Geometric Properties and Iterates of Hesse Derivatives of Cubic Curves

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key-words: Elliptic curves, Hessian curves, geometry of cubic curves, halving formulae, discrete dynamical system

2020 Mathematics Subject Classification: **11G05** 37N99

Abstract

The Hesse curve or Hesse derivative হΓ_f of a cubic curve Γ_f given by a homogeneous polynomial f is the set of points P such that $\det(H_f(P)) = 0$, where $H_f(P)$ is the Hesse matrix of f evaluated at P. Then হΓ_f is again a cubic curve. We show that for any point $P \in$ হΓ_f, all contact points of tangents from P to both curves Γ_f and হΓ_f are intersection points of two straight lines ℓ_1^P and ℓ_2^P (meeting on হΓ_f) with Γ_f and হΓ_f, where the product of ℓ_1^P and ℓ_2^P is the degenerate polar conic of Γ_f at P. Furthermore, the operator হ defines an iterative discrete dynamical system on the set of cubic curves. We identify the two fixed points of this system, investigate orbits that end in the fixed points, and discuss the closed orbits of the dynamical system.

1 Introduction

We will work with cubic curves in the real projective plane $\mathbb{RP}^2$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by capital letters, the components with small letters, and the equivalence class by $[X] := \{\lambda X \mid \lambda \in \mathbb{R} \setminus \{0\}\}$. However, since we mostly work with representatives, we often omit the square brackets in the notation.

Let f be a homogeneous polynomial in the variables x_1, x_2, x_3 of degree 3. Then f defines the projective cubic curve

$$\Gamma_f := \left\{[X] \in \mathbb{RP}^2 \mid f(X) = 0\right\}.$$

The Hesse matrix of f is the symmetric 3×3 matrix $H_f = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Observe that $\det(H_f)$ is again a homogeneous cubic polynomial. Therefore, we can define the *Hesse derivative* of Γ_f, denoted হΓ_f[1], as the cubic curve

$$\text{হ}\Gamma_f := \Gamma_{\det(H_f)} = \left\{[X] \in \mathbb{RP}^2 \mid \det(H_f(X)) = 0\right\}.$$

[1]In an extraordinary twist of mathematical fate, we find ourselves compelled to introduce the Bengali alphabet হ (pronounced "Haw") as a notation in this paper to denote the Hesse derivative. In a desperate search for suitable symbols beyond the scope of English, Greek and Latin, we have been left with no choice but to embark on this linguistic expedition. Fun Fact: "Hesse" in Bengali means "to laugh"!

The polar conic of Γ_f with respect to the pole P is given by the equation

$$\mathcal{C}_f(P):\ \langle X, H_f(P)X\rangle = 0. \tag{1}$$

Here $H_f(P)X$ denotes the product of the Hesse matrix of f evaluated at P with the vector X, and $\langle\cdot\,,\,\cdot\rangle$ denotes the Euclidean inner product. An equivalent formulation is given by

$$\mathcal{C}_f(P):\ \langle \nabla f(X), P\rangle = 0, \tag{2}$$

where ∇ denotes the gradient operator. Indeed, a simple but somewhat tedious calculation shows that $\langle X, H_f(P)X\rangle = 2\langle \nabla f(X), P\rangle$ (see [4] for details and a general formula of this type for homogeneous polynomials of any degree). It is clear from (2) that the contact points of the tangents from P to Γ_f are precisely the intersection points of $\mathcal{C}_f(P)$ with Γ_f (see Figure 1).

If there is no danger of confusion, we will omit the index and briefly write Γ instead of Γ_f. Moreover, we will use the notation H_Γ instead of H_f, and $\mathcal{C}_\Gamma(P)$ instead of $\mathcal{C}_f(P)$ if the polynomial f is determined by the context or if a general but unique polynomial is meant. We would like to mention that the Hesse derivative हΓ is also known as *Hessian curve*, denoted Hess(Γ) (see, e.g.,[5, § 4.12 , p. 111]). However, we prefer the notation हΓ because we want to interpret ह as an operator whose iterations we want to study. Whenever convenient, we will use x, y, z instead of x_1, x_2, x_3 for the coordinates. The figures below of the various projective curves show images of the curves in the affine plane $x_3 = 1$ embedded in $\mathbb{RP}^2$.

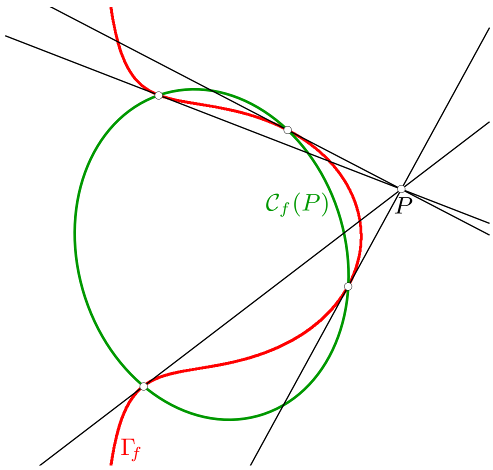

Figure 1: A cubic curve Γ_f, its polar conic $\mathcal{C}_f(P)$ with respect to the pole P, and the tangents from P to the curve Γ_f.

It is well known that the polar conic is the product of two projective lines ℓ_1^P and ℓ_2^P, if and only if the determinant of the Hesse matrix evaluated at P is equal to 0, i.e.,

$$\mathcal{C}_\Gamma(P) = \langle X, \ell_1^P\rangle\langle X, \ell_2^P\rangle \iff \det\left(H_\Gamma(P)\right) = 0.$$

In particular, we obtain the following well-known result (see Figure 2):

Fact 1. *For any point $P \in$ हΓ, the polar conic is the product of the two lines ℓ_1^P and ℓ_2^P, and the tangents from P to Γ touch Γ precisely at the points $\Gamma \cap \ell_1^P$ and $\Gamma \cap \ell_2^P$.*

In the next section we compute the contact points of the tangents from a point $P \in$ हΓ to the curve हΓ. In Section 3 we show that these contact points lie also on the degenerate polar conic $\mathcal{C}_f(P)$ and that the intersection point of the two lines of the conic $\mathcal{C}_f(P)$ lies on the curve हΓ (see

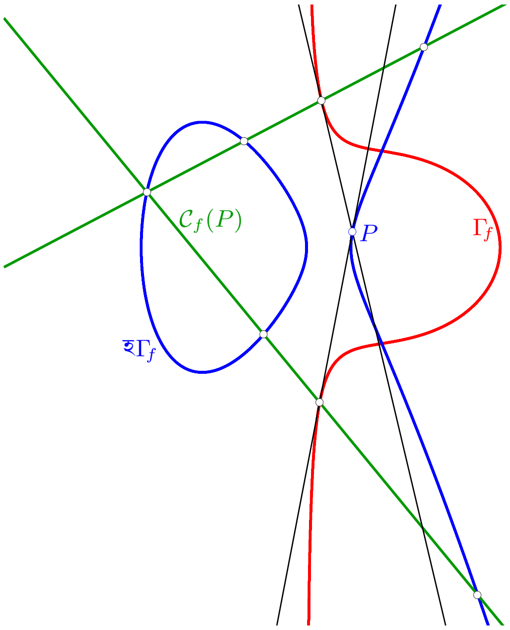

Figure 2: A cubic curve Γ_f, its degenerate polar conic $\mathcal{C}_f(P)$ with respect to the pole P on the Hesse derivative हΓ_f, and the tangents from P to the curve Γ_f.

