Integer triangles with a rational ratio of circumcircle radius to excircle radius

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Abstract

We consider the problem of finding integer triangles with R/r a positive rational, where R and r are the radii of the circumcircle and an excircle, respectively. We show that for general triangles $R/r > 1/4$ applies. The equation $R/r = N$ turns out to be related to the elliptic curve $\mathcal{E}_N$ given by $v^2 = u^3 + 2(2N^2 + 2N - 1)u^2 - (4N - 1)u$. If $N > 1/4$ is rational, then the torsion group of $\mathcal{E}_N$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ if $N(N + 2)$ is a square and $\mathbb{Z}/6\mathbb{Z}$ otherwise. We show that a rational triangle with rational ratio $R/r = N$ exists if and only if $N > 1/4$ and there exists a rational non-torsion point on the curve $\mathcal{E}_N$ which satisfies a certain condition. Furthermore, we show that the rank of $\mathcal{E}_N$ is positive when $N = m^2 \pm 1 > 1/4$ for a rational m. We also show that on every curve $\mathcal{E}_N$ whose rank is positive, there are infinitely many rational points which lead to infinitely many non-similar integer triangles with $R/r = N$.

1 Introduction

Consider a triangle with sides of integer length f, g, h. Prominent circles associated with the triangle are the circumcircle which passes through the three vertices, the incircle, and the three excircles which have the three sides as tangents. The radii of the circumcircle C and one of the three excircles E are denoted in the following by R and r, respectively (see Figure 1). Let d be the distance between the centres of these two circles. It is a standard result in triangle geometry that

$$d^2 = R(R + 2r) \tag{1}$$

(see, e.g., [2, Theorem 295]).

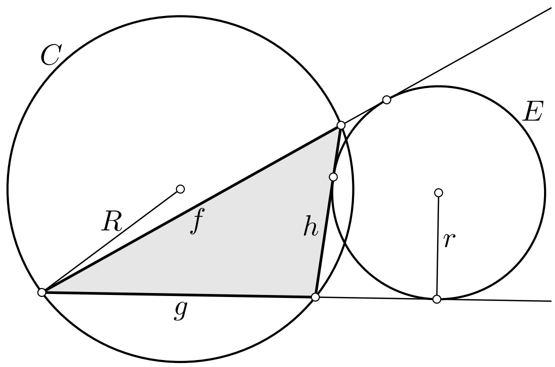

Figure 1: A triangle with sides f, g, h, circumcircle C with radius R, excircle E touching h from outside with radius r.

As a consequence of equation (1), we obtain the following.

Lemma 1. *$R/r > 1/4$.*

Proof. If r is the radius of an excircle and $r > R$, then, since the vertices of the triangle lie outside the excircle, we have $d + R > r$, i.e., $d > r - R$. By (1), we obtain

$$d^2 = R^2 + 2Rr > R^2 - 2Rr + r^2,$$

which implies $4R > r$, i.e., $R/r > 1/4$. *q.e.d.*

Equally well known are the formulas

$$R = \frac{fgh}{4\Delta}, \quad r = \sqrt{\frac{s(s-f)(s-g)}{s-h}} \tag{2}$$

where r is the radius of the excircle touching h from outside, $s = (f+g+h)/2$ is the semi-perimeter, and $\Delta = \sqrt{s(s-f)(s-g)(s-h)}$ the area of the triangle. Hence, we have

$$\frac{R}{r} = \frac{2fgh}{(f+g+h)(f-g-h)(g-h-f)}. \tag{3}$$

Observe that for non-degenerate triangles, the triangle inequality applies, i.e., the denominator on the right hand side of (3) is different from 0.

