COMBINATORIAL PROPERTIES OF SETS OF PARTITIONS

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Au départ, l'art du puzzle semble un art bref, un art mince, tout entier contenu dans un maigre enseignement de la Gestalt-theorie: l'objet visé n'est pas une somme d'éléments qu'il faudrait d'abord isoler et analyser, mais un ensemble, c'est-à-dire une forme, une structure: l'élément ne préexiste pas à l'ensemble, il n'est ni plus immédiat ni plus ancien, ce ne sont pas les éléments qui déterminent l'ensemble, mais l'ensemble qui détermine les éléments…

(Georges Perec,
La Vie mode d'emploi)

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Introduction

Combinatorics, including infinitary combinatorics, is a broad field of Mathematics which is quite difficult to describe properly. Nevertheless, let us start with a definition of combinatorics which shall be suitable for our purpose:

> *Combinatorics is the branch of mathematics which studies collections of objects that satisfy certain criteria, and is in particular concerned with deciding how large or how small such collections might be.*

In the following we give a few examples which should illustrate some aspects of infinitary combinatorics appearing later in this work. Let us start with an example from graph theory.

EXAMPLE 1 (König's Lemma). A *tree* is a connected undirected graph without circuits one of whose vertices is designated as the origin. A tree is *infinite* if its set of vertices is countable infinite and it is *finitely branching* if each vertex has only finitely many successors. A *branch* in a tree is a maximal path beginning at the origin. Now, König's Lemma [**43**, VI, §2, Satz 6] states that every infinite, finitely branching tree contains an infinite branch. Notice that finitely branching is necessary to assure that the tree is infinitely high.

Even though this fact looks quite obvious, in order to prove it one must use some kind of choice. The full *Axiom of Choice* AC states that a Cartesian product of non-empty sets is non-empty, or equivalently, that every set of non-empty sets has a choice function. It is easy to see that König's Lemma follows from AC. On the other hand, König's Lemma – which is a purely combinatorial result – is equivalent to the statement $\mathsf{AC}_{\omega,<\omega}$, which says that every countable family of non-empty finite sets has a choice function (*cf.* [**35**, Form 10]). It is well known that not only AC, but also $\mathsf{AC}_{\omega,<\omega}$ and many other weakened forms of AC are independent of the axioms of *Zermelo-Fraenkel Set Theory*, denoted by ZF.

At this point, let us briefly explain the meaning of "independent" and "consistent". Let Σ be any set of statements, or axioms, then Σ is called *consistent* if we cannot derive a contradiction from Σ, which is – by Gödel's completeness theorem – equivalent to the fact that Σ has a model (*e.g.*, the set of permutations of three objects is a model for the axioms of group theory). Further, a statement φ is called *independent* of Σ if either set $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ is consistent.

Let us turn back to our example. There are models of ZF in which AC – and consequently $\mathsf{AC}_{\omega,<\omega}$ – is true, but there are also models of ZF in which $\mathsf{AC}_{\omega,<\omega}$ – and consequently AC – fails. Moreover, there are also models of ZF in which $\mathsf{AC}_{\omega,<\omega}$ is true but AC fails. Thus, we can conclude that even basic combinatorial statements like König's Lemma may depend on the underlying model of set theory.

This first example shows that – depending on the set theoretical axioms we are starting with – some *objects, satisfying certain criteria*, might or might not exist.

Throughout this work, we will always assume AC, so, our basic axiom system will be ZFC, which is ZF + AC. This also means that we will never discuss how much of AC is needed to get certain results.

The next example can be seen as a problem in *infinitary extremal combinatorics*. The word "extremal" comes from the nature of problems this field deals with, and refers to the second part of our definition: *how large or how small collections satisfying certain criteria might be*.

For example, how many people must be on a party to be sure that there are three people who all either know each other or don't know each other? Or, given a finite set of non-zero integers S. How large can a set $A \subseteq S$ be such that A does not contain the sum of any two of its members. It turns out that (independent of the given set S) there is always an A which contains at least one-third of the numbers in S.

If the objects considered are infinite, then the answer how large or how small certain sets can be might depend on the underlying model of set theory, as the next example shows.

EXAMPLE 2 (reaping number). A family $\mathcal{R}$ of infinite subsets of the natural numbers ω is called *reaping* (also called *unsplitting*), if for every coloring of ω with two colors there exists a monochromatic set in $\mathcal{R}$. The *reaping number* $\mathfrak{r}$ is the minimal size of a reaping family. Now we can ask: How large is $\mathfrak{r}$?

It is easy to see that a reaping family cannot be countable. Indeed, let $\mathcal{A} = \{A_i : i \in \omega\}$ be any countable family of infinite subsets of ω. For each $i \in \omega$, pick n_i and m_i from the set A_i in such a way that for all $i \in \omega$, $n_i < m_i < n_{i+1}$. Each n_i ($i \in \omega$) gets colored blue and all other numbers red. For this coloring, there is no monochromatic set in $\mathcal{A}$, and hence, $\mathcal{A}$ cannot be a reaping family. Consequently, assuming the *Continuum Hypothesis* CH, any reaping family must have the same cardinality as the continuum, denoted by $\mathfrak{c}$, and we get the same assuming *Martin's Axiom* MA. On the other hand, with the forcing technique – invented by Paul Cohen in the early 1960's (*cf.* [**13**]) – one can show that the minimal size of a reaping family is independent of ZFC. In other words, there are models of ZFC in which $\mathfrak{r} = \mathfrak{c}$, but there are other models in which $\mathfrak{r} < \mathfrak{c}$.

So, the second example shows that we get different answers – depending on the additional axioms of set theory we start with – when we try to *decide how large or how small certain collections might be*.

Another field of combinatorics is the so-called Ramsey Theory, and since many results in this work are "partition-versions" of classical Ramsey-type theorems, let us give a brief description of Ramsey Theory.

Loosely speaking, *Ramsey Theory* (which can be seen as a part of extremal combinatorics) is that branch of combinatorics which deals with structures preserved under partitions, or colorings. Typically, one looks at the following kind of question: If a particular object (*e.g.*, algebraic, geometric or combinatorial) is arbitrarily colored with finitely many colors, what kinds of monochromatic structures can we find?

For example, van der Waerden's Theorem tells us that if the integers are colored with finitely many colors, then there are arbitrarily long monochromatic arithmetic progressions. Or, for any coloring of the points in the Euclidean plane with finitely many colors, there are three monochromatic points which are the verices of a right-angled triangle of unit area.

The most famous result in Ramsey Theory is surely Ramsey's Theorem. In fact, there are two versions of Ramsey's Theorem, an infinite version [**57**, Theorem A] and a finite version [**57**, Theorem B], but because the second one follows from the first one, we consider Theorem A as "Ramsey's Theorem", also called "Ramsey Theorem":

EXAMPLE 3 (Ramsey's Theorem). For any positive integer n, let $[\omega]^n$ denote the set of all n-element subsets of the natural numbers. Now, Ramsey's Theorem tells us that if we color $[\omega]^n$ with finitely many colors, we find an infinite subset $H \subseteq \omega$ such that all n-element subsets of H have the same color, and such a set H we call *homogeneous*.

The following is just a consequence of Ramsey's Theorem:

FINITARY RAMSEY THEOREM. For all positive integers m, n, r, where $n \leq m$, there exists a number $N \in \omega$ such that for every coloring of $[N]^n$ with r colors, we find a set $H \in [N]^m$ such that $[H]^n$ is monochromatic.

For example the "party-problem" mentioned above is a typical problem in Ramsey theory and an easy Ramsey-type argument shows that at least six persons must be on the party. On the other hand, if we ask how many people must get invited to a party to make sure that there are five people who all either know each other or don't know each other, then the answer is not known, but it is conjectured that at least 43 persons must be invited (see [**56**]).

Ramsey's theorems have applications to many different fields such as Banach space theory (*cf.* [**51**]), and set theory without the axiom of choice (see *e.g.*, [**30**, Proposition 7.3.1]).

Sometimes, we also get Ramsey-type (or anti Ramsey-type) results even for a partition into infinitely many classes. For example, there is a coloring of the points in the Euclidean plane with countably many colors, such that no two points of any "copy of the rational line" get the same color (see [**42**]). This result can be seen as an anti Ramsey-type theorem (since we are far away from "monochromatic structures"), and it shows that Ramsey-type theorems cannot be generalized arbitrarily. On the other hand, one can consider just these colorings which "behave well", or which have some nice monochromatic structures, and investigate how complicated such colorings may be. Such an approach leads to combinatorial properties, as the next example illustrates.

EXAMPLE 4 (Ramsey property). Let $[\omega]^\omega$ denote the set of all infinite subsets of ω, and for $H \in [\omega]^\omega$, let $[H]^\omega$ denote the set of all infinite subsets of H. A set $\mathcal{A} \subseteq [\omega]^\omega$ has the *Ramsey property* if there is an $H \in [\omega]^\omega$ such that either $[H]^\omega \subseteq \mathcal{A}$ or $[H]^\omega \cap \mathcal{A} = \emptyset$. In other words, if we color all infinite subsets of ω with two colors, and we find an infinite subset of ω, all of whose infinite subsets have the same color, then the coloring has the Ramsey property.

With the aid of AC it is not hard to construct a set $\mathcal{A} \subseteq [\omega]^\omega$ which does not have the Ramsey property. On the other hand, one can show that all analytic sets have the Ramsey property and it is consistent with ZF that each $\mathcal{A} \subseteq [\omega]^\omega$ has the Ramsey property. Further, assuming the existence of an inaccessible cardinal, one can show that it is consistent with ZFC that all projective sets have the Ramsey property, but it is not known if the assumption of an inaccessible cardinal is necessary.

Let us turn back to Ramsey's Theorem which tells us that for every coloring $\pi : [\omega]^2 \to \{0,1\}$ there is an infinite homogeneous set $H \subseteq \omega$. But it does not tell us where to find such set H. If there would be an ultrafilter over ω such that the homogeneous set always belongs to the ultrafilter, this would be useful, especially from a combinatorial point of view. This leads to the following:

EXAMPLE 5 (Ramsey ultrafilters). Let $\mathcal{U}$ be an ultrafilter over ω, then $\mathcal{U}$ is called a *Ramsey ultrafilter* if for every coloring $\pi : [\omega]^2 \to \{0,1\}$ there is an infinite homogeneous set $H \in \mathcal{U}$.

One can show that either CH or MA implies the existence of Ramsey ultrafilters. On the other hand, it is consistent with ZFC that there are no Ramsey ultrafilters at all. Ramsey ultrafilters, together with Mathias forcing, play an important role in the investigation of the Ramsey property, and the beautiful interaction between Ramsey ultrafilters, Mathias forcing and the Ramsey property was the main motivation to investigate the corresponding theory for sets of partitions.

The aim of this work is to investigate combinatorial properties of sets of partitions along the guideline given by the preceding examples. Since, from the category theoretical point of view, partitions are the duals of subsets, going from subsets to partitions is called "dualization". The main difference between subsets of ω and partitions of ω is that partitions do not have a proper complement. If they would have, there would be nothing to do than replace the word "subset" by "partition". But this is not the case, and sometimes, it is not even straightforward to find the right dualization.

For example, consider the spaces $\beta\omega$ (which is the space of ultrafilters over ω) and $\beta\omega \setminus \omega$ (the space of non-principal ultrafilters over ω). If we want to dualize these two spaces, we have to dualize first the notion of ultrafilters, which gives us the notion of partition-ultrafilters, defined as maximal partition-filters. It turns out that there are two natural ways to do this, so we get two sets of partition-ultrafilters. Now, we have to define a topology on each of these two sets of partition-ultrafilters, and it turns out that we have again two possibilities to do this. Thus, we end up with four topological spaces of partition-ultrafilters, but none of them is homeomorphic to $\beta\omega$ or to $\beta\omega \setminus \omega$. Other difficulties and asymmetries occur when we try to dualize Ramsey's Theorem (see Chapter IV) or some cardinal characteristics of the continuum (see Chapter II), or if we try to find a dual form of Ramsey ultrafilters (see Chapter VII).

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As mentioned above, the following work can be seen as a dualization – in terms of partitions – of the combinatorics of sets of subsets of ω, and consists mainly of the papers [**22**], [**23**], [**24**] and [**27**], which are all published in refereed journals. The only exception is Chapter IV (where a theorem is given, which can be seen as the partition form – rather than the dual form – of Ramsey's Theorem). Let us now briefly summarize the content of each chapter:

In Chapter I we introduce our terminology and give the basic definitions of partitions of ω. Further, it is shown that from the category theoretical point of view, partitions are the duals of subsets, which motivates the term "dualization" for the process of going from subsets to partitions.

Henceforth, for any property, like the Ramsey property, or cardinal characteristic of the continuum, like the reaping number $\mathfrak{r}$, *etc.*, the dual Ramsey property or the dual-reaping cardinal $\mathfrak{R}$, *etc.*, refers to the corresponding partition form of the Ramsey property and the reaping number, *etc.*

In Chapter II we dualize some well-known cardinal characteristics of the continuum like the reaping number $\mathfrak{r}$ (see Example 2) and the splitting number $\mathfrak{s}$. It will be shown that the dual forms of these cardinal characteristics do in general not agree with their standard form. For example, it is consistent with ZFC that the dual-splitting cardinal $\mathfrak{S}$ is strictly bigger than $\mathfrak{s}$, which would be obvious if $\mathfrak{S} = \mathfrak{c}$, but it is also consistent that $\mathfrak{S}$ is strictly smaller than the continuum. Moreover, one can show that – no matter in which model of ZFC we are – the dual tower number is always ω_1 (the first uncountable cardinal), which is smaller than or equal to the classic tower number; and that a maximal almost orthogonal family – which corresponds to a maximal almost disjoint family – has always the same size as the continuum, and therefore, such a family can be strictly greater than its classical relative. Thus, dual cardinal numbers can be fixed, whereas their classical relatives can be consistently moved. On the other hand, there is also a cardinal characteristics of the continuum which is fixed – like the cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ such that for every infinite subset of ω there is a disjoint set in $\mathcal{F}$, which has always the same size as the continuum – whereas its dualization $\mathfrak{D}$ can be proved to be greater than or equal to $\mathfrak{p}$ (the so-called pseudo-intersection number) and less than or equal to $\mathfrak{i}$ (the so-called independent number). Further, it is provable in ZFC that the dual-reaping cardinal $\mathfrak{R}$ is less than or equal to $\min\{\mathfrak{r}, \mathfrak{D}\}$, but it is greater than or equal to $\mathfrak{p}$. Summarizing the previous facts, the dual form of cardinal characteristics of the continuum is completely asymmetric to the classical ones. The results of this chapter can also be found in [**22**].

In Chapter III we investigate the four topological spaces mentioned above which can be seen as the dualizations of the spaces $\beta\omega$ and $\beta\omega \setminus \omega$, which are both compact Hausdorff. Even though all four topological spaces are natural dualizations of $\beta\omega$ or $\beta\omega \setminus \omega$, none of these four spaces is homeomorphic to $\beta\omega$ or $\beta\omega \setminus \omega$. To prove this, we will be using some combinatorial tools like König's Lemma (see Example 1). In particular, it will be shown that two of these four spaces are Hausdorff but not compact, and the other two are not Hausdorff but countable compact. Further, the dualization and the existence of P-points will be discussed. For a slightly more general approach in terms of filters on semilattice see [**27**].

