Solution of Riddle 12246

The Logic Coffee Circle*

Abstract

We solve the following riddle:

12246. *Proposed by Seán Stewart, Bomaderry, Australia.* Let ζ be the Riemann zeta function, defined for $n \geq 2$ by $\zeta(n) = \sum_{k=1}^{\infty} 1/k^n$. Let H_n be the nth harmonic number, defined by $H_n = \sum_{k=1}^{n} 1/k$. Prove

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{n^2} + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) H_n}{n} = \frac{\pi^2}{6}.$$

Solution 1

First, we rearrange the two series on the left-hand side:

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{n^2} = \sum_{k=1}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^2 k^n} \right) = \left(\sum_{n=2}^{\infty} \frac{1}{n^2} \right) + \sum_{k=1}^{\infty} \underbrace{\left(\sum_{n=2}^{\infty} \frac{1}{n^2 (k+1)^n} \right)}_{=:a_k} = \big(\zeta(2) - 1\big) + \sum_{k=1}^{\infty} a_k$$

and

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) H_n}{n} = \sum_{k=1}^{\infty} \underbrace{\left(\sum_{n=2}^{\infty} (-1)^n \frac{H_n}{n k^n} \right)}_{=:b_k} = \sum_{k=1}^{\infty} b_k.$$

Modifying the well-known power series representation $\ln(1+t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}$, $|t| < 1$, we get

$$\frac{-\ln(1-t)}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \tag{1}$$

as well as

$$\frac{\ln(1+t)}{(1+t)t} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{(-t)^{n-1}}{(1+t)} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=n}^{\infty} (-t)^{m-1} \right) =$$

$$\sum_{m=1}^{\infty} (-t)^{m-1} \left(\sum_{n=1}^{m} \frac{1}{n} \right) = \sum_{m=1}^{\infty} (-t)^{m-1} H_m. \tag{2}$$

Hence, for every natural number $k \geq 1$,

$$a_k + \frac{1}{k+1} \overset{(1)}{=} \int_0^{\frac{1}{k+1}} \frac{-\ln(1-t)}{t} \, dt = \int_0^{\frac{1}{k}} \frac{\ln(1+s)}{(1+s)s} \, ds \overset{(2)}{=} -b_k + \frac{1}{k},$$

where we substituted $t = \frac{s}{1+s}$ in the second equation. It follows that $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} \big(\frac{1}{k} - \frac{1}{k+1} \big) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \mp \cdots = 1$. In conjunction with Euler's famous result $\zeta(2) = \frac{\pi^2}{6}$, this finishes the proof.

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Solution 2

First notice that

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n^2}+\sum_{n=2}^{\infty}\frac{(-1)^n \mathrm{H}_n}{n}\zeta(n)=\sum_{k=1}^{\infty}\left(\sum_{n=2}^{\infty}\frac{1}{n^2}\left(\frac{1}{k}\right)^n\right)+\left(\sum_{n=2}^{\infty}\frac{(-1)^n \mathrm{H}_n}{n}\left(\frac{1}{k}\right)^n\right). \tag{3}$$

Since for $n=1$, $\frac{1}{n^2}+\frac{(-1)^n \mathrm{H}_n}{n}=0$, sum (3) is equal to

$$\sum_{k=1}^{\infty}\left(\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{k}\right)^n}_{=:\mathrm{Li}_2\left(\frac{1}{k}\right)}+\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n \mathrm{H}_n}{n}\left(\frac{1}{k}\right)^n}_{=:\mathrm{Hi}_1\left(\frac{1}{k}\right)}\right). \tag{4}$$

Note that Li_2 is the so-called dilogarithm. It is well known that $\mathrm{Li}_2(1)=\frac{\pi^2}{6}$ and that

$$\mathrm{Li}_2(z)=-\int_0^z \frac{\ln(1-t)}{t}\,dt\,. \tag{5}$$

Now we will show that $-\mathrm{Hi}_1\left(\frac{1}{k}\right)=\mathrm{Li}_2\left(\frac{1}{k+1}\right)$ for all $k\geq 1$. To prove this, we use the following result (see AMM 69 (1962), Solution to Problem 4946): If $f(x)=\sum_{n=1}^{\infty}a_n x^n$ converges, then

$$\sum_{n=1}^{\infty}a_n \mathrm{H}_n x^n=\int_0^1 \frac{f(x)-f(tx)}{1-t}\,dt\,.$$

For $a_n:=\frac{(-1)^{n+1}}{n}$ we have $f(x)=\ln(1+x)$, and for $x=\frac{1}{k}$ we obtain

$$-\mathrm{Hi}_1\left(\frac{1}{k}\right)=\sum_{n=1}^{\infty}a_n \mathrm{H}_n\left(\frac{1}{k}\right)^n=\int_0^1\frac{\ln\left(1+\frac{1}{k}\right)-\ln\left(1+\frac{t}{k}\right)}{1-t}\,dt=$$
$$\int_0^1\frac{-\ln\left(\frac{k+t}{k+1}\right)}{1-t}\,dt=\int_0^{\frac{1}{k+1}}\frac{-\ln(1-s)}{s}\,ds\overset{(5)}{=}\mathrm{Li}_2\left(\frac{1}{k+1}\right),$$

where we substituted $t=1-(k+1)s$. So, sum (4) equals $\mathrm{Li}_2(1)=\frac{\pi^2}{6}$, which completes the proof.