Solution of Riddle 12208

The Logic Coffee Circle*

Abstract

We solve the following riddle:

12208. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* (In memory of John Horton Conway, 1937–2020.) Three wise women, Alice, Beth, and Cecily, sit around a table. A card with a positive integer on it is attached to each woman's forehead, so she can see the other two numbers but not her own. The women know that one of the three integers is equal to the sum of the other two. The same question, "Can you determine the number on your forehead?," is addressed to the wise women in the following order: Alice, Beth, Cecily, Alice, Beth, Cecily, ... The answer is either "No" or "Yes, the number is ___," and the other wise women hear the answer. The questioning ends as soon as the positive answer is obtained. (Assume that the women are logical and honest, they all know this, they all know that they all know this, and so on.)

(a) Prove that whichever numbers are assigned to the wise women, an affirmative answer is obtained eventually.

(b) Suppose that Alice's second answer is "Yes, the number is 50." Determine the numbers assigned to Beth and Cecily.

(c) Suppose the numbers assigned to Alice, Beth, and Cecily are 1492, 1776, and 284, respectively. Determine who will give the affirmative answer and how many negative answers she will give before that.

1 Solution

Let $\{a, b, c\} \subseteq \mathbb{Z}_{>0}$ such that one of the integers is equal to the sum of the other two. Define $S_{(a,b,c)} \in \mathbb{Z}_{>0} \cup \{\infty\}$ to be the minimal number of steps the women need with the optimal strategy until one of the women gives the positive answer. For example, for all $a > 0$ we have that

$$S_{(a,a,2a)} = 3, \ S_{(a,2a,a)} = 2 \text{ and } S_{(2a,a,a)} = 1. \tag{1}$$

a) We want to show that $S_{(a,b,c)} \in \mathbb{Z}_{>0}$. Assume towards a contradiction that $S_{(a,b,c)}$ is not always finite. Choose numbers $a, b, c \in \mathbb{Z}_{>0}$ with minimal $a+b+c$ such that $S_{(a,b,c)} = \infty$. Without loss of generality assume that $a < b \leq c$ (if $a = b$ we are done by (1)). Cecily knows that her number is either $a+b$ or $b-a$. So when in $S_{(a,b,b-a)}$ steps no one gives a positive answer, she knows that her number is $a + b$. So we have that

$$S_{(a,b,a+b)} \leq S_{(a,b,b-a)} + 3.$$

Since $a + b + (b - a) < a + b + (a + b)$ we have that $S_{(a,b,b-a)} < \infty$. This is a contradiction.

b) and c) First we want to show that always the woman with the sum gives the positive answer. Assume not. Then there exist $a, b \in \mathbb{Z}_{>0}$ and, without loss of generality, Alice finds out her number in round $S_{(a,b,a+b)}$ where additionally $S_{(a,b,a+b)}$ is minimal. Hence, if Alice sees that Beth and Cecily have the numbers b and $a + b$, respectively, then Alice concludes that her number is a whenever

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the other women (and Alice) answers "no" before round $S_{(a,b,a+b)}$. Thus, if the women have the same number except Alice is attached $a+2b$ to her forehead, then it cannot happen that all the women are answering "no" before round $S_{(a,b,a+b)}$. Therefore, we get

$$S_{(a+2b,b,a+b)} < S_{(a,b,a+b)}.$$

Moreover, the positive answer must come from Beth or Cecily where both do not have the sum and this is a contradiction to the minimality of $S_{(a,b,a+b)}$.

Now we want to show that for all $1 \leq a < b$ we have that

(α) $S_{(a,b,a+b)} = S_{(a,b,b-a)} + 1$

(β) $S_{(b,a+b,a)} = S_{(b,b-a,a)} + 1$

(γ) $S_{(a+b,a,b)} = S_{(b-a,a,b)} + 1$

(δ) $S_{(a,a+b,b)} = S_{(a,b-a,b)} + 2$

(ε) $S_{(b,a,a+b)} = S_{(b,a,b-a)} + 2$

(ζ) $S_{(a+b,b,a)} = S_{(b-a,b,a)} + 2$

All these formulas can be proven similarly. Therefore, we will only prove (δ). Every woman knows that her number is either the sum of the two numbers she sees or the positive difference. In particular, Beth knows that her number is either $a+b$ or $b-a$. As we have seen above, for the problem $(a, b-a, b)$, Cecily gives the positive answer in step $S_{(a,b-a,b)}$ because she is the one with the sum. So in the problem $(a, a+b, b)$, when Cecily says, "no" in step $S_{(a,b-a,b)}$, Beth cannot know until the next round that she does not have the number $b-a$ and must therefore give the positive answer in step $S_{(a,b-a,b)} + 2$. This proves formula (δ).

With the formulas (α)-(ζ) and (1), we can easily solve parts b) and c) of the riddle. For b) we have that

$$S_{(50,20,30)} = S_{(10,20,30)} + 1 = S_{(10,20,10)} + 2 = 2 + 2 = 4.$$

So Beth's number is 20 and Cecily's number is 30 and uniqueness of the solution follows directly by the above formulas and the fact that 50 has to be the sum of the other two numbers.

For c) we get that

$$S_{(1492,1776,284)} = S_{(1492,1208,284)} + 1 = \cdots = S_{(4,8,4)} + 36 = 2 + 36 = 38.$$

So Beth gives the positive answer after saying $12 = \lfloor \frac{38}{3} \rfloor$ times "No".

A python program for computing $S_{(a,b,c)}$

```
def S(a,b,c):
    if (a+b-c)*(a-b+c)*(-a+b+c)!=0:
        print "No number is the sum of the other two!"
    else:
        return s(0,[a,b,c],1)

def s(j,li,i):
    if li[j%3]==li[(j+1)%3]+li[(j+2)%3]:
        d=li[(j+1)%3]-li[(j+2)%3]
        if d==0:
            return i+(j%3)
        else:
            li[j%3]=abs(d)
            return s((j+1)%3,li,i+(3+sgn(d))/2)
    else:
        return s((j+1)%3,li,i)
```

2 A generalization of the riddle

Assume that $n \geq 3$ women are playing the same game. I.e. every woman has a number $a_i \in \mathbb{Z}_{>0}$ on her forehead and one of the integers is equal to the sum of the other integers. Let $S_{(a_1,\ldots,a_n)}$ be the minimal number of steps the women need with the optimal strategy until one of the women gives the positive answer. Define i_0 and choose i_1 such that

$$a_{i_0} = \max\{a_i \mid 1 \leq i \leq n\} \text{ and } a_{i_1} = \max\{a_i \mid 1 \leq i \leq n \wedge i \neq i_0\}.$$

Let $d := a_{i_1} - \sum_{\substack{i=1 \\ i \notin \{i_0, i_1\}}}^{n} a_i$. With the same arguments we used in the case $n = 3$, we can prove the following formulas:

1. If $d \leq 0$ we have that
$$S_{(a_1,\ldots,a_n)} = i_0.$$
2. Otherwise, if $d > 0$, we have that
$$S_{(a_1,\ldots,a_n)} = S_{(a_1,\ldots,a_{i_0-1},d,a_{i_0+1},\ldots,a_n)} + k,$$
where k is minimal such that $(i_1 + k) \bmod n = i_0$.