Construction of a quadrilateral from its sides and the diagonal angle

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1 Introduction

In [2], the authors challenged the readers by posing the task of constructing a convex quadrilateral from its four vertex angles and the angle between its diagonals. This problem was solved in [4]. A similar open problem was recently posed in [1]: Construct a quadrilateral from its sides and the angle between its diagonals. Allowed are the classical tools compass and ruler. In the present short note, we show such a construction. As explained in [1], if the diagonals are orthogonal, then there is a continuum of solutions for given side lengths and the construction is elementary. We will therefore assume that the diagonals are not orthogonal.

2 The setup

We choose a Cartesian coordinate system with origin in a vertex such that one diagonal lies on the x-axis. The quadrilateral may be non-convex or self-intersecting (see Figure 1). The sides and their

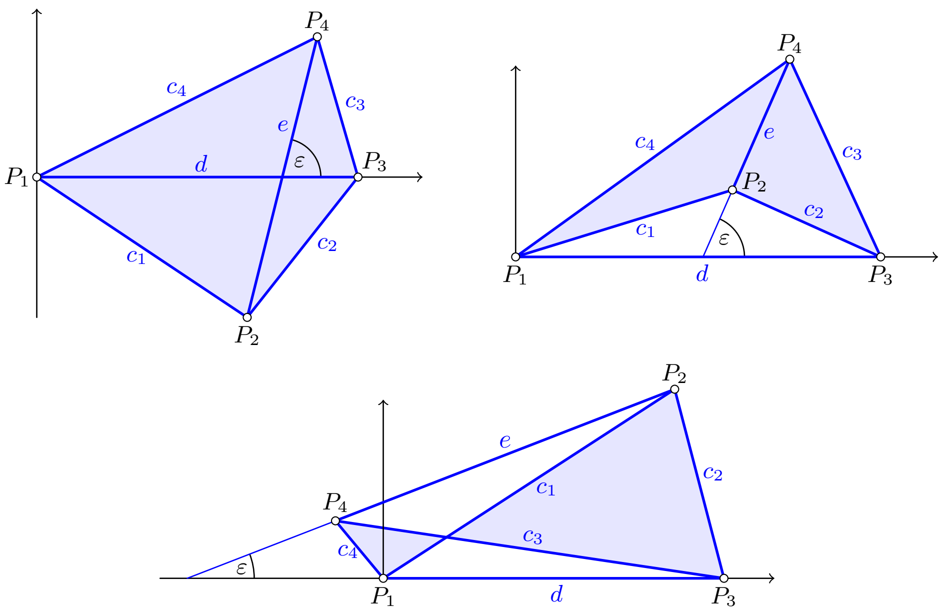

Figure 1: A coordinate system adapted to the quadrilateral.

lengths are denoted by c_1, c_2, c_3, c_4, the diagonals and their lengths by d and e. The coordinates of the vertices of the quadrangle are $P_1 = (0, 0)$, $P_2 = (x_2, y_2)$, $P_3 = (d, 0)$ and $P_4 = (x_4, y_4)$. The angle between the diagonals is ε. By considering a mirror image of the quadrilateral if necessary, we may assume that $\varepsilon < \pi/2$.

We consider first the case of a convex quadrilateral. Then we have $y_2 < 0 < y_4$. The aim is to find an algebraic expression for d which involves only square roots and arithmetic operations on the side lengths c_i and $\tan\varepsilon$. Then, d and subsequently the quadrilateral can be constructed by ruler and compass. We start with the equations

$$c_4^2 = x_4^2 + y_4^2, \tag{1}$$
$$c_3^2 = (d - x_4)^2 + y_4^2. \tag{2}$$

We can solve the difference of (1) and (2) for x_4 and obtain

$$x_4 = \frac{1}{2d}(c_4^2 - c_3^2 + d^2). \tag{3}$$

If we plug in this expression for x_4 in equation (1) and solve for y_4, we get

$$y_4 = \sqrt{c_4^2 - \frac{1}{4d^2}(c_4^2 - c_3^2 + d^2)^2} \tag{4}$$

Similarly, we find

$$x_2 = \frac{1}{2d}(c_1^2 - c_2^2 + d^2) \tag{5}$$

and

$$y_2 = -\sqrt{c_1^2 - \frac{1}{4d^2}(c_1^2 - c_2^2 + d^2)^2}. \tag{6}$$

Using the expressions in (3)–(6) we can compute

$$\tan\varepsilon = \frac{y_4 - y_2}{x_4 - x_2} = \frac{\sqrt{4c_1^2d^2 - (c_1^2 - c_2^2 + d^2)^2} + \sqrt{4c_4^2d^2 - (c_4^2 - c_3^2 + d^2)^2}}{c_2^2 - c_1^2 + c_4^2 - c_3^2}. \tag{7}$$

Solving (7) directly for d by hand or a computer algebra system leads to an extremely unpleasant expression. We are therefore taking a different approach.