After a short introduction to Ramsey Theory, we present in Chapter IV a partition form of Ramsey's Theorem (see Example 3), which will be used to define Ramseyan ultrafilters in Chapter VII. Ramsey's Theorem says that if we color the n-element subsets of ω with finitely many colors, then we find an infinite homogeneous set. So, in a dual form of Ramsey's Theorem – which was introduced by Timothy Carlson in [**11**] – we would expect that if we color the n-part partitions of ω with finitely many colors, then we find an infinite homogeneous partition. But there is a coloring of the 2-part partitions of ω with just two colors, such that there is no infinite homogeneous partition of ω. So, the dual form of Ramsey's Theorem is not as general as the classical version. On the other hand, if we replace the n-element subsets of ω by n-part partitions of integers $k \in \omega$, then the corresponding partition form of Ramsey's Theorem has similar features as the classical version, even though it is not the proper dualization (see Chapter I.5).

In Chapter V we begin to investigate the dual Ramsey property (see Example 4). In this context, the only important cardinal (also used in Chapter VI) is the dual-shattering cardinal $\mathfrak{H}$, which is the dualization of the shattering number $\mathfrak{h}$. Firstly, it will be shown how $\mathfrak{H}$ is related to the dual Ramsey property. In particular, we will see that $\mathfrak{H} = add(\boldsymbol{R}_0^\flat) = cov(\boldsymbol{R}_0^\flat)$, where $\boldsymbol{R}_0^\flat$ denotes the ideal of completely dual Ramsey null sets, and *add* and *cov* denote the additivity and the covering numbers, respectively. Secondly, we investigate $\mathfrak{H}$ itself. One can show that $\mathfrak{H} \leq \mathfrak{h}$ and that it is consistent with ZFC that $\mathfrak{H} < \mathfrak{h}$ (even under MA). This would be obvious if $\mathfrak{H} = \omega_1$, but we will see that $\mathfrak{H} > \omega_1$ as well as $\mathfrak{H} > cov(\boldsymbol{B}_0)$ (where $\boldsymbol{B}_0$ denotes the ideal of meager sets) is consistent with ZFC. The results of this chapter can be found again in [**22**].

Finally, after discussing asymmetries in the dualization process, we look in Chapter VI at the symmetries between the Ramsey property and the dual Ramsey property. Some results about the dual Ramsey property are straightforward dualizations of results about the Ramsey property. But as a matter of fact we will see that most proofs in the dual case are much more involved than the classic ones. The reason leading to more sophisticated proofs is that a partition – unlike a subset – does not have a proper complement. It will be shown that the dual Ramsey property is closed under a generalized Suslin operation involving the dual-shattering cardinal $\mathfrak{H}$. Further, the notion of game-families and game-filters will be introduced and dual Mathias forcing (restricted to these game-filters) will be investigated. In particular, it will be shown that an ω_2-iteration of dual Mathias forcing with countable support starting from *Gödel's constructible universe* yields a model in which every $\boldsymbol{\Sigma}^1_2$-set has the dual Ramsey property, but not every $\boldsymbol{\Delta}^1_2$-set has the Baire property. A similar model exists with respect to the Ramsey property. Almost all results of this chapter can be found in [**23**].

In Chapter VII we define an ordering on the set of partition-filters which is similar to the Rudin-Keisler ordering on $\beta\omega$. Further, we introduce a partition form (which is not the dual form!) of Ramsey ultrafilters (see Example 5), called Ramseyan ultrafilters. The Rudin-Keisler ordering on $\beta\omega$ is defined as follows: $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U}$ is the image of $\mathcal{V}$ under the canonical extension $\beta f : \beta\omega \to \beta\omega$ of some map $f : \omega \to \omega$. Now, Ramsey ultrafilters over ω build the minimal points of the Rudin-Keisler ordering on $\beta\omega \setminus \omega$. It will be shown that a similar result is true for Ramseyan ultrafilters with respect to the ordering on the set of partition-filters, and that CH implies the existence of $2^{\mathfrak{c}}$ pairwise non-equivalent Ramseyan ultrafilters. Further, it will be shown that dual Mathias forcing restricted to a Ramseyan ultrafilter has the same features as Mathias forcing restricted to a Ramsey ultrafilter. In particular, it has the homogeneity property, has pure decision and can be decomposed. Ramsey ultrafilters can also be described as happy families that are also filters, and so, we also dualize the notion of happy families and show that the so-called relatively happy families have a similar characterization in terms of games as their classic relatives. Finally, we consider the dual form of some cardinal characteristics of the continuum which are to some extend related to Ramseyan ultrafilters. This chapter is essentially [**24**].

CHAPTER I

Partitions, a Dual Form of Sets

Most of our set-theoretical terminology is standard and can be found in textbooks like [**3**], [**36**] and [**44**]. However, let us recall some frequently used notation.

1. Some basic definitions

Let S be a set. $|S|$ denotes the **cardinality** of the set S, which is the least ordinal number α such that there exists a bijection between S and α. In particular, ω denotes the least infinite ordinal, ω_1 denotes the least uncountable ordinal, and so on. Let $\mathcal{P}(S)$ denote the **power-set** of S. For a cardinal number κ, let $[S]^\kappa := \{T \in \mathcal{P}(S) : |T| = \kappa\}$ and $[S]^{<\kappa} := \{T \in \mathcal{P}(S) : |T| < \kappa\}$.

The least infinite ordinal number is denoted by $\omega = \{0, 1, 2, \dots\}$ which is the **set of natural numbers**, where a **natural number** $n = \{k \in \omega : k < n\}$ (in particular, $0 = \emptyset$). Further, let $\mathfrak{c} := |\mathcal{P}(\omega)|$ denote the **cardinality of the continuum**.

For our purpose, without loss of generality we consider the set $[\omega]^\omega$ as the set of irrational numbers, and the set $[\omega]^{<\omega}$ as the set of rationals. However, sometimes it is more convenient to identify the reals with the set ${}^\omega\omega$ (the set of all functions from ω to ω) or with the set ${}^\omega 2$ (the set of all functions from ω to $\{0, 1\}$).

2. Partitions of $\boldsymbol{\omega}$

The main objects of this work will be *partitions* of ω. A **partition** X of ω is a subset of $\mathcal{P}(\omega)$ such that the following holds:

(i) if $b \in X$, then $b \neq \emptyset$,
(ii) if $b_1, b_2 \in X$ and $b_1 \neq b_2$ then $b_1 \cap b_2 = \emptyset$,
(iii) $\bigcup X = \omega$.

In other words, a partition of ω is a set of pairwise disjoint, non-empty subsets of ω such that the union is all of ω. The set of all partitions of ω is denoted by $(\omega)^{\leq\omega}$.

A partition means always a partition of ω. If X is a partition and $b \in X$, then we call b a **block** of X. If a partition has infinitely many blocks (or equivalently, if X is infinite) we call X an **infinite partition**. The set of all infinite partitions is denoted by $(\omega)^\omega$. Further, the set of all **finite partitions** is denoted by $(\omega)^{<\omega}$.

A **partial partition** X' is a subset of $\mathcal{P}(\omega)$ such that (i) and (ii) hold but instead of (iii) we have

(iii)' $\bigcup X' =: \operatorname{dom}(X') \subseteq \omega$.

Note that a partition is always also a partial partition. If $\mathrm{dom}(X') \in \omega$, then X' is a partition of some $n \in \omega$. The set of all partial partitions X' where $\mathrm{dom}(X') \in \omega$ is denoted by $(\mathbb{N})$. Further, for $s \in (\mathbb{N})$, s^* denotes the partial partition $s \cup \{\{\mathrm{dom}(s)\}\}$.

3. Notation

Throughout this work we will usually denote:

- elements of ω by lower case letters like $n, m, k, h \ldots$
- elements of $[\omega]^\omega$ by lower case letters like $x, y \ldots$
- partitions by upper case letters like $X, Y \ldots$
- finite subsets of $[\omega]^\omega$ by lower case letters like $a, b \ldots$
- elements of $(\mathbb{N})$ by lower case letters like $s, t \ldots$
- subsets of $[\omega]^\omega$ by calligraphic letters like $\mathcal{F}, \mathcal{S}, \mathcal{U} \ldots$
- sets of partitions by even more calligraphic letters like $\mathscr{F}, \mathscr{S}, \mathscr{U} \ldots$
- cardinal characteristics of the continuum which are related to $[\omega]^\omega$ by lower case fracture letters like $\mathfrak{h}, \mathfrak{r}, \mathfrak{s} \ldots$
- cardinal characteristics of the continuum which are related to partitions by upper case fracture letters like $\mathfrak{H}, \mathfrak{R}, \mathfrak{S} \ldots$

4. Relations on the set of partitions

Let X_1, X_2 be two partial partitions. We say that X_1 is **coarser** than X_2, or that X_2 is **finer** than X_1, and write $X_1 \sqsubseteq X_2$, if for all blocks $b \in X_1$ the set $b \cap \mathrm{dom}(X_2)$ is the union of some sets $b_i \cap \mathrm{dom}(X_1)$, where each b_i is a block of X_2. In particular, $\{\{\omega\}\}$ is the **coarsest partition** and $(\omega) := \{\{n\} : n \in \omega\}$ is the **finest partition**. Let $X_1 \sqcap X_2$ denote the finest partial partition which is coarser than X_1 and X_2 such that $\mathrm{dom}(X_1 \sqcap X_2) = \mathrm{dom}(X_1) \cup \mathrm{dom}(X_2)$, and let $X_1 \sqcup X_2$ denote the coarsest partial partition which is finer than X_1 and X_2 such that $\mathrm{dom}(X_1 \sqcup X_2) = \mathrm{dom}(X_1) \cup \mathrm{dom}(X_2)$.

If $p \in [\omega]^{<\omega}$ is a finite subset of ω, then $\{p\}$ is a partial partition with $\mathrm{dom}(\{p\}) = p$. For two partial partitions X_1 and X_2 we write $X_1 \sqsubseteq^* X_2$ if there is a finite set $p \subseteq \mathrm{dom}(X_1)$ such that $X_1 \sqcap \{p\} \sqsubseteq X_2$ and say that X_1 is **almost coarser** than X_2, or that X_2 is **almost finer** than X_1. If $X_1 \sqsubseteq^* X_2$, $X_2 \sqsubseteq^* X_1$ and $\mathrm{dom}(X_1) = \mathrm{dom}(X_2)$, then we write $X_1 \overset{*}{=} X_2$. If $X \overset{*}{=} (\omega)$ or $X = \{\{\omega\}\}$, then X is called **trivial**; in other words, X is trivial if X is either the one-block-partition or all blocks of X are finite and just finitely many blocks contain more than one element.

Let X_1, X_2 be two partial partitions. If each block of X_1 can be written as the intersection of a block of X_2 and $\mathrm{dom}(X_1)$, then we write $X_1 \prec X_2$. Note that $X_1 \prec X_2$ implies $\mathrm{dom}(X_1) \subseteq \mathrm{dom}(X_2)$.

If X is a partial partition, then

$$\mathrm{Min}(X) := \{n \in \omega : \exists b \in X (n = \min(b))\},$$

where $\min(b) := \bigcap b$. If we order the blocks of X by their least element, then $X(n)$ denotes the n^{th} block with respect to this ordering and $X(n)(k)$ denotes the k^{th} element (with respect to the natural ordering) of $X(n)$.

5. Partitions as the dual form of subsets

One can think of the duality between subsets and partitions in a category-theoretic way.

Let N be an arbitrary set. For our purposes, N will be just ω. Consider one-to-one functions into N from arbitrary domains, and call two such functions, say $f : A \to N$ and $g : B \to N$ equivalent if there is a bijection $h : A \to B$ such that $f = gh$. Then the equivalence classes can be identified with the subsets of N, because f and g are equivalent if and only if they have the same image.

In fact, in general categories, we can define a "subobject" to be such an equivalence class. For this, we need category-theoretic definitions of "one-to-one" and "bijection": A bijection is a map with a two-sided inverse (with respect to composition), and a map is one-to-one if and only if it is cancellable on the left.

Now we apply the general category-theoretic notion of duality: Reverse the direction of all arrows and (therefore) reverse the order of composition. "Bijection" is self-dual, but the dual of "one-to-one map" is "right-cancellable map" which amounts to (in the category of sets) "onto map". So, the dual of a subobject of N would be an equivalence class of surjections $f : N \to A$ (for arbitrary sets A); here $f : N \to A$ and $g : N \to B$ are equivalent if and only if there is a bijection $h : B \to A$ such that $f = hg$. Untangling the definitions, we find that f and g are equivalent if and only if the partitions they induce on N (the pre-images of singletons in A and in B) are the same. In other words, dualizing the notion of subset (or, more precisely, dualizing a category-theoretic description of subsets) gives (a category-theoretic description of) partitions.

Further, the inclusion relation on subsets admits a category-theoretic description in terms of the one-to-one maps; it just says that $f = gh$ for some h, not necessarily a bijection. Dualizing, you get a description, in terms of surjections, of the "coarser than" relation on partitions. So, the dualization of the inclusion relation between subsets is the "coarser than" relation between partitions.

Similarly, where finite sets occur in some theory, we would expect partitions with finitely many pieces in the dual theory, because both say that the A (or B) above is finite.

With the concept of dualization we can seek for dualizations of cardinal characteristics of the continuum (see [**12**]) or for a dual form of Ramsey's Theorem (see [**11**]). On the other hand, from the combinatorial point of view it is sometimes appropriate to look for a "partition form" of certain combinatorial theorems, which might be different from the corresponding dual form (see for example Chapter IV).

CHAPTER II

Dualizations of Cardinal Characteristics

The dualization of some cardinal characteristics of the continuum was first investigated by Jacek Cichoń, Adam Krawczyk, Barbara Majcher-Iwanow and Bogdanin Węglorz in [**12**]. In this chapter we proceed their work.

Sometimes, it will be convenient to consider infinite partitions such that at least one block is infinite, thus, let $(\omega)^{\omega'}$ denote the set of all those partitions.

Two partitions $X_1, X_2 \in (\omega)^\omega$ are called **almost orthogonal**, denoted $X_1 \perp_* X_2$, if $X_1 \sqcap X_2 \notin (\omega)^\omega$; otherwise, they are called **compatible**, denoted $X_1 | X_2$. If $X_1 \sqcap X_2 = \big\{\{\omega\}\big\}$, then they are called **orthogonal**, denoted $X_1 \perp X_2$.

Recall that $\mathfrak{c} := |\mathcal{P}(\omega)|$ denotes the **cardinality of the continuum**.

