Solving the Quartic by Conics

Lorenz Halbeisen

Department of Mathematics, ETH Zentrum, Rämistrasse101, 8092 Zürich, Switzerland
ORCID 0000-0001-6078-7237, lorenz.halbeisen@math.ethz.ch

Norbert Hungerbühler

Department of Mathematics, ETH Zentrum, Rämistrasse101, 8092 Zürich, Switzerland
ORCID 0000-0001-6191-0022, norbert.hungerbuehler@math.ethz.ch

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Abstract

Two conic sections that pass through two given points can generally have two further points of intersection. It is shown how these can be constructed using a compass and ruler. The idea of the construction is then used to reduce a general quartic equation to a cubic equation and to solve it.

1 Introduction

We investigate the problem of finding the intersection points of two conic sections from a geometric and from an algebraic point of view. The conics are given by quadratic equations in the variables x and y, in the form

$$a_{xx}x^2 + a_{xy}xy + a_{yy}y^2 + a_x x + a_y y + a = 0, \tag{1}$$

where the coefficients are real numbers. A system of two such quadratic equations is equivalent to two quartic equations, one in x alone and one in y alone. This can be seen by applying the Buchberger algorithm to the two quadratic equations to determine a Gröbner basis. The coefficients of the two quartic equations are polynomials in the coefficients of the two quadratic equations. If a point P of one of the conics is known, one can apply a projective transformation to send P to a point at infinity, and we can therefore assume that one conic is a parabola. Note that in general the solution of a quadratic equation is required to find such a point P. We may then assume, by a suitable affine transformation, that one conic is the normal parabola $y = x^2$, and the other conic is given by (1). In this form the quartic equation in x becomes apparent.

This article is organized as follows. In Section 2, we present a construction of the intersections of two conics by ruler and compass, provided two intersections are known. In Section 3, we use the geometric idea to show how a quartic equation can be reduced to a cubic equation by considering the pencil of two associated conics.

2 Construction of the intersection of two conic sections

As discussed in the introduction, the problem of finding the intersection points of two conic sections corresponds to a quartic equation. It is therefore in general impossible to construct the intersections by ruler and compass. However, if two points of intersection are known, the problem reduces to a quadratic equation, and a construction should be feasible. In [2] such a construction was used to intersect a circle and a hyperbola in a special position. In this section we want to present a construction, which works for two arbitrary conics for which two common points are known.

Before we describe the construction, we will prove a lemma concerning quadruples of points forming harmonic ranges. Recall that four points A, B, X, Y on a line in the real projective plane form a *harmonic range*, denoted $(A, B; X, Y)$, if their cross ratio satisfies $\mathrm{cr}(A, B; X, Y) = -1$.

Lemma. *Let C_1 and C_2 be two conics which meet in the four points I_1, I_2, J_1, J_2 (see Figure 1). Let $r := I_1I_2$, let P_1 and Q_2 be the poles of the polar line r with respect to C_1 and C_2, respectively, let p_2 be the polar line of P_1 with respect to C_2, and let q_1 be the polar line of Q_2 with respect to C_1. Furthermore, let $s := P_1Q_2$, let R, T_1, T_2 be the intersection points of s with r, q_1, p_2, respectively, and let A_1, B_1 and A_2, B_2 be the intersection points of s with C_1 and C_2, respectively. Finally, let H_1 and H_2 be the intersection points of I_1J_2 with I_2J_1, and I_1J_1 with I_2J_2, respectively.*

Then we have the following:

(a) The points $A_1, B_1, A_2, B_2, H_1, H_2$ are collinear.

(b) $(P_1, R; A_1, B_1)$ and $(Q_2, T_1; A_1, B_1)$.

(c) $(P_1, T_2; A_2, B_2)$ and $(Q_2, R; A_2, B_2)$.

(d) $(A_1, B_1; H_1, H_2)$ and $(A_2, B_2; H_1, H_2)$.

Proof. By a projective transformation we can assume that the four points I_1, I_2, J_1, J_2 form a isosceles trapezoid, where the lines I_1I_2 and J_1J_2 are parallel. In this situation, the entire configuration is mirror symmetric with respect to s. In particular, the lines r, p_2, q_1 are parallel.