Figure 3). In Section 4 and 5 we investigate iterated Hesse derivatives of cubic curves in Hesse form, and then compute in Section 6 the number of chains and loops of given length of iterated Hesse derivatives. Finally, in Section 7 it is shown how the results from Section 6 for cubic curves in Hesse form can be transformed into curves in other normal forms.

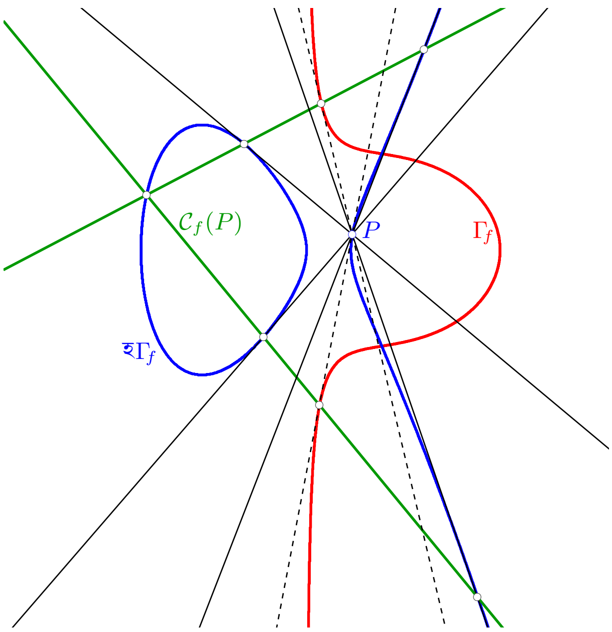

Figure 3: A cubic curve Γ_f, its degenerate polar conic $\mathcal{C}_f(P)$ with respect to the pole P on the Hesse derivative हΓ_f, and the tangents from P to the curves Γ_f and हΓ_f.

2 Halving formulae for points on হΓ

In this section, we compute the contact points of the tangents from a point $P \in$ হΓ to the curve হΓ. By a suitable projective transformation, we may assume that the curve is of the form $E_{a,b}$ defined by

$$E_{a,b}:\ y^2 = x^3 + a\,x^2 + b\,x \tag{3}$$

in the affine plane, where $a,\, b \in \mathbb{R}$.

Proposition 2. *Let $E_{a,b}$ be a non-singular elliptic curve over $\mathbb{C}$ defined by*

$$E_{a,b}:\ y^2 = x^3 + a\,x^2 + b\,x$$

where $a,\, b \in \mathbb{C}$, and let $P = (x_0, y_0)$ be a point on $E_{a,b}$.

Let

$$e_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad e_2 = \frac{-a - \sqrt{a^2 - 4b}}{2},$$

and let

$$\gamma = \sqrt{x_0}, \qquad \alpha = \sqrt{x_0 - e_1}, \qquad \beta = \sqrt{x_0 - e_2}.$$

Then, $E_{a,b}$ is of the form

$$y^2 = x(x - e_1)(x - e_2)$$

and the x-coordinates of the contact points of the tangents of P with $E_{a,b}$ are

$$\begin{aligned} x_{11} &= (\alpha + \gamma)(\beta + \gamma), \\ x_{12} &= (\alpha - \gamma)(\beta - \gamma), \\ x_{21} &= (\alpha + \gamma)(-\beta + \gamma), \\ x_{22} &= (\alpha - \gamma)(-\beta - \gamma). \end{aligned}$$

Proof. Since $-e_1 - e_2 = a$ and $e_1 e_2 = b$, the curve $E_{a,b}$ is of the form $y^2 = x(x - e_1)(x - e_2)$. In order to show that $x_{11}, x_{12}, x_{21}, x_{22}$ are the x-coordinates of the contact points of the tangents of P with $E_{a,b}$, it is enough to show that the x-coordinate of the points $2 * Q_{ij}$, where $i, j \in \{1, 2\}$, $Q_{ij} := (x_{ij}, y_{ij})$ and $y_{ij}^2 = x_{ij}^3 + a\,x_{ij}^2 + b\,x_{ij}$, is equal to x_0. In other words, for the four points Q_{ij} we have $2 * Q_i = -P$, i.e., $\frac{P}{2} = -Q_{ij}$. Here $2 * Q_{ij} = Q_{ij} + Q_{ij}$ is the usual elliptic curve operation on $E_{a,b}$ (see, e.g., [3]). Now, the x-coordinate x_{2ij} of the point $2 * Q_{ij}$ is given by the formula

$$x_{2ij} = \frac{x_{ij}^4 - 2b\,x_{ij}^2 + b^2}{4(x_{ij}^3 + a\,x_{ij}^2 + b\,x_{ij})} = \frac{(x_{ij}^2 - b)^2}{4x_{ij}(x_{ij}^2 + a\,x_{ij} + b)}.$$

Furthermore, we have $a = \alpha^2 + \beta^2 - 2\gamma^2$ and $b = (\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)$, and if we write x_{ij}, a, b in terms of γ, α, β, it is not hard to verify that

$$(x_{ij}^2 - b)^2 = 4x_{ij}\gamma^2(x_{ij}^2 + a\,x_{ij} + b),$$

which shows that $x_{2ij} = x_0$. *q.e.d.*

3 Intersection of $\mathcal{C}_\Gamma(P)$ with হΓ for $P \in$ হΓ

In this section, we combine the results from Section 2 with the property that for every point $P \in$ হΓ, the polar conic of Γ with respect to the pole P is the product of two lines, ℓ_1^P and ℓ_2^P (see

Fact 1). In particular, we will show that the intersection of the two lines ℓ_1^P and ℓ_2^P lies on the curve $\text{ह}\Gamma$ and that the points x_{ij}, $i, j = 1, 2$ correspond to the other intersection points of ℓ_1^P and ℓ_2^P with $\text{ह}\Gamma$ (see Figure 4).

For this, we consider the projective cubic curve

$$\Gamma_{a,b}: \; ax^3 + 3xy^2 + 3bx^2z - b^2z^3 = 0$$

with $a, b \in \mathbb{R}$, $b \neq 0$. We first show that every regular cubic curve Γ can be transformed to the curve $\Gamma_{a,b}$. By a suitable projective transformation, we may assume that Γ is of the form $E_{A,B} : y^2 = x^3 + A\,x^2 + B\,x$. Then, the projective transformation given by the matrix

$$\begin{pmatrix} A & 0 & -(A^2-3B)^2 \\ 0 & 1 & 0 \\ -3 & 0 & 0 \end{pmatrix}$$

transforms the curve $E_{A,B}$ to the curve $\Gamma_{a,b}$ with $a = -2A^3 + 9AB$ and $b = (A^2 - 3B)^3$.

Now, for $\Gamma_{a,b}$, we get

$$H_{\Gamma_{a,b}} := \begin{pmatrix} 6ax + 6bz & 6y & 6bx \\ 6y & 6x & 0 \\ 6bx & 0 & -6b^2z \end{pmatrix}$$

and hence

$$\text{ह}\Gamma_{a,b}: \; y^2z = x^3 + ax^2z + bxz^2.$$

In other words, $\text{ह}\Gamma_{a,b} = E_{a,b}$, as introduced in (3) in the previous section.