1. On the dual-splitting cardinals $\mathfrak{S}$ and $\mathfrak{S}'$

Let X_1, X_2 be two partitions. We say X_1 **splits** X_2 if $X_1 | X_2$ and there is a partition $Y \sqsubseteq X_2$ such that $X_1 \perp Y$. A family $\mathscr{S} \subseteq (\omega)^\omega$ is called **splitting** if for each non-trivial $X \in (\omega)^\omega$ there exists an $S \in \mathscr{S}$ such that S splits X. The **dual-splitting cardinal** $\mathfrak{S}$ ($\mathfrak{S}'$, respectively) is the least cardinal number κ for which there exists a splitting family $\mathscr{S} \subseteq (\omega)^\omega$ ($\mathscr{S} \subseteq (\omega)^{\omega'}$, respectively) of cardinality κ.

It is obvious that $\mathfrak{S} \leq \mathfrak{S}'$. In the following, we compare first the dual-splitting number $\mathfrak{S}'$ with the well-known unbounding number $\mathfrak{b}$ (a definition of $\mathfrak{b}$ can be found in [**65**]).

THEOREM II.1.1. $\mathfrak{b} \leq \mathfrak{S}'$.

PROOF. Assume there exists a family $\mathscr{S} = \{S_\iota : \iota < \kappa < \mathfrak{b}\} \subseteq (\omega)^{\omega'}$ which is splitting. Let $\mathcal{B} = \{b_\iota : \iota < \kappa\} \subseteq [\omega]^\omega$ a set of infinite subsets of ω such that $b_\iota \in S_\iota$ (for all $\iota < \kappa$). Let $f_{b_\iota} \in {}^\omega\omega$ be the (unique) increasing function such that range$(f_{b_\iota})=b_\iota$. Because $\kappa < \mathfrak{b}$, the set $\{f_{b_\iota} : \iota < \kappa\}$ is not unbounded. Therefore, there exists a one-to-one function $d \in {}^\omega\omega$ such that $f_{b_\iota} <^* d$ (for all $\iota < \kappa$). With the function d we construct an infinite partition D. First we define an infinite set of pairwise disjoint finite sets p_i $(i \in \omega)$:

$$p_i := \left[d^i(0), d^{i+1}(0)\right),$$

where d^i denote the i-fold composition of d. Now, the blocks of D are defined as follows:

$$n \textit{ is in the } k^{th} \textit{ block of } D \;\Leftrightarrow\; n \in p_i \textit{ and } i - \max\big\{l(l+1)/2 < i : l \in \omega\big\} = k.$$

Because d dominates $\mathcal{B}$, for all $b_\iota \in \mathcal{B}$ there exists a natural number m_ι such that for all $i > m_\iota$ we have $d^i(0) \leq b_\iota(d^i(0)) < d^{i+1}(i)$ (*cf.* [**65**, page 121]). So, for all $i > m_\iota$, $p_i \cap b_\iota \neq \emptyset$ and therefore by the construction of the blocks of D, b_ι intersects each block of D. But this implies that D is not compatible with any element of $\mathscr{S}$ and hence, $\mathscr{S}$ cannot be a splitting family. ⊣

Let us now compare $\mathfrak{S}'$ with the splitting number $\mathfrak{s}$ (*cf.* [**65**]).

Corollary II.1.2. It is consistent with ZFC that $\mathfrak{s} < \mathfrak{S}'$.

Proof. Because $\mathfrak{b} \leq \mathfrak{S}'$ is provable in ZFC, it is enough to show that $\mathfrak{s} < \mathfrak{b}$ is consistent with ZFC, which is proved by Saharon Shelah in [**58**]. ⊣

Now we show that $cov(\boldsymbol{B}_0) \leq \mathfrak{S}$ (where $\boldsymbol{B}_0$ denotes the ideal of meager sets). In [**12**] it is shown that if $\kappa < cov(\boldsymbol{B}_0)$ and $\{X_\alpha : \alpha < \kappa\} \subseteq (\omega)^\omega$ is a family of partitions, then there exists $Y \in (\omega)^\omega$ such that $Y \perp X_\alpha$ for each $\alpha < \kappa$. This implies the following

Corollary II.1.3. $cov(\boldsymbol{B}_0) \leq \mathfrak{S}$.

Proof. Let $S, Y \in (\omega)^\omega$. If $S \perp Y$, then S does not split Y and therefore a family of cardinality less than $cov(\boldsymbol{B}_0)$ can not be splitting. ⊣

As a corollary we get again a consistency result:

Corollary II.1.4. It is consistent with ZFC that $\mathfrak{s} < \mathfrak{S}$.

Proof. After an ω_1-iteration of Cohen forcing with finite support starting from a model $V \models cov(\boldsymbol{B}_0) = \omega_2 = \mathfrak{c}$, we get a model in which $\omega_1 = \mathfrak{s} < cov(\boldsymbol{B}_0) = \omega_2 = \mathfrak{c}$. Hence, by Corollary II.1.3, this is a model for $\omega_1 = \mathfrak{s} < \mathfrak{S} = \omega_2$. ⊣

Until now we have $\max\big\{cov(\boldsymbol{B}_0), \mathfrak{b}\big\} \leq \mathfrak{S}'$, which would be trivial if one could show that $\mathfrak{S}' = \mathfrak{c}$. But this is not the case (*cf.* [**12**]). To construct a model in which $\mathfrak{S}' < \mathfrak{c}$ we will use a modified version of a forcing notion introduced in [**12**].

Let $\mathcal{F}$ be an arbitrary but fixed ultrafilter over ω. Let $\mathbb{Q}$ be the notion of forcing defined as follows: The conditions of $\mathbb{Q}$ are pairs $\langle s, A\rangle$ such that $s \in (\mathbb{N})$ (called the stem of the condition), $A \in (\omega)^{<\omega}$, $A(0) \in \mathcal{F}$ and $s \prec A$, stipulating $\langle s, A\rangle \leq \langle t, B\rangle$ if and only if $t \prec s$ and $B \sqsubseteq A$. If $\langle s, A_1\rangle, \langle s, A_2\rangle$ are two $\mathbb{Q}$-conditions, then $\langle s, A_1 \sqcap A_2\rangle \leq \langle s, A_1\rangle, \langle s, A_2\rangle$. Hence, two $\mathbb{Q}$-conditions with the same stem are compatible and because there are only countably many stems, the forcing notion $\mathbb{Q}$ is σ-centered.

Now we will see that $\mathbb{Q}$ adds an infinite partition which is compatible with all old infinite partitions but is not almost finer than any old partition. (So, the forcing notion $\mathbb{Q}$ is in a sense like the dualization of Cohen forcing.)

Lemma II.1.5. Let G be $\mathbb{Q}$-generic over V. Then $G \in (\omega)^{\omega'}$ and $V[G] \models \forall X \in (\omega)^\omega \cap V\big(G|X \wedge \neg(X \sqsubseteq^* G)\big)$.

PROOF. Let $X \in V$ be an arbitrary, infinite partition. Then for every $n \in \omega$, the set D_n is dense in $\mathbb{Q}$, where D_n is a set of $\mathbb{Q}$-conditions $\langle s, A\rangle$, defined as follows:
(1) $s(0)$ has more than n elements,
(2) at least n blocks of X are unions of blocks of A,
(3) there are at least n different blocks $b_i \in X$, such that $\bigcup b_i \in s \sqcap X$.
Therefore, at least one block of G is infinite, because of (1), G is compatible with X, because of (2), and X is not coarser* than G, because of (3). Now, because X was arbitrary, the $\mathbb{Q}$-generic partition G has the desired properties. ⊣

Because the forcing notion $\mathbb{Q}$ is σ-centered and each $\mathbb{Q}$-condition can be encoded by a real number, forcing with $\mathbb{Q}$ does neither collapse any cardinals nor change the cardinality of the continuum. Thus, following [**12**], we get:

PROPOSITION II.1.6. It is consistent with ZFC that $\mathfrak{S}' < \mathfrak{c}$.

PROOF. Take an ω_1-iteration of $\mathbb{Q}$ with finite support, starting from a model in which $\mathfrak{c} = \omega_2$, then the ω_1 generic objects form a splitting family. ⊣

Even though a partition does not have a complement, for each non-trivial partition X we can define a non-trivial partition Y, such that $X \perp Y$: Let $X = \{b_i : i \in \omega\} \in (\omega)^\omega$ and assume that the blocks b_i are ordered by their least element and that each block is ordered by the natural order. A block is called trivial, if it is a singleton. With respect to this ordering define for each non-trivial partition X the partition $X^{\angle}$ as follows:

If $X \in (\omega)^{\omega'}$, then n is in the i^{th} block of $X^{\angle}$ *iff* n is the i^{th} element of a block of X, and if $X \notin (\omega)^{\omega'}$, then n, m are in the same block of $X^{\angle}$ *iff* n, m are both least elements of blocks of X.

It is not hard to see that for each non-trivial $X \in (\omega)^\omega$, $X \perp X^{\angle}$.

A family $\mathscr{W} \subseteq (\omega)^{\omega'}$ is called **weakly splitting**, if for each partition $X \in (\omega)^\omega$, there is a $W \in \mathscr{W}$ such that W splits X or W splits $X^{\angle}$. The cardinal number $w\mathfrak{S}$ is the least cardinal number κ for which there exists a weakly splitting family of cardinality κ. (It is obvious that $w\mathfrak{S} \leq \mathfrak{S}'$.)

A family $\mathcal{U}$ is called a $\boldsymbol{\pi}$**-base** for a free ultrafilter $\mathcal{F}$ over ω provided for every $x \in \mathcal{F}$ there is a $u \in \mathcal{U}$ such that $u \subseteq x$. Define

$$\pi\mathfrak{u} := \min\left\{|\mathcal{U}| : \mathcal{U} \subseteq [\omega]^\omega \textit{ is a } \pi\textit{-base for a free ultra-filter over } \omega\right\}.$$

In [**2**] it is shown that $\pi\mathfrak{u} = \mathfrak{r}$, where $\mathfrak{r}$ is the reaping number defined in the introduction (see [**69**] for more results concerning $\mathfrak{r}$).

Now we can give an upper and a lower bound for the size of $w\mathfrak{S}$.

THEOREM II.1.7. $w\mathfrak{S} \leq \mathfrak{r}$.

PROOF. We will show that $w\mathfrak{S} \leq \pi\mathfrak{u}$. Let $\mathcal{U} := \{u_\iota \in [\omega]^\omega : \iota < \pi\mathfrak{u}\}$ be a π-basis for a free ultrafilter $\mathcal{F}$ over ω. Without loss of generality we may assume that all the $u_\iota \in \mathcal{U}$ are co-infinite. Let $\mathscr{Y} = \left\{Y_u \in (\omega)^\omega : u \in \mathcal{U} \wedge Y_u = \{u_i : u_i = u \vee (u_i = \{n\} \wedge n \notin u)\}\right\}$.

Now we take an arbitrary $X = \{b_i : i \in \omega\} \in (\omega)^\omega$ and define for every $u \in \mathcal{U}$ the sets $I_u := \{i : b_i \cap u \neq \emptyset\}$ and $J_u := \{j : b_j \cap u = \emptyset\}$. It is clear that for every u, $I_u \cup J_u = \omega$.

If we find a $u \in \mathcal{U}$ such that $|I_u| = |J_u| = \omega$, then Y_u splits X. To see this, define the two infinite partitions

$$Z_1 := \big\{a_k : a_k = \bigcup_{i \in I_u} b_i \ \vee \ \exists j \in J_u (a_k = b_j)\big\}$$

and

$$Z_2 := \big\{a_k : a_k = \bigcup_{j \in J_u} b_j \ \vee \ \exists i \in I_u (a_k = b_i)\big\}.$$

We have $X \sqcap Y_u = Z_1$ (therefore $Z_1 \sqsubseteq X, Y_u$) and $Z_2 \sqsubseteq X$ but $Z_2 \perp Y_u$.

If we find an $x \in \mathcal{F}$ such that $|I_x| < \omega$ (and therefore $|J_x| = \omega$), then we find an $x' \subseteq x$, such that $I_{x'} = \{i\}$ and $|b_i \setminus x'| = \omega$ (this is because $\mathcal{F}$ is a free ultra-filter). Now take a $u \in \mathcal{U}$ such that $u \subseteq x'$, and since $X \in (\omega)^{\omega'}$, Y_u splits $X^\angle$.

If we find an $x \in \mathcal{F}$ such that $|J_x| < \omega$ (and therefore $|I_x| = \omega$), let $I(n)$ be an enumeration of I_x and define $y := x \cap \bigcup_{k \in \omega} b_{I(2k)}$. Then $y \subseteq x$ and $|x \setminus y| = \omega$. Hence, either y or $\omega \setminus y$ is a superset of some $u \in \mathcal{U}$. But now $|J_u| = \omega$ and we are in a former case. ⊣

A lower bound for $w\mathfrak{S}$ is $cov(\boldsymbol{B}_0)$:

THEOREM II.1.8. $cov(\boldsymbol{B}_0) \leq w\mathfrak{S}$.

PROOF. Let $\kappa < cov(\boldsymbol{B}_0)$ and $\mathscr{W} = \{W_\iota : \iota < \kappa\} \subseteq (\omega)^{\omega'}$. Assume that for each $W_\iota \in \mathscr{W}$ the blocks are ordered by their least element and each block is ordered by the natural order. Further assume that $b_{i(\iota)}$ is the first block of W_ι which is infinite. Now, for each $\iota < \kappa$ the set D_ι of functions $f \in {}^\omega\omega$ such that

$$\begin{aligned} \forall n, m, k : \quad & \exists t_n \in b_n \, \exists t_m \in b_m \, \exists h \in \omega \, \exists t_h, t'_h \in b_h \, \exists s \in b_{i(\iota)} \\ & f(t_n) = f(t_h) \wedge f(t_m) = f(t'_h) \wedge |\{s' \leq s : f(s') = f(s)\}| = k+1. \end{aligned}$$

is the intersection of countably many open dense sets and therefore the complement of a meager set. Because $\kappa < cov(\boldsymbol{B}_0)$, we find an unbounded function $g \in {}^\omega\omega$ such that $g \in \bigcap_{\iota<\kappa} D_\iota$. The partition $G = \{g^{-1}(n) : n \in \omega\} \in (\omega)^{\omega'}$ is orthogonal to each member of $\mathscr{W}$ and for each $W_\iota \in \mathscr{W}$ and each $k \in \omega$, there exists an $s \in b_{i(\iota)}$, such that s is the k^{th} element of a block of G. Hence, $\mathscr{W}$ can not be a weakly splitting family. ⊣

2. On the dual-reaping cardinals $\mathfrak{R}$ and $\mathfrak{R}'$

A family $\mathscr{R} \subseteq (\omega)^\omega$ is called **reaping** (**reaping$'$**, respectively), if for each partition $X \in (\omega)^\omega$ ($X \in (\omega)^{\omega'}$, respectively) there exists a partition $R \in \mathscr{R}$ such that $R \perp X$ or $R \sqsubseteq^* X$. The **dual-reaping cardinal** $\mathfrak{R}$ ($\mathfrak{R}'$, respectively) is the least cardinal number κ for which there exists a reaping (reaping$'$, respectively) family of cardinality κ.