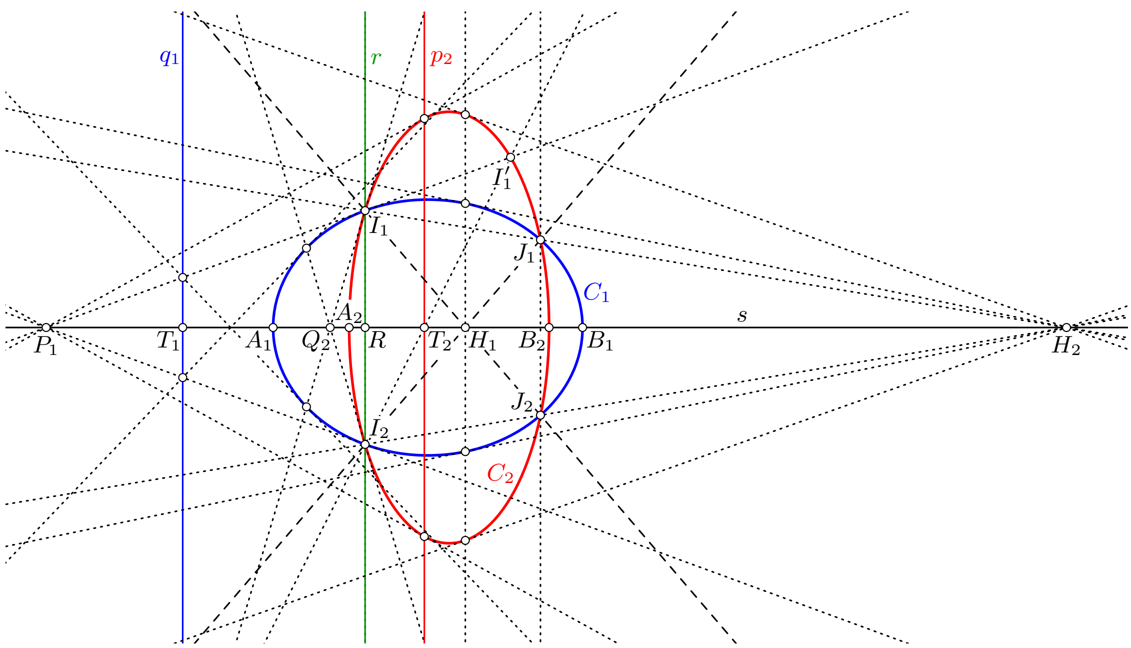

Figure 1

(a) follows directly from the symmetry. (b), (c) and (d) follow from the following fact (see, e.g., [1, Satz 4.9]): Let C be a conic, let P be a point not on C, let p be the polar line of P with respect to C, and let s be a line through P with intersects p at T and the conic C at the two points A and B. Then the four points P, T, A, B on s form a harmonic range $(P, T; A, B)$. *q.e.d.*

Now, we are ready to present a construction of the intersections of two conics C_1 and C_2 by ruler and compass, provided that the two conics have four intersection points and that two intersections I_1 and I_2 are known.

Construction. Since a conic is determined by five points, assume that C_1 and C_2 are given by the points $I_1, I_2, C_{1,1}, C_{1,2}, C_{1,3}$ and $I_1, I_2, C_{2,1}, C_{2,2}, C_{2,3}$, respectively. Then, we construct the following points and lines.

1. **Point P_1**: By Pascal's Theorem, we can construct by ruler alone the tangents to the conic C_1 at the points I_1 and I_2 (see Figure 2). The intersection of these two tangents is the point P_1.

