Twins of Conic Hexagons

Lorenz Halbeisen
Department of Mathematics, ETH Zentrum, Rämistrasse101, 8092 Zürich, Switzerland
lorenz.halbeisen@math.ethz.ch

Norbert Hungerbühler
Department of Mathematics, ETH Zentrum, Rämistrasse101, 8092 Zürich, Switzerland
norbert.hungerbuehler@math.ethz.ch

key-words: Pascal's mysticum, incidence theorems, geometry of conics

2020 Mathematics Subject Classification: **51M05** 51M15

Abstract

Six points on a conic section define 60 different hexagons and therefore 60 Pascal lines. Each Pascal line passes through three of the 45 intersections of connecting lines of the six given points. Instead of searching for collinear triples (Pascal lines) among these 45 points, we identify and classify all six-tuples among the 45 points which lie on a conic section. These six-tuples will be called Pascal twins of the given six points. It turns out that there are also six-tuples that lie on a conic section which have two points in common with the given six points. These six-tuples are called Siamese Pascal twins for evident reasons.

1 Introduction

One of the most fundamental theorems of projective geometry is Pascal's Hexagon Theorem. The result is often referred to as the *Hexagrammum Mysticum Theorem*. It states the following. Let $P_1, P_2, \dots, P_6$ be arbitrary points on a non-degenerate conic C and σ a permutation of the set $\{1, 2, \dots, 6\}$. Then the three pairs of opposite sides of the conic hexagon $P_{\sigma(1)}, P_{\sigma(2)}, \dots, P_{\sigma(6)}$ (extended if necessary) meet at three points which lie on a straight line, called the Pascal line of the hexagon. Modulo cyclic renumbering of the points or reversal of the order, there are $\frac{6!}{6\cdot 2} = 60$ different hexagons. In general, the 60 resulting Pascal lines are different from each other. The Swiss mathematician Jakob Steiner found that these Pascal lines concur in threes in 20 points, which we today call Steiner nodes. Two decades after Steiner's discovery, Thomas Kirkman announced that the Pascal lines also concur in threes at 60 more points, now known as the Kirkman nodes. This is only the beginning of a cascade of further incidences: Three of the Kirkman nodes and one Steiner node lie on one of 20 Cayley lines. The Steiner nodes lie in fours on 15 Plücker lines. Four Cayley lines concur in one of 15 Salmon nodes. We refer to [2] for a wonderful presentation of all these incidences.

Six pairwise distinct points $P_1, P_2, \dots, P_6$ on a non-degenerate conic C, called a *conic hexa-set*, define $\binom{6}{2} = 15$ lines which in turn yield, in general, 45 intersection points different from the points P_i. The intersection of the lines P_iP_j and P_kP_l will be denoted by P_{ijkl}. Let S be the set of these 45 points P_{ijkl}. Instead of chasing collinear points among the points in S, like Pascal did, we ask in this article, if there are six points in S which lie on a non-degenerate conic. Such a hexagon with vertices in S will be called a *Pascal twin* of the original hexagon with vertices P_i (see Figures 1 to 4). A hexagon with $k \geq 1$ vertices among the points $P_1, P_2, \dots, P_6$ and $6 - k$ vertices in S will be called a *Siamese Pascal twin* of the original hexagon with vertices P_i (see Figures 5 to 8).

We will use a rational model to computationally detect and classify Pascal twins and Siamese Pascal twins. These incidence relations are then proven in general by classical methods. The paper is organized as follows. In Section 2 we identify all possible Pascal twins of a conic hexagon. The main result will be that essentially only four such twins exist. In Section 3 we identify all possible Siamese Pascal twins of a conic hexagon. It will turn out that a Siamese twin necessarily has exactly two points in common with the original hexagon, and that again essentially only four such Siamese twins exist. Section 4 will be devoted to the proofs of the results for the Siamese Pascal twins, and the final Section 5 contains the proofs of the results for the Pascal twins.

2 Candidates for Pascal twins

In order to determine the possible candidates for Pascal twins of a conic hexagon, we proceed as follows. We chose six different points $P_1, P_2, \ldots, P_6$ with rational coordinates on a non-degenerate conic C in such a way, that S consists of 45 different points, also with rational coordinates. By a computer search, using exact rational numbers, we check for all $\binom{45}{6}$ possible hexa-sets of points in S whether they lie on a non-degenerate conic. This results in 255 such conic hexa-sets. However, many of these conic hexa-sets are combinatorially the same in the following sense: Suppose T is a conic hexa-set with points $P_{i_n j_n k_n l_n}$ (for $n = 1, 2, \ldots, 6$) lying on a conic, where $P_{i_n j_n k_n l_n}$ is the intersection of the lines $P_{i_n}P_{j_n}$ and $P_{k_n}P_{l_n}$ Let T' be another conic hexa-set of points which is obtained by a permutation σ of the points $P_1, P_2, \ldots, P_6$, *i.e.*, T' consists of the points $P_{\sigma(i_n)\sigma(j_n)\sigma(k_n)\sigma(l_n)}$, $n = 1, 2, \ldots, 6$. We will then say, that T and T' are equivalent. The tedious task to identify the equivalence classes can be delegated to a computer program. One finds exactly four equivalence classes. The following figures show one representative in each class $e_1, \ldots, e_4$. The brown triangles are only for better orientation.

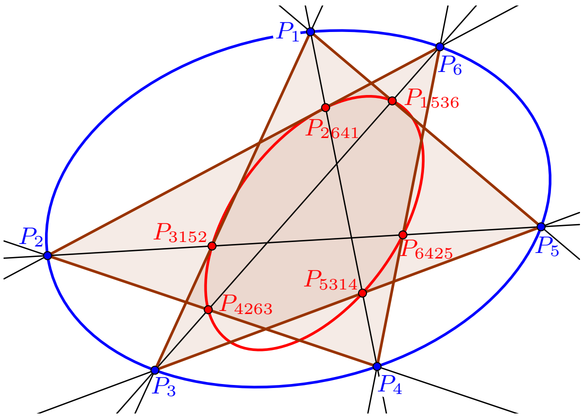

Figure 1: A Pascal twin representative of equivalence class e_1. From one point P_{ijkl} counterclockwise to the next one, apply the permutation (123456) to each index.