It is clear that $\mathfrak{R}' \leq \mathfrak{R}$. Further, by finite modifications of the elements of a reaping family we may replace $\sqsubseteq^*$ by $\sqsubseteq$ in the definition above.

If we cancel in the definition of the reaping number the expression "$R \sqsubseteq^* X$", we get the definition of an orthogonal family:

A family $\mathscr{O} \subseteq (\omega)^\omega$ is called **orthogonal** (**orthogonal**$'$, respectively), if for each non-trivial partition $X \in (\omega)^\omega$ (for each partition $X \in (\omega)^{\omega'}$, respectively) there exists a partition $O \in \mathscr{O}$ such that $O \perp X$. The **dual-orthogonal cardinal** $\mathfrak{O}$ ($\mathfrak{O}'$, respectively) is the least cardinal number κ, for which there exists a orthogonal (orthogonal$'$, respectively) family of cardinality κ. It is obvious that $\mathfrak{O}' \leq \mathfrak{O}$. Note that $\mathfrak{o} = \mathfrak{c}$, where $\mathfrak{o}$ is defined like $\mathfrak{O}$ but for infinite subsets of ω instead of infinite partitions. (Take the complements of the members of an almost disjoint family of cardinality $\mathfrak{c}$. Because an orthogonal family must avoid all these complements, it must have at least the cardinality $\mathfrak{c}$.)

It is also clear that each orthogonal$^{(\prime)}$ family is also a reaping$^{(\prime)}$ family and therefore $\mathfrak{R}^{(\prime)} \leq \mathfrak{O}^{(\prime)}$. Further one can show that $\mathfrak{R}'$ is uncountable (*cf.* [**12**]). Now we show that $\mathfrak{O}' \leq \mathfrak{d}$, where $\mathfrak{d}$ is the well-known dominating number (for a definition *cf.* [**65**]), and that $cov(\boldsymbol{B}_0) \leq \mathfrak{O}'$.

THEOREM II.2.1. $\mathfrak{O}' \leq \mathfrak{d}$.

PROOF. Let $\{d_\iota \in {}^\omega\omega : \iota < \mathfrak{d}\}$ be a dominating family. Then it is not hard to see that the family $\{D_\iota : \iota < \kappa\} \subseteq (\omega)^\omega$, where each D_ι is constructed from d_ι like D from d in the proof of Theorem II.1.1, is an orthogonal family. $\dashv$

Let $\mathfrak{i}$ be the least cardinality of an independent family (a definition and some results can be found in [**44**]), then

THEOREM II.2.2. $\mathfrak{O} \leq \mathfrak{i}$.

PROOF. Let $\mathcal{I} \subseteq [\omega]^\omega$ be an independent family of cardinality $\mathfrak{i}$. Let $\mathcal{I}' := \{r \in [\omega]^\omega : r \stackrel{*}{=} \bigcap \mathcal{A} \setminus \bigcup \mathcal{B}\}$, where $\mathcal{A}, \mathcal{B} \in [\mathcal{I}]^{<\omega}$, $\mathcal{A} \neq \emptyset$, $\mathcal{A} \cap \mathcal{B} = \emptyset$, and $r \stackrel{*}{=} x$ means $|(r \setminus x) \cup (x \setminus r)| < \omega$. It is not hard to see that $|\mathcal{I}'| = |\mathcal{I}| = \mathfrak{i}$. Now let $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$ where $\mathscr{I}_1 := \big\{X_r \in (\omega)^\omega : r \in \mathcal{I}' \wedge X_r = \{b_i : b_i = r \vee (b_i = \{n\} \wedge n \notin r)\}\big\}$ and $\mathscr{I}_2 := \big\{Y_r : \exists X_r \in \mathscr{I}_1 \, (Y_r = X_r^{\angle})\big\}$. We see that $\mathscr{I} \subseteq (\omega)^\omega$ and $|\mathscr{I}| = \mathfrak{i}$. It remains to show that $\mathscr{I}$ is an orthogonal family.

Let $Z \in (\omega)^\omega$ be arbitrary and let $r := \mathrm{Min}(Z)$. If $r \in \mathcal{I}'$, then $X_r \perp Z$ (where $X_r \in \mathscr{I}_1$). And if $r \notin \mathcal{I}'$, then there exists an $r' \in \mathcal{I}'$ such that $r \cap r' = \emptyset$. But then $Y_{r'} \perp Z$ (for $Y_{r'} \in \mathscr{I}_2$). $\dashv$

Because $\mathfrak{R} \leq \mathfrak{O}$, the cardinal number $\mathfrak{i}$ is also an upper bound for $\mathfrak{R}$. But for $\mathfrak{R}$, we also find another upper bound.

THEOREM II.2.3. $\mathfrak{R} \leq \mathfrak{r}$.

PROOF. Like in Theorem II.1.7 we show that $\mathfrak{R} \leq \pi\mathfrak{u}$. Let $\mathcal{U} := \{u_\iota \in [\omega]^\omega : \iota < \pi\mathfrak{u}\}$ be as in the proof of Theorem II.1.7 and let

$$\mathscr{Y} = \big\{Y_u \in (\omega)^\omega : u \in \mathcal{U} \wedge Y_u = \{u_i : u_i = \omega \setminus u \vee (u_i = \{n\} \wedge n \in u)\}\big\}.$$

Take an arbitrary partition $X \in (\omega)^\omega$. Let $r := \operatorname{Min}(X)$ and $r_1 := \big\{n \in r : \{n\} \in X\big\}$. If we find a $u \in \mathcal{U}$ such that $u \subseteq r_1$, then $Y_u \sqsubseteq X$. Otherwise, we find a $u \in \mathcal{U}$ such that either $u \subseteq \omega \setminus r$ or $u \subseteq r \setminus r_1$ and in both cases $Y_u \perp X$. ⊣

Now we will show that it is consistent with ZFC that $\mathfrak{D}$ can be small. For this we first show that a Cohen real encode an infinite partition which is orthogonal to each old non-trivial infinite partition. (This result is in fact a corollary of [**12**, Lemma 5].)

LEMMA II.2.4. If $c \in {}^\omega\omega$ is a Cohen real over V, then $C := \{c^{-1}(n) : n \in \omega\} \in (\omega)^{\omega'} \cap V[c]$ and $\forall X \in (\omega)^\omega \cap V\big(\neg(X \stackrel{*}{=} \{\omega\}) \to C \perp X\big)$.

PROOF. We will consider the Cohen-conditions as finite sequences of natural numbers, $s = \{s(i) : i < n < \omega\}$. Let $X = \{b_i : i \in \omega\} \in V$ be an arbitrary, non-trivial infinite partition. The set $D_{n,m}$ of Cohen-conditions s such that

(i) $|\{i : s(i) = 0\}| \geq n$,
(ii) $\exists k > n\, \exists i (s(i) = k)$,
(iii) $\exists a_n \in b_n\, \exists a_m \in b_m\, \exists l\, \exists a_1, a_2 \in b_l \big(s(a_n) = s(a_1) \wedge s(a_m) = s(a_2)\big)$,

is dense for all $n, m \in \omega$. Note that because of (i), $C \in (\omega)^{\omega'}$. Now, because X was arbitrary, the infinite partition C is orthogonal to each infinite partition which is in V. ⊣

We now can show that $\mathfrak{D}$ can be small:

PROPOSITION II.2.5. It is consistent with ZFC that $\mathfrak{D} < cov(\boldsymbol{B}_0)$.

PROOF. Take an ω_1-iteration of Cohen forcing with finite support, starting from a model in which we have $\mathfrak{c} = \omega_2 = cov(\boldsymbol{B}_0)$, then the ω_1 generic objects form an orthogonal family. Because this ω_1-iteration of Cohen forcing does not change the cardinality of $cov(\boldsymbol{B}_0)$, we have a model in which $\omega_1 = \mathfrak{D} < cov(\boldsymbol{B}_0) = \omega_2$ holds. ⊣

Because $\mathfrak{R} \leq \mathfrak{D}$, we also get the relative consistency of $\mathfrak{R} < cov(\boldsymbol{B}_0)$. Note that this is not true for $\mathfrak{r}$.

As a lower bound for $\mathfrak{R}'$ we find $\mathfrak{p}$, where $\mathfrak{p}$ is the pseudo-intersection number (a definition of $\mathfrak{p}$ can be found in [**65**]).

THEOREM II.2.6. $\mathfrak{p} \leq \mathfrak{R}'$.

PROOF. In [**4**] it is proved that $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$, where

$$\mathfrak{m}_{\sigma\text{-centered}} = \min\{\kappa : \text{“}\mathsf{MA}(\kappa) \text{ for } \sigma\text{-centered posets” fails}\}.$$

Let $\mathscr{R} = \{R_\iota : \iota < \kappa < \mathfrak{p}\}$ be a set of infinite partitions. Now remember that the forcing notion $\mathbb{Q}$ (defined in Section 1) is σ-centered and because $\kappa < \mathfrak{p}$ we find an $X \in (\omega)^{\omega'}$ such that $\mathscr{R}$ does not reap X. ⊣

As a corollary we get:

Corollary II.2.7. If we assume MA, then $\mathfrak{R}' = \mathfrak{c}$.

Proof. If we assume MA, then $\mathfrak{p} = \mathfrak{c}$. $\dashv$

3. What about towers and orthogonal families?

Let κ_{mao} be the least cardinal number κ for which there exists an infinite maximal almost orthogonal family of cardinality κ, and let κ_{tower} be the least cardinal number κ for which there exists a family $\mathscr{F} \subseteq (\omega)^\omega$ of cardinality κ, such that $\mathscr{F}$ is well-ordered by $\sqsubseteq^*$ and $\neg\exists Y \in (\omega)^\omega \forall X \in \mathscr{F}(Y \sqsubseteq^* X)$.

Krawczyk proved that $\kappa_{mao} = \mathfrak{c}$ (*cf.* [**12**]) and Carlson proved that $\kappa_{tower} = \omega_1$ (*cf.* [**46**]). So, these cardinals are interesting. But what happens if we cancel the word "almost" in the definition of κ_{mao}? In fact nothing happens since Otmar Spinas has shown in [**62**] that an infinite maximal orthogonal family has always cardinality $\mathfrak{c}$.

4. The diagrams of the results

Now we summarize the results proved in this chapter together with some other known results.

The dual-splitting number:

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, dash] \arrow[dd, dash] & \mathfrak{S}' \arrow[rr, dash] \arrow[d, dash] & & \mathfrak{c} \arrow[dd, dash] \\
& \mathfrak{S} \arrow[d, dash] & & \\
\omega_1 \arrow[r, dash] & cov(\boldsymbol{B}_0) \arrow[r, dash] & w\mathfrak{S} \arrow[r, dash] & \mathfrak{r}
\end{tikzcd}$$

The dual-reaping number and the dual-orthogonal number:

$$\begin{tikzcd}
& & \mathfrak{d} \arrow[r, dash] \arrow[d, dash] & \mathfrak{i} \arrow[r, dash] \arrow[d, dash] & \mathfrak{c} \arrow[dd, dash] \\
& & \mathfrak{D}' \arrow[r, dash] \arrow[d, dash] & \mathfrak{D} \arrow[d, dash] & \\
\omega_1 \arrow[r, dash] & \mathfrak{p} \arrow[r, dash] & \mathfrak{R}' \arrow[r, dash] & \mathfrak{R} \arrow[r, dash] & \mathfrak{r}
\end{tikzcd}$$

In the diagrams, the cardinal characteristics grow larger as one moves up or to the right.

Consistency results:

- $\mathfrak{s} < \mathfrak{S}$
- $\mathfrak{S}' < \mathfrak{c}$
- $\mathfrak{D} < cov(\boldsymbol{B}_0)$

CHAPTER III

Topologies on the Set of Partition-Ultrafilters

In this chapter we define four topologies on the set of partition-ultrafilters over ω and show that none of these topological spaces is homeomorphic to $\beta\omega$ or $\beta\omega \setminus \omega$. For a slightly more general approach in terms of semilattices see [**27**].

1. Partition-ultrafilters

In this chapter we will consider just **homogeneous partitions**, *i.e.*, partitions of ω all of whose blocks are infinite, but we do not introduce a new notation, thus, throughout this chapter, $(\omega)^{\leq\omega}$ denotes the set of all homogeneous partitions.

We can define partition-filters in two different ways:
A set $\mathscr{F} \subseteq (\omega)^{\leq\omega}$ is a **$\sqsubseteq$-partition-filter**, if the following holds:

(a) $\big\{\{\omega\}\big\} \notin \mathscr{F}$.
(b) For any $X, Y \in \mathscr{F}$ we have $X \sqcap Y \in \mathscr{F}$.
(c) If $X \in \mathscr{F}$ and $X \sqsubseteq Y \in (\omega)^{\leq\omega}$, then $Y \in \mathscr{F}$.

A set $\mathscr{F} \subseteq (\omega)^{\leq\omega}$ is a **$\sqsupseteq$-partition-filter**, if the following holds:

(a) For any $X, Y \in \mathscr{F}$ we have $X \sqcup Y \in \mathscr{F}$.
(b) If $X \in \mathscr{F}$ and $X \sqsupseteq Y \in (\omega)^{\leq\omega}$, then $Y \in \mathscr{F}$.

A $\sqsubseteq$-partition-filter $\mathscr{F} \subseteq (\omega)^{\leq\omega}$ is called **principal**, if there is a partition $X \in (\omega)^{\leq\omega}$ such that $\mathscr{F} = \{Y : X \sqsubseteq Y\}$. A set $\mathscr{U} \subseteq (\omega)^{\leq\omega}$ is a **partition-ultrafilter** (of some type), if $\mathscr{U}$ is a partition-filter which is not properly contained in any other partition-filter (of the same type).

Notice that a $\sqsubseteq$-partition-ultrafilter $\mathscr{U}$ which does not contain a finite partition is always non-principal, and vice versa, a principal partition-ultrafilter always contains a finite partition, in fact, it contains a 2-block partition (see [**27**, Fact 3.1]). Thus, if $\mathscr{U}$ is a non-principal $\sqsubseteq$-partition-ultrafilter, $X \in \mathscr{U}$ and $X \sqsubseteq^* Y$, then $Y \in \mathscr{U}$. Similarly, if $\mathscr{U}$ is a $\sqsupseteq$-partition-ultrafilter, $X \in \mathscr{U}$ and $Y \sqsubseteq^* X$, then $Y \in \mathscr{U}$.

Let $\mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ denote the set of all **$\sqsubseteq$-partition-ultrafilters** and **$\sqsupseteq$-partition-ultrafilters**, respectively, on ω.

2. Topologies on $\mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$

In the following, we will define two topologies on $\mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ as well as on $\mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$, but let us do it just for $\mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$. First define for each $X \in (\omega)^{\leq\omega}$

two sets

$$(X)^+ := \{\mathscr{U} \in \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big) : X \in \mathscr{U}\}$$

and

$$(X)^- := \{\mathscr{U} \in \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big) : X \notin \mathscr{U}\} = \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big) \setminus (X)^+ .$$

Set $\mathcal{O}^+ := \{(X)^+ : X \in (\omega)^{\leq\omega}\}$ and $\mathcal{O}^- := \{(X)^- : X \in (\omega)^{\leq\omega}\}$ and call the topology generated by $\mathcal{O}^+$ the **positive topology** τ^+ and the topology generated by $\mathcal{O}^-$ the **negative topology** τ^-. Note that $\mathcal{O}^+$ is a base for τ^+, but $\mathcal{O}^-$ is not a base for τ^-. This difference accounts for some of the asymmetries. In the same way we can define the negative and positive topology on $\mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$.