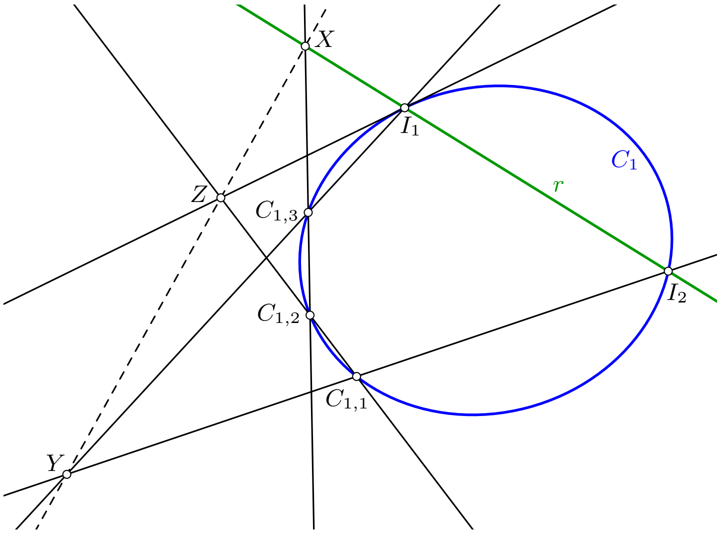

Figure 2: Construction of the tangent in I_1: XY is the pascal line in the hexagon $I_1I_2C_{1,1}C_{1,2}C_{1,3}I_1$. Hence ZI_1 is the tangent to C_1 in I_1.

2. **Point Q_2**: The point Q_2 is constructed as above as the intersection of the tangents in I_1 and I_2 with respect to the conic C_2.

3. **Line s**: Joining the points P_1 and Q_2, we obtain the line s.

4. **Point R**: Intersecting the line $r = I_1I_2$ with s gives the point R.

5. **Point T_2**: By Pascal's Theorem we can construct the intersection point I_1' of C_2 with the line P_1I_1. Then T_2 is the intersection of I_2I_1' and s. See Figure 1 and [1, Satz 4.10]).

6. **Point T_1**: The point T_1 is constructed in the same way as T_2, above, with the point Q_2 in place of P_1 and the conic C_1 in place of C_2.

7. **Points A_1, B_1**: By item (b) of the above lemma we have the harmonic ranges $(P_1, R; A_1, B_1)$ and $(Q_2, T_1; A_1, B_1)$. Hence, the points A_1 and B_1 are determined by the points P_1, R, Q_2, T_1—which we have already constructed—and can be constructed using a compass and ruler (see Figure 3, and [1, p. 78]).

8. **Points A_2, B_2**: The points A_2, B_2 are constructed as above with respect to the points P_1, T_2, Q_2, R and using item (c) of the lemma.

9. **Points H_1, H_2**: The points H_1, H_2 are constructed as above with respect to the points A_1, B_1, A_2, B_2 and using item (d) of the lemma.

10. **Points J_1, J_2**: Finally, the other two intersection points J_1, J_2 of C_1 and C_2 are obtained as follows: J_1 is the intersection of the lines I_1H_2 and I_2H_1, and J_2 is the intersection of the lines I_1H_1 and I_2H_2.

3 Solving the quartic by conics

It is well known that a quartic equation in the variable z can be reduced to a cubic equation. The usual procedure is to first get rid of the cubic term in the quartic equation. This is done

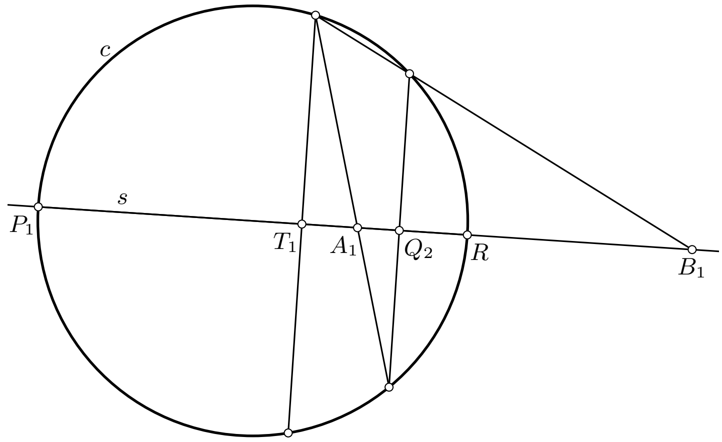

Figure 3: Construction of the points A_1, B_1. The circle c has P_1R as a diameter. The segments in T_1 and Q_2 are orthogonal to s.

by a substitution $z = x - \mu$, leading to the depressed quartic in the variable x. Then one can follow Ferrari's solution, for example, which leads to a cubic equation (see, e.g., [5, §3.2]). Another way is to employ Galois theory and factorization to reduce the quartic to a cubic problem (see, e.g., [4]). In this section, we want to show an alternative way which is inspired by the geometric considerations of Section 2 to reduce the quartic to a cubic equation. The idea is that the pairs of lines I_1J_1, I_2J_2 and I_1J_2, I_2J_1 are degenerate conics in the pencil spanned by C_1 and C_2. And to compute theses degenerate conics from the equations of C_1, C_2 is only a cubic problem. To make this idea work, we only need to reformulate a given quartic equation as a system of two quadratic equations in two variables which represent the conics C_1 and C_2. Before we describe the general case, we illustrate the method with the following example.