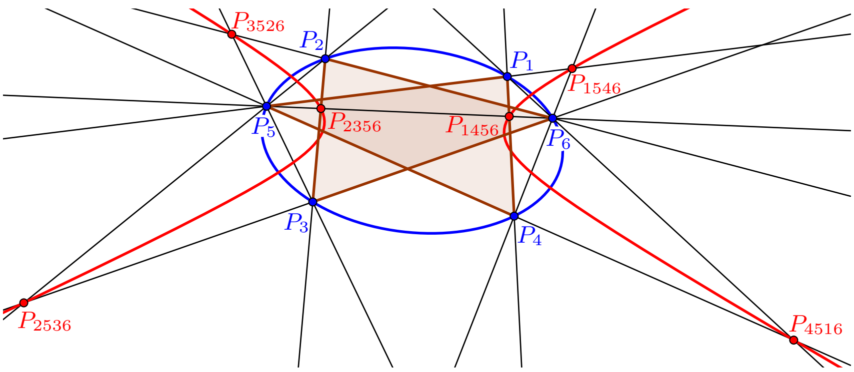

Figure 2: A Pascal twin representative of equivalence class e_2.

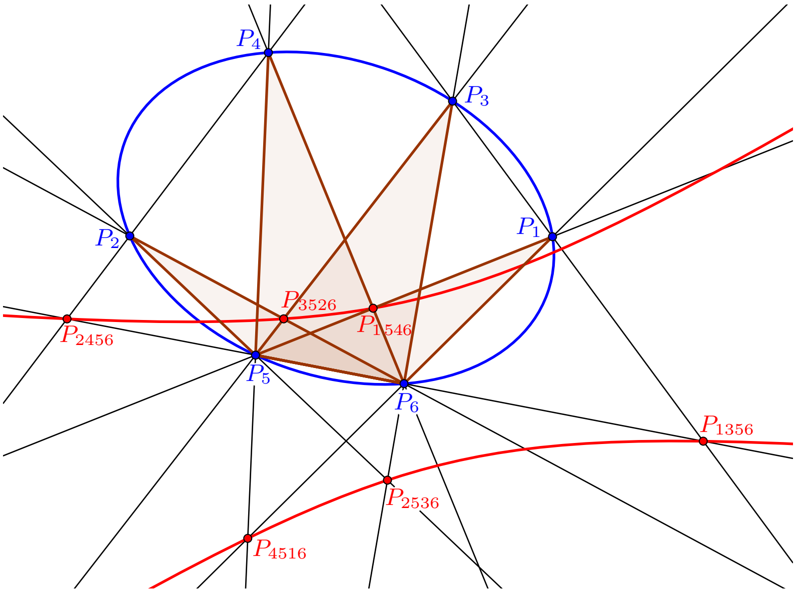

Figure 3: A Pascal twin representative of equivalence class e_3.

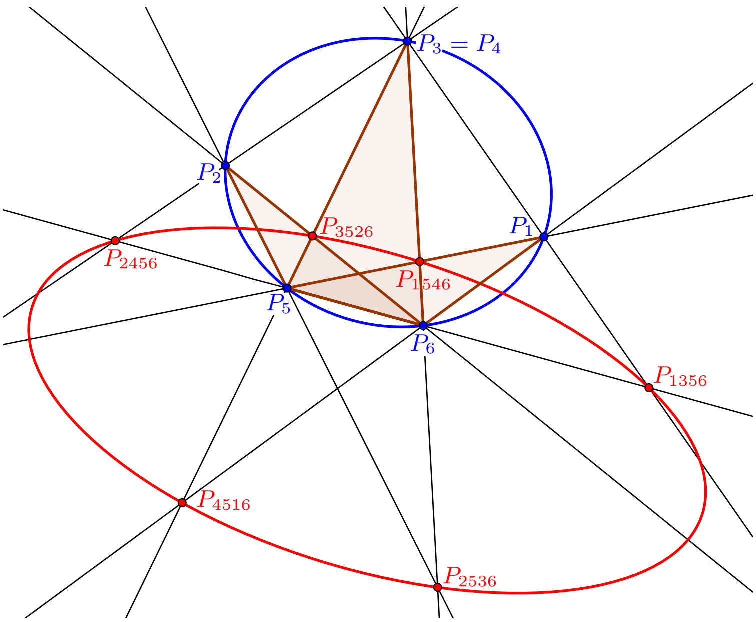

Figure 4: A Pascal twin representative of equivalence class e_4. Notice that Class e_4 is essentially the same as Class e_3, where the points P_3 and P_4 are identified. In particular, there are just 5 different points involved.

Notice that so far, these are only results for the rational points $P_1, P_2, \ldots, P_6$ which we have initially chosen. In Section 5 we will actually prove that these twins exist for an arbitrary choice of points $P_1, P_2, \ldots, P_6$ on a conic C.

3 Candidates for Siamese Pascal twins

The procedure to identify candidates for Siamese Pascal twins is identical to the one in the previous section. We start with six rational points $P_1, P_2, \ldots, P_6$ on a conic C and compute the set S of the 45 rational intersection points P_{ijkl}. Then we fix the points $P_1, P_2, \ldots, P_k$ with $k = 1, 2, \ldots, 4$, and complete them with all possible combinations of points from S to form a hexa-set. We check for each resulting hexa-set if it lies on a non-degenerate conic. Notice that we can restrict the search to $k \leq 4$, since a conic is defined by five points. It turns out that only for $k = 2$ such Siamese Pascal twins exist, and that modulo renumbering, again only four equivalence classes $f_1, \ldots, f_4$ exist. For better readability we denote the two points which lie on both conics by X and Y, and the four remaining points on the original conic are denoted by $P_1, \ldots, P_4$. The Figures 5 to 8 show one representative of each equivalence class.

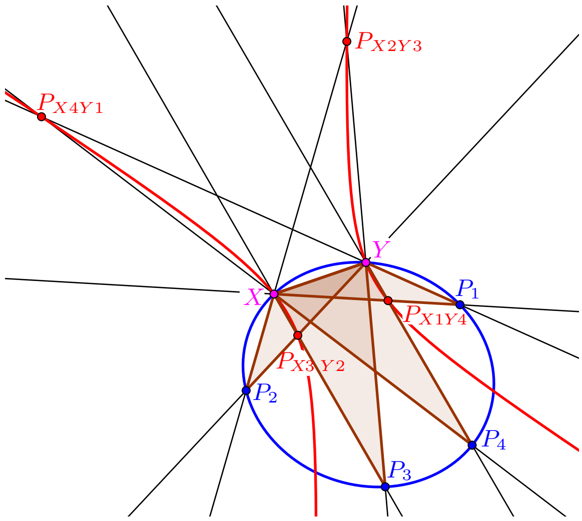

Figure 5: A Siamese Pascal twin representative of equivalence class f_1.