It should be noted that we can always scale a triangle with rational sides to a similar triangle with integer sides without changing the value of R/r. It is interesting to speculate whether the ratio R/r could ever be an integer N for a triangle with integer sides. In [3], MacLeod studied a similar problem for such triangles with R/ρ an integer, where R and ρ are the radii of the circumcircle and incircle, respectively. He showed that these triangles are relatively rare, and by brute force found only a handful such integer triangles. It is interesting to see that our problem, although very similar to MacLeod's, behaves quite differently. Indeed, we will see in Section 2 that our problem leads to finding certain rational non-torsion points on the elliptic curve $\mathcal{E}_N$ given by $v^2 = u^3 + 2(2N^2 + 2N - 1)u^2 - (4N - 1)u$. In other words, the existence of integer triangles with rational ratio $R/r = N$ is equivalent to the existence of special non-torsion points on the curve $\mathcal{E}_N$. The torsion group of $\mathcal{E}_N$ and the torsion points which are needed later are determined in Section 3. Section 4 identifies the non-torsion points that correspond to real triangles. So, a necessary condition for the existence of such triangles is that the rank of $\mathcal{E}_N$ is positive. We will show in Section 5 that this is in particular the case for the values $N = m^2 \pm 1$ for any rational number $m \geq 2$. We then show in Section 6 that whenever the rank of $\mathcal{E}_N$ is positive for a rational $N > 1/4$, infinitely many pairwise non-similar integer triangles with $R/r = N$ exist. We conclude in Section 7 with an application of our results to Poncelet's porism.

2 The construction of the elliptic curve $\mathcal{E}_N$

Suppose we have a triangle with rational sides f, g, h such that the ratio of the radius R of the circumcircle, and the radius r of the excircle touching h from outside

$$N = \frac{R}{r} = \frac{2fgh}{(f+g+h)(f-g-h)(g-h-f)} \tag{4}$$

is a rational number. If we multiply (4) by the denominator and move all terms to the left, we can rewrite (4) as a cubic equation in the variables f, g, h:

$$Nf^3 - N(g-h)f^2 - (g^2N + 2gh(N-1) + h^2N)f + N(g-h)(g+h)^2 = 0.$$

Now, we replace h by $2s - f - g$ and obtain the following quadratic equation in f:

$$gf^2 + \big(g^2 + 2s(2N-1)g - 4s^2N\big)f - 4(g-s)s^2N = 0. \tag{5}$$

Since f is rational, the quadratic equation (5) must have a discriminant which is a rational square, so that there must be a rational d with

$$d^2 = g^4 + 4s(2N-1)g^3 + 4s^2(4N^2 - 2N + 1)g^2 - 32N^2s^3g + 16N^2s^4.$$

If we substitute $d = s^2y$ and $g = sx$, we get the quartic equation

$$\mathcal{C}_N : y^2 = x^4 + 4(2N-1)x^3 + 4(4N^2 - 2N + 1)x^2 - 32N^2x + 16N^2. \tag{6}$$

Now we apply the birational transformation

$$x = -\frac{4N(2Nu+v)}{(u-1)(4N+u-1)}, \tag{7}$$

$$y = -\frac{4N\,(4N+u^2-1)\,(8N^2u+4Nu+4Nv-4N+u^2-2u+1)}{(u-1)^2(4N+u-1)^2} \tag{8}$$

to $\mathcal{C}_N$. This transformation leads to a cubic curve in u and v which is given by

$$\mathcal{E}_N : v^2 = u^3 + 2(2N^2+2N-1)u^2 - (4N-1)u. \tag{9}$$

The curve $\mathcal{E}_N$ is regular for positive rational $N \neq 1/4$. Conversely, we get from $\mathcal{E}_N$ to $\mathcal{C}_N$ with the birational transformation

$$u = -\frac{8N^2x+2Nx^2-8N^2+2Ny-x^2}{x^2}, \tag{10}$$

$$v = -\frac{2N(x-2)(8N^2x+2Nx^2-8N^2+2Ny-x^2)}{x^3}. \tag{11}$$

Thus, starting with the rational triangle with sides f, g, h and rational ratio $R/r = N$, we can find a rational point (u, v) on the cubic curve $\mathcal{E}_N$. Using the equations in the opposite direction, we can also reconstruct the sides f, g, h from (u, v), at least up to scaling. To do so, use (7) and (8) to find (x, y) on $\mathcal{C}_N$ and then the formulas in Theorem 3 to compute the sides of the triangle. However, not every rational point (u, v) on $\mathcal{E}_N$ leads to numbers f, g, h which are the sides of a triangle. So our next task is to identify those rational points (u, v) on $\mathcal{E}_N$ that lead to *positive* values of f, g, h which satisfy the *triangle inequalities* $f+g > h, g+h > f$ and $h+f > g$. The three triangle inequalities can be equivalently expressed by the single inequality

$$(f+g-h)(f-g+h)(-f+g+h)(f+g+h) > 0$$

or, also equivalently,

$$(f^2+g^2+h^2)^2 > 2(f^4+g^4+h^4).$$

Observe that if we replace N by $-N$ in (9), we get MacLeod's curve [3, eq. (9)]. Thus, our discussion also covers MacLeod's problem in the case when N is a rational number.