Example. Suppose we want to solve the quartic

$$z^4 + z^3 - 45z^2 - 97z + 140 = 0. \tag{2}$$

To have it easier later on, we first remove the quadratic term in the equation. If we substitute $z = x - \mu$ we find the coefficient $3(2\mu^2 - \mu - 15)$ for x^2. This quadratic expression vanishes for $\mu = 3$. The resulting quartic equation in x is then

$$x^4 - 11x^3 + 92x + 80 = 0. \tag{3}$$

Now we consider the two conics

$$C_1 : \quad y - x^2 = 0 \tag{4}$$

$$C_2 : \quad y^2 - 11xy + 92x + 80 = 0 \tag{5}$$

Clearly, if x, y is a solution of the system (4)–(5), then x solves (3), and vice versa, if x is a solution of (3), then $x, y = x^2$ is a solution of (4)–(5). We can write the first conic (4) as $\langle X, AX\rangle = 0$, where $X = (x, y, 1)^\top$ and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. Similarly, the second conic (5) is given by the equation $\langle X, BX\rangle = 0$ with

$$B = \begin{pmatrix} 0 & -11 & 92 \\ -11 & 2 & 0 \\ 92 & 0 & 160 \end{pmatrix}.$$

The pencil of the two conics, i.e., the conics whose equations are linear combinations of (4) and (5), are then given by

$$\langle X, (\lambda A + B)X\rangle = 0. \tag{6}$$

Observe that all conics of the pencil pass through the intersections of C_1 and C_2. The idea is now to find two values of λ such that the matrix $\lambda A + B$ is singular and hence the corresponding conic of the pencil degenerates to two straight lines. Then, the intersections of the conics C_1 and C_2 can simply be computed by intersecting these lines, see Figure 4.

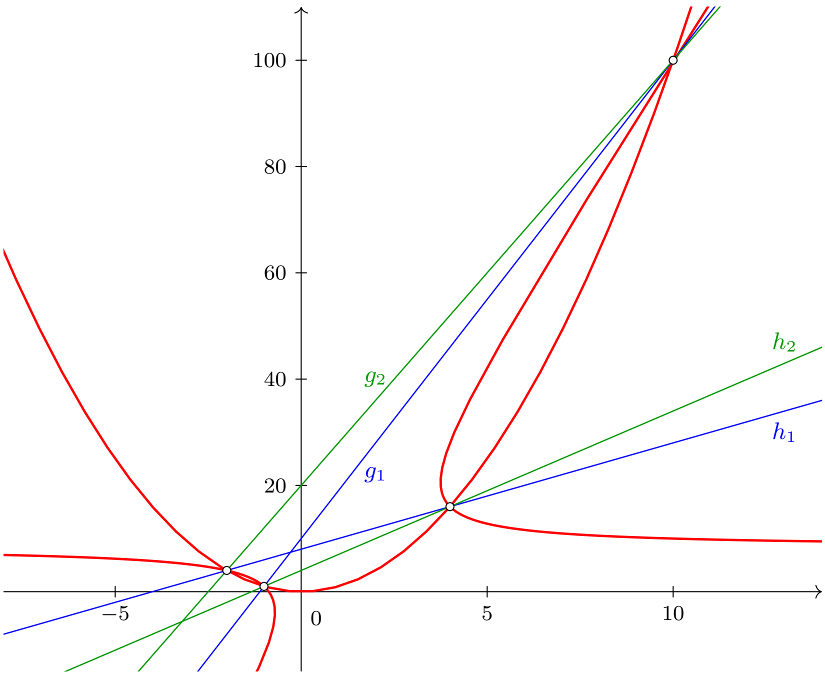

Figure 4: The parabola C_1 and the hyperbola C_2 are shown in red. The degenerate conics in the pencil of C_1, C_2 are the blue lines g_1, h_1 and the green lines g_2, h_2.

