point is $C_{i}=$ $\frac{Z_{i+1} Z_{i-1}}{Z_{i}} \in K$. Using the expressions we found for the points $P_{i}, Q_{i}$ and $R_{i}$ it is then easy to verify that

$$
\begin{aligned}
\left(C_{i}-P_{i-1}\right)\left(\bar{C}_{i}-\bar{P}_{i-1}\right) & =\left(C_{i}-P_{i+1}\right)\left(\bar{C}_{i}-\bar{P}_{i+1}\right) \\
& =\left(C_{i}-R_{i-1}\right)\left(\bar{C}_{i}-\bar{R}_{i-1}\right) \\
& =\left(C_{i}-R_{i+1}\right)\left(\bar{C}_{i}-\bar{R}_{i+1}\right) \\
& =\left(C_{i}-Q_{i}\right)\left(\bar{C}_{i}-\bar{Q}_{i}\right)=\frac{\left(Z_{i-2}+Z_{i+2}\right)^{2}}{Z_{i-2} Z_{i+2}}
\end{aligned}
$$

Hence the five points $P_{i-1}, P_{i+1}, R_{i-1}, R_{i+1}$ and $Q_{i}$ lie on a circle with center $C_{i}$ and radius $\left|Z_{i-2}+Z_{i+2}\right|$.

Notice that if

$$
\begin{equation*}
Z_{i-2} Z_{i+1}=-Z_{i+2} Z_{i-1} \tag{14}
\end{equation*}
$$

then

$$
P_{i-2}=P_{i+2}=Q_{i-2}=Q_{i+2}=R_{i}=Z_{i},
$$

which means that the circle $K_{i}$ degenerates to a point. Vice versa, $P_{i-2}=P_{i+2}=Z_{i}$ implies (14).

Before we formulate our results as a theorem, we turn our attention to the second quintuple of circles which we identified in Lemma 2. It will turn out that these circles carry a twin of the original Five Circles Theorem.

### 2.2 The quintuple (II) and the Twin of the Five Circles Theorem

Let us consider the five circles $K_{1}, \ldots, K_{5}$ having centers $Z_{1}, \ldots, Z_{5}$ on $K=\mathcal{B}_{0,1}^{1}$, and $K_{i-2} \cap K_{i+2}=\left\{P_{i}, Q_{i}\right\}$ with

$$
P_{i}=\frac{Z_{i} Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}}, \quad Q_{i}=Z_{i-2}+Z_{i+2}-\bar{P}_{i} Z_{i-2} Z_{i+2}=Z_{i-2}+Z_{i+2}-\frac{Z_{i+1} Z_{i-1}}{Z_{i}} .
$$

Let, as before, $l_{i}$ denote the line through $Q_{i-2}, Q_{i+2}, \infty$ given by (10), and $h_{i}$ the line through $Z_{i}, P_{i}, \infty$ given by (11). Then the points

$$
S_{i, i-1}=Z_{i}\left(\frac{Z_{i-2}}{Z_{i-1}}-\frac{Z_{i+2}}{Z_{i+1}}+1\right) \text { and } S_{i, i+1}=Z_{i}\left(\frac{Z_{i+2}}{Z_{i+1}}-\frac{Z_{i-2}}{Z_{i-1}}+1\right)
$$

are the intersections of $h_{i}$ with $l_{i-1}$ and $l_{i+1}$, respectively, which can easily be checked by inserting these expressions in (10) and (11). We now claim that $S_{i, i-1}$ and $S_{i, i+1}$ are points on $K_{i}$. Indeed, if we insert $S_{i, i-1}$ for $z$ in (13) we obtain for the bracket factors

$$
\begin{aligned}
Z_{i}-S_{i, i-1} & =\frac{Z_{i}\left(Z_{i+1} Z_{i+3}-Z_{i+2} Z_{i+4}\right)}{Z_{i+1} Z_{i+4}} \\
\bar{Z}_{i}-\bar{S}_{i, i-1} & =-\frac{Z_{i+1} Z_{i+3}-Z_{i+2} Z_{i+4}}{Z_{i} Z_{i+2} Z_{i+3}} \\
Z_{i}-P_{i+2} & =\frac{Z_{i}\left(Z_{i+1} Z_{i+3}-Z_{i+2} Z_{i+4}\right)}{Z_{i+1} Z_{i+3}} \\
\bar{Z}_{i}-\bar{P}_{i+2} & =-\frac{Z_{i+1} Z_{i+3}-Z_{i+2} Z_{i+4}}{Z_{i} Z_{i+2} Z_{i+4}}
\end{aligned}
$$