Also notice that if R/r is rational for a rational triangle for just one of its three excircles, then all ratios of radii of the three excircles, the incircle, and the circumcircle are rational. Another remark is that the formula (4) is symmetric in f and g. Hence, if we start with a rational point (u, v) on $\mathcal{E}_N$ which corresponds to a triangle with sides f, g, h, then we can find another rational point (u', v') on $\mathcal{E}_N$ which belongs to the mirrored triangle with sides g, f, h.

3 Torsion points of $\mathcal{E}_N$

One can easily check that the torsion group of $\mathcal{E}_N$ for rationals $N > 1/4$ contains $\mathbb{Z}/6\mathbb{Z}$ with torsion points

$$T_2 = (0,0),\ T_3^{\pm} = (1, \pm 2N),\ T_6^{\pm} = (1-4N, \pm 2N(4N-1))$$

of orders 2, 3, 6 respectively, see [3, Section 2]. However, T_2 corresponds via (7) to $x = 0$ and hence $g = 0$, which does not give a real triangle. For $T_3^{\pm}$ and $T_6^{\pm}$, the expression (7) is not defined. Thus, we must look at the second type of rational points—those of infinite order. In particular, since the ranks of $\mathcal{E}_N$ are zero for the following integers N, no integer or rational triangle with $R/r = N$ exists:

$$N = 1, 2, 4, 6, 7, 9, 12, 14, 16, 18, 19, 20, 21, 22, 25, 28, 30, \ldots$$

Also notice that the ranks of the curves $\mathcal{E}_1$, $\mathcal{E}_{1/2}$ and $\mathcal{E}_{1/3}$ are zero, and recall that $R/r > 1/4$. Therefore no rational triangle exists with r/R an integer.

Let us now consider the case when $N > 1/4$ is a rational and $N(N+2) = M^2$ for some $M \in \mathbb{Q}$. In this case, N cannot be an integer, but, for example, for $N = 2/3$ we get $N(N+2) = (4/3)^2$. If $N(N+2) = M^2$, then $\mathcal{E}_N$ has the two additional torsion points of order 2

$$(1 - 2N(N+1) \pm 2NM, 0)\,,$$

which shows that the torsion group of $\mathcal{E}_N$ in the case when $N(N+2) = M^2$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Finally, let us consider the case when $N > 1/4$ is rational and $N(N+2)$ is not a square. Then the torsion group of $\mathcal{E}_N$ contains $\mathbb{Z}/6\mathbb{Z}$ and is different from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Thus, it is either $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$. Suppose that $P = (u, v)$ is a point of order 12. This is equivalent to saying that the point $[2]P$ is of order 6, i.e.,

$$\frac{(u^2 + 4N - 1)^2}{4\big(u^3 + 2(2N^2 + 2N - 1)u^2 - (4N-1)u\big)} = 1 - 4N.$$

Replacing u by $u + (1 - 4N)$ this corresponds to the quartic equation

$$u^4 + 8N(1 - 6N + 8N^2)u^2 - 32(1-4N)^2N^2u + 64(1-4N)^2N^3 = 0. \tag{12}$$

Using the criterion given in the table on page 53 in [1], where

$$\delta = 1048576(1-4N)^6(2N^7 + N^8)$$

and

$$L = -1024N^3(-1+4N)^3(-1 + N + 16N^2 + 8N^3),$$

it is easy to check that for $N > 1/4$ we have $\delta > 0$ and $L \leq 0$, which implies that the equation (12) does not have real roots for $N > 1/4$. Hence, the torsion group $\mathbb{Z}/12\mathbb{Z}$ is excluded.

In summary, we obtain the following result:

Proposition 2. *Let $N > 1/4$ be a rational number. Then the torsion group of $\mathcal{E}_N$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ if $N(N+2) = M^2$ for some $M \in \mathbb{Q}$, and $\mathbb{Z}/6\mathbb{Z}$ otherwise.*

4 The triangle inequalities

To simplify the notation, we will use the following abbreviations

$$\begin{aligned} A_1 &:= -x^2 - 2(2N-1)x + 4N, & A_3 &:= x^2 - 4Nx + 4N, \\ A_2 &:= -x^2 + 2(2N+1)x - 4N, & A_4 &:= x^2 + 4Nx - 4N, \end{aligned}$$

and

$$B := x^4 + 4(2N-1)x^3 + 4(4N^2 - 2N + 1)x^2 - 32N^2x + 16N^2.$$

The following theorem now provides information on the rational points on $\mathcal{C}_N$ which correspond to real triangles.