By definition, if $P = (x_0, y_0, z_0) \in \text{ह}\Gamma_{a,b}$, then $\det\big(H_{\Gamma_{a,b}}(P)\big) = 0$, which implies that the conic section

$$\mathcal{C}_{\Gamma_{a,b}}(P): \; (ax_0 + bz_0)x^2 + x_0y^2 + 2y_0xy + 2bx_0xz - b^2z_0z^2 = 0$$

can be written as the product of two lines ℓ_1^P and ℓ_2^P. In the following lemma, we will compute these two lines in terms of e_1 and e_2 defined in the previous section.

Lemma 3. *The lines ℓ_1^P and ℓ_2^P are given by*

$$\ell_1^P: \; ux + vy + wz = 0$$
$$\ell_2^P: \; rx + sy + tz = 0$$

where

$$\begin{aligned} u &= -x_0 - \sqrt{(e_1 - x_0)(e_2 - x_0)} \\ v &= -\sqrt{x_0} \\ w &= e_1e_2 \\ r &= -x_0 + \sqrt{(e_1 - x_0)(e_2 - x_0)} \\ s &= \sqrt{x_0} \\ t &= e_1e_2\,. \end{aligned}$$

Proof. If we replace y_0 by $\sqrt{x_0^3 + ax_0^2 + bx_0}$, and a, b by $-e_1 - e_2$ and e_1e_2, respectively, then we have

$$\mathcal{C}_{\Gamma_{a,b}}: \; (e_1e_2 - e_1x_0 - e_2x_0)\,x^2 + x_0\,y^2 + 2\sqrt{x_0^3 - e_1x_0^2 - e_2x_0^2 + e_1e_2x_0}\;xy + 2e_1e_2x_0\,x - e_1^2e_2^2.$$

Now, in order to show that $\mathcal{C}_{\Gamma_{a,b}} = \ell_1^P \cdot \ell_2^P$, we just have to check that

$$\begin{aligned}
ur &= -e_1e_2 + e_1x_0 + e_2x_0 \\
us + vr &= -2\sqrt{x_0^3 - e_1x_0^2 - e_2x_0^2 + e_1e_2x_0} \\
vs &= -x_0^2 \\
ut + wr &= -2e_1e_2x_0 \\
wt &= e_1^2e_2^2 \\
vt + ws &= 0
\end{aligned}$$

which is easy to see. *q.e.d.*

Before we compute the intersection points of ℓ_1^P and ℓ_2^P with the curve $E_{a,b}$, we show that the two lines intersect on the curve $E_{a,b}$.

Lemma 4. *Let $P = (x_0, y_0, z_0)$ be a point on $\text{ह}\Gamma_{a,b}$. Then the point $\ell_1^P \cap \ell_2^P =: S = (x_S, y_S, z_S)$ lies on the same curve $\text{ह}\Gamma_{a,b} = E_{a,b}$.*

Proof. First, let us assume $x_0 \neq 0$ and rewrite the lines as

$$\begin{aligned}
\ell_1^P:\ y &= \frac{-u}{v}x + \frac{-w}{v} \\
\ell_2^P:\ y &= \frac{-r}{s}x + \frac{-t}{s}.
\end{aligned}$$

Then the intersection $S = (x_S, y_S, 1)$ of the two lines is given by

$$\begin{aligned}
x_S &= \frac{tv - ws}{us - rv} = \frac{e_1e_2}{x_0} = \frac{b}{x_0} \\
y_S &= \frac{-u}{v}x_S + \frac{-w}{v} = -\frac{by_0}{x_0^2}.
\end{aligned}$$

Now, we use the fact that

$$y_0^2 = x_0^3 + ax_0^2 + bx_0$$

to show that

$$\Big(\frac{b}{x_0}\Big)^3 + a\Big(\frac{b}{x_0}\Big)^2 + b\Big(\frac{b}{x_0}\Big) = \Big(\frac{-by_0}{x_0^2}\Big)^2$$

which means that $S \in E_{a,b}$, as claimed.

On the other hand, if $x_0 = 0$, then $P = (0,0,1)$ or $P = (0,1,0)$. In the first case, we have $\mathcal{C}_{\Gamma_{a,b}} : x^2 - bz^2 = (x + \sqrt{b}z)(x - \sqrt{b}z) = 0$, and the two lines intersect in $S = (0,1,0)$ on $E_{a,b}$. In the second case, we have $\mathcal{C}_{\Gamma_{a,b}} : xy = 0$, and the two lines intersect in $S = (0,0,1)$ on $E_{a,b}$. *q.e.d.*

Lemma 5. *The map $\text{ह}\Gamma_{a,b} \to \text{ह}\Gamma_{a,b}$, $P = (x_0, y_0, z_0) \mapsto S = (x_S, y_S, z_S)$, is an involution.*

Proof. For $x_0 \neq 0$, we just have to check that

$$\frac{b}{\frac{b}{x_0}} = x_0 \qquad \text{and} \qquad \frac{-b \cdot \frac{-by_0}{x_0^2}}{\frac{b^2}{x_0^2}} = y_0,$$

which is easy to see. For the case $x_0 = 0$ see the end of the proof of Lemma 4. *q.e.d.*

Now, we show that the other intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points x_{ij}, $i, j = 1, 2$ from Proposition 2.

Lemma 6. *Besides the point S, the intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points Q_i, $i = 1, 2, 3, 4$, with $2 * Q_i = -P$. More precisely, the points x_{11} and x_{12} are on the line ℓ_1^P and the points x_{21} and x_{22} are on ℓ_2^P.*

Proof. To find the x-coordinate of the intersection points of ℓ_i^P with $E_{a,b}$ we eliminate y form the equations for $E_{a,b}$ and ℓ_1^P, and ℓ_2^P, respectively. The resulting equations are of degree 3 in x, but since we already know the root $\frac{b}{x_0}$, the problem reduces to a quadratic equation

$$x^2 - 2\left(x_0 \pm \alpha\beta\right)x + e_1e_2 = 0.$$

The solutions are

$$x_0 + \alpha\beta \pm \sqrt{\left(x_0 + \alpha\beta\right)^2 - e_1e_2} \qquad \text{and} \qquad x_0 - \alpha\beta \pm \sqrt{\left(x_0 - \alpha\beta\right)^2 - e_1e_2}$$

and one checks easily that these expressions agree with the formulas for x_{ij} from Proposition 2.

It remains to show that the y-coordinates match as well. To see that, let us denote by A and B the points at which the tangents from P to $E_{a,b}$ meet $E_{a,b}$. So, for our claim to be true, we have

$$\begin{aligned} A + B &= -S \\ 2 * A &= -P \\ 2 * B &= -P \end{aligned}$$

which implies

$$2 * P = 2 * S$$

and hence it is enough to show that this is indeed the case. To do so, we note that a formula for doubling the point $P = (x_0, y_0)$ on $E_{a,b}$ is given by

$$2 * P = \left(\frac{\left(x_0^2 - e_1e_2\right)^2}{4y_0^2}, \frac{\left(x_0^2 - e_1e_2\right)\left(e_1e_2 - 2e_1x_0 + x_0^2\right)\left(e_1e_2 - 2e_2x_0 + x_0^2\right)}{8y_0^3} \right).$$