In the sequel, the topological space $\big\langle \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big), \tau^+\big\rangle$ is denoted by $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ and $\big\langle \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big), \tau^-\big\rangle$ is denoted by $\mathtt{PUF}^-_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$. Similarly, $\big\langle \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big), \tau^+\big\rangle$ is denoted by $\mathtt{PUF}^+_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ and $\big\langle \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big), \tau^-\big\rangle$ by $\mathtt{PUF}^-_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$.

Let $\mathtt{UF}\big(\mathcal{P}(\omega)\big)$ denote the set of ultrafilters over ω and let $\mathtt{UF}\big([\omega]^\omega\big)$ denote the set of non-principal ultrafilters over ω. Following the construction above, one can define four topologies on $\mathtt{UF}\big(\mathcal{P}(\omega)\big)$, namely $\mathtt{UF}^+_{\subseteq}\big(\mathcal{P}(\omega)\big)$, $\mathtt{UF}^-_{\subseteq}\big(\mathcal{P}(\omega)\big)$, $\mathtt{UF}^+_{\supseteq}\big(\mathcal{P}(\omega)\big)$ and $\mathtt{UF}^-_{\supseteq}\big(\mathcal{P}(\omega)\big)$, but each of these topological spaces is homeomorphic to $\beta\omega$, the space of ultrafilters over ω. Further, one can also define four topologies on $\mathtt{UF}\big([\omega]^\omega\big)$ (which is the set of non-principal ultrafilters over ω), namely $\mathtt{UF}^+_{\subseteq}\big([\omega]^\omega\big)$, $\mathtt{UF}^-_{\subseteq}\big([\omega]^\omega\big)$, $\mathtt{UF}^+_{\supseteq}\big([\omega]^\omega\big)$ and $\mathtt{UF}^-_{\supseteq}\big([\omega]^\omega\big)$, but each of these topological spaces is homeomorphic to $\beta\omega \setminus \omega$, the space of non-principal ultrafilters over ω.

FACT III.2.1. The spaces $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^-_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^+_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}^-_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ are all T_1 spaces (*i.e.*, all singletons are closed).

PROOF. For any singleton $\{\mathscr{U}\}$ look at $\bigcup_{X\notin\mathscr{U}}(X)^+$ for the positive topology and $\bigcup_{X\in\mathscr{U}}(X)^-$ for the negative topology. A simple argument using the maximality of partition-ultrafilters shows that these sets are just the complement of $\{\mathscr{U}\}$. But since they are open in the respective topologies, $\{\mathscr{U}\}$ is closed in either topology. ⊣

3. The spaces $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^-_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^+_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}^-_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$

3.1. Principal spaces. We shall call a topological space **principal** if it contains an open set with just one element. Being principal is obviously a property preserved under homeomorphisms, so it is a topological invariant. Concerning $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, we like to mention the following:

FACT III.3.1. If $\mathscr{U} \in \mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathscr{U}$ contains a finite partition, then there is a 2-block partition X such that $\mathscr{U} = \{Y \in (\omega)^{\leq\omega} : X \sqsubseteq Y\}$, and hence, $\mathscr{U}$ is principal.

PROOF. Let $m := \min\{n : \exists Y \in \mathscr{U}(|Y| = n)\}$. This minimum exists by assumption. Let $X \in \mathscr{U}$ be such that $|X| = m$.

First we show that for all $Y \in \mathscr{U}$ we have $X \sqsubseteq Y$. Suppose this is not the case for some $Y \in \mathscr{U}$, then we have $X \neq X \sqcap Y \in \mathscr{U}$ (since $\mathscr{U}$ is a filter), which implies $|X \sqcap Y| < |X| = m$ and contradicts the definition of m. On the other hand, there is a 2-block partition Z with $Z \sqsubseteq X$, and because $Z \sqsubseteq X$ we get $Z \sqsubseteq Y$ for any $Y \in \mathscr{U}$. Therefore, since $\mathscr{U}$ is an ultrafilter, we get $Z = X$, which implies $\{Y \in (\omega)^{\leq\omega} : X \sqsubseteq Y\} = \mathscr{U}$ and $m = 2$. ⊣

This leads to the following observation:

FACT III.3.2. The space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is a principal topological space, whereas the space $\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ is non-principal.

PROOF. That $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is principal follows directly from Fact III.3.1. For the second assertion we note that for every partition $Y \in (\omega)^{\leq\omega}$ we find $Z_1, Z_2 \in (\omega)^{\leq\omega}$ such that $Y \sqsubseteq Z_1$, $Y \sqsubseteq Z_2$ and $Z_1 \sqcup Z_2 \notin (\omega)^{\leq\omega}$, and therefore, we find $\mathscr{U}_1, \mathscr{U}_2 \in \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ with $Z_1 \in \mathscr{U}_1$ and $Z_2 \in \mathscr{U}_2$, which implies that $\mathscr{U}_1$ and $\mathscr{U}_2$ both belong to $(Y)^+$. So, for each $Y \in (\omega)^{\leq\omega}$, the set $(Y)^+$ is not a singleton. In fact, by this argument, $\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ does not have any finite open sets. ⊣

3.2. The space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$. First notice that like in the space $\beta\omega$, the principal $\sqsubseteq$-partition-ultrafilters form a dense set in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, but since there are continuum many 2-block partitions (one for each subset of ω), they cannot witness that the space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is separable. Moreover, we get the following

OBSERVATION III.3.3. The space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is not separable.

PROOF. Spinas proved in **[62]** that there is an uncountable set $\{X_\iota : \iota \in I\} \subseteq (\omega)^\omega$ of infinite partitions such that $X_\iota \sqcap X_{\iota'} = \big\{\{\omega\}\big\}$ whenever $\iota \neq \iota'$ (see the end of Chapter II). Thus, $(X_\iota)^+ \cap (X_{\iota'})^+ = \emptyset$ (for $\iota \neq \iota'$), which implies that there is no countably dense set in the space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$. ⊣

PROPOSITION III.3.4. The space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is a Hausdorff space.

PROOF. Let $\mathscr{U}$ and $\mathscr{V}$ be two distinct $\sqsubseteq$-partition-ultrafilters. Because $\mathscr{U} \neq \mathscr{V}$ and both are maximal $\sqsubseteq$-partition-filters, we find partitions $X \in \mathscr{U}$ and $Y \in \mathscr{V}$ such that $X \sqcap Y = \big\{\{\omega\}\big\}$. Thus, we get $\mathscr{U} \in (X)^+$, $\mathscr{V} \in (Y)^+$ and $(X)^+ \cap (Y)^+ = \emptyset$. ⊣

For two partitions $X, Y \in (\omega)^{\leq\omega}$ we write $X \perp_{\sqsubseteq} Y$ if $X \sqcap Y = \big\{\{\omega\}\big\}$. Before we prove the next proposition, we state the following useful

LEMMA III.3.5. If $X_0, \dots, X_n \in (\omega)^{\leq\omega}$ is a finite set of non-trivial partitions, then there is a non-trivial partition $Y \in (\omega)^{\leq\omega}$ such that $Y \perp_{\sqsubseteq} X_i$ for all $i \leq n$.

PROOF. Let $Z_0 := \mathrm{Min}(X_0)$. If Z_i is such that $Z_i \cap X_{i+1}(k) \neq \emptyset$ for every $k \leq |X_{i+1}|$, then $Z_{i+1} = Z_i$. Otherwise, we define $Z_{i+1} \supseteq Z_i$ as follows: If $Z_i \cap X_{i+1}(k) \neq \emptyset$, then $Z_{i+1} \cap X_{i+1}(k) = Z_i \cap X_{i+1}(k)$; and if $Z_i \cap X_{i+1}(k) = \emptyset$, then $Z_{i+1} \cap X_{i+1}(k) = \min(X_{i+1}(k))$. It is easy to see that $\omega \setminus Z_i$ is infinite for every $i \leq n$. Finally, let $Y = \{Y(0), Y(1)\} \in (\omega)^{\leq\omega}$ be such that $Z_n \subseteq Y(0)$ and by construction we get $Y \perp_{\sqsubseteq} X_i$ for all $i \leq n$. ⊣

PROPOSITION III.3.6. The space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ is not compact.

Proof. Let $\mathcal{A} = \{(X)^+ : X \in (\omega)^\omega\}$, then it is easy to see that $\bigcup \mathcal{A} = \mathtt{PUF}^+_\sqsubseteq\big((\omega)^{\leq\omega}\big)$. We will show that $\mathcal{A}$ is a cover with no finite subcovers. Assume to the contrary that there are finitely many infinite partitions $X_0, \dots, X_n \in (\omega)^\omega$ such that $(X_0)^+ \cup \ldots \cup (X_n)^+ = \mathtt{PUF}^+_\sqsubseteq\big((\omega)^{\leq\omega}\big)$. By Lemma III.3.5 we find a non-trivial partition $Y \in (\omega)^{\leq\omega}$ such that $Y \perp_\sqsubseteq X_i$ (for all $i \leq n$). Let $\mathscr{U} \in \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ be such that $Y \in \mathscr{U}$, then $X_i \notin \mathscr{U}$ (for all $i \leq n$), which contradicts the assumption. ⊣

3.3. The space $\mathtt{PUF}^-_\sqsubseteq\big((\omega)^{\leq\omega}\big)$.

Proposition III.3.7. The space $\mathtt{PUF}^-_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ is not a Hausdorff space.

Proof. Let $\mathscr{U}$ and $\mathscr{V}$ be two distinct $\sqsupseteq$-partition-ultrafilters. Take any non-trivial partitions $X_0, \dots, X_k$, $Y_0, \dots, Y_\ell \in (\omega)^{\leq\omega}$ such that

$$\mathscr{U} \in (X_0)^- \cap \ldots \cap (X_k)^- \quad\text{and}\quad \mathscr{V} \in (Y_0)^- \cap \ldots \cap (Y_\ell)^- .$$

Now, by Lemma III.3.5, there is a non-trivial partition Z such that $Z \perp_\sqsubseteq X_i$ (for $i \leq k$) and $Z \perp_\sqsubseteq Y_j$ (for $j \leq \ell$), which implies $Z \in \bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq \ell}(Y_j)^-$. Hence, $\bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq \ell}(Y_j)^-$ is not empty. ⊣

Proposition III.3.8. The space $\mathtt{PUF}^-_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ is countably compact.

Proof. Let $\mathcal{A} = \{\bigcap A_i : i \in \omega\}$ be such that $\bigcup \mathcal{A} = \bigcup_{i\in\omega}(\bigcap A_i) = \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$, where each A_i is a finite set of open sets of the form $(X)^-$ for some $X \in (\omega)^{\leq\omega}$. Assume $\bigcup_{i \in I}(\bigcap A_i) \neq \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ for every finite set $I \subseteq \omega$. If $A_i = \{(X^i_0)^-, \dots, (X^i_n)^-\}$ and $A_j = \{(X^j_0)^-, \dots, (X^j_m)^-\}$ and $\bigcap A_i \cup \bigcap A_j \neq \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$, then we find a $\mathscr{U} \in \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ such that $\mathscr{U} \in \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big) \setminus \big(\bigcap A_i \cup \bigcap A_j\big)$. Hence, there are $k \leq n$ and $\ell \leq m$ such that X^i_k and X^j_ℓ are both in $\mathscr{U}$, which implies $X^i_k \sqcap X^j_\ell \neq \big\{\{\omega\}\big\}$. We define a tree T as follows: For $n \in \omega$ the sequence $\langle s_0, \dots, s_n\rangle$ belongs to T if and only if for every $i \leq n$ there is an $(X^i_k)^- \in A_i$ such that $s_i = X^i_k$ and $(s_0 \sqcap \ldots \sqcap s_n) \neq \big\{\{\omega\}\big\}$. The tree T, ordered by inclusion, is by construction (and by our assumption) a tree of height ω and each level of T is finite. Therefore, by König's Lemma, the tree T contains an infinite branch. Let $\langle X^i : i \in \omega\rangle$ be an infinite branch of T, where $X^i \in A_i$. By construction of T, for every finite $I = \{\iota_0, \dots, \iota_n\} \subseteq \omega$ we have $X^{\iota_0} \sqcap \ldots \sqcap X^{\iota_n} \neq \big\{\{\omega\}\big\}$. Thus, the partitions constituting the branch have the finite intersection property and therefore we find a $\mathscr{U} \in \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$ such that $X^i \in \mathscr{U}$ for every $i \in \omega$. Now, $\mathscr{U} \notin \bigcup_{i\in\omega}(X^i)^-$ which implies that $\mathscr{U} \notin \bigcup \mathcal{A}$, but this contradicts $\bigcup \mathcal{A} = \mathtt{PUF}_\sqsubseteq\big((\omega)^{\leq\omega}\big)$. ⊣

3.4. The space $\mathtt{PUF}^+_\sqsupseteq\big((\omega)^{\leq\omega}\big)$.

Proposition III.3.9. The space $\mathtt{PUF}^+_\sqsupseteq\big((\omega)^{\leq\omega}\big)$ is a Hausdorff space.

Proof. Let $\mathscr{U}$ and $\mathscr{V}$ be two distinct $\sqsupseteq$-partition-ultrafilters. Because $\mathscr{U} \neq \mathscr{V}$ and both are maximal filters, we find partitions $X \in \mathscr{U}$ and $Y \in \mathscr{V}$ such that $X \sqcup Y \notin (\omega)^{\leq\omega}$. Hence we get $\mathscr{U} \in (X)^+$, $\mathscr{V} \in (Y)^+$ and $(X)^+ \cap (Y)^+ = \emptyset$. ⊣

For two partitions $X, Y \in (\omega)^{\leq\omega}$ we write $X \perp_{\sqsupseteq} Y$ if $X \sqcap Y \notin (\omega)^{\leq\omega}$. Before we prove the next proposition, we state the following useful

LEMMA III.3.10. If $X_0, \ldots, X_n \in (\omega)^{<\omega}$ is a finite set of non-trivial, finite partitions, then there is a finite partition $Y \in (\omega)^{<\omega}$ such that $Y \perp_{\sqsupseteq} X_i$ for all $i \leq n$.