Theorem 3. *A rational triangle with rational quotient $R/r = N > 1/4$ exists if and only if $\mathcal{C}_N$ has a rational point (x, y) such that $0 < x < 1$. In this case, the sides of the triangle are given by*

$$f = \frac{s}{2x}(A_1 - \sqrt{B}), \ g = sx, \ h = \frac{s}{2x}(A_2 + \sqrt{B}),$$

for $0 < s \in \mathbb{Q}$.

Proof. Let f, g, h be sides of a rational triangle with $R/r = N$. Then, as we have seen in Section 2, (x, y) must be a rational point on $\mathcal{C}_N$. From $g = sx$ in (5), we infer that $x > 0$. On the other hand, we have from the triangle inequality

$$0 < f - g + h \iff 2g < f + g + h \iff x = \frac{g}{s} = \frac{2g}{f+g+h} < 1.$$

For the opposite direction, assume we have a rational point (x, y) on $\mathcal{C}_N$ with $0 < x < 1$. We may also assume that $y > 0$. For $g = sx$, the equation (5) has the following roots for f:

$$f_\pm = \frac{s}{2x}(A_1 \pm \sqrt{B}).$$

Note that by (6), $0 < \sqrt{B} = y \in \mathbb{Q}$.

By $h_\mp = 2s - f_\pm - g$, we get the following two values:

$$h_\mp = \frac{s}{2x}(A_2 \mp \sqrt{B}),$$

and hence

$$\begin{aligned} f_\pm + g - h_\mp &= \frac{s}{x}(A_3 \pm \sqrt{B}), \\ f_\pm + h_\mp - g &= -2s(x-1), \\ g + h_\mp - f_\pm &= \frac{s}{x}(A_4 \mp \sqrt{B}). \end{aligned} \tag{13}$$

We observe that

$$B - A_2^2 = 16Nx(x-1)^2 > 0 \Longrightarrow B > A_2^2 \Longrightarrow \sqrt{B} > A_2 \Longrightarrow h_- < 0.$$

Hence, the root $f = f_+$ does not lead to a triangle. So, we will take $f = f_-$ and $h = h_+$. In this case, we have the following.

- Since $N > 0$ and $0 < x < 1$, we have $A_1 > 0$. Then

$$\begin{aligned} f > 0 &\Longleftrightarrow A_1 - \sqrt{B} > 0 \Longleftrightarrow A_1 > \sqrt{B} \\ &\Longleftrightarrow A_1^2 > B \Longleftrightarrow A_1^2 - B = -16Nx(x-1) > 0, \end{aligned}$$

  which is the case.

- Since $B > A_2^2$, it follows that $h = h_+ > 0$.
- Trivially, we have $g = sx > 0$.

It remains to check that f, g, h satisfy the triangle inequalities.

- Since $N > 0$ and $0 < x < 1$, we have $A_3 > 0$. Then we have

$$\begin{aligned} f + g - h > 0 &\Longleftrightarrow A_3 - \sqrt{B} > 0 \Longleftrightarrow A_3 > \sqrt{B} \\ &\Longleftrightarrow A_3^2 > B \Longleftrightarrow A_3^2 - B = -4x^2(x-1)(4N-1) > 0, \end{aligned}$$

  which is the case.

- To show $g + h - f > 0$ it is enough to show that $A_4 + \sqrt{B} > 0$. We have:

$$\begin{aligned} B - A_4^2 = -4x^2(x-1) > 0 &\Longrightarrow (\sqrt{B} - A_4)(\sqrt{B} + A_4) > 0 \\ &\Longrightarrow \sqrt{B} - A_4 > 0, \sqrt{B} + A_4 > 0 \text{ or } \sqrt{B} - A_4 < 0, \sqrt{B} + A_4 < 0. \end{aligned}$$

  But the second case cannot happen, since then $0 < \sqrt{B} < A_4$ implies $B - A_4^2 < 0$, a contradiction. Therefore, the first case happens and hence $g + h - f > 0$.

- Finally, we have $f + h - g > 0$ by (13).

q.e.d.

We can carry over the previous result from $\mathcal{C}_N$ to the elliptic curve $\mathcal{E}_N$.