and we see that the product of the first two and the product of the last two expressions agree. A similar calculation shows that $S_{i, i+1}$ also lies on $K_{i}$. Notice that in the classical Five Circles Theorem carried by the quintuple (I) the intersection of $h_{i}$ and $l_{i-1}$ agrees with the intersection of $h_{i}$ and $l_{i+1}$. For the quintuple (II) of circles this is no longer the case. Indeed, we have:

Lemma 4. For the quintuple (II) there holds $S_{i, i-1} \neq S_{i, i+1}$ for all $i \in\{1, \ldots, 5\}$, unless $K_{i}$ degenerates to a point.

Proof. Assume by contradiction that $S_{i, i-1}=S_{i, i+1}$, i.e.,

$$
S_{i, i-1}-S_{i, i+1}=2 Z_{i}\left(\frac{Z_{i+3}}{Z_{i+4}}-\frac{Z_{i+2}}{Z_{i+1}}\right)=0 .
$$

This is equivalent to

$$
Z_{i}=\frac{Z_{i} Z_{i+2} Z_{i+4}}{Z_{i+1} Z_{i+3}}=P_{i+2},
$$

where we have used (3) for the last equality. But this would mean that $K_{i}$ degenerates to a point.

Let us again consider the points $C_{i}=\frac{Z_{i+1} Z_{i-1}}{Z_{i}}$ which we obtain by reflecting $Z_{i}$ at the perpendicular bisector of $Z_{i-1}$ and $Z_{i+1}$. Then, using as usual that $\bar{Z}_{i}=1 / Z_{i}$, it is easy to check that

$$
\begin{aligned}
\left(C_{i}-P_{i-1}\right)\left(\bar{C}_{i}-\bar{P}_{i-1}\right) & =\left(C_{i}-P_{i+1}\right)\left(\bar{C}_{i}-\bar{P}_{i+1}\right) \\
& =\left(C_{i}-S_{i+1, i}\right)\left(\bar{C}_{i}-\bar{S}_{i+1 i}\right) \\
& =\left(C_{i}-S_{i-1, i}\right)\left(\bar{C}_{i}-\bar{S}_{i-1, i}\right)=2-\frac{Z_{i-2}}{Z_{i+2}}-\frac{Z_{i+2}}{Z_{i-2}} \neq 0,
\end{aligned}
$$

since $Z_{i-2} \neq Z_{i+2}$. Thus, the four points $P_{i-1}, P_{i+1}, S_{i+1, i}$ and $S_{i-1, i}$ lie on a circle with center $C_{i} \in K$.

Similarly, for the points $D_{i}=-C_{i}$ we have

$$
\begin{aligned}
\left(D_{i}-P_{i-1}\right)\left(\bar{D}_{i}-\bar{P}_{i-1}\right) & =\left(D_{i}-P_{i+1}\right)\left(\bar{D}_{i}-\bar{P}_{i+1}\right) \\
& =\left(D_{i}-S_{i-1, i-2}\right)\left(\bar{D}_{i}-\bar{S}_{i-1, i-2}\right) \\
& =\left(D_{i}-S_{i+1, i+2}\right)\left(\bar{D}_{i}-\bar{S}_{i+1, i+2}\right) \\
& =\left(D_{i}-Q_{i}\right)\left(\bar{D}_{i}-\bar{Q}_{i}\right)=\frac{\left(Z_{i-2}+Z_{i+2}\right)^{2}}{Z_{i-2} Z_{i+2}}
\end{aligned}
$$

Hence, the five points $P_{i-1}, P_{i+1}, S_{i-1, i-2}, S_{i+1, i+2}$ and $Q_{i}$ lie on a circle with center $D_{i} \in K$ with radius $\left|Z_{i-2}+Z_{i+2}\right|$. It follows that the line through the points $C_{i}, D_{i}, \infty$ is the perpendicular bisector of the points $P_{i-1}, P_{i+1}$.