On the other hand, since $S = \left(\frac{e_1e_2}{x_0}, \frac{-e_1e_2y_0}{x_0^2}\right)$, we have

$$2 * S = \left(\frac{\left(\frac{e_1^2e_2^2}{x_0^2} - e_1e_2\right)^2}{4 \cdot \frac{e_1^2e_2^2y_0^2}{x_0^4}}, \frac{\left(\frac{e_1^2e_2^2}{x_0^2} - e_1e_2\right)\left(e_1e_2 - 2e_1 \cdot \frac{e_1e_2}{x_0} + \frac{e_1^2e_2^2}{x_0^2}\right)\left(e_1e_2 - 2e_2 \cdot \frac{e_1e_2}{x_0} + \frac{e_1^2e_2^2}{x_0^2}\right)}{-8 \cdot \frac{e_1^3e_2^3y_0^3}{x_0^6}} \right).$$

These two expressions for $2 * P$ and $2 * S$ obviously coincide. *q.e.d.*

Theorem 7. *Let Γ be a cubic curve and let $P \in$ हΓ. Then, all the contact points of tangents from P to the curves Γ and हΓ are intersection points of ℓ_1^P and ℓ_2^P with Γ and हΓ. In addition, the intersection Q of ℓ_1^P and ℓ_2^P lies on हΓ (see Figure 4).*

4 Hesse Form of Cubic Curves

In this section, we consider a cubic curve in its Hesse form

$$\Gamma_c: \; x^3 + y^3 + z^3 + c\,xyz = 0$$

with $c \in \mathbb{R}$. Notice Γ_{-3} is a degenerate curve. Formally, we put

$$\Gamma_\infty: \; xyz = 0.$$

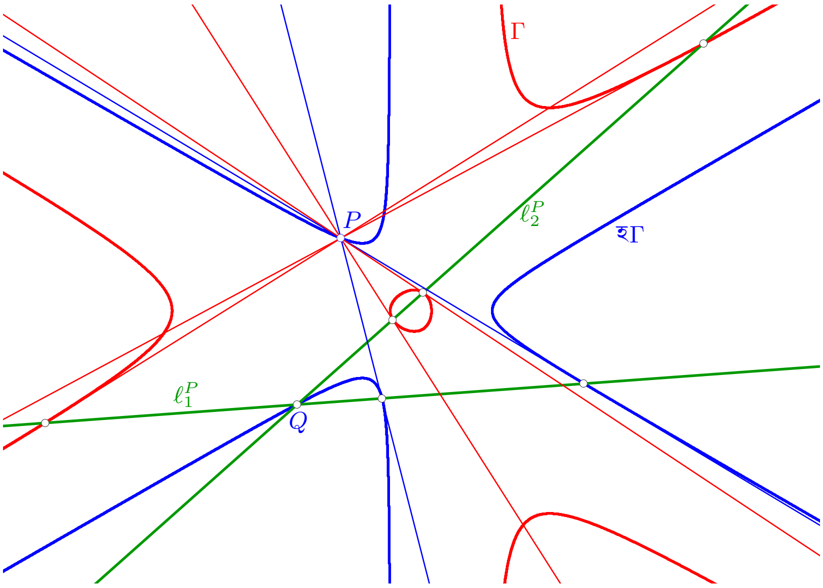

Figure 4: Illustration for Theorem 7 for a cubic curve Γ with the symmetry group of an equilateral triangle (see [1]). The Hesse derivative हΓ has the same symmetry. The curve Γ is given by $2\sqrt{3}x^3 + 9(\sqrt{3}+1)(x^2+y^2)z - 6\sqrt{3}xy^2 - 9z^3 = 0$ and has the property that ह$^2\Gamma = \Gamma$ (see Section 6).

Lemma 8. *Let $c_0 \neq 0$. Then the Hesse derivative of Γ_{c_0} is ह$\Gamma_{c_0} = \Gamma_{c_1}$ where*

$$c_1 = -\frac{108 + c_0^3}{3c_0^2}.$$

The Hesse derivative of Γ_0 is ह$\Gamma_0 = \Gamma_\infty$, and the Hesse derivative of Γ_∞ is ह$\Gamma_\infty = \Gamma_\infty$.

Proof. We have

$$H_{\Gamma_{c_0}}(x,y,z) := \begin{pmatrix} 6x & c_0 z & c_0 y \\ c_0 z & 6y & c_0 x \\ c_0 y & c_0 x & 6z \end{pmatrix}.$$

This yields $\det H_{\Gamma_{c_0}}(x,y,z) = -6c^2(x^3+y^3+z^3) + 2(108+c^3)xyz$, and the claim follows for $c_0 \in \mathbb{R} \setminus \{0\}$. The cases Γ_0 and Γ_∞ are also easily checked. *q.e.d.*

An immediate corollary is

Corollary 9. *Let $c_0 \neq 0$. Then, the $(n+1)$-th Hesse derivative of Γ_{c_0} is given by*

$$ह^{n+1}\Gamma_{c_0} : \; x^3 + y^3 + z^3 + c_{n+1}xyz = 0$$

where $c_{n+1} = -\frac{108+c_n^3}{3c_n^2}$ for every $n \geq 0$, as long as $c_n \neq 0$.

5 Analysis of iterates

Motivated by Lemma 8, we consider the function

$$h(x) = \frac{a + x^3}{bx^2} \tag{4}$$

for $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$.

Fact 10. *The function h defined in (4) has a pole at $x = 0$, and an oblique asymptote $y = \frac{x}{b}$. For $b \neq -1$*

$$\varphi := \sqrt[3]{\frac{a}{b-1}}$$

is the unique real fixed point of h. The function h has the unique critical point $\kappa := \sqrt[3]{2a}$ with critical value

$$h(\kappa) = \frac{3a^{1/3}}{2^{2/3}b}.$$

The proof is elementary.

Remark: In our case, $b = -3$, we have

$$h(\varphi) = \varphi = -\sqrt[3]{\frac{a}{4}} = -\sqrt[3]{\frac{a^3}{4a^2}} = h(\kappa).$$

This case also gives us the crucial property for the partition of $\mathbb{R} \setminus \{\varphi\}$ into the intervals $N = (-\infty, \varphi)$ and $P = (\varphi, \infty)$. Namely, we have $x \in P \setminus \{0\}$ iff $h(x) \in N$, and $x \in N$ iff $h(x) \in P$. For the next propositions, we will assume $b = -3$ and $a \neq 0$, and hence $\varphi \neq 0$. For now, we refrain from imposing a specific value of a, since, as we will see below, most of the results are independent of the value of a. We will specify $a = 108$ in the next section.

Proposition 11. *Let $b = -3$ and $a \neq 0$. If we define*

$$h^{(n)} := \underbrace{h \circ h \circ \ldots h}_{n \text{ times}}$$

then, $y = \frac{x}{b^n}$ is an oblique asymptote of $h^{(n)}$. Furthermore, if κ_n is a critical point of $h^{(n)}$, we have

$$h^{(n)}(\kappa_n) = \varphi = -\sqrt[3]{\frac{a}{4}}.$$

Conversely, if $h^{(n)}(x) = \varphi$, then either $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$.

Proof. Since we already know the oblique asymptote of h from Fact 10, we can now inductively argue that

$$\begin{aligned}
\lim_{x\to\pm\infty} \left| h^{(n+1)}(x) - \frac{x}{b^{n+1}} \right| &= \lim_{x\to\pm\infty} \left| \frac{a + \left(h^{(n)}(x)\right)^3}{b\left(h^{(n)}(x)\right)^2} - \frac{\frac{x}{b^n}}{b} \right| \\
&= \lim_{x\to\pm\infty} \left| \frac{a}{b\left(h^{(n)}(x)\right)^2} + \frac{h^{(n)}(x) - \frac{x}{b^n}}{b} \right| = 0
\end{aligned}$$

using the fact that $\left(h^{(n)}(x)\right)^2 \to \infty$ for $x \to \pm\infty$. Note that in this part we did not need the assumption $b = -3$.