PROOF. Define an equivalence relation on ω as follows:

$$s \approx t \iff \forall i, k\big(s \in X_i(k) \leftrightarrow t \in X_i(k)\big)$$

Because every partition X_i is finite and we only have finitely many partitions X_i, at least one of the equivalence classes must be infinite, say I. Since each block of each partition X_i is infinite and the partitions have been assumed to be non-trivial, we also must have $\omega \setminus I$ is infinite. Let $I_{-1} := I$ and define $I_{i+1} := I_i \dot{\cup} \{s_{i+1}\}$ in such a way that for any $t \in I$ we have $s_{i+1} \in X_{i+1}(k) \to t \notin X_{i+1}(k)$. Let $Y := \{I_n, \omega \setminus I_n\}$, then $Y \in (\omega)^{\leq\omega}$ and for every $i \leq n$, $Y \sqcup X_i$ contains a finite block and therefore, $Y \perp_{\sqsupseteq} X_i$ (for all $i \leq n$). ⊣

PROPOSITION III.3.11. The space $\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ is not compact.

PROOF. Let $\mathcal{A} = \{(X)^+ : X \in (\omega)^{<\omega}\}$, then it is easy to see that $\bigcup \mathcal{A} = \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$. Assume to the contrary that there are finitely many finite partitions $X_0, \ldots, X_n \in (\omega)^{<\omega}$ such that $(X_0)^+ \cup \ldots \cup (X_n)^+ = \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$. By Lemma III.3.10 we find a $Y \in (\omega)^{<\omega}$ such that $Y \perp_{\sqsupseteq} X_i$ (for all $i \leq n$). Let $\mathscr{U} \in \mathtt{PUF}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ be such that $Y \in \mathscr{U}$, then $X_i \notin \mathscr{U}$ (for all $i \leq n$), which contradicts the assumption. ⊣

3.5. The space $\mathtt{PUF}^{-}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$.

PROPOSITION III.3.12. The space $\mathtt{PUF}^{-}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ is not a Hausdorff space.

PROOF. We first show that if $\mathscr{U} \in (X)^-$ for some $X \in (\omega)^\omega$, then there is an $X' \in (\omega)^{<\omega}$ such that $X' \sqsubseteq X$ (and therefore $(X')^- \subseteq (X)^-$) and $\mathscr{U} \in (X')^-$. Since $\mathscr{U} \in (X)^-$, there is a $Y \in p$ such that $Y \sqcup X \notin (\omega)^{\leq\omega}$, which is equivalent to the following statement (recall that we only allowed infinite blocks): There are $y \in Y$ and $x \in X$ such that $x \cap y$ is a non-empty, finite set. Now, for $X' := \{x, \omega \setminus x\}$ we obviously have $X' \sqsubseteq X$ and $p \in (X')^-$.

Let $\mathscr{U}$ and $\mathscr{V}$ be two distinct $\sqsupseteq$-partition-ultrafilters and take any partitions $X_0, \ldots, X_k, Y_0, \ldots, Y_l \in (\omega)^{\leq\omega}$ such that $\mathscr{U} \in (X_0)^- \cap \ldots \cap (X_k)^-$ and $\mathscr{V} \in (Y_0)^- \cap \ldots \cap (Y_l)^-$. By the fact mentioned above we may assume that the X_i's as well as the Y_i's are finite partitions. Now, by Lemma III.3.10, there is a finite partition Z such that $Z \perp_{\sqsupseteq} X_i$ (for $i \leq k$) and $Z \perp_{\sqsupseteq} Y_j$ (for $j \leq l$), which implies $Z \in \bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq l}(Y_j)^-$. Hence, $\bigcap_{i \leq k}(X_i)^- \cap \bigcap_{j \leq l}(Y_j)^-$ is not empty. ⊣

PROPOSITION III.3.13. The space $\mathtt{PUF}^{-}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ is countably compact.

PROOF. Replacing "$\sqcap$" by "$\sqcup$" and "$\sqsubseteq$" by "$\sqsupseteq$", one can simply copy the proof of Proposition III.3.8. ⊣

3.6. Conclusion. Now we are ready to state the main result of this paper.

THEOREM III.3.14. None of the spaces $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^{-}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}^{-}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ is homeomorphic to $\beta\omega$ or $\beta\omega\setminus\omega$. Moreover, no two of the spaces $\beta\omega$, $\beta\omega\setminus\omega$, $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$, $\mathtt{PUF}^{-}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$ and $\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$ are homeomorphic.

PROOF. The proof is given in the following table which is just the compilation of the results from the previous sections. The separation property T_1 holds for all spaces and thus does not help to discern any two of these spaces; it is just included for completeness.

	$\beta\omega$	$\beta\omega\setminus\omega$	$\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$	$\mathtt{PUF}^{-}_{\sqsubseteq}\big((\omega)^{\leq\omega}\big)$	$\mathtt{PUF}^{+}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$	$\mathtt{PUF}^{-}_{\sqsupseteq}\big((\omega)^{\leq\omega}\big)$
principal	YES	NO	YES		NO	
T_1	YES	YES	YES	YES	YES	YES
Hausdorff	YES	YES	YES	NO	YES	NO
ctb. compact	YES	YES		YES		YES
compact	YES	YES	NO		NO	

⊣

4. About the space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$

In the following we investigate the space $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$, where $\mathtt{PUF}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ denotes the set of all non-principal $\sqsubseteq$-partition-ultrafilters.

Notice first that for $X, Y \in (\omega)^{\omega}$, in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ we have $(X)^{+} \subseteq (Y)^{+}$ if and only if $X \sqsubseteq^{*} Y$.

4.1. The height of tree π-bases of $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$. We first give the definition of the dual-shattering cardinal $\mathfrak{H}$, which will be further investigated in Chapter V. A family $\mathscr{A} \subseteq (\omega)^{\omega}$ is called **maximal almost orthogonal** (*mao*) if $\mathscr{A}$ is a maximal family of pairwise orthogonal partitions (see also the end of Chaper II). As a matter of fact we like to mention that every infinite *mao* family has the cardinality of the continuum (*cf.* [**12**] or [**62**]). A family $\mathscr{A}\!\!\!\mathscr{A}$ of *mao* families of partitions **shatters** a partition $X \in (\omega)^{\omega}$, if there is an $\mathscr{A} \in \mathscr{A}\!\!\!\mathscr{A}$ and two distinct partitions in $\mathscr{A}$ which are both compatible (*i.e.*, not orthogonal) with X. A family of *mao* families of partitions is **shattering** if it shatters each partition of $(\omega)^{\omega}$. The **dual-shattering cardinal** $\mathfrak{H}$ is the least cardinal number κ for which there is a shattering family of cardinality κ.

The dual-shattering cardinal $\mathfrak{H}$ is a dualization of the well-known shattering number $\mathfrak{h}$ introduced by Bohuslav Balcar, Jan Pelant and Petr Simon in [**1**] and the letter $\mathfrak{h}$ comes from the word "height". In [**1**] it is proved that

$$\mathfrak{h} = \min\{\kappa : \text{there is a tree } \pi\text{-base for } \beta\omega\setminus\omega \text{ of } \mathfrak{h}\text{eight } \kappa\}\,,$$

where a family $\mathcal{B}$ of non-empty open sets is called a **π-base** for a space S provided every non-empty open set of S contains a member of $\mathcal{B}$, and a **tree π-base** T is a π-base which is a tree when considered as a partially ordered set under reverse inclusion (*i.e.*, for every $t \in T$ the set $\{s \in T : s \supseteq t\}$ is well-ordered by $\supseteq$). The height of an element $t \in T$ is the ordinal α such that $\{s \in T : s \supsetneq t\}$ is of order type α, and the **height** of a tree T is the smallest ordinal α such that no element of T has height α.

One can show that $\mathfrak{H} \leq \mathfrak{h}$ and $\mathfrak{H} \leq \mathfrak{S}$, where $\mathfrak{S}$ is the dual-splitting cardinal (*cf.* [**12**] or see Chapter V).

It is consistent with ZFC that $\mathfrak{H} = \aleph_2 = 2^{\aleph_0}$ (see Chapter V or [**22**]) and also that $\aleph_1 = \mathfrak{H} < \mathfrak{h} = \aleph_2$ (*cf.* [**62**]). Further, it is consistent with $\mathsf{ZFC} + \mathsf{MA} + 2^{\aleph_0} = \aleph_2$ that $\mathfrak{H} = \aleph_1 < \mathfrak{h} = \aleph_2$ (*cf.* [**10**]).

Following Balcar, Pelant and Simon, it is not hard to prove the following

Proposition III.4.1. Let $\mathfrak{H}$ be the dual-shattering cardinal defined as above, then

$$\mathfrak{H} = \min\big\{\kappa : \text{there is a tree } \pi\text{-base for } \mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big) \text{ of height } \kappa\big\}\,.$$

Proof. Bearing in mind that for every countable decreasing sequence of basic open sets $(X_0)^+ \supseteq (X_1)^+ \supseteq \ldots \supseteq (X_n)^+ \supseteq \ldots$ there is a basic open set $(Y)^+$ such that for all $i \in \omega$ we have $(Y)^+ \subseteq (X_i)^+$ (*cf.* [**46**, Proposition 4.2]), one can follow the proof of the Base Matrix Lemma 2.11 of [**1**]. $\dashv$

Because the shattering number and the dual-shattering cardinal can be different, this gives us an asymmetry between the two spaces $\beta\omega \setminus \omega$ and $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$.

4.2. On *P*-points in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$. In this section we give a sketch of the proof that P-points exist in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ if we assume CH, and that in general, both existence and non-existence of P-points are consistent with the axioms of set theory.

An $\sqsubseteq$-partition-ultrafilter $\mathscr{U}$ in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ is a ***P*-point** if the intersection of any family of countably many neighbourhoods of $\mathscr{U}$ is a (not necessarily open) neighbourhood of $\mathscr{U}$.

First we show that a P-point in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ induces in a canonical way a P-point in $\beta\omega \setminus \omega$.

Lemma III.4.2. If there is a P-point in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$, then there is a P-point in $\beta\omega \setminus \omega$ as well.

Proof. Let $\mathscr{U}$ be a P-point in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$, then it is not hard to see that the filter generated by $\{\mathrm{Min}(X) : X \in p\}$ is a P-point in $\beta\omega \setminus \omega$. $\dashv$

Proposition III.4.3. It is consistent with ZFC that $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ does not contain any P-point.

Proof. Saharon Shelah proved in [**60**, Chapter VI, §4] that it is consistent with ZFC that there are no P-points in $\beta\omega \setminus \omega$. But in a model of ZFC in which there are no P-points in $\beta\omega \setminus \omega$, there are also no P-points in $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$ by Lemma III.4.2. $\dashv$

Let $\mathbb{W} = \langle(\omega)^{\omega}, \leq\rangle$ be the partial order defined as follows:

$$X \leq Y \iff X \sqsubseteq^* Y\,.$$

The forcing notion $\mathbb{W}$ is a natural dualization of $\mathcal{P}(\omega)/\mathit{fin}$.

LEMMA III.4.4. If G_p is $\mathbb{W}$-generic over $\mathbf{V}$, then G_p is a P-point in $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^\omega\big)$ in the model $\mathbf{V}[G_p]$.

PROOF. First notice that the forcing notion $\mathbb{W}$ is σ-closed (*cf.* [**46**, Proposition 4.2]) and hence, $\mathbb{W}$ does not add new reals. For every countable set of neighbourhoods $\{N_i : i \in \omega\}$ of the filter G_p we find a countable set of partitions $\{X_i : i \in \omega\} \subseteq G_p$ such that $(X_i)^+ \subseteq N_i$ and $X_i \sqsubseteq^* X_j$ for $i \geq j$. Now, since every partition $X \in (\omega)^\omega$ can be encoded by a real number and $\mathbb{W}$ does not add new reals, there is a $\mathbb{W}$-condition Y which forces that the sequence $X_0 \,{}^*\!\sqsupseteq X_1 \,{}^*\!\sqsupseteq \dots$ belongs to $\mathbf{V}$, and since $\mathbb{W}$ is σ-closed we find an infinite partition $Z \sqsubseteq Y$ such that $Z \sqsubseteq^* X_i$ for every $i \in \omega$. Hence, Z forces that $(Z)^+$ belongs to $\bigcap_{i\in\omega} N_i$ and that Z belongs to G_p. ⊣

PROPOSITION III.4.5. Assume CH, then there is a P-point in $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^\omega\big)$.

PROOF. Assume $\mathbf{V} \models \mathsf{CH}$. Let χ be large enough such that $\mathcal{P}((\omega)^\omega) \in H(\chi)$, *i.e.*, the power-set of $(\omega)^\omega$ (in $\mathbf{V}$) is hereditarily of size $< \chi$. Let $\mathbf{N}$ be an elementary submodel of $\langle H(\chi), \in\rangle$ containing all the reals of $\mathbf{V}$ such that $|\mathbf{N}| = 2^{\aleph_0}$. We consider the forcing notion $\mathbb{W}$ in the model $\mathbf{N}$. Since $|\mathbf{N}| = 2^{\aleph_0}$, in $\mathbf{V}$ there is an enumeration $\{D_\alpha \subseteq (\omega)^\omega : \alpha < 2^{\aleph_0}\}$ of all dense sets of $\mathbb{W}$ which lie in $\mathbf{N}$. Since $\mathbb{W}$ is σ-closed and because $\mathbf{V} \models \mathsf{CH}$, $\mathbb{W}$ is $2^{\aleph_0}$-closed in $\mathbf{V}$ and therefore we can construct a descending sequence $\{X_\alpha : \alpha < 2^{\aleph_0}\}$ in $\mathbf{V}$ such that $X_\alpha \in D_\alpha$ for each $\alpha < 2^{\aleph_0}$. Let $G_p := \{X \in (\omega)^\omega : X_\alpha \sqsubseteq X \text{ for some } X_\alpha\}$, then G_p is $\mathbb{W}$-generic over $\mathbf{N}$. By Lemma III.4.4 we have $\mathbf{N}[G_p] \models$ "there is a P-point in $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^\omega\big)$" and because $\mathbf{N}$ contains all reals of $\mathbf{V}$ and every countable descending sequence of basic open sets $(Y_i)^+$ can be encoded by a real number, the P-point G_p in the model $\mathbf{N}[G_p]$ is also a P-point in $\mathtt{PUF}^+_{\sqsubseteq}\big((\omega)^\omega\big)$ in the model $\mathbf{V}$, which completes the proof. ⊣

CHAPTER IV

The Partition Form of Ramsey's Theorem

In this chapter we present a generalized version of Carlson's Lemma which can be seen as the partition form of Ramsey's Theorem.

1. Historical background

The earliest results in Ramsey Theory are the theorems of Bartel L. van der Waerden and Frank P. Ramsey. We begin by discussing van der Waerden's Theorem:

VAN DER WAERDEN'S THEOREM. For all $n, r \in \omega$ there exists an $N \in \omega$ such that for every coloring of $\{0, \ldots, N\}$ with $r + 1$ colors, there exists a monochromatic arithmetic progression of length $n + 1$.

This result was first proved by van der Waerden in [**67**] (for a short but not easy proof see [**20**] and for a description of how van der Waerden found his proof we refer the reader to [**68**]). Almost 40 years after van der Waerden's proof, Alfred W. Hales and Robert I. Jewett found a proof for a proper combinatorial statement which implies van der Waerden's Theorem. To state the Hales-Jewett Theorem, we first have to give the definition of a combinatorial line.