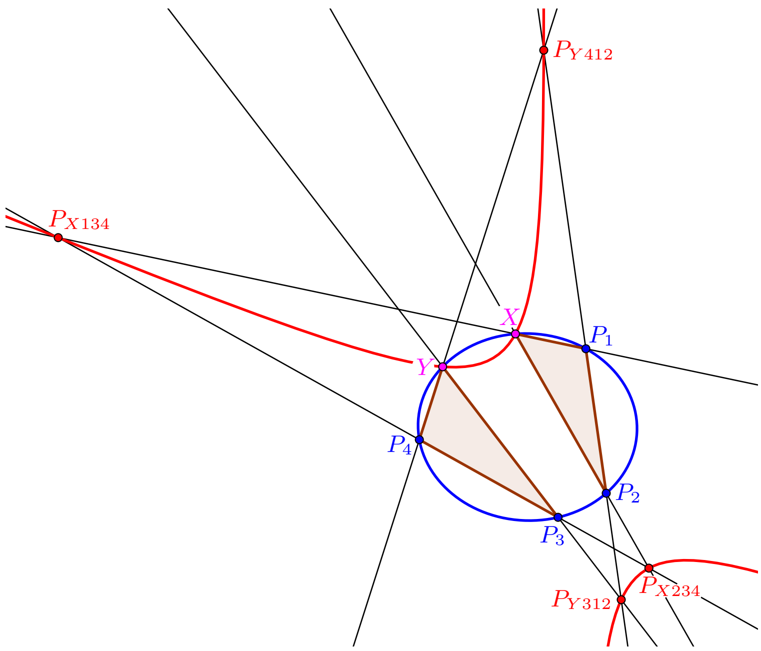

Figure 6: A Siamese Pascal twin representative of equivalence class f_2.

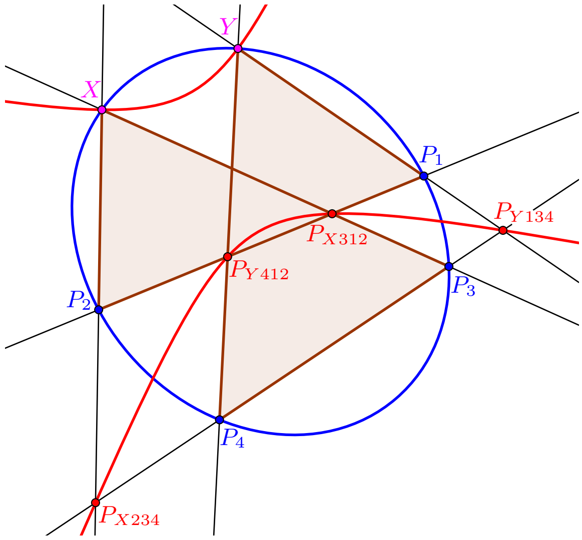

Figure 7: A Siamese Pascal twin representative of equivalence class f_3.

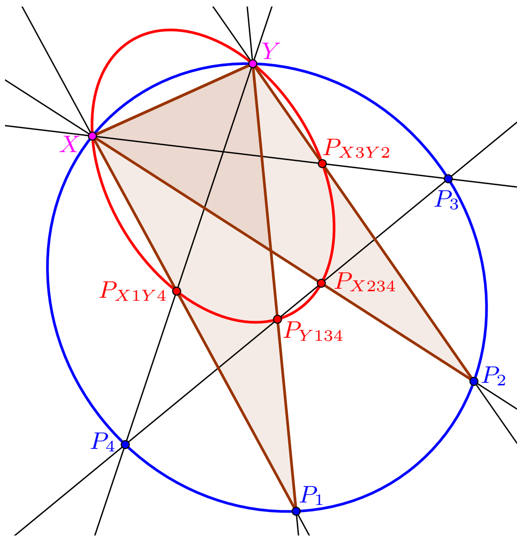

Figure 8: A Siamese Pascal twin representative of equivalence class f_4.

4 Incidence results for Siamese Pascal twins

As a short hand notation, we will use $P - Q$ to denote the line joining two points P, Q, and $g \wedge h$ to denote the intersection of two lines g, h. Recall that Pascal's Theorem states that six points, numbered ①, ②, ③, ④, ⑤, ⑥, lie on a conic iff the three points

$$① - ② \wedge ④ - ⑤, \quad ② - ③ \wedge ⑤ - ⑥, \quad ③ - ④ \wedge ⑥ - ①$$

are collinear. Another useful tool will be Carnot's Theorem (see [1, no. 396]): The the six points $\textcircled{1}, \ldots, \textcircled{6}$ lie on a conic iff the lines

$$\left(\textcircled{1} - \textcircled{2} \wedge \textcircled{3} - \textcircled{4}\right) - \left(\textcircled{4} - \textcircled{5} \wedge \textcircled{6} - \textcircled{1}\right)$$
$$\textcircled{2} - \textcircled{5}$$
$$\textcircled{3} - \textcircled{6}$$

are concurrent. Since Carnot's Theorem is independent of the enumeration of six points and since every enumeration of six points $P_1, \ldots, P_6$ leads to three lines, we get that these three lines are concurrent iff the six points $P_1, \ldots, P_6$ lie on a conic. For example, if we enumerate the six points $P_1, \ldots, P_6$ by $\textcircled{2}, \textcircled{4}, \textcircled{6}, \textcircled{1}, \textcircled{3}, \textcircled{5}$ and define $h_1 = P_4 - P_1$, $h_2 = P_5 - P_2$, $h_3 = P_2 - P_6$, $h_4 = P_3 - P_4$, $Q_1 = h_1 \wedge h_2$, $Q_2 = h_3 \wedge h_4$, then the six points $P_1, \ldots, P_6$ lie on a conic iff the three lines $Q_1 - Q_2$, $P_1 - P_6$, $P_5 - P_3$, are concurrent. For better readability we denote this as follows, where for simplicity we omit the circles around the numbers:

	P_1	P_2	P_3	P_4	P_5	P_6	concurrent lines
enumeration	2	4	6	1	3	5	$Q_1 - Q_2$, $P_1 - P_6$, $P_5 - P_3$

Theorem 1 (Class f_1)**.** *Let $X, Y, P_1 \ldots, P_4$ be points in the projective plane, and P_{XiYj} be the intersection of the lines $X - P_i$ and $Y - P_j$. Then the points $X, Y, P_1, \ldots, P_4$ lie on a conic iff the points $X, Y, P_{X1Y4}, P_{X3Y2}, P_{X2Y3}, P_{X4Y1}$ lie on a conic.*

In the following theorems, all conics are assumed to be non-degenerate.