The point is that finding the degenerate conics in the pencil of C_1 and C_2 corresponds to the *cubic* equation $\det(\lambda A + B) = -2\lambda^3 + 2664\lambda - 36288 = 0$. Observe that there is no quadratic term, which makes it quite easy to find the roots in general (see below). In our model case we find the values $\lambda_1 = 18$ and $\lambda_2 = 24$. For these values (6) factors in two straight lines as follows (see below for the general case):

$$\langle X, (18A + B)X\rangle = \langle X, g_1\rangle\langle X, h_1\rangle, \quad \langle X, (24A + B)X\rangle = \langle X, g_2\rangle\langle X, h_2\rangle,$$

with

$$g_1 = \begin{pmatrix} 9 \\ -1 \\ 10 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 4 \\ -2 \\ 16 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 8 \\ -1 \\ 20 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix}.$$

The intersections of these lines can be computed by the respective cross products:

$$g_1 \times g_2 = -\begin{pmatrix} 10 \\ 100 \\ 1 \end{pmatrix}, \quad g_1 \times h_2 = -12\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad h_1 \times g_2 = 12\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}, \quad h_1 \times h_2 = 4\begin{pmatrix} 4 \\ 16 \\ 1 \end{pmatrix}.$$

We read off the solutions $x_1 = 10$, $x_2 = -1$, $x_3 = -2$, $x_4 = 4$ of the equation (3). Hence, the solutions of the original equation (2) are $z_1 = 7$, $z_2 = -4$, $z_3 = -5$, $z_4 = 1$.

The general case of a quartic equation. Let uns now solve the general quartic equation

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + z^4 = 0.$$

The coefficients are allowed to be complex numbers, and all subsequent computations are carried out in $\mathbb{C}$. Substituting $z = x - \mu$ yields for the quadratic term of x^2 the expression $a_2 - 3a_3\mu + 6\mu^2$. This is a quadratic equation, and we can choose one of its solutions to obtain a quartic equation without quadratic term. So, from now on we assume that the quartic has the form

$$a_0 + a_1 x + a_3 x^3 + x^4 = 0. \tag{7}$$

Consider the two conics

$$C_1 : \quad y - x^2 = 0 \tag{8}$$

$$C_2 : \quad a_0 + a_1 x + a_3 xy + y^2 = 0. \tag{9}$$

As in the example above we have that if x, y is a solution of the system (8)–(9), then x solves (7), and vice versa, if x is a solution of (7), then $x, y = x^2$ is a solution of (8)–(9). As before, the first conic (8) is given by $\langle X, AX\rangle = 0$, and the second conic by $\langle X, BX\rangle = 0$, where now

$$B = \begin{pmatrix} 0 & a_3 & a_1 \\ a_3 & 2 & 0 \\ a_1 & 0 & 2a_0 \end{pmatrix}.$$

The cubic equation to determine the degenerate conics in the pencil of C_1, C_2 is

$$\det(\lambda A + B) = \underbrace{a_1^2 + a_0 a_3^2}_{=:p} + \underbrace{(a_1 a_3 - 4a_0)}_{=:q}\lambda + \lambda^3 = 0. \tag{10}$$

If $p = 0$, then $\lambda = 0$ is a solution. This means that B is singular and the left-hand side of (9) is the product of two linear terms, namely $(-a_1/a_3 + a_3 x + y)(a_1/a_3 + y)$ if $a_3 \neq 0$ and $(y + \sqrt{-a_0})(y - \sqrt{-a_0})$ if $a_3 = 0$. The problem is therefore reduced to determining the intersection points of the parabola C_1 and straight lines. So let us assume now that $p \neq 0$. Then, (10) hat at least two different solutions. Observe that every complex number $\lambda \neq 0$ can be written as $\lambda = \alpha + \beta$ with $\alpha^3 + \beta^3 = -p$. Indeed, for $\beta = \lambda - \alpha$ we have $\beta^3 = \lambda^3 - 3\lambda^2\alpha + 3\lambda\alpha^2 - \alpha^3$ and hence

$$-p = \alpha^3 + \beta^3 = \lambda(\lambda^2 - 3\lambda\alpha + 3\alpha^2).$$

This is a quadratic equation for $\alpha \in \mathbb{C}$ with a solution if $\lambda \neq 0$. In particular, a root λ of (10) can be written in the form $\lambda = \alpha + \beta$ with $\alpha^3 + \beta^3 = -p$. Using this in (10), we find