Notice that if

$$
\begin{equation*}
Z_{i-2} Z_{i+1}=Z_{i+2} Z_{i-1} \tag{15}
\end{equation*}
$$

then

$$
P_{i+2}=P_{i-2}=Q_{i-2}=Q_{i+2}=S_{i, i-1}=S_{i, i+1}=Z_{i}
$$

which means that the circle $K_{i}$ degenerates to a point. Vice versa, $P_{i+2}=P_{i-2}=Z_{i}$ implies (15).

We can now combine our findings and formulate the following theorem which contains the classical Five Circles Theorem and its twin. Since our calculations carry over to Miquelian Möbius planes obtained from a separable quadratic field extension, the theorem is valid in this more general framework (see Section 3).

Theorem 5. Let $Z_{1}, \ldots, Z_{5}$ be five different points on the circle $K$ given by the equation $z \bar{z}=1$, and let $C_{i}=\frac{Z_{i-1} Z_{i+1}}{Z_{i}} \in K$. Then there are two families of five circles $K_{1}, \ldots, K_{5}$, where $Z_{i}$ is the center of $K_{i}$ and such that $K_{i-2}$ and $K_{i+2}$ intersect at $P_{i} \in K$ and at $Q_{i}$ for each $i \in\{1, \ldots, 5\}$. Let $l_{i}$ denote the line through $Q_{i-2}, Q_{i+2}, \infty$, and $h_{i}$ the line through $Z_{i}, P_{i}, \infty$. Then,

- in one family the three lines $h_{i}, l_{i+1}, l_{i-1}$, meet in a point $R_{i} \in K_{i}$ and the five points $P_{i-1}, P_{i+1}, R_{i-1}, R_{i+1}$ and $Q_{i}$ lie on a circle with center $C_{i}$ (see Figure 4).
- In the other family the lines $h_{i}$ and $l_{i-1}$ meet in $S_{i, i-1} \in K_{i}$ and the lines $h_{i}$ and $l_{i+1}$ meet in $S_{i, i+1} \in K_{i}$. Moreover, the four points $P_{i-1}, P_{i+1}, S_{i+1, i}$ and $S_{i-1, i}$ lie on a circle with center $C_{i}$ and the five points $P_{i-1}, P_{i+1}, S_{i-1, i-2}, S_{i+1, i+2}$ and $Q_{i}$ lie on a circle with center $D_{i}=-C_{i}$. (see Figure 5).


Figure 4: The enhanced Five Circles Theorem. In order not to overload the figure, from the five additional circles only the dotted one with center $C_{4}$ is drawn.

Recall that the point $C_{i}$ is geometrically obtained by reflecting $Z_{i}$ at the perpendicular bisector of $Z_{i-1}$ and $Z_{i+1}$, and $D_{i}$ is the antipode of $C_{i}$ on $K$.


Figure 5: The twin of the Five Circles Theorem. In order not to overload the figure, from the ten additional circles only the dotted ones with the centers $C_{5}$ and $D_{1}$ are drawn.

## 3 Generalization to Miquelian Möbius planes

A Möbius plane is an incidence structure consisting of points $\mathbb{P}$ and blocks $\mathbb{B}$ which satisfies the following axioms (see, e.g., [4, Chapter 6] or [1]):
(M1) For any three points $P, Q, R, P \neq Q, P \neq R$ and $Q \neq R$, there exists a unique block $C$ with $P \in C, Q \in C$ and $R \in C$.
(M2) For any block $C$, and points $P, Q$ with $P \in C$ and $Q \notin C$, there exists a unique block $D$ such that $P \in D$ and $Q \in D$, but for all points $R$ with $R \in C, P \neq R$, we have $R \notin D$.
(M3) There are four points $P_{1}, P_{2}, P_{3}, P_{4}$ such that for all blocks $C$, we have $P_{i} \notin C$ for at least one $i \in\{1,2,3,4\}$. Moreover, for all blocks $C$ there exists a point $P$ with $P \in C$.