For $n, N \in \omega$ where $N > 0$, a set $\{x_0, \ldots, x_n\} \subseteq \{0, \ldots, n\}^N$ is called a **combinatorial line** iff for each $m < N$ we have $x_i(m) = i$ (for all $i \leq n$) or $x_i(m) = x_{i+1}(m)$ (for all $i < n$); and the former case is true for at least one $m < N$. Now we can formulate the

HALES-JEWETT THEOREM. For all $n, r \in \omega$, there exists a positive natural number N such that for every coloring of $\{0, \ldots, n\}^N$ with $r + 1$ colors, there exists a monochromatic combinatorial line.

This result was first proved by Hales and Jewett in [**31**] (a very sophisticated proof providing a primitive recursive bound for the Hales-Jewett function is given by Shelah in [**59**]).

On December 16th, 1929, Ramsey's article "On a problem of formal logic" was issued (*cf.* [**57**]). This article begins with two combinatorial theorems which are Theorem A and Theorem B. Because the second one follows from the first one, we consider Theorem A as the "Ramsey Theorem", also called "Ramsey's Theorem", and call Theorem B the "Finitary Ramsey Theorem", because it is the finite version of Theorem A.

In order to state these two theorems of Ramsey, we have to give again some notations.

For $n \in \omega$, we denote the set of all n-element sets of natural numbers by $[\omega]^n$. Further, for any set of natural numbers H and $n \in \omega$, $[H]^n$ denotes the set of all n-element subsets of H. Ramsey's Theorem states as follows.

RAMSEY'S THEOREM. For every $n \in \omega$ and for every coloring of $[\omega]^n$ with finitely many colors, there exists an infinite set $H \subseteq \omega$ such that $[H]^n$ is monochromatic.

The finite version of Ramsey's Theorem, which is Theorem B of [**57**], is the following:

FINITARY RAMSEY THEOREM. For all $m, n, r \in \omega$, where $n \leq m$, there exists an $N \in \omega$ such that for every coloring of $[N]^n$ with $r+1$ colors, there exists a set $H \in [N]^m$ such that $[H]^n$ is monochromatic.

The Finitary Ramsey Theorem was discovered and proved independently by Paul Erdös and George Szekerés (see [**16**]). They arrived at it in the following geometrical context.

ERDÖS-SZEKERÉS THEOREM. For every $n \in \omega$, there exists an $N \in \omega$ with the following property: If P is a set of N points of the Euclidean plane without 3 colinear points, then P contains n points which form the vertices of a convex n-gon.

The Hales-Jewett Theorem and the Finitary Ramsey Theorem are commonly considered as the two main roots of Ramsey Theory. Both results are coloring theorems of the same type, so it is surprising that they remained quite unrelated for a long time until Ronald L. Graham and Bruce L. Rothschild extended in [**19**] the Hales-Jewett Theorem in a remarkable way. Using the notion of n-parameter sets, they proved a result ([**19**, Corollary 10]) from which one can derive both the Hales-Jewett Theorem and the Finitary Ramsey Theorem (see also [**55**]). For any set X and $n \in \omega$, let $(X)^n$ denote the set of all partitions of X containing exactly n pieces.

GRAHAM-ROTHSCHILD RESULT. For any $m, n, r \in \omega$, where $m \geq n$, there exists a natural number N such that for every coloring of $(N)^n$ with $r+1$ colors, there exists a partition $P \in (N)^n$ such that $(P)^n$ is monochromatic.

This result looks very similar to the Finitary Ramsey Theorem. The relation becomes clearer if we consider an n-element subset of N as an injective function from n into N, and similarly, a partition of N containing n pieces as a surjective function from N onto n, where we identify in both cases two functions if they are equal modulo a permutation of n. Therefore, partitions with n pieces are a dual form of sets with n elements (see also Chapter I).

For more background and further results in Ramsey Theory we refer the reader to [**50**] and [**21**].

Ten years after Graham and Rothschild proved their combinatorial result, Steve G. Simpson tried to prove a dual version of the Ramsey Theorem and succeeded with the help of Timothy J. Carlson. The original motivation of Simpson to prove such a dualization of Ramsey's Theorem was to find a combinatorial statement which is like the theorem of Leo Harrington and Jeff Paris (*cf.* [**52**]), but stronger in the sense that it cannot be proved in reasonably strong subsystems of second-order arithmetic. The crucial point in the proof of the so-called Dual Ramsey Theorem, which is Theorem 1.2 of [**11**], is the Lemma 2.4 of [**11**], which was proved in a slightly stronger form by Carlson (*cf.* [**11**, Theorem 6.3]). In the following we give a slightly more general version of Carlson's Lemma which can be seen as the partition form of Ramsey's Theorem.

2. The partition form of Ramsey's Theorem

Remember that for $s \in (\mathbb{N})$, s^* denotes the partition $s \cup \big\{\{\mathrm{dom}(s)\}\big\}$, and notice that $|s^*| = |s| + 1$.

For $s \in (\mathbb{N})$ and $X \in (\omega)^\omega$ with $s \sqsubseteq X$, let

$$(s, X)^\omega := \{Y \in (\omega)^\omega : s \prec Y \sqsubseteq X\}\,.$$

A set $(s, X)^\omega$, where s and X are as above, is called a **dual Ellentuck neighborhood** (*cf.* [**11**, p. 275] or Chapter V).

For a natural number n, let $(\omega)^{n*}$ denote the set of all $u \in (\mathbb{N})$ such that $|u| = n$. Further, for $n \in \omega$ and $X \in (\omega)^\omega$ let

$$(X)^{n*} := \{u \in (\mathbb{N}) : |u| = n \wedge u^* \sqsubseteq X\}\,;$$

and if $s \in (\mathbb{N})$ is such that $|s| \leq n$ and $s \sqsubseteq X$, let

$$(s, X)^{n*} := \{u \in (\mathbb{N}) : |u| = n \wedge s \prec u \wedge u^* \sqsubseteq X\}\,.$$

With the notation given above, we can state our main result as follows:

Theorem IV.2.1. For any coloring of $(\omega)^{n*}$ with $r + 1$ colors, where $r, n \in \omega$ and $n > 0$, there exists an infinite partition $X \in (\omega)^\omega$ such that $(X)^{n*}$ is monochromatic.

To prove the Theorem IV.2.1, we will make use of Carlson's Lemma (see [**11**, Lemma 2.4]). In our notation it reads as follows.

Carlson's Lemma. For any coloring of $(\omega)^{n*}$ with $r + 1$ colors, where $r, n \in \omega$ and $n > 0$, and for any dual Ellentuck neighborhood $(s, X)^\omega$, where $|s| \leq n$, there exists a $Y \in (s, X)^\omega$ such that $(s, Y)^{n*}$ is monochromatic.

With this result we are prepared to give the

Proof of Theorem IV.2.1. The proof is by induction on n. For $n = 1$, Theorem IV.2.1 follows immediately from Carlson's Lemma. So, let $n, r \in \omega$ be given such that $1 < n$ and assume that Theorem IV.2.1 is already proved for all $n' \in \omega$ with $n' < n$.

Fix an arbitrary coloring $\pi : (\omega)^{n*} \to r + 1$. Let $X_0 \in (\omega)^\omega$ and let $s_0 \in (\mathbb{N})$ be such that $|s_0| = n - 1$ and $s_0^* \prec X_0$. Further assume we already have constructed $X_i \in (\omega)^\omega$ and $s_i \in (\mathbb{N})$ such that $|s_i| = (n - 1) + i$ and $s_i^* \prec X_i$.

We construct partitions s_{i+1} and X_{i+1} with the same properties as above. As a byproduct, the construction yields a partial mapping χ from $(\omega)^{(n-1)*}$ to $r + 1$.

Let $\{t^i_k \in (\mathbb{N}) : k \leq h_i\}$ be an enumeration of all $t \sqsubseteq s_i$ with $\operatorname{dom}(t) = \operatorname{dom}(s_i)$ and $|t| = n-1$. Let $Y^i_{-1} := X_i$, then by Carlson's Lemma, for each k there exists a $Y^i_k \in (s^*_i, Y^i_{k-1})^\omega$ such that $\pi|_{((t^i_k)^*, Y^i_k)^{n*}}$ is constant, say

$$\pi|_{((t^i_k)^*, Y^i_k)^{n*}} =: \chi(t^i_k)\,.$$

Let $X_{i+1} := Y^i_{h_i}$ and let $s_{i+1} \in (\mathbb{N})$ be such that $s^*_{i+1} \prec X_{i+1}$ and $|s_{i+1}| = (n-1)+(i+1)$. Finally, let $Y \in (\omega)^\omega$ be the unique partition such that for all $i \in \omega$ we have $s_i \prec Y$. For each $u \in (Y)^{n*}$ there exist exactly two numbers $i, k \in \omega$ such that $(t^i_k)^* \prec u$, and we can define

$$\chi(u) := \chi(t^i_k)\,.$$

Notice that $\chi(u)$ is well defined for every $u \in (Y)^{n*}$. By the induction hypothesis we find a $Z \in (Y)^\omega$ such that $\chi|_{(Z)^{(n-1)*}}$ is constant, say $\chi|_{(Z)^{(n-1)*}} = \{j\}$. Let $s^* \sqsubseteq Z$ be such that $|s| = n$ and let $s^*_0 \prec s$ be such that $|s_0| = n-1$. The domain of s_0, $\operatorname{dom}(s_0)$, corresponds with $\operatorname{dom}(s_i)$ for some $i \in \omega$. Consider now the partition X_{i+1}. By the construction of X_{i+1} we know that for all $t \sqsubseteq s_i$ with $|t| = n-1$ and $\operatorname{dom}(t) = \operatorname{dom}(s_i)$ we have $\pi|_{(t^*, X_{i+1})^{n*}}$ is constant and by the construction of Z, this constant value is j, thus $\pi|_{(t^*, X_{i+1})^{n*}} = \{j\}$ and in particular $\pi(s^*_0) = j$. Hence, because $(s, Z)^\omega \subseteq (s^*_0, X_{i+1})^\omega$, we get $\pi(s) = j$, which completes the proof. ⊣

3. A weakened form of the Halpern-Läuchli Theorem

One can show that the Finitary Ramsey Theorem, the Ramsey Theorem as well as the Hales-Jewett Theorem, the Graham-Rothschild Result and a weakened form of the Halpern-Läuchli Theorem are derivable from Theorem IV.2.1. We just give the proof for the weakened Halpern-Läuchli Theorem (the full form, proved by James D. Halpern and Hans Läuchli, can be found in [**32**]).

To state this weakened form, we have to give first some notations: For $k \in \omega$, let ${}^k2$ be the set of all functions $\mu : k \to 2$ and let $2^{<\omega} := \bigcup_{k\in\omega} {}^k2$. A set $T \subseteq 2^{<\omega}$ is called a **tree** if for every $\mu \in T$ and $k \leq \operatorname{dom}(\mu)$ we have $\mu|_k \in T$. So, the set $2^{<\omega}$ itself forms a tree. For a tree $T \subseteq 2^{<\omega}$ and $l \in \omega$ let

$$T(l) := \{\mu \in T : \operatorname{dom}(\mu) = l\}\,.$$

If $T^d = T_0 \times \ldots \times T_{d-1} \subseteq \left(2^{<\omega}\right)^d$ where $d \in \omega$ is a product of trees $T_i \subseteq 2^{<\omega}$, then for $l \in \omega$ let

$$T^d(l) := \{\mu \in T^d : \mu \in T_0(l) \times \ldots \times T_{d-1}(l)\}\,.$$

A tree $T \subseteq 2^{<\omega}$ is called **perfect** if for each $\mu \in T$ there exist two distinct functions $\nu_0, \nu_1 \in T$ such that $\operatorname{dom}(\nu_0) = \operatorname{dom}(\nu_1)$ and $\nu_0|_{\operatorname{dom}(\mu)} = \nu_1|_{\operatorname{dom}(\mu)} = \mu$.

Corollary IV.3.1. For every positive $d \in \omega$, and for every coloring of $\bigcup_{l\in\omega} \left({}^l2\right)^d$ with finitely many colors, there exists a product of perfect trees $T^d = T_0 \times \ldots \times T_{d-1}$ and an infinite set $H \subseteq \omega$ such that $\bigcup_{l\in H} T^d(l)$ is monochromatic.

PROOF. Let d be a fixed positive natural number and let $n := 2^d$. Because $|{}^d2| = 2^d$, there exists a one-to-one correspondence ζ between n and ${}^d2$. For any $l \in \omega$, an element $\langle \mu_0, \dots, \mu_{d-1} \rangle \in \left({}^l2\right)^d$ is a sequence of length d of functions $\mu_i : l \to 2$. For any $l \in \omega$ we define the function $\xi : \left({}^l2\right)^d \to \left({}^d2\right)^l$ as follows:

$$\xi(\langle \mu_0, \dots, \mu_{d-1} \rangle) := \langle \nu_0, \dots, \nu_{l-1} \rangle \text{ where } \nu_j(i) := \mu_i(j) \,.$$

Notice that for each $l \in \omega$, the function ξ is a one-to-one function from $\left({}^l2\right)^d$ onto $\left({}^d2\right)^l$. Now we define the function $\eta : (\omega)^{n*} \to \left(2^{<\omega}\right)^d$ by

$$\eta(s) := \xi^{-1}(\langle \nu_0, \dots, \nu_{\mathrm{dom}(s)-1} \rangle) \,,$$

where $\nu_j(i) := \zeta(k)(i)$ for $j \in s(k)$. Note that $\eta(s) \in \left({}^{\mathrm{dom}(s)}2\right)^d$. Finally, for any coloring $\pi : \bigcup_{l \in \omega} \left({}^l2\right)^d \to r+1$, where $r \in \omega$, we define the coloring $\tau : (\omega)^{n*} \to r+1$ by stipulating $\tau(s) := \pi(\eta(s))$. Let $X \in (\omega)^\omega$ be as in the conclusion of Theorem IV.2.1 (w.r.t. the coloring τ). Let $s_n^* \prec X$ be such that $|s_n| = n$ and let $H := \mathrm{Min}(X) \setminus \mathrm{Min}(s_n)$. Further let

$$S_n := \{t \in (\mathbb{N}) : t \prec s_n \,\vee\, (s_n \prec t \sqsubseteq X \wedge |t| = n)\}$$

and define

$$T^d := \{\mu : \exists t \in S_n(\mu = \eta(t))\} \,.$$

We leave it to the reader to check that T^d and H are as desired and that they have the desired properties. ⊣

4. The "dual form" of Ramsey's Theorem versus its "partition form"

Let us compare Theorem IV.2.1 with the so-called Dual Ramsey Theorem of Carlson and Simpson (*cf.* [**11**, Theorem 1.2]). The following notations are used to state their Dual Ramsey Theorem.