Proof. We have to show that $X, Y, P_1 \ldots, P_4$ lie on a conic iff the points $X, Y, Q_1, \ldots, Q_4$ lie on a conic, where $Q_1 = P_{X1Y4}$, $Q_2 = P_{X3Y2}$, $Q_3 = P_{X2Y3}$, $Q_4 = P_{X4Y1}$ (see Figure 9):

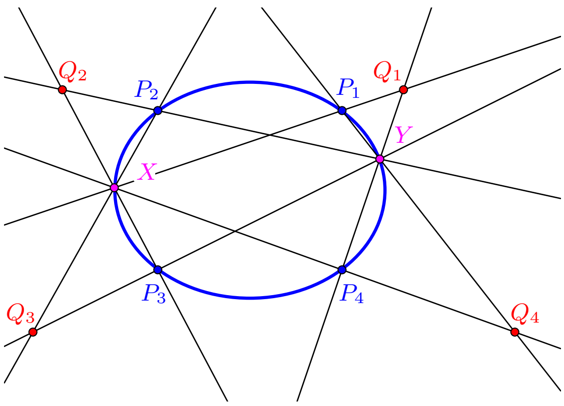

Figure 9: Proof of Theorem 1.

By Carnot's Theorem with respect to the 6 points $X, Y, P_1 \ldots, P_4$ we obtain:

	X	Y	P_1	P_2	P_3	P_4	concurrent lines
enumeration	1	4	2	5	6	3	$Q_3 - Q_4$, $P_1 - P_2$, $P_3 - P_4$
enumeration	1	4	3	6	5	2	$Q_1 - Q_2$, $P_1 - P_2$, $P_3 - P_4$

Furthermore, by Carnot's Theorem with respect to the 6 points $X, Y, Q_1 \ldots, Q_4$ we have:

	X	Y	Q_1	Q_2	Q_3	Q_4	concurrent lines
enumeration	1	4	6	3	2	5	$P_1 - P_2$, $Q_1 - Q_2$, $Q_3 - Q_4$
enumeration	1	4	5	2	3	6	$P_3 - P_4$, $Q_1 - Q_2$, $Q_3 - Q_4$

This shows that $X, Y, P_1 \ldots, P_4$ lie on a conic if and only if $X, Y, Q_1 \ldots, Q_4$ lie on a conic. *q.e.d.*

Theorem 2 (Class f_2)**.** *Let $X, Y, P_1 \ldots, P_4$ be points in the projective plane. Then the points $X, Y, P_1, \ldots, P_4$ lie on a conic iff the points $X, Y, P_{X134}, P_{X234}, P_{Y312}, P_{Y412}$ lie on a conic.*

Proof. We have to show that $X, Y, P_1 \ldots, P_4$ lie on a conic iff the points $X, Y, Q_1, \ldots, Q_4$ lie on a conic, where $Q_1 = P_{X134}$, $Q_2 = P_{X234}$, $Q_3 = P_{Y312}$, $Q_4 = P_{Y412}$ (see Figure 10):

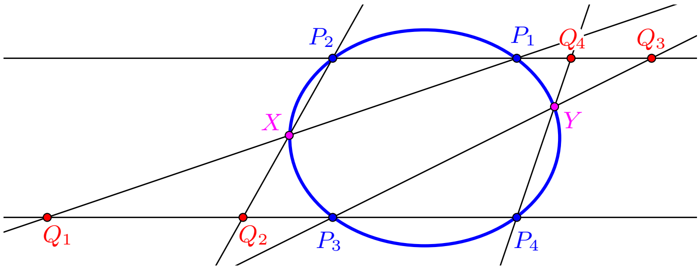

Figure 10: Proof of Theorem 2.

By Carnot's Theorem with respect to the 6 points $X, Y, P_1 \ldots, P_4$ we obtain:

	X	Y	P_1	P_2	P_3	P_4	concurrent lines
enumeration	3	6	5	4	2	1	$Q_2 - Q_4$, $P_3 - P_1$, $X - Y$

Furthermore, by Carnot's Theorem with respect to the 6 points $X, Y, Q_1 \ldots, Q_4$ we have:

	X	Y	Q_1	Q_2	Q_3	Q_4	concurrent lines
enumeration	3	6	4	5	1	2	$P_1 - P_3$, $Q_4 - Q_2$, $X - Y$

This shows that $X, Y, P_1 \ldots, P_4$ lie on a conic if and only if $X, Y, Q_1 \ldots, Q_4$ lie on a conic. *q.e.d.*

Theorem 3 (Class f_3)**.** *Let $X, Y, P_1 \ldots, P_4$ be points in the projective plane. Then the points $X, Y, P_1, \ldots, P_4$ lie on a conic iff the points $X, Y, P_{Y412}, P_{X312}, P_{X234}, P_{Y134}$ lie on a conic.*

Proof. We have to show that $X, Y, P_1 \ldots, P_4$ lie on a conic iff the points $X, Y, Q_1, \ldots, Q_4$ lie on a conic, where $Q_1 = P_{Y134}$, $Q_2 = P_{X234}$, $Q_3 = P_{X312}$, and $Q_4 = P_{Y412}$ (see Figure 11):

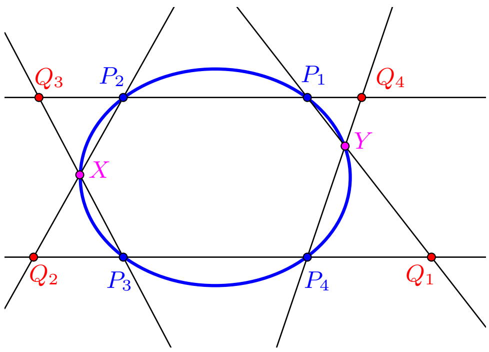

Figure 11: Proof of Theorem 3.