For the next part, observe first that we have

$$h(x) = \varphi \iff (x - \varphi)(x - \kappa)^2 = 0.$$

This equation has only two solutions, namely $x_1 = \varphi$ and $x_2 = \kappa$. By chain rule we obtain

$$\begin{aligned}
\frac{d}{dx}h^{(n)}(x) = \prod_{r=0}^{n-1} h'\left(h^{(r)}(x)\right) = 0 \qquad (5)\\
\iff h'\left(h^{(r)}(x)\right) = 0 \text{ for some } r \in \{0, 1, \ldots, n-1\} \\
\iff h^{(r)}(x) = \kappa \text{ for some } r \in \{0, 1, \ldots, n-1\}.
\end{aligned}$$

So, it follows immediately that $\frac{d}{dx}h^{(n)}(x) = 0$ implies $h^{(n)}(x) = \varphi$.

Now we prove the converse, as stated in the lemma. For $n = 0$, the statement is trivially true. Assume that for some $n \geq 1$ we have that $h^{(n)}(x) = \varphi$ implies that either $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$. Then we have for $h^{(n+1)}(x) = \varphi$ that $h^{(n)}(x) = \kappa$ or $h^{(n)}(x) = \varphi$. On the other hand, $\frac{d}{dx}h^{(n+1)}(x) = h'(h^{(n)}(x))\frac{d}{dx}h^{(n)}(x)$. If $h^{(n)}(x) = \kappa$, then the first factor in this product is zero and the derivative of $\frac{d}{dx}h^{(n+1)}(x)$ vanishes. If $h^{(n)}(x) = \varphi$, then, by induction, $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$, and again the derivative of $\frac{d}{dx}h^{(n+1)}(x)$ is zero. *q.e.d.*

Proposition 12. *Let χ_n be the number of critical points of $h^{(n)}$. Then, the sequence $\{\chi_n\}$ is given by*

$$\chi_{2r+1} = 2 \times 3^r - 1 \quad \textit{and} \quad \chi_{2r} = 3^r - 1$$

for all integers $r \geq 0$. This corresponds to OEIS A062318.

Proof. We only carry out the case $\varphi < 0$. The proof for $\varphi > 0$ is essentially the same.

Let $N = (-\infty, \varphi)$ and $P = (\varphi, \infty)$. Observe first that for given $y \in N$ the equation

$$y = h(x) \quad \text{or equivalently} \quad x^3 + 3yx^2 - 4\varphi^3 = 0$$

has three distinct real roots. Indeed, the discriminant $\Delta = 27 \times 16\varphi^3\left(y^3 - \varphi^3\right)$ is strictly positive for $y \in N$. Moreover, if $x, y \in N$ then the expression $x^3 + 3yx^2 - 4\varphi^3$ is strictly negative, hence the three solutions of $y = h(x)$ must lie in P. Thus, the preimage $h^{-1}(y)$ of a point $y \in N$ has cardinality 3, and lies in P. Similarly, the preimage $h^{-1}(y)$ of a point $y \in P$ has cardinality 1, and lies in N.

Now, for $n = 1$, the set of critical points of h is $C_1 = \{\kappa\} \subset P$. Let $S_1 := h^{-1}(C_1)$, and $S_k := h^{-1}(S_{k-1})$ for $k > 1$. In other words, S_k is the preimage of $\{\kappa\}$ under $h^{(k)}$. Observe that $S_n \subset N$ if n is odd, and $S_n \subset P$ if n is even. Also note that two sets S_k and S_j, $k > j$ are disjoint. Indeed, if $x \in S_j$, then $h^{(j)}(x) = \kappa$, and hence $h^{(k)}(x) = \varphi \neq \kappa$. It follows that $\operatorname{card} S_{2n} = \operatorname{card} S_{2n+1} = 3^n$. For $n > 1$ we can read off from equation (5) that the set of critical points of $h^{(n)}$ is the set $C_n = C_{n-1} \cup S_{n-1}$. In particular, the sets C_n and S_n are disjoint for all n since the sets S_n are disjoint. Hence, we have $\operatorname{card} C_{2n} = \operatorname{card} C_{2n-1} + 3^{n-1}$, and $\operatorname{card} C_{2n+1} = \operatorname{card} C_{2n} + 3^n$. This corresponds to the sequence OEIS A062318. *q.e.d.*

Proposition 13. *Let Φ_n be the number of fixed points of $h^{(n)}$. Then, the sequence $\{\Phi_n\}$ is given by*

$$\Phi_{2r+1} = 1 \quad \textit{and} \quad \Phi_{2r} = 2\chi_{2r} - 1 = 2 \times 3^r - 3$$

for all integers $r \geq 0$.

Proof. Since h maps N to P and vice versa, the only fixed point of $h^{(2r+1)}$ is φ.

For the fixed points of $h^{(2r)}$, we begin by assuming without loss of generality that $a > 0$, $\varphi < 0$, and recalling that h has a pole at $x = 0$, it is decreasing in the intervals $(-\infty, 0)$, (κ, ∞) and increasing in $(0, \kappa)$. We also know that the only critical point of h is at κ and it has a local maximum there. So, for convenience, we define

$$\tilde{h}(x) := h(x + \varphi) - \varphi$$

i.e., we shift the point (φ, φ) to the origin. The number of fixed points of $h^{(n)}$ is the same as the number of fixed points of $\tilde{h}^{(n)}$. Consider the sets $\tilde{N} = (-\infty, 0)$, $\tilde{P} = (0, \infty)$, and $A = (0, -\varphi)$, $B = (-\varphi, -\varphi + \kappa)$, $C = (-\varphi + \kappa, \infty)$. Then, $\tilde{h}$ maps $\tilde{N}$ bijectively to $\tilde{P}$, and A, B and C each bijectively to $\tilde{N}$. Hence $\tilde{h}$ maps $\tilde{N}, \tilde{P}$ to $\tilde{P}$ and tree copies of $\tilde{N}$. So after $2r$ iterations, the range of $\tilde{h}^{(2r)}$

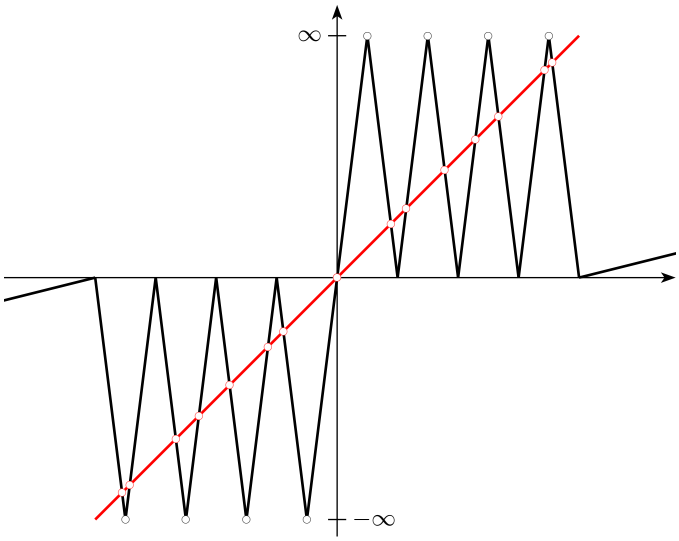

Figure 5: Schematic profile of the function $\tilde{h}^{(2r)}$: We have $(3^r-1)/2$ spikes on the positive x-axis and $(3^r-1)/2$ on the negative x-axis.

consists of 3^r copies of $\tilde{N}$ and 3^r copies of $\tilde{P}$. Figure 5 shows schematically the behaviour of $\tilde{h}^{(2r)}$.