Theorem 4. *A rational triangle with rational quotient $R/r = N > 1/4$ exists if and only if $\mathcal{E}_N$ has a rational non-torsion point (u, v) such that*

$$0 < -\frac{4N(v + 2Nu)}{(u-1)(u+4N-1)} < 1. \tag{14}$$

In this case, the sides are given by

$$f = \frac{s}{2x}(A_1 - \sqrt{B}),\ g = sx,\ h = \frac{s}{2x}(A_2 + \sqrt{B}),$$

where

$$x = -\frac{4N(v + 2Nu)}{(u-1)(u+4N-1)}$$

and $0 < s \in \mathbb{Q}$.

Example 1. *For $N = 1$ the rank of $\mathcal{E}_N$ is zero which does not lead to a triangle. For $N = 3$, the rank of $\mathcal{E}_N$ is 1, being generated by $P = (-44, 66)$. Since the coordinates of the point P do not satisfy the conditions of Theorem 4, there is no triangle corresponding to this point. However, by using the point $[2]P = (3481/16, -226029/64)$, we see that its coordinates satisfy the conditions of Theorem 4 and hence we get (after scaling)*

$$f = 98315,\ g = 55696,\ h = 52371.$$

We note that there are more triangles for $N = 3$. For example, the points

$$[4]P = (16322076723481/363300329536,\\ -93684940203017164611/218977093825846784)$$

and

$$-P - T_3^+ = (-11/9, 242/27),$$

respectively lead to the triangles

$$(f, g, h) = (468221204113406697 69, 393521352504713274 56,\\ 15634506390670773305),$$

and

$$(f, g, h) = (25, 27, 8).$$

Note that he point $P + T_6^+ = (9, -66)$ also gives the latter triangle.

Example 2. *Among the integer numbers $1 \leq N \leq 50$, the values*

3, 5, 8, 10, 11, 13, 15, 17, 23, 24, 26, 27, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 46, 48, 50

lead to positive rank elliptic curves $\mathcal{E}_N$. If we want to find just one triangle for one of these values of N, we can proceed as follows. Let P be a generator of $\mathcal{E}_N$, and $u(P), v(P)$ its u- and v-coordinate.

- *If $u(P)v(P) < 0$, we use one of the points $P + T_6^+$ or $-P - T_6^+$ depending on $u(P + T_6^+)v(P + T_6^+)$ is negative or positive respectively.*

- *If $u(P)v(P) > 0$, we use one of the points $-P + T_6^+$ or $P - T_6^+$ depending on $u(-P + T_6^+)v(-P + T_6^+)$ is negative or positive respectively.*

Using the formulas in Theorem 4 we then get the triangles exhibited in Table 1.

Table 1: Examples for $1 \leq N \leq 50$

N	f	g	h
3	25	27	8
5	121	147	40
8	49	50	6
10	121	128	15
11	471969	591976	142885
13	24037	35000	11913
15	243	245	16
17	4107	4205	272
23	363	368	17
24	242	243	10
26	1568	1587	65
27	24900840	26234439	1866059
29	256	261	11
31	130355	126736	6171
32	84568	73947	11830
33	1323	1352	55
34	529	640	119
35	847	845	24
36	5043	5415	448
37	31423	31205	888
38	150544	164331	15895
39	116467264	143721781	28780245
41	158251147734128961	179454792712801424	23209487182638905
42	841	864	35
43	2989137	2805275	219128
46	18910493440839	18598307793911	571951808000
48	675	676	14
50	2401	2535	160

In Section 6 we intend to find not only one triangle, but an infinite family of triangles for a given rational value of $N = R/r$ for which the rank of $\mathcal{E}_N$ is positive. To do so, we first notice that the condition (14) in Theorem 4 can be transformed into a nicer form.

Theorem 5. *A rational triangle with rational quotient* $R/r = N > 1/4$ *exists if and only if* $\mathcal{E}_N$ *has a rational non-torsion point* (u, v) *such that*

$$1 - 4N < u < 0 \quad \textit{or} \quad u > 1. \tag{15}$$

In this case, the sides are given by

$$f = \frac{s}{2x}(A_1 - \sqrt{B}), \; g = sx, \; h = \frac{s}{2x}(A_2 + \sqrt{B}),$$

where

$$x = -\frac{4N(v + 2Nu)}{(u-1)(u+4N-1)}$$

and $0 < s \in \mathbb{Q}$.