Equation (7) is an equation for $x := d^2$ of the form

$$a = \sqrt{bx - (c+x)^2} + \sqrt{fx - (g+x)^2} \tag{8}$$

with

$$a = (c_2^2 - c_1^2 + c_4^2 - c_3^2)\tan\varepsilon, \qquad b = 4c_1^2, \qquad c = c_1^2 - c_2^2, \qquad f = 4c_4^2, \qquad g = c_4^2 - c_3^2. \tag{9}$$

Lemma. *A solution $x > 0$ of $a = \sqrt{bx - (c+x)^2} \pm \sqrt{fx - (g+x)^2}$ is also a solution of the quadratic equation $A + Bx + Cx^2 = 0$ with*

$$\begin{aligned} A &= 2g^2(a^2 - c^2) + (a^2 + c^2)^2 + g^4, \\ B &= 2(a^2(2c - b - f + 2g) + (c^2 - g^2)(2c - b + f - 2g)), \\ C &= 4a^2 + (b - 2c - f + 2g)^2. \end{aligned}$$

Proof. Set $u + v = bx - (c+x)^2, u - v = fx - (g+x)^2$, i.e.,

$$u = \frac{1}{2}(x(b - 2c + f - 2g) - 2x^2 - c^2 - g^2), \tag{10}$$

$$v = \frac{1}{2}(x(b - 2c - f + 2g) - c^2 + g^2). \tag{11}$$

Observe that if $a = \sqrt{u+v} \pm \sqrt{u-v}$ then a is a solution of

$$a^4 - 4a^2u + 4v^2 = 0 \tag{12}$$

(plug in, or see [3]). Using the expressions (10) and (11) for u and v in (12) and expanding, we get a quadratic equation $A + Bx + Cx^2 = 0$ with the desired coefficients A, B, C. $\square$

Suppose that the given side lengths and the diagonal angle come from a convex quadrilateral. Then we can now formulate the construction:

Construction 1. Choose a unit length l. Then proceed as follows:

1. Construct with the intercept theorems line segments of lengths a, b, c, f, g as given in (9).

2. Construct with the intercept theorems line segments of lengths $\frac{A}{C}, \frac{B}{C}$ for the values A, B, C as given in the Lemma.

3. Construct the solutions of $\frac{A}{C} + \frac{B}{C}x + x^2 = 0$ using the intersecting chords theorem or the intersecting secants theorem (see [3]). Obtain one or two solutions in form of a line segment of length x.

4. Transform the rectangle with sides x and l into a square of equal area with side length d, using the right triangle altitude theorem.

5. Construct the quadrilateral with diagonal d and sides c_1, c_2, c_3, c_4.

Note that there is only one solution if and only if the quadrilateral is cyclic (see Corollary 1 in [1]).

For the non-convex or the self-intersecting case, the sign in the numerator on the right of equation (7) changes to a minus:

$$\tan\varepsilon = \frac{y_4 - y_2}{x_4 - x_2} = \frac{\sqrt{4c_1^2 d^2 - (c_1^2 - c_2^2 + d^2)^2} - \sqrt{4c_4^2 d^2 - (c_4^2 - c_3^2 + d^2)^2}}{c_2^2 - c_1^2 + c_4^2 - c_3^2}. \tag{13}$$

Since the lemma also covers this case, the formulas and the construction remain unchanged.

References

[1] Hans Humenberger. Two further characterizations of orthodiagonal quadrilaterals. *Elem. Math.*, to appear.

[2] Hans Humenberger. Similarity of quadrilaterals as starting point for a geometric journey to orthocentric systems and conics. *Elem. Math.*, 79(4):148–158, 2024. With an appendix by Ivan Izmestiev and Arseniy Akopyan.

[3] Norbert Hungerbühler. An alternative quadratic formula. *Math. Semesterber.*, 67(1):85–95, 2020.

[4] Norbert Hungerbühler and Juan Läuchli. Construction of a quadrilateral from its vertex angles and the diagonal angle. *Elem. Math.*, 79(4):159–166, 2024.

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