The blocks generalize the lines and circles of the classical Möbius plane. Note, however,
that the term "center of a circle" does not appear in the axioms.
A Möbius plane is called Miquelian if in addition the Six Circles Theorem of Miquel [7, Théorème I] holds:
Theorem 6 (Miquel). If one can assign 8 points $P_{1}, \ldots, P_{8}$ to the corners of a cube in such a way that the points assigned to five of its faces each lie on a circle, then this is also the case for the points assigned to the 6th face (see Figure 6).


Figure 6: The Six Circles Theorem of Miquel

A famous result by Chen [2] states that a Miquelian Möbius plane is isomorphic to a Möbius plane $\mathfrak{M}(K, q)$ over a field $K$ where $q(z)=z^{2}+a_{0} z+b_{0}$ is an irreducible polynomial with $a_{0}, b_{0} \in K$. Here, the set of points in $\mathfrak{M}(K, q)$ is

$$
\mathbb{P}:=K^{2} \cup\{\infty\}
$$

where $\infty \notin K$, and the set of blocks $\mathbb{B}$ consists of

- lines, i.e., the sets of solutions $\left(x_{1}, x_{2}\right)$ of the equations $u x_{1}+v x_{2}+w=0$ for $u, v, w \in K,(u, v) \neq(0,0)$, and the element $\infty$, and
- circles, i.e., the sets of solutions $\left(x_{1}, x_{2}\right)$ of the equations $x_{1}^{2}+a_{0} x_{1} x_{2}+b_{0} x_{2}^{2}+u x_{1}+$ $v x_{2}+w=0$ for $u, v, w \in K$, if this set of solutions consists of more than one point.

A point is incident with a block, if it is an element of the block. Let $E$ be the splitting field of $q$. Hence there are $\alpha_{1}, \alpha_{2} \in E$ such that $q(z)=\left(z+\alpha_{1}\right)\left(z+\alpha_{2}\right)$, and $E$ is a two dimensional vector space over $K$ with basis $\left\{1, \alpha_{1}\right\}$ or $\left\{1, \alpha_{2}\right\}$. Since every point $\left(x_{1}, x_{2}\right) \in K^{2}$ can be represented by $z=x_{1}+\alpha_{1} x_{2}$ or $z=x_{1}+\alpha_{2} x_{2}$, we can identify $K^{2}$ with $E$. If $q$ is separable, i.e. $\alpha_{1} \neq \alpha_{2}$, then the mapping

$$
{ }^{-}: E \rightarrow E, \quad z=x_{1}+\alpha_{1} x_{2} \mapsto \bar{z}=x_{1}+\alpha_{2} x_{2}=x_{1}+a_{0} x_{2}-\alpha_{1} x_{2}
$$

is an involutorial automorphism of $E$ (observe that $\alpha_{1}+\alpha_{2}=a_{0}$ ). Hence we have

$$
x_{1}=\frac{\alpha_{1} \bar{z}-\alpha_{2} z}{\alpha_{1}-\alpha_{2}}, \quad x_{2}=\frac{z-\bar{z}}{\alpha_{1}-\alpha_{2}},
$$

and the equation of a line $u x_{1}+v x_{2}+w=0$ can be written in the form $\bar{c} z+c \bar{z}=r$ with $c \in$ $E \backslash\{0\}$ and $r \in K$. Similarly, the equation of a circle $x_{1}^{2}+a_{0} x_{1} x_{2}+b_{0} x_{2}^{2}+u x_{1}+v x_{2}+w=0$ can be written as a quadratic equation of the form $(z-c)(\bar{z}-\bar{c})=r$ for $r \in K \backslash\{0\}$, and $c \in E$ (use $x_{1}^{2}+a_{0} x_{1} x_{2}+b_{0} x_{2}^{2}=z \bar{z}$ for $z=x_{1}+\alpha_{1} x_{2}$ ). Hence, in this case the center $c$ can be assigned to the circle. For $K=\mathbb{R}$ and $q(z)=z^{2}+1$ we have $E=\mathbb{C}$ and we are in the situation of the classical model of the Möbius plane as describes in the previous section. Another example is the Galois field $K=G F(t)$ for an odd prime power $t=p^{n}$, and $q(z)=z^{2}-\alpha$ for a non-square $\alpha \in G F(t)$. Then, $G F(t)(\alpha) \cong G F\left(t^{2}\right)$ and the conjugation is given by the Frobenius automorphism $z \mapsto \bar{z}=z^{t}$.