Observe that the oblique asymptote of $\tilde{h}^{(2r)}$ is given by $y = x/3^{2r}$, hence the line $y = x$ does not intersect the leftmost and the rightmost branch of the graph of $\tilde{h}^{(2r)}$. Hence the number of fixed points of $\tilde{h}^{(2r)}$ is $2(3^r - 1) - 1 = 2 \times 3^r - 3$. *q.e.d.*

The number of zeros of $h^{(n)}$ is now calculated in the same way as the number of fixed points.

Proposition 14. *Let ρ_n be the number of zeros of $h^{(n)}$. Then, the sequence $\{\rho_n\}$ is given by*

$$\rho_{2r} = \rho_{2r+1} = 3^r$$

for all integers $r \geq 0$.

Proof. We still assume $\varphi < 0$, the case $\varphi > 0$ is similar. The number of zeros of $h^{(n)}$ equals the number of solutions of the equation $\tilde{h}^{(n)} = -\varphi$. Let us first assume that $n = 2r$ is even. Then, as in the proof of Proposition 13, $\tilde{h}^{(2r)}$ maps $\tilde{N}$, $\tilde{P}$ to 3^r copies of $\tilde{N}$ and 3^r copies of $\tilde{P}$. Since $-\varphi \in \tilde{P}$, the number of solutions of $\tilde{h}^{(n)} = -\varphi$ is 3^r.

If $n = 2r+1$ is odd, then, $\tilde{h}^{(2r+1)}$ maps $\tilde{N}$, $\tilde{P}$ to 3^{r+1} copies of $\tilde{N}$ and 3^r copies of $\tilde{P}$. Hence, we have again 3^r solutions of $\tilde{h}^{(n)} = -\varphi$. *q.e.d.*

6 Loops and chains of Hesse derivatives

We return to considering the curves in Hesse form, i.e.,

$$\Gamma_c : \ x^3 + y^3 + z^3 + c\,xyz = 0, \qquad \Gamma_\infty : \ xyz = 0 \tag{6}$$

for $c \in \mathbb{R}$. Depending on the value of c, the curve Γ_c has either one or two components in the affine plane $z = 1$ (see Figure 6). Notice that Γ_c is unchanged under the transformation $(x, y) \mapsto (y, x)$,

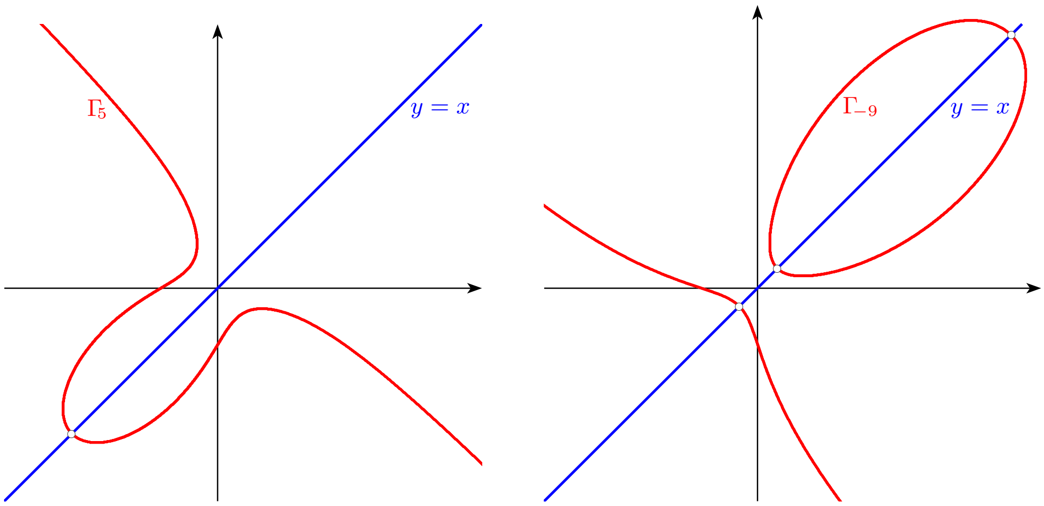

Figure 6: Hesse curves Γ_5 and Γ_{-9}.

and hence it is symmetric with respect to the line $y = x$. For $c = -3$ the curve degenerates to the line $x + y + 1 = 0$ and the point $(1, 1)$, otherwise it is smoothly embedded. If the curve has one component, it intersects the line $y = x$ in one point, if it has two components we have three intersection points.

In this section, while referring to the function h from the previous section, we will assume $a = 108$, $b = -3$, $\varphi = -3$, $\kappa = 6$ (see Lemma 8 and Fact 10).

We will begin with the following observation.

Lemma 15. *Let $c \in \mathbb{R}$. Then the curve Γ_c has two components in the affine plane $z = 1$ if $c < -3$ and only one component if $c > -3$. Moreover, for $c \neq 0$, if Γ_c has one component, then $\text{ह}\Gamma_c$ has two components, and if Γ_c has two components, then $\text{ह}\Gamma_c$ has one component.*

Proof. As explained above, we can determine the number of components of Γ_c by computing the number of intersections with the line $y = x$, i.e., by counting the zeros of the equation

$$2x^3 + cx^2 + 1 = 0.$$

The discriminant of this cubic equation is

$$-27 \cdot 4 - 4c^3$$

and hence the equation has three real roots if $c < -3$, and one if $c > -3$.

Recall that $\text{ह}\Gamma_c = \Gamma_{h(c)}$ if $c \neq 0$ (see Lemma 8), and that h maps $N = (-\infty, -3)$ to $P = (-3, \infty)$ and P to N (see the proof of Proposition 12). Hence, if Γ_c has two components, we have $c \in N$ and hence $h(c) \in P$, which means that $\text{ह}\Gamma_c = \Gamma_{h(c)}$ has one component—and the same applies the other way around. *q.e.d.*

The operator ह defines via $\Gamma \mapsto \text{ह}\Gamma$ an iterative discrete dynamical system on the set of the cubic curves (6) in Hesse form. The dynamics is given by Lemma 8. The system has exactly two fixed points, namely Γ_{-3} and Γ_∞. We are now interested in orbits of a given length which end in one of the fixed points, and in closed orbits of a given length. We call the former *Hesse chains* and the latter *Hesse loops*. So, a Hesse chain is given by

$$\text{ह}^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{-3} \qquad \text{or} \qquad \text{ह}^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_\infty$$

where we call the minimal n with this property the length of the chain. Similarly, a Hesse loop is given by

$$হ^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{c_0}$$

where the minimal $n > 0$ with this property is the length of the loop.

The number of Hesse chains ending in Γ_{-3} of length n is easy to calculate as shown in the following Lemma.