Proof. We first note that the curve $\mathcal{E}_N$ intersects u-axis in the points

$$u_1 = 0, \; u_2 = -2N^2 - 2N + 1 - 2N\sqrt{N(N+2)}, \; u_3 = -2N^2 - 2N + 1 + 2N\sqrt{N(N+2)},$$

see Figure 2. According to (7), we observe that if $x = 0$, then $v = -2Nu$ and if $x = 1$, then

$$v = -\frac{u^2 + 2(4N^2 + 2N - 1)u - (4N - 1)}{4N}.$$

Hence the region of points (u, v) satisfying condition (14) is the green region between the two curves

$$c_1 : \; v = -2Nu,$$
$$c_2 : \; v = -\frac{u^2 + 2(4N^2 + 2N - 1)u - (4N - 1)}{4N}.$$

The three curves c_1, c_2, and $\mathcal{E}_N$ meet in the torsion points

$$T_6^- = (1 - 4N, -2N(1 - 4N)) \text{ and } T_3^- = (1, -2N), \text{ an inflection point of } \mathcal{E}_N.$$

The curves c_2 and $\mathcal{E}_N$ actually have a common tangent in T_6^-. Moreover, c_1 and $\mathcal{E}_N$ meet in $T_2 = (0, 0)$. Hence $\mathcal{E}_N$ crosses the boundary of the green region only in T_6^-, T_3^-, and in the origin. It follows that exactly the points (u, v) on $\mathcal{E}_N$ with $1 - 4n < u < 1$ or $u > 1$ satisfy condition (14). *q.e.d.*

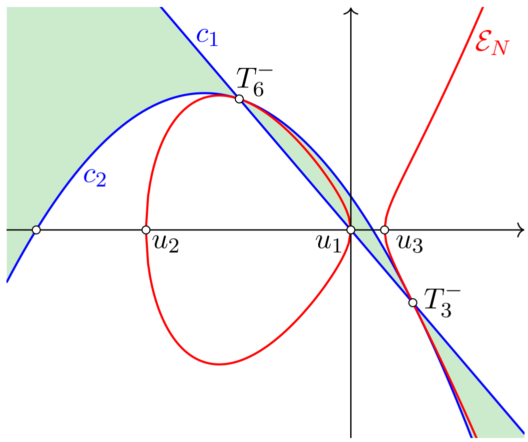

Figure 2: Only the points on $\mathcal{E}_N$ inside the green region satisfy condition (14).

5 The cases $N = m^2 \pm 1$

The previous section suggests that if $N > 1/4$ is a positive rational number for which the rank of $\mathcal{E}_N$ is positive, we may be able to construct integer triangles with $R/r = N$. So we are facing two problems: Find rational numbers $N > 1/4$ for which the rank of $\mathcal{E}_N$ is positive, and find rational points on such curves that satisfy condition (15). In Section 6 we will show that whenever $\mathcal{E}_N$ has positive rank, the curve has infinitely many rational points satisfying condition (15), and that these points lead to infinitely many non-similar integer triangles with $R/r = N$. In this section we first show that for each $N = m^2 \pm 1 > 1/4$, $m \in \mathbb{Q}$, the rank of $\mathcal{E}_N$ is positive. We provide explicit formulas for the sides f, g, h of integer triangles with $R/r = N \in \mathbb{Q}$.

Theorem 6. *(i) If $N = m^2 + 1$ for a rational number $m > 1$ the rank of $\mathcal{E}_N$ is positive with a non-torsion point*

$$P = \Big(\frac{1}{m^2}, \frac{3m^2+1}{m^3}\Big)$$

such that the point

$$-P - T_6^+ = \Big(-\frac{(4m^2+3)(m+1)^2}{(2m^2-m+1)^2}, \frac{2(m+1)(4m^2+3)(m^2+1)(2m^3+m^2+2m-1)}{(2m^2-m+1)^3}\Big)$$

leads to the triangle

$$f = (m-1)(2m^2+m+1)^2,\ g = (m+1)(2m^2-m+1)^2,\ h = 4m(m^2+1).$$

(ii) If $N = m^2 - 1 > 1/4$ for a rational number $m > 1$ the rank of $\mathcal{E}_N$ is positive with a non-torsion point