Let

$$S := X - P_2 \wedge Y - P_4 \quad \text{and} \quad T := X - P_3 \wedge Y - P_1 .$$

By Carnot's Theorem with respect to the 6 points $X, Y, P_1 \ldots, P_4$ we obtain:

	X	Y	P_1	P_2	P_3	P_4	concurrent lines
enumeration	1	4	5	2	6	3	$S - T$, $P_2 - P_1$, $P_4 - P_3$

Furthermore, since $S = X - Q_2 \wedge Y - Q_4$ and $T = X - Q_3 \wedge Y - Q_1$, by Carnot's Theorem with respect to the 6 points $X, Y, Q_1 \ldots, Q_4$ we have:

	X	Y	Q_1	Q_2	Q_3	Q_4	concurrent lines
enumeration	1	4	5	2	6	3	$S - T$, $Q_2 - Q_1$, $Q_4 - Q_3$

Since $P_2 - P_1 = Q_4 - Q_3$ and $P_4 - P_3 = Q_2 - Q_1$, this shows that $X, Y, P_1 \dots, P_4$ lie on a conic if and only if $X, Y, Q_1 \dots, Q_4$ lie on a conic. *q.e.d.*

Theorem 4 (Class f_4)**.** *Let $X, Y, P_1 \dots, P_4$ be points in the projective plane. Then the points $X, Y, P_1, \dots, P_4$ lie on a conic iff the points $X, Y, P_{X1Y4}, P_{X3Y2}, P_{X234}, P_{Y134}$ lie on a conic.*

Proof. We have to show that $X, Y, P_1 \dots, P_4$ lie on a conic iff the points $X, Y, Q_1, \dots, Q_4$ lie on a conic, where $Q_1 = P_{X1Y4}$, $Q_2 = P_{X3Y2}$, $Q_3 = P_{X234}$, and $Q_4 = P_{Y134}$ (see Figure 12):

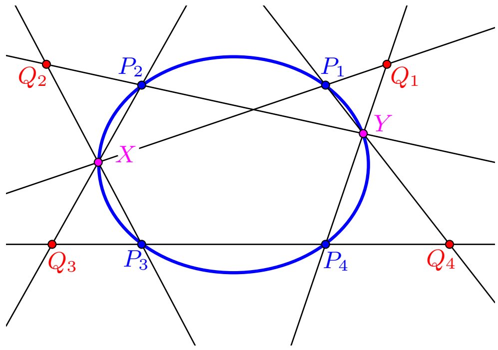

Figure 12: Proof of Theorem 4.

By Carnot's Theorem with respect to the 6 points $X, Y, P_1 \dots, P_4$ we obtain:

	X	Y	P_1	P_2	P_3	P_4	concurrent lines
enumeration	1	4	6	3	2	5	$Q_2 - Q_1$, $P_3 - P_4$, $P_2 - P_1$

Furthermore, by Carnot's Theorem with respect to the 6 points $X, Y, Q_1 \dots, Q_4$ we have:

	X	Y	Q_1	Q_2	Q_3	Q_4	concurrent lines
enumeration	1	4	2	5	6	3	$P_1 - P_2$, $Q_1 - Q_2$, $Q_4 - Q_3$

Since $P_3 - P_4 = Q_4 - Q_3$, this shows that $X, Y, P_1 \dots, P_4$ lie on a conic if and only if $X, Y, Q_1 \dots, Q_4$ lie on a conic. *q.e.d.*

5 Incidence results for Pascal twins

We start with the proof for the Class e_1. This proof is again based on a multiple nested application of the theorems of Pascal and Carnot.

Theorem 5 (Class e_1)**.** *Let $P_1, P_2, \dots, P_6$ be points in the projective plane, and P_{ijkl} be the intersection of the lines $P_i - P_j$ and $P_k - P_l$. Then the points $P_1, P_2, \dots, P_6$ lie on a conic iff the points $P_{1426}, P_{2531}, P_{3642}, P_{4153}, P_{5264}, P_{6315}$ lie on a conic. The two hexa-sets $P_1, P_2, \dots, P_6$ and $P_{1426}, P_{2531}, P_{3642}, P_{4153}, P_{5264}, P_{6315}$, share two common Pascal lines.*

Proof. We have to show that $P_1, \dots, P_6$ lie on a conic iff the points $Q_1, \dots, Q_6$ lie on a conic, where $Q_1 = P_{1426}$, $Q_2 = P_{2531}$, $Q_3 = P_{3642}$, $Q_4 = P_{4153}$, $Q_5 = P_{5264}$, $Q_6 = P_{6315}$ (see Figure 13):

By Pascal's Theorem with respect to the 6 points $P_1 \dots, P_6$ we obtain:

	P_1	P_2	P_3	P_4	P_5	P_6	collinear points
enumeration	3	6	1	4	5	2	$\begin{cases} P_3 - P_6 \wedge P_4 - P_5 =: R \\ P_6 - P_1 \wedge P_5 - P_2 =: S \\ P_1 - P_4 \wedge P_2 - P_3 =: T \end{cases}$

In particular, the three points R, S, T lie on a Pascal line of the hexa-set $P_1, \dots, P_6$.

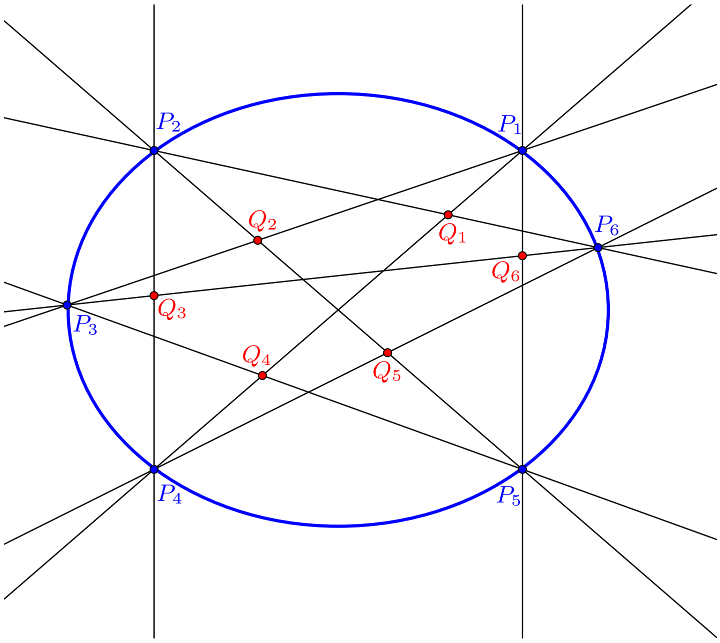

Figure 13: Proof of Theorem 5.

Furthermore, by Carnot's Theorem with respect to the 6 points $P_1 \ldots, P_6$ we obtain:

	P_1	P_2	P_3	P_4	P_5	P_6	concurrent lines
enumeration	1	4	6	2	5	3	$Q_1 - Q_2,\ \underbrace{P_4 - P_5,\ P_6 - P_3}_{\text{meet in } R}$
enumeration	2	6	4	1	3	5	$Q_4 - Q_3,\ \underbrace{P_1 - P_6,\ P_5 - P_2}_{\text{meet in } S}$
enumeration	2	6	3	5	1	4	$Q_6 - Q_5,\ \underbrace{P_1 - P_4,\ P_3 - P_2}_{\text{meet in } T}$

Finally, by Pascal's Theorem with respect to the 6 points $Q_1 \ldots, Q_6$ we obtain:

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	collinear points
enumeration	1	2	5	6	3	4	$\begin{cases} \underbrace{Q_1 - Q_2 \wedge \overbrace{Q_6 - Q_3}^{=P_6 - P_3}}_{=R} \\ \underbrace{\overbrace{Q_2 - Q_5}^{=P_2 - P_5} \wedge Q_3 - Q_4}_{=S} \\ \underbrace{Q_5 - Q_6 \wedge \overbrace{Q_4 - Q_1}^{=P_4 - P_1}}_{=T} \end{cases}$

This shows that the hexagon $P_3P_6P_1P_4P_5P_2$ (in this order) lie on a conic with Pascal line $R-S-T$ if and only if the hexagon $Q_1Q_2Q_5Q_6Q_3Q_4$ (in this order) lie on a conic with the same Pascal line.