$$0 = p + q(\alpha + \beta) + (\alpha + \beta)^3 = (\alpha + \beta)(3\alpha\beta + q)$$

and hence, since $\alpha + \beta \neq 0$, it follows that $\beta = -\frac{q}{3\alpha}$. This yields

$$-p = \alpha^3 + \beta^3 = \alpha^3 - \frac{q^3}{27\alpha^3}$$

which is a quadratic equation in α^3, and hence we can consider (10) as solved. We pick two solutions λ_1, λ_2. For these values the conics given by $\langle X, (\lambda_j A + B)X\rangle = 0$ are the product of two straight lines

$$0 = \langle X, (\lambda_j A + B)X\rangle = \langle X, g_j\rangle\langle X, h_j\rangle, \tag{11}$$

say

$$g_j = \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix}, \quad h_j = \begin{pmatrix} u_j \\ v_j \\ w_j \end{pmatrix}.$$

Writing out the matrix we have

$$\lambda_j A + B = \begin{pmatrix} 2\lambda_j & a_3 & a_1 \\ a_3 & 2 & -\lambda_j \\ a_1 & -\lambda_j & 2a_0 \end{pmatrix}.$$

If we compare the coefficients in (11), we obtain:

$$\begin{array}{cc} au = 2\lambda_j & du + av = 2a_3 \\ bv = 2 & cu + aw = 2a_1 \\ cw = 2a_0 & cv + bw = -2\lambda_j \end{array}$$

One finds the following solution (up to a nonzero factor):

$$\begin{aligned} (u_j, v_j, w_j) &= (2\lambda_j, a_3 + \sqrt{\mu_j}, a_1 + q_j\sqrt{\nu_j}) \\ (a_j, b_j, c_j) &= (2\lambda_j, a_3 - \sqrt{\mu_j}, a_1 - q_j\sqrt{\nu_j}) \end{aligned}$$

where

$$\mu_j = a_3^2 - 4\lambda_j, \quad \nu_j = a_1^2 - 4a_0\lambda_j$$

are negative minors of $\lambda_j A + B$. Depending on the choice of the roots of μ_1 and ν_j the sign q_j is

$$q_j = \frac{a_1a_3 + 2\lambda_j^2}{\sqrt{\mu_j}\sqrt{\nu_j}} \in \{-1, 1\}.$$

Observe that

$$0 = -2\lambda_j \det(\lambda_j A + B) = (a_1a_3 + 2\lambda_j^2)^2 - \mu_j\nu_j.$$

If $\mu_j\nu_j = 0$, one can choose either $q_j = 1$ or $q_j = -1$. Now that the lines g_j and h_j are determined, their intersections and hence the solutions x_j are easily computed by the respective cross products, as in the example.

Of course, there are other ways to find the factorization in (11). For example, one can diagonalize the matrix $\lambda_1 A + B$ with an orthogonal matrix T such that

$$T^\top(\lambda_1 A + B)T = \operatorname{diag}(\xi_1, \xi_2, 0) =: D$$

with eigenvalues ξ_1, ξ_2. In this form, we have

$$0 = \langle X, DX\rangle = \xi_1 x^2 + \xi_2 y^2 = (\sqrt{\xi_1}x + i\sqrt{\xi_2}y)(\sqrt{\xi_1}x - i\sqrt{\xi_2}y).$$

The two lines are therefore

$$\left\langle \begin{pmatrix} \sqrt{\xi_1} \\ \pm i\sqrt{\xi_2} \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right\rangle = 0.$$

Backtransformation gives the line

$$\left\langle T \begin{pmatrix} \sqrt{\xi_1} \\ \pm i\sqrt{\xi_2} \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right\rangle = 0.$$

Hence the lines g_1, h_1 are given by $\sqrt{\xi_1}n_1 \pm i\sqrt{\xi_2}n_2$, where n_1, n_2 are the first two columns of T, i.e., the two eigenvectors of T that corrrespond to the eigenvalues ξ_1, ξ_2. The same can be done with the matrix $\lambda_2 A + B$ in order to determine the lines g_2, h_2.

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