Lemma 16. *If* $চ_n^{(-3)}$ *denotes the number*[2] *of Hesse chains ending in* Γ_{-3} *of length* n*, then*

$$চ_{2r}^{(-3)} = চ_{2r-1}^{(-3)} = 3^{r-1}$$

for integers $r \geq 1$*.*

Proof. Notice that $হ^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{-3}$ implies $h^{(n)}(c_0) = \varphi = -3$. By Proposition 11 it follows that either $c_0 = -3$ or that c_0 is a critical point of $h^{(n)}$. Recall that the set of critical points of $h^{(n)}$ contains the set of critical points of $h^{(n-1)}$ (see the proof of Proposition 12). Hence, by the minimality of n, it follows from Proposition 12 that

$$চ_{2r}^{(-3)} = \chi_{2r} - \chi_{2r-1} = 3^{r-1}$$

and

$$চ_{2r-1}^{(-3)} = \chi_{2r-1} - \chi_{2r-2} = 3^{r-1}.$$

q.e.d.

Lemma 17. *For any positive* $B \in \mathbb{R}$*, there exists a* $c > B$ *and a* $d < -B$ *and an* $n \in \mathbb{N}$ *such that* $হ^n\Gamma_c = \Gamma_{-3}$ *and* $হ^n\Gamma_d = \Gamma_{-3}$*.*

Proof. Observe first that $\Gamma_6 = \Gamma_{-3}$. Also note that for $c \geq 6$ the solution $\bar{c}$ of the equation $h(\bar{c}) = c$ satisfies $\bar{c} < -3c$. On the other hand, for $c \leq -6$, the largest of the three solutions $\bar{c}$ of the equation $h(\bar{c}) = c$ satisfies $\bar{c} > -3c - 1$. Hence by backward iteration and choosing always the solution with the largest absolute value, we can construct an orbit ending in Γ_{-3} and starting at some Γ_c or Γ_d with $c > B$ or $d < -B$. *q.e.d.*

Similarly as before, we consider the Hesse chains ending in Γ_∞.

Lemma 18. *If* $চ_n^\infty$ *denotes the number of Hesse chains of length* n *ending in* Γ_∞*, then*

$$চ_{2r}^\infty = চ_{2r-1}^\infty = 3^{r-1}$$

for integers $r \geq 1$*.*

Proof. Since $হ\Gamma_0 = \Gamma_\infty$, the number of Hesse chains ending in Γ_∞ of length n is the number of zeros of $h^{(n-1)}$. Therefore the claim follows from Lemma 14. *q.e.d.*

Now we turn our attention towards Hesse loops.

Proposition 19. *The only Hesse loop of odd length is the trivial loop* $হ\Gamma_{-3} = \Gamma_{-3}$*.*

[2] We again find ourselves at a loss of expressions to notate the "ch" sound used in "chain" as such an alphabet is not present in English, Latin or Greek script. We will use this excuse to use another Bengali alphabet, namely চ, pronounced as *"chaw"*.

Proof. This follows immediately from Lemma 15. *q.e.d.*

Now, we want to determine the number of Hesse loops of length n for even n. We start with the following observation.

Proposition 20. *For every even n, there is at least one Hesse loop of length n.*

Proof. Recall that Φ_n denotes the number of fixed points of $h^{(n)}$ (see Proposition 13). Now, note that the value of Φ_n already includes the trivial fixed point -3. So let $\Phi'_n := \Phi_n - 1$ denote the number of non-trivial fixed points of $h^{(n)}$. Furthermore, for any r, a fixed point of $h^{(r)}$ is also a fixed point of $h^{(mr)}$ for any m. Also, any loop of length r consists of r elements and contributes this number to Φ'_n. It is enough to show that the quantity

$$\Phi'_{2r} - \sum_{k=1}^{r-1} \Phi'_{2k}$$

is strictly positive. This follows from the fact that for $r > 1$, we have

$$\sum_{k=1}^{r-1} \Phi'_{2k} = \sum_{k=1}^{r-1} \left(2 \times 3^k - 4\right) = 3^r - 4r + 1,$$

and this is indeed strictly smaller than $\Phi'_{2r} = 2 \times 3^r - 4$. *q.e.d.*

Remark. The above calculation also shows that we must at least have

$$\left\lceil \frac{1}{2r} \left(\Phi'_{2r} - \sum_{k=1}^{r-1} \Phi'_{2k} \right) \right\rceil = \left\lceil \frac{3^r - 5}{2r} \right\rceil + 2$$

loops of length $2r$.

Proposition 21. *If Λ_n denotes the number of Hesse loops of length n, then the sequence $\{\Lambda_{2r}\}$ is strictly increasing.*

Proof. Since Φ'_{2r} includes the two elements of the only 2-loop, the value of Λ_{2r} can be at most

$$\left\lfloor \frac{\Phi'_{2r} - 2}{2r} \right\rfloor = \left\lfloor \frac{3^r - 3}{r} \right\rfloor$$

and hence

$$\left\lceil \frac{3^r - 5}{2r} \right\rceil + 2 \leq \Lambda_{2r} \leq \left\lfloor \frac{3^r - 3}{r} \right\rfloor$$

for $r > 1$. So, to prove that the sequence $\{\Lambda_{2r}\}$ is strictly increasing, it is enough to show that

$$\left\lceil \frac{3^r - 5}{2r} \right\rceil > \left\lfloor \frac{3^{r-1} - 3}{r - 1} \right\rfloor.$$

This follows from

$$\frac{3^x - 5}{2x} > \frac{3^{x-1} - 3}{x - 1}$$

which is true for $x \geq 3$ as then we have

$$3^{x-1} \cdot x + 5 + x > 3^x$$

which completes the proof for $r \geq 3$.

The cases $r = 1, 2$ can be checked by hand. *q.e.d.*

We close this discussion by an explicit formula for the number of loops of length $n = 2r$.

Theorem 22. *The number of loops of length $2r$ is*

$$\Lambda_{2r} = \frac{1}{2r} \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}$$

where $\Phi'_{2d} = 2 \times 3^d - 4$, and μ is the Möbius function.

Proof. Let the even divisors of $2r$ be $d_1 = 2, d_2, \ldots, d_k = 2r$. Since each loop of length d_m contains exactly d_m elements, the total number of fixed points $\neq -3$ of $h^{(2r)}$ is given by

$$\Phi'_{2r} = \sum_{\substack{d|2r \\ d \text{ even}}} d \cdot \Lambda_d.$$

The even divisors of $2r$ are twice the divisors of r. Hence we may write

$$\Phi'_{2r} = \sum_{d|r} 2d \cdot \Lambda_{2d}.$$

Using the Möbius inversion formula, we obtain

$$2r \cdot \Lambda_{2r} = \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}$$

hence completing the proof. *q.e.d.*

The sequence (Λ_{2r}) starts as follows:

$$\Lambda_2 = 1,\ \Lambda_4 = 3,\ \Lambda_6 = 8,\ \Lambda_8 = 18,\ \Lambda_{10} = 48,\ \Lambda_{12} = 116,\ \Lambda_{14} = 312,\ \Lambda_{16} = 810,\ \ldots$$

The Hesse loop of length 2 is shown in Figure 4.

Remark. Let $\lambda_r := \Lambda_{2r}$. Then we have for $r \geq 1$

$$\lambda_r = \frac{1}{2r} \sum_{d|r} \mu\left(\frac{r}{d}\right) (2 \cdot 3^d - 4) = \underbrace{\frac{1}{r} \sum_{d|r} \mu\left(\frac{r}{d}\right) \cdot 3^d}_{=:a_r} - 2 \cdot \underbrace{\frac{1}{r} \sum_{d|r} \mu\left(\frac{r}{d}\right)}_{=\delta_{1,r}},$$

where $\delta_{1,r} = 1$ if $r = 1$ and $\delta_{1,r} = 0$ for all other values of r. Notice that the sequence a_r represents the number of aperiodic necklaces with r beads of 3 colors, see OEIS A027376. Another sequence that agrees with λ_r for $r \geq 2$ is OEIS A185171.