In order to find the second common Pascal line of the two conics, we proceed as follows: By Pascal's Theorem with respect to the 6 points $P_1 \ldots, P_6$ we obtain:

	P_1	P_2	P_3	P_4	P_5	P_6	collinear points
enumeration	2	5	6	3	4	1	$\begin{cases} P_2 - P_5 \wedge P_3 - P_4 =: R' \\ P_5 - P_6 \wedge P_4 - P_1 =: S' \\ P_6 - P_3 \wedge P_1 - P_2 =: T' \end{cases}$

Furthermore, by Carnot's Theorem with respect to the 6 points $P_1 \ldots, P_6$ we obtain:

	P_1	P_2	P_3	P_4	P_5	P_6	concurrent lines
enumeration	6	3	5	1	4	2	$Q_6 - Q_1, \underbrace{P_3 - P_4,\ P_5 - P_2}_{\text{meet in } R'}$
enumeration	2	5	1	3	6	4	$Q_2 - Q_3, \underbrace{P_5 - P_6,\ P_1 - P_4}_{\text{meet in } S'}$
enumeration	4	1	3	5	2	6	$Q_4 - Q_5, \underbrace{P_1 - P_2,\ P_3 - P_6}_{\text{meet in } T'}$

Finally, by Pascal's Theorem with respect to the 6 points $Q_1 \ldots, Q_6$ we obtain:

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	collinear points
enumeration	6	1	4	5	2	3	$\begin{cases} \underbrace{Q_6 - Q_1 \wedge \overbrace{Q_5 - Q_2}^{=P_5 - P_2}}_{=R'} \\ \underbrace{\overbrace{Q_1 - Q_4}^{=P_1 - P_4} \wedge Q_2 - Q_3}_{=S'} \\ \underbrace{Q_4 - Q_5 \wedge \overbrace{Q_3 - Q_6}^{=P_3 - P_6}}_{=T'} \end{cases}$

This shows that the conic hexagon $P_2P_5P_6P_3P_4P_1$ has the same Pascal line $R' - S' - T'$ as the conic hexagon $Q_6Q_1Q_4Q_5Q_2Q_3$. *q.e.d.*

Remark. By a computer search we found that the Siamese Pascal twins in the Classes f_1, f_2, f_3, f_4 share $4, 6, 3, 3$ Pascal lines. In the Class f_2, 4 of the 6 common Pascal lines meet in the point P_{1234}. The Pascal twins in the Classes e_2, e_3, e_4 have no common Pascal line.

Before we consider the Classes e_2, e_3 and e_4, we prove two auxiliary results. The first one is a quantified version of Pascal's Theorem:

Lemma 6. *Let P_1, P_2, P_3, P_4, X, Y be points on a conic, and let $\ell = X - Y$. If $A = P_1 - P_2 \wedge \ell$, $B = P_1 - P_4 \wedge \ell$, $A' = P_3 - P_2 \wedge \ell$, and $B' = P_3 - P_4 \wedge \ell$, then the cross ratios (X, Y, A, B) and (X, Y, A', B') are equal. Vice versa, if $(X, Y', A, B) = (X, Y', A', B')$ for some Y' on ℓ, then either $Y' = X$ or $Y' = Y$.*

Proof. By a projective transformation we may assume that $P_1P_2P_3P_3$ is a rectangle and X a point on its circumcircle C (see [3, proof of Satz 7.10]). Now, observe that the set of the four blue lines $P_1P_2, P_1 - X, P_1 - P_4, P_1 - Y$ in Figure 14 is congruent to the set of the four green lines $P_3 - P_2, P_3 - X, P_3 - P_4, P_3 - Y$. For example, the angles $\sphericalangle Y P_1 P_4$ and $\sphericalangle Y P_3 P_4$ agree as angles over the same arc $\overset{\frown}{YP_4}$ on the circle C. The first set of lines intersects the line ℓ in the points A, X, B, Y, the second set of lines intersects the line ℓ in the points $A', X, B'Y$. Hence the corresponding cross ratios agree.

Vice versa, using the definition of the cross ratio, a short calculation shows that the only solutions of the equation $(X, Y', A, B) = (X, Y', A', B')$ are $Y' = X$ and $Y' = Y$. *q.e.d.*

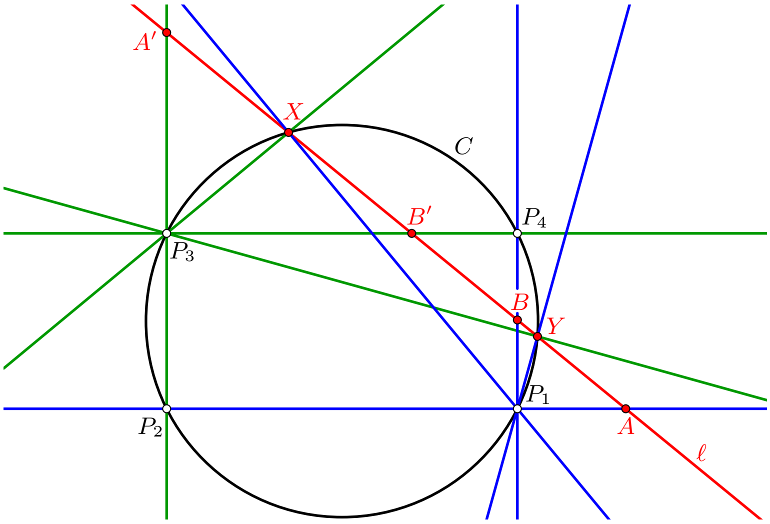

Figure 14: Equality of the cross ratios $(X, Y, A, B) = (X, Y, A', B')$ in the proof of Lemma 6.