If $q$ is separable, the proofs of the previous section carry over verbatim to the Möbius plane $\mathfrak{M}(K, q)$. Notice also, that every finite extension of a finite field is separable. Hence, Theorem 5 is valid in each Miquelian Möbius plane $\mathfrak{M}(K, q)$ if $q$ is separable, and in particular in every finite Miquelian Möbius plane. Notice however that by (3), over a field $K$ of characteristic 2, the two families of five circles in Theorem 5 coincide: The twin is identical to the classical statement in this case.

## 4 Some closing remarks

The reader may notice that there are other incidences hidden in the Five Circles configuration:

- Each one of the quadruples $Z_{i}, P_{i+1}, Q_{i+2}, \infty$ and $Z_{i}, P_{i-1}, Q_{i-2}, \infty$ are contained in a block.
- The line $l_{i}$ and the line $l_{i}^{\prime}$ through the points $Z_{i-1}, Z_{i+1}, \infty$ are touching at $\infty$ (i.e., the lines are parallel).

However, both observations are general properties of Miquelian Möbius planes obtained from a separable quadratic field extension and not limited to the Five Circles configuration as we show in the following two propositions.
Proposition 7. Let $K$ be a circle, $P, X, Y \in K, K_{X}$ the circle with center $X$ through $P$, and $K_{Y}$ the circle with center $Y$ through $P$. Let $Q \neq P$ be the second intersection of $K_{X}$ and $K_{Y}$, and $P_{X} \neq P$ and $P_{Y} \neq P$ the second intersections of $K$ with $K_{X}$ and $K_{Y}$, respectively. Then the points $X, Q, P_{Y}, \infty$ and the points $Y, Q, P_{X}, \infty$ each lie on a line (see Figure 7, left).

Proof. We may again assume that $K$ is the unit circle with the equation $z \bar{z}=1$. Using the maps $\varphi_{X}$ and $\varphi_{Y}$, we find $P_{X}=X^{2} \bar{P}$ and $P_{Y}=Y^{2} \bar{P}$. It is then elementary to check that

$$
Q=X+Y-X Y \bar{P}
$$



Figure 7: Two general incidence relations in Miquelian Möbius planes

If we use that $\bar{X}=1 / X, \bar{Y}=1 / Y$ and $\bar{P}=1 / P$, a short calculation shows that $z=Q$ solves the equations

$$
\begin{aligned}
(X-z)(\bar{X}-\bar{z}) & =(X-P)(\bar{X}-\bar{P}) \\
(Y-z)(\bar{Y}-\bar{z}) & =(Y-P)(\bar{Y}-\bar{P})
\end{aligned}
$$

of $K_{X}$ and $K_{Y}$, respectively.

Proposition 8. Let $K$ be a circle, $X, Y, Z \in K, K_{X}$ a circle with center $X, K_{Y}$ a circle with center $Y$, and $K_{Z}$ a circle with center $Z$, such that $K_{X} \cap K_{Y}=\{P, Q\}$ with $P \in K$ and $K_{X} \cap K_{Z}=\{R, S\}$ with $R \in K$. Then the lines through $Y, Z, \infty$ and through $Q, S, \infty$ are touching at $\infty$ (see Figure 7, right).

Proof. We assume again that $K$ is the unit circle $z \bar{z}=1$. Using the map $\varphi_{X}$ we find that $R=X^{2} \bar{P}$. Then we infer from the proof of Proposition 7 that $Q=X+Y-X Y \bar{P}$ and $S=X+Z-\bar{X} Z P$. Thus the lines through $R, S, \infty$ and through $Y, Z, \infty$ are given by the equations

$$
\begin{aligned}
(Q-z)(\bar{S}-\bar{z}) & =(\bar{Q}-\bar{z})(S-z) \\
(Y-z)(\bar{Z}-\bar{z}) & =(\bar{Y}-\bar{z})(Z-z)
\end{aligned}
$$

It is now easy to check that $(Q-S)(\bar{Z}-\bar{Y})-(\bar{Q}-\bar{S})(Z-Y)=0$, and hence the two lines are indeed touching at $\infty$.

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