7 Hesse derivatives of other normal forms

So far, we just considered Hesse derivatives of cubic curves in Hesse form, i.e., of curves Γ_c. The reason was that the Hesse derivative of a curve in Hesse form is again a curve in Hesse form, which is in general not the case for curves, for example, in Weierstrass normal form (WNF).

Below, we first provide examples of curves in WNF such that their Hesse derivatives are not in WNF, and then we provide cubic curves with a D_3-symmetry, whose Hesse derivatives have the same symmetry — like the cubics in Figure 4.

Curves in Weierstrass normal form

Let $\Gamma_c:\ x^3+y^3+z^3+c\,xyz=0$ be a cubic curve in Hesse form. Then, as described in [2, Sec. 3], by a projective transformation, for

$$c=-\frac{2q^3+1}{q^2}\,,$$

the curve Γ_c can be transformed to the curve

$$E_{a,b}:y^2=x^3+a\,x^2+b\,x$$

where

$$b=\frac{(q-1)^3}{q+q^2+q^3}\qquad\text{and}\qquad a=\frac{b^2-6b-3}{4}\,.$$

For example, for $c_0=-3(\sqrt{3}+1)$ we obtain

$$q_0=\frac{\sqrt{3}-1}{2},\qquad b_0=3-2\sqrt{3},\qquad a_0=0\,.$$

As a matter of fact we would like to mention that over the quadratic field $\mathbb{Q}(\sqrt{3})$, the torsion group of E_{a_0,b_0} is $\mathbb{Z}/6\mathbb{Z}$ with the generating point $(-2\sqrt{3}+3,5\sqrt{3}-9)$. Furthermore, we have

$$\text{ह}E_{a_0,b_0}=7-4\sqrt{3}+(-3+2\sqrt{3})x^2-xy^2=0$$

and $\text{ह}^2E_{a_0,b_0}=E_{a_0,b_0}$.

Curves in D_3-symmetric form

In [1, Sec. 2], a D_3-symmetric form of cubic curves was introduced. With a suitable projective transformation, every regular cubic curve can be brought into the form

$$\Gamma_b:x^3-3xy^2+b(x^2+y^2)z+z^3=0,\tag{7}$$

with $b\in\mathbb{R},b\neq-\frac{3}{\sqrt[3]{4}}$. The corresponding curve in the affine plane $z=1$ has the symmetry of an equilateral triangle, like the curves in Figure 4. For $b=-\frac{3}{\sqrt[3]{4}}$ the curve degenerates to three lines:

$$x^3-3xy^2-\frac{3}{\sqrt[3]{4}}(x^2+y^2)z+z^3=\frac{1}{6}(2x+\sqrt[3]{2})(\sqrt{3}x+3y-\sqrt[3]{2}\sqrt{3})(\sqrt{3}x-3y-\sqrt[3]{2}\sqrt{3}).$$

This class of cubic curves shows a very similar behaviour under the operator ह as the class of Hesse curves which we investigated before.

Proposition 23. *The class of cubic curves (7) is invariant under the operator* ह*. We have*

$$\text{ह}\Gamma_b=\Gamma_{\tilde{b}},\ \textit{with}\ \tilde{b}=-\frac{27+b^3}{3b^2}.$$

Proof. For $f(x,y,z)=x^3-3xy^2+b(x^2+y^2)z+z^3$, we have

$$H_f=\frac{1}{2}\begin{pmatrix}3x+bz&-3y&bx\\-3y&-3x+bz&by\\bx&by&3z\end{pmatrix}.$$

Taking the determinant gives the desired result. *q.e.d.*

In order to find a Hesse loop of length 2 in D_3-symmetric form, we have to find b_0 and b_1, such that

$$b_1 = -\frac{27+b_0^3}{3b_0^2} \qquad \text{and} \qquad b_0 = -\frac{27+b_1^3}{3b_1^2},$$

which holds for

$$b_0 = -3\sqrt[3]{\frac{3\sqrt{3}+5}{2}} \qquad \text{and} \qquad b_1 = 3\sqrt[3]{\frac{3\sqrt{3}-5}{2}}.$$

Figure 7 shows the two cubics Γ_{b_0} and Γ_{b_1} of the Hesse loop of length 2 in D_3-symmetric form together with degenerate polar conics at point $P = (0,1,0)$, which is one of the three real intersecting points of the two cubics. Notice that since the intersecting points of the curves are also the points of inflection of the curves, the two polar conics share a line which meets the two conics in four points, and the other two lines of the two conics are the tangent lines at the inflection point at Γ_{b_0} and Γ_{b_1}, respectively, and are tangent to Γ_{b_1} and Γ_{b_0}, respectively.

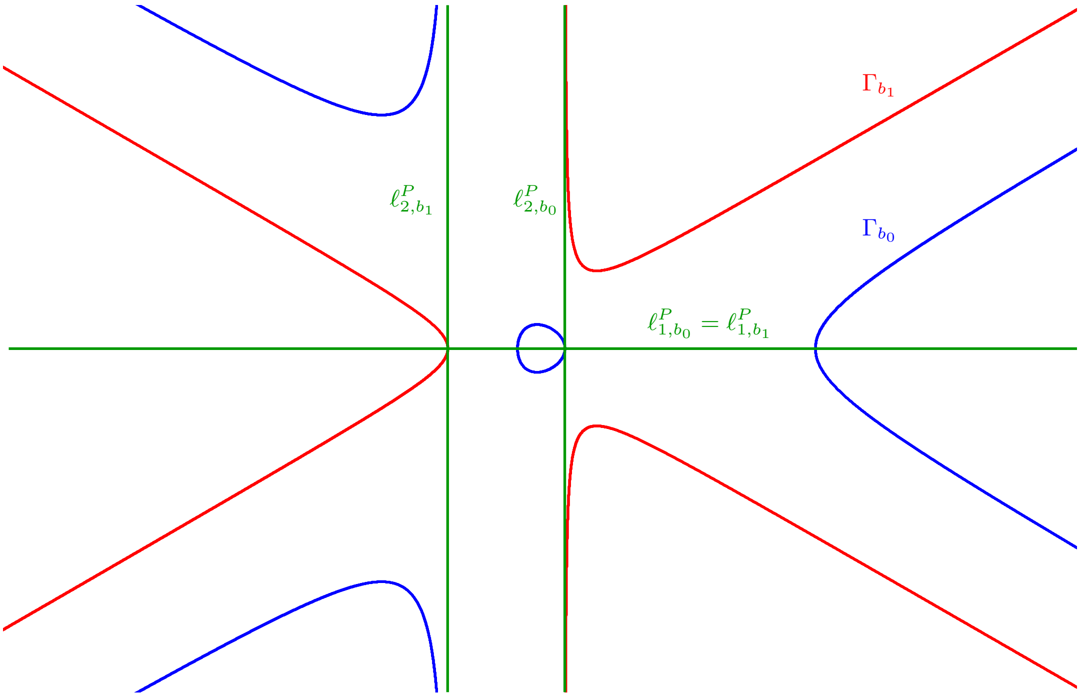

Figure 7: The Hesse loop of length 2 of two curves in D_3-symmetric form, together with the two degenerate polar conics at the intersection point $P = (0,1,0)$.

Acknowledgement

We are grateful to the referee for the valuable comments that helped improve this article.

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