The next result can be seen as a dynamic version of Class f_1:

Pascal Twin Porism. *Let $X, Y, P_1 \ldots, P_4$ and $X, Y, P_{X1Y4}, P_{X3Y2}, P_{X2Y3}, P_{X4Y1}$ be two conic hexagons. Furthermore, let X' be a point on the line $\ell = X - Y$, let C' be the conic defined by the 5 points $X', P_1 \ldots, P_4$, and let Y' be the other intersection point of the line ℓ with C'. Then the conic C'' defined by the 5 points $X', P_{X1Y4}, P_{X3Y2}, P_{X2Y3}, P_{X4Y1}$ passes also through Y' (see Figure 15).*

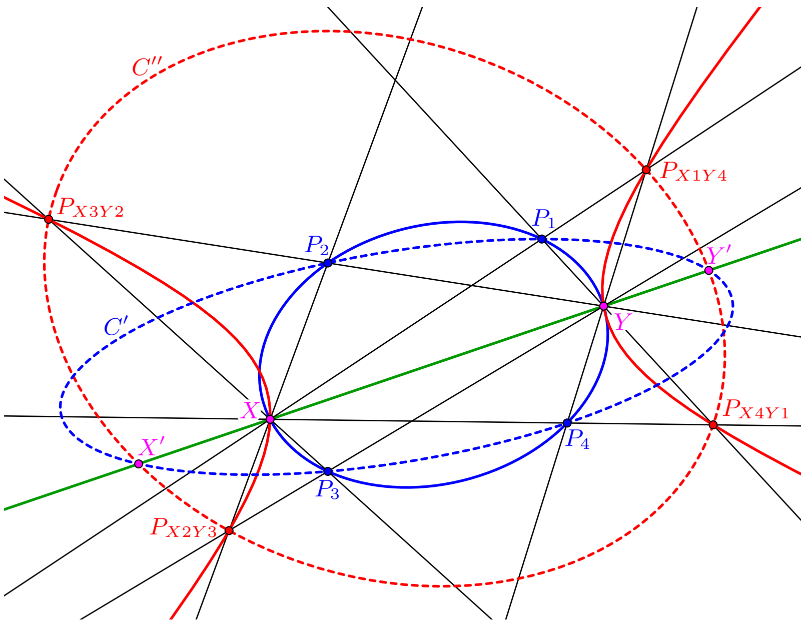

Figure 15: Proof of the Pascal Twin Porism.

Proof. We will use twice Lemma 6 in the proof to locate the point Y', once with respect to the conic C' and once with respect to the conic C''. To do so, we need the following points:

$$\begin{aligned} A &= P_1 - P_2 \wedge \ell & E &= P_{X4Y1} - P_{X2Y3} \wedge \ell \\ B &= P_1 - P_4 \wedge \ell & F &= P_{X4Y1} - P_{X1Y4} \wedge \ell \\ A' &= P_3 - P_2 \wedge \ell & E' &= P_{X3Y2} - P_{X2Y3} \wedge \ell \\ B' &= P_3 - P_4 \wedge \ell & F' &= P_{X3Y2} - P_{X3Y2} \wedge \ell \end{aligned}$$

By a projective transformation, we may assume that

$$\begin{aligned} P_1 &= (0,1,1) & A' &= (1,0,1) \\ P_2 &= (1,1,1) & X &= (0,0,1) \end{aligned}$$

The other points have the coordinates

$$\begin{aligned} P_3 &= (1,c,1) & X' &= (u,0,1) \\ P_4 &= (a,b,1) & Y' &= (v,0,1) \end{aligned}$$

and $\ell = (0,1,0)$. The recall that on the level of coordinates, the intersection of lines and the line joining two points is realized by the cross product in $\mathbb{R}^3$. Concretely, this results in the following coordinates for the individual points:

$$\begin{aligned} A &= \ell \times (P_1 \times P_2) = (1,0,0) & B &= \ell \times (P_1 \times P_4) = (a,0,1-b) \\ A' &= \ell \times (P_3 \times P_2) = (1,0,1) & B' &= \ell \times (P_3 \times P_4) = (ac-b,0,c-b) \end{aligned}$$

Using the equation $(X,Y,A,B) = (X,Y,A',B')$ from Lemma 6, we find for Y the coordinates

$$Y = (\underbrace{ac(a-b) + b(b-1)}_{=:\alpha},\ 0,\ (b-ac)(b-1)).$$

Using this, we obtain

$$\begin{aligned} P_{X4Y1} &= (Y \times P_1) \times (X \times P_4) = (\alpha a,\ \alpha b,\ ac(a-b^2) + b(a+b)(b-1)) \\ P_{X2Y3} &= (Y \times P_3) \times (X \times P_2) = (\alpha,\ \alpha,\ a(a+c-bc-1) + b(b-1)) \\ P_{X3Y2} &= (Y \times P_2) \times (X \times P_3) = (\alpha,\ \alpha c,\ b^2 + ac(1+(a-1)c) - b(1+ac)) \\ P_{X1Y4} &= (Y \times P_4) \times (X \times P_1) = (0,\ \alpha b,\ b(a-1)(1-b+ac)) \end{aligned}$$

and hence

$$\begin{aligned} E &= \ell \times (P_{X4Y1} \times P_{X2Y3}) = (\alpha(a-b),\ 0,\ a(a-b)(c-b)) \\ F &= \ell \times (P_{X4Y1} \times P_{X1Y4}) = (\alpha ab,\ 0,\ b(b-1)(b(2a+b-1) - ac(a+b))) \\ E' &= \ell \times (P_{X3Y2} \times P_{X2Y3}) = (\alpha,\ 0,\ b(b-1) - ac(a+b-2)) \\ F' &= \ell \times (P_{X3Y2} \times P_{X1Y4}) = (\alpha b,\ 0,\ b(b-1)(b-c)). \end{aligned}$$

It turns out that for $X' \neq Y'$, both equations

$$(X',Y',A,B) = (X',Y',A',B') \quad \text{and} \quad (X',Y',E,F) = (X',Y',E',F')$$

can be reduced to

$$b(ac(u+v-1) - ((c+1)uv) + u + v - 1) + c(a-u)(a-v) + b^2(u-1)(v-1) = 0\,.$$

Hence, both equations yield the same point Y' for a given point X' on ℓ. *q.e.d.*

Now, we are ready to proof the Classes e_2, e_3 and e_4. In fact, it will turn out that these classes just follow from the *Pascal Twin Porism* starting with a Siamese Pascal twin in Class f_1.