$$P = \Big(\frac{1}{m^2}, \frac{m^2-1}{m^3}\Big)$$

such that the point

$$-P - T_6^+ = \Big(-\frac{4m^2-5}{(2m-1)^2}, \frac{2(m-1)(4m^2-5)(2m^2+m+1)}{(2m-1)^3}\Big)$$

leads to the triangle

$$f = (m-1)(2m+1)^2,\ g = (m+1)(2m-1)^2,\ h = 4m.$$

Proof. (i) By enforcing $u = N-1$ on $\mathcal{E}_N$, the expression $N-1$ must be a square, say m^2 for some rational number m. Then, $N = m^2+1$ and the corresponding elliptic curve $\mathcal{E}_N$ owns the rational non-torsion point

$$P = \Big(\frac{1}{m^2}, \frac{3m^2+1}{m^3}\Big).$$

By the addition law, we compute

$$\begin{aligned} -P - T_6^+ = \Big(&-\frac{(4m^2+3)(m+1)^2}{(2m^2-m+1)^2}, \\ &\frac{2(m+1)(4m^2+3)(m^2+1)(2m^3+m^2+2m-1)}{(2m^2-m+1)^3}\Big). \end{aligned}$$

By taking $(u, v) = -P - T_6^+$ we observe that condition (15) holds, hence by Theorem 3 we get

$$f = (m-1)(2m^2+m+1)^2,\ g = (m+1)(2m^2-m+1)^2,\ h = 4m(m^2+1).$$

(ii) In a similar fashion, by enforcing $u = N+1$ on $\mathcal{E}_N$, the expression $N+1$ must be a square, say m^2 for some rational number m. Then, $N = m^2-1$ and the corresponding elliptic curve $\mathcal{E}_N$ owns the rational non-torsion point

$$P = \Big(\frac{1}{m^2}, \frac{m^2-1}{m^3}\Big).$$

By the addition law, we compute

$$-P - T_6^+ = \Big(-\frac{4m^2-5}{(2m-1)^2}, \frac{2(m-1)(4m^2-5)(2m^2+m+1)}{(2m-1)^3}\Big).$$

By taking $(u, v) = -P - T_6^+$ we observe that condition (15) holds, hence by Theorem 3 we get

$$f = (m-1)(2m+1)^2,\ g = (m+1)(2m-1)^2,\ h = 4m.$$

q.e.d.

6 Infinitely many triangles for curves of positive rank

We start by determining just one rational point on $\mathcal{E}_N$ that fulfills condition (15). This point will then be the stepping stone for an infinite number of such points.

Lemma 7. *If $\mathcal{E}_N$ has positive rank for a rational number $N > 1/4$, then there are rational points on $\mathcal{E}_N$ that satisfy condition* (15).

Proof. Let $R = (u_R, v_R)$ be a rational non-torsion point on $\mathcal{E}_N$ that does not satisfy condition (15). If $u_R < 1-4n$, then $R+T_3^-$ is a rational non-torison point satisfying condition (15). If $0 < u_R < 1$, then $-(R+T_6^-)$ is a rational non-torison point satisfying condition (15). See Figure 3. *q.e.d.*

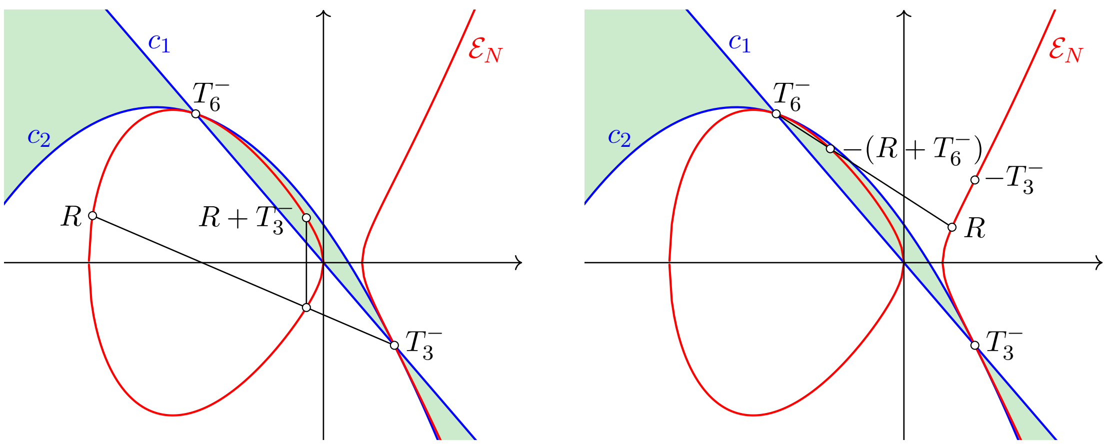

Figure 3: Rational points on $\mathcal{E}_N$, satisfying condition (15).