Theorem 7 (Class e_2)**.** *Let $P_1, P_2, \ldots, P_6$ be points in the projective plane, and let P_{ijkl} be the intersection of the lines P_iP_j and P_kP_l. Then the points $P_1, P_2, \ldots, P_6$ lie on a conic iff the points $P_{1546}, P_{3526}, P_{2536}, P_{4516}, P_{2356}, P_{1456}$ lie on a conic.*

Proof. We have to show that $P_1, \ldots, P_6$ lie on a conic iff the points $Q_1, \ldots, Q_6$ lie on a conic, where $Q_1 = P_{1546}$, $Q_2 = P_{3526}$, $Q_3 = P_{2536}$, $Q_4 = P_{4516}$, $Q_5 = P_{2356}$, $Q_6 = P_{1456}$ (see Figure 16). Furthermore, let $X = P_5$ and let $Y = P_6$.

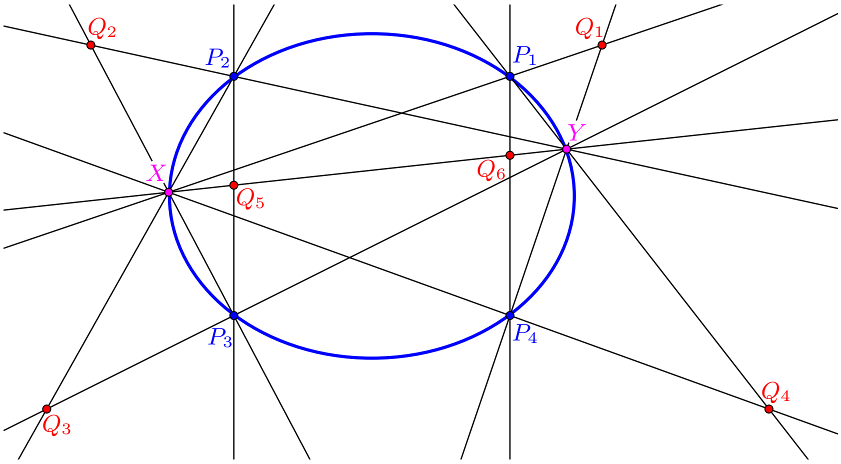

Figure 16: Proof of Theorem 7.

By Theorem 1, we know that $X, Y, P_1, \ldots, P_4$ lie on a conic iff the points $X, Y, Q_1, \ldots, Q_4$ lie on a conic. Thus, by the *Pascal Twin Porism*, for any points X', Y' on the line $X - Y$ we have that $X', Y', P_1, \ldots, P_4$ lie on a conic iff $X', Y', Q_1, \ldots, Q_4$ lie on a conic. In particular, for $X' = Q_5$ we have that the three points P_2, Q_5, P_3 are collinear, which implies that the conic through the five points P_1, P_2, P_3, P_4, Q_5 falls apart into two lines. One line contains the points P_2, Q_5, P_3, and the other line contains the points P_1, P_4 and $Q_6 = Y'$. Now, by the *Pascal Twin Porism* we conclude that the six points $P_1, \ldots, P_6$ lie on a conic if and only if $Q_1, \ldots, Q_6$ lie on a conic, which completes the proof. *q.e.d.*

Theorem 8 (Class e_3)**.** *Let $P_1, P_2, \ldots, P_6$ be points in the projective plane, and let P_{ijkl} be the intersection of the lines P_iP_j and P_kP_l. Then the points $P_1, P_2, \ldots, P_6$ lie on a conic iff the points $P_{1546}, P_{3526}, P_{2536}, P_{4516}, P_{2456}, P_{1356}$ lie on a conic.*

Proof. We have to show that $P_1, \ldots, P_6$ lie on a conic iff the points $Q_1, \ldots, Q_6$ lie on a conic, where $Q_1 = P_{1546}$, $Q_2 = P_{3526}$, $Q_3 = P_{2536}$, $Q_4 = P_{4516}$, $Q_5 = P_{2456}$, $Q_6 = P_{1356}$ (see Figure 17). Furthermore, let $X = P_5$ and let $Y = P_6$.

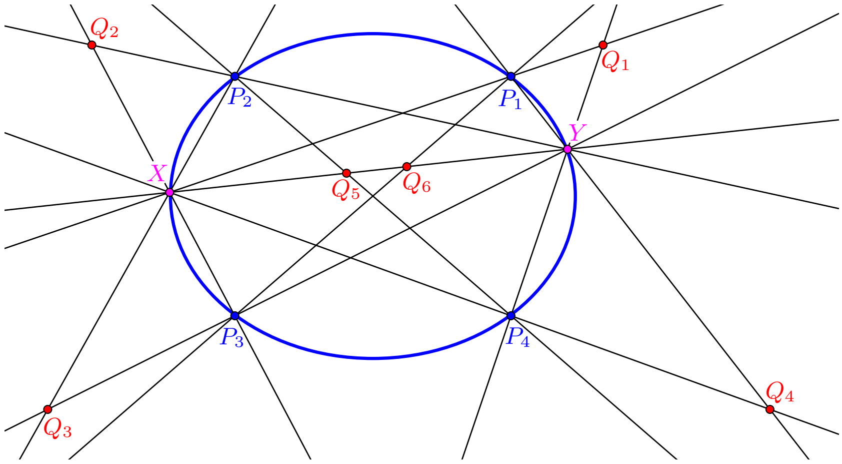

Figure 17: Proof of Theorem 8.

The proof is essentially the same as the proof for Class e_2, the only difference is that we move X' on the line $X - Y$ to the point $Q_5 = X - Y \wedge P_2 - P_4$. Since the three points P_2, Q_5, P_4 are collinear, the corresponding point Y' on the line $X - Y$ must be on the line $P_1 - P_3$, which implies that $Y' = Q_6$. Therefore, by the *Pascal Twin Porism* we have that the six points $P_1, \ldots, P_6$ lie on a conic if and only if $Q_1, \ldots, Q_6$ lie on a conic, which completes the proof. *q.e.d.*

Theorem 9 (Class e_4)**.** *Let P_1, P_2, P_X, P_5, P_6 be five points in the projective plane such that no three points are on a line. Then the points $P_{15X6}, P_{X526}, P_{25X6}, P_{X516}, P_{2X56}, P_{1X56}$ lie on a conic.*

Proof. This class follows immediately from Class e_3 by identifying the two points P_3 and P_4 with the point P_X (see also Figure 4). *q.e.d.*

References

[1] Lazare Carnot. *Géométrie de Position.* J. B. M. Duprat, 1803.

[2] John Conway and Alex Ryba. The Pascal mysticum demystified. *Math. Intelligencer*, 34(3):4–8, 2012.

[3] Lorenz Halbeisen, Norbert Hungerbühler, and Juan Läuchli. *Mit harmonischen Verhältnissen zu Kegelschnitten. Perlen der klassischen Geometrie.* Berlin: Springer Spektrum, 2nd edition edition, 2021.