Lemma 8. *If $\mathcal{E}_N$ has positive rank for a rational number $N > 1/4$, then there are rational points $R = (u_R, v_R)$ on $\mathcal{E}_N$ that satisfy condition* (15) *and $u_R > 1$.*

Proof. According to Lemma 7, there are rational non-torsion points that satisfy condition (15). Now, suppose $1 - 4N < u_R < 0$, then $R + T_6^- - T_3^-$ satisfies condition (15) and the u-coordinate of this point is larger than 1. *q.e.d.*

Lemma 9. *If $\mathcal{E}_N$ has positive rank for a rational number $N > 1/4$, then there are infinitely many rational points $R = (u_R, v_R)$ on $\mathcal{E}_N$ that satisfy $u_R > 1$.*

Proof. According to Lemma 8, there are rational non-torsion points that satisfy condition (15) and whose u-coordinates are larger than 1. If $R_0 = (u_0, v_0)$ is such a point, then the point $-(2R_0 + T_3^-)$ is also a point that satisfy condition (15) and whose u-coordinate is larger than 1. Iterating this process yields a sequence

$R_0, R_k = (-1)^k(2^k R_0 + J_k T_3^-)$, where $(J_k)_k$ is the Jacobsthal sequence modulo 3, i.e., $J_k \equiv -k \mod 3$. Recall that T_3^- is a torsion point of order 3. In particular, the sequence R_k is not periodic, and infinitely many non-similar triangles result from the sequence of points R_k. *q.e.d.*

As a result, we obtain the following.

Corollary 10. *If $\mathcal{E}_N$ has positive rank for a rational number $N > 1/4$, then there are infinitely many rational and integer triangles with $R/r = N$.*

7 Poncelet's theorem

Our results now allow us to make an interesting statement with regard to Poncelet's closure theorem.

Corollary 11. *Let C and E be two circles such that the ratio of their radii is rational and such that a triangle with rational sides has C as circumcircle and E as one of its excircles. Then there exist infinitely many rational non-similar triangles with rational sides that have C as circumcircle and E as excircle.*

Figure 4 shows three such Poncelet triangles with $R = 5/2, r = 2$:

$$\begin{array}{lllll} f_1 = 5 & g_1 = 4 & h_1 = 3 & u = -\dfrac{1}{4} & v = \dfrac{5}{4} \\ f_2 = \dfrac{204}{65} & g_2 = \dfrac{416}{85} & h_2 = \dfrac{700}{221} & u = \dfrac{169}{64} & v = -\dfrac{4355}{512} \\ f_3 = \dfrac{6757}{5513} & g_3 = \dfrac{37697}{8621} & h_3 = \dfrac{126540}{34717} & u = -\dfrac{64009}{22201} & v = \dfrac{53116085}{6615898} \end{array}$$

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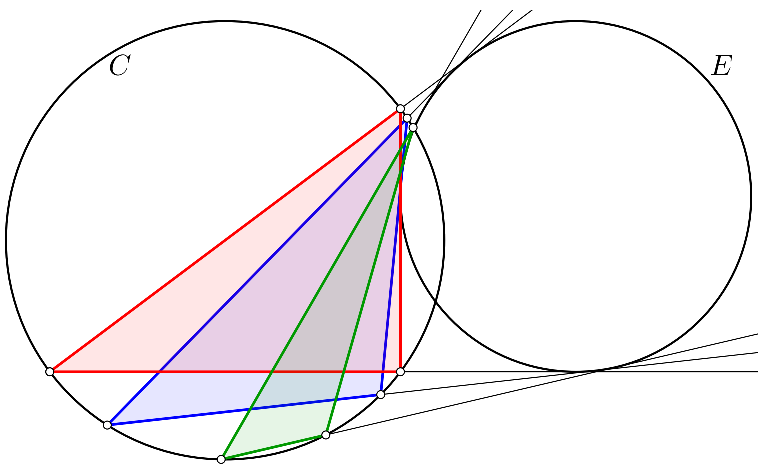

Figure 4: Poncelet's theorem for triangles.