For $n \in \omega$ let $(\omega)^n$ denote the set of all partitions of ω containing exactly n blocks and for $X \in (\omega)^\omega$ let

$$(X)^n := \{Y \in (\omega)^n : Y \sqsubseteq X\} \,.$$

For $s \in (\mathbb{N})$ let $\mathcal{O}_s := \{X \in (\omega)^n : s \prec X\} \subseteq (\omega)^n$. For a finite set $S \subseteq (\mathbb{N})$ define $\mathcal{B}_S := \bigcap_{s \in S} \mathcal{O}_s$, then the set of all $\mathcal{B}_S$, where $S \subseteq (\mathbb{N})$ is finite, forms a basis of a topology on $(\omega)^n$. Now we can formulate the

DUAL RAMSEY THEOREM. If $\pi : (\omega)^n \to r+1$, where $n, r \in \omega$, is such that for each $i \leq r$, $\pi^{-1}(i)$ is a Borel set (with respect to the product topology), then there exists an $X \in (\omega)^\omega$ such that $\pi|_{(X)^n}$ is constant.

A restriction on the coloring is necessary because one can show – using AC – that there exists a coloring of $(\omega)^2$ with 2 colors such that for no infinite partition X, $(X)^2$ is monochromatic.

As mentioned above, the Graham-Rothschild result is – in terms of partitions – the analogue of the Finitary Ramsey Theorem, but stronger in the sense that it also implies the Hales-Jewett Theorem (which deals in some sense also with partitions). Further, the Graham-Rothschild result, the Finitary Ramsey Theorem, the Hales-Jewett Theorem, are completely finite results. On the "infinite" side we have the Dual Ramsey Theorem, which is in some sense the analogue – in terms of partitions – of the Galvin-Prikry result. Putting all together, we get the following diagram, where the "partition-results" are on the left, results dealing with "sets of singletons" are on the right, and where an arrow means an implication:

$$\begin{tikzcd}
\text{Dual Ramsey Theorem} \arrow[d] & \\
??? \arrow[r] \arrow[d] & \text{Ramsey Theorem} \arrow[d] \\
\text{Graham-Rothschild} \arrow[r] \arrow[d] & \text{Finitary Ramsey Theorem} \\
\text{Hales-Jewett} &
\end{tikzcd}$$

What is missing in this diagram is a "partition version" of the Ramsey Theorem, or equivalently, an infinite version of the Graham-Rothschild result. Now, Theorem IV.2.1 fills this gap, and even though it is just a consequence of the Dual Ramsey Theorem, one can define in a natural way its associated filters, which will play an important rôle in Chapter VII (see also [**24**]). (Notice that such a construction does not exist with respect to the Dual Ramsey Theorem.) These partition-filters can be seen as a strengthened version of the well-studied Ramsey filters over ω, and they are important in the investigation of the combinatorics of Dual Mathias forcing, which is the "partition version" of Mathias forcing (*cf.* Chapter VII).

As mentioned above, the Dual Ramsey Theorem does not hold for arbitrary colorings. This is similar to the case when the infinite subsets of ω are colored: One can show – using AC – that there is a coloring of $[\omega]^\omega$ with 2 colors, such that for no $S \in [\omega]^\omega$, $[S]^\omega$ is monochromatic. This yields to the following property of colorings of $[\omega]^\omega$.

Ramsey Property: A finite coloring of $[\omega]^\omega$ has the Ramsey property, if there is a set $S \in [\omega]^\omega$ such that $[S]^\omega$ is monochromatic.

Fred Galvin and Karel Prikry proved in [**17**] that every Borel-coloring has the Ramsey property. Moreover, Jack Silver has shown in [**61**] that this holds also for every analytic coloring.

There is a natural analogue of the Galvin-Prikry result in terms of partitions, namely the so-called Dual Galvin-Prikry Theorem (see [**11**, Theorem 1.3]), and similar to the Galvin-Prikry result, the Dual Galvin-Prikry Theorem yields to the dual

Ramsey property (introduced in [**11**]). Further, in Chapter VI (see also [**23**]) we will see that also every analytic coloring has the dual Ramsey property.
With these results, we get the following diagram:

$$
\begin{tikzcd}
\text{Dual Galvin-Prikry} \arrow[r] \arrow[d] & \text{Galvin-Prikry} \arrow[d] \\
\text{Theorem IV.2.1} \arrow[r] \arrow[d] & \text{Ramsey Theorem} \arrow[d] \\
\text{Graham-Rothschild} \arrow[r] & \text{Finitary Ramsey Theorem}
\end{tikzcd}
$$

Considering these symmetries between the Ramsey Theorem and Theorem IV.2.1, it is reasonable to consider Theorem IV.2.1 as the partition form of Ramsey's Theorem. Another Ramsey type theorem which is slightly stronger than Theorem IV.2.1 can be found in [**28**].

CHAPTER V

The Shattering Cardinal and the Dual Ramsey Property

In Chapter III Section 4.1 we defined the dual-shattering cardinal $\mathfrak{H}$ as the minimum height of a tree π-base of $\mathtt{PUF}^{+}_{\sqsubseteq}\big((\omega)^{\omega}\big)$. In this chapter we will give some equivalent definitions of $\mathfrak{H}$ and show that $\mathfrak{H} > \omega_1$ is consistent with ZFC.

1. The dual Ellentuck topology and the dual Ramsey property

First we define a topology on the set of infinite partitions: Let $X \in (\omega)^{\omega}$ and $s \in (\mathbb{N})$ such that $s \sqsubseteq X$, then

$$(s, X)^{\omega} := \{Y \in (\omega)^{\omega} : s \prec Y \wedge Y \sqsubseteq X\}$$

and

$$(X)^{\omega} := (\emptyset, X)^{\omega}\,.$$

Now, let the basic open sets on $(\omega)^{\omega}$ be the sets $(s, X)^{\omega}$ (where X and s as above). These sets are called the **dual Ellentuck neighborhoods**. The topology on $(\omega)^{\omega}$ induced by the dual Ellentuck neighborhoods is called the **dual Ellentuck topology** (*cf.* [**12**]).

Let $\mathscr{C} \subseteq (\omega)^{\omega}$ be a set of partitions, then we say that $\mathscr{C}$ has the **dual Ramsey property**, or that $\mathscr{C}$ is **dual Ramsey**, if there is a partition $X \in (\omega)^{\omega}$ such that $(X)^{\omega} \subseteq \mathscr{C}$ or $(X)^{\omega} \cap \mathscr{C} = \emptyset$. If for each dual Ellentuck neighborhood $(s, Y)^{\omega}$ there is an $X \in (s, Y)^{\omega}$ such that $(s, X)^{\omega} \subseteq \mathscr{C}$ or $(s, X)^{\omega} \cap \mathscr{C} = \emptyset$, we call $\mathscr{C}$ **completely dual Ramsey**. If for each dual Ellentuck neighborhood the latter case holds, we say that $\mathscr{C}$ is **completely dual Ramsey null**.

Remark 1. In [**11**] it is proved that a set is completely dual Ramsey if and only if it has the Baire property with respect to the dual Ellentuck topology, and that it is completely dual Ramsey null if and only if it is meager with respect to the dual Ellentuck topology. From this it follows that a set is completely dual Ramsey null if and only if the complement contains a dense and open subset (with respect to the dual Ellentuck topology).

Let $\boldsymbol{R}_0^{\flat}$ be the set of sets of partitions which are completely dual Ramsey null. The set $\boldsymbol{R}_0^{\flat} \subseteq \mathcal{P}\big((\omega)^{\omega}\big)$ is an ideal which is not prime. Let us consider the **additivity number** $add(\boldsymbol{R}_0^{\flat})$ and the **covering number** $cov(\boldsymbol{R}_0^{\flat})$ of the ideal $\boldsymbol{R}_0^{\flat}$: $add(\boldsymbol{R}_0^{\flat})$ is the smallest cardinal κ such that there exists a family $\mathscr{J} = \{\mathscr{J}_{\alpha} \in \boldsymbol{R}_0^{\flat} : \alpha < \kappa\}$ with $\bigcup \mathscr{J} \notin \boldsymbol{R}_0^{\flat}$; and $cov(\boldsymbol{R}_0^{\flat})$ is the smallest cardinal κ such that there exists a family $\mathscr{J} = \{\mathscr{J}_{\alpha} \in \boldsymbol{R}_0^{\flat} : \alpha < \kappa\}$ with $\bigcup \mathscr{J} = (\omega)^{\omega}$.

Because $(\omega)^\omega \notin \boldsymbol{R}_0^\flat$, it is clear that $add(\boldsymbol{R}_0^\flat) \leq cov(\boldsymbol{R}_0^\flat)$. Further, it is easy to see that $\omega_1 \leq add(\boldsymbol{R}_0^\flat)$. Later we will see that $add(\boldsymbol{R}_0^\flat) = cov(\boldsymbol{R}_0^\flat)$.

2. The distributivity number $dsb(\mathbb{W})$

A complete Boolean algebra $\langle B, \leq \rangle$ is called κ-distributive, where κ is a cardinal, if and only if for every family $\langle u_{\alpha i} : i \in I_\alpha, \alpha < \kappa \rangle$ of members of B the following holds:

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha i} = \sum_{f \in \prod\limits_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha f(\alpha)}.$$

It is well known (*cf.* [**36**]) that for a forcing notion $\langle P, \leq \rangle$ the following statements are equivalent:

- r.o.(P) is κ-distributive (where "r.o." stands for "regular open").
- The intersection of κ open dense sets in P is dense.
- Every family of κ maximal anti-chains of P has a common refinement.
- Forcing with P does not add a new subset of κ.

Let the forcing notion $\mathbb{W} = \langle (\omega)^\omega, \sqsubseteq^* \rangle$ be defined as at the end of Chapter III, and let the **distributivity number** $dsb(\mathbb{W})$ be the least cardinal κ for which the Boolean algebra r.o.$(\mathbb{W})$ is not κ-distributive.

3. The four cardinals are equal

Now we will show that the four cardinals $\mathfrak{H}$, $add(\boldsymbol{R}_0^\flat)$, $cov(\boldsymbol{R}_0^\flat)$ and $dsb(\mathbb{W})$ are all equal. This is a similar result as in the case when we consider infinite subsets of ω instead of infinite partitions (*cf.* [**54**] and [**1**]).

FACT V.3.1. If $\mathscr{T} \subseteq (\omega)^\omega$ is an open and dense set with respect to the dual Ellentuck topology, then it contains a *mao* family.

PROOF. First choose an almost orthogonal family $\mathscr{A} \subseteq \mathscr{T}$ which is maximal in $\mathscr{T}$. Now for an arbitrary $X \in (\omega)^\omega$, $\mathscr{T} \cap (X)^\omega \neq \emptyset$. So, X must be compatible with some $A \in \mathscr{A}$ and therefore $\mathscr{A}$ is *mao*. $\dashv$

LEMMA V.3.2. $\mathfrak{H} \leq add(\boldsymbol{R}_0^\flat)$.

PROOF. Let $\langle \mathscr{S}_\alpha : \alpha < \lambda < \mathfrak{H} \rangle$ be a sequence of completely dual Ramsey null sets and let $\mathscr{T}_\alpha \subseteq (\omega)^\omega \setminus \mathscr{S}_\alpha$ $(\alpha < \lambda)$ be such that $\mathscr{T}_\alpha$ is open and dense with respect to the dual Ellentuck topology (which is always possible by Remark 1). For each $\alpha < \lambda$ let

$$T_\alpha^* := \big\{X \in (\omega)^\omega : \exists Y \in T_\alpha \big(X \sqsubseteq^* Y \wedge \neg(X \stackrel{*}{=} Y)\big)\big\}.$$

It is easy to see that for each $\alpha < \lambda$ the set T_α^* is open and dense with respect to the dual Ellentuck topology.

Let $U_\alpha \subseteq T_\alpha^*$ $(\alpha < \lambda)$ be *mao*. Because $\lambda < \mathfrak{H}$, the set $\langle U_\alpha : \alpha < \lambda \rangle$ cannot be shattering. Let for $\alpha < \lambda$ $U_\alpha^* := \{X \in (\omega)^\omega : \exists Z_\alpha \in U_\alpha (X \sqsubseteq^* Z_\alpha)\}$, then $U_\alpha^* \subseteq T_\alpha$ and $\bigcap_{\alpha<\lambda} U_\alpha^*$ is open and dense with respect to the dual Ellentuck topology:

$\bigcap_{\alpha<\lambda} U^*_\alpha$ is open: clear.
$\bigcap_{\alpha<\lambda} U^*_\alpha$ is dense: Let $(s,Z)^\omega$ be arbitrary. Because $\langle U_\alpha : \alpha < \lambda\rangle$ is not shattering, there is a $Y \in (s,Z)^\omega$ such that $\forall\alpha < \lambda\,\exists X_\alpha \in U_\alpha(Y \sqsubseteq^* X_\alpha)$. Hence, $Y \in \bigcap_{\alpha<\lambda} U^*_\alpha$.

Further we have by construction

$$\bigcap_{\alpha<\lambda} U^*_\alpha \cap \bigcup_{\alpha<\lambda} S_\alpha = \emptyset,$$

which completes the proof. ⊣

Lemma V.3.3. $\mathfrak{H} \leq dsb(\mathbb{W})$.

Proof. Let $\langle T_\alpha : \alpha < \lambda < \mathfrak{H}\rangle$ be a sequence of open and dense sets with respect to the dual Ellentuck topology. Now the set $\bigcap_{\alpha<\lambda} U^*_\alpha$, constructed as in Lemma V.3.2, is dense (and even open) and a subset of $\bigcap_{\alpha<\lambda} T_\alpha$. Therefore $\mathfrak{H} \leq dsb(\mathbb{W})$. ⊣

Lemma V.3.4. $add(\boldsymbol{R}^\flat_0) \leq \mathfrak{H}$.

Proof. Let $\langle \mathscr{R}_\alpha : \alpha < \mathfrak{H}\rangle$ be a shattering family and for $\alpha < \mathfrak{H}$ let

$$\mathscr{D}_\alpha := \{X : \exists Y \in \mathscr{R}_\alpha(X \sqsubseteq^* Y)\}\,.$$

For each $\alpha < \mathfrak{H}$, $\mathscr{D}_\alpha$ is dense and open with respect to the dual Ellentuck topology:
$\mathscr{D}_\alpha$ is open: clear.
$\mathscr{D}_\alpha$ is dense: Let $(s,Z)^\omega$ be arbitrary and $X \in (s,Z)^\omega$. Because $\mathscr{R}_\alpha$ is *mao*, there is a $Y \in \mathscr{R}_\alpha$ such that $X' := X \sqcap Y \in (\omega)^\omega$. Let $X'' \stackrel{*}{=} X'$ such that $X'' \in (s,Z)^\omega$, then $X'' \sqsubseteq^* Y$.

Now we show that $\bigcap_{\alpha<\mathfrak{H}} P_\alpha = \emptyset$ and therefore $\bigcup_{\alpha<\mathfrak{H}} \big((\omega)^\omega \setminus \mathscr{D}_\alpha\big) = (\omega)^\omega$. Assume there is an $X \in \bigcap_{\alpha<\mathfrak{H}} \mathscr{D}_\alpha$, then $\forall\alpha < \mathfrak{H}\,\exists Y_\alpha \in \mathscr{R}_\alpha(X \sqsubseteq^* Y_\alpha)$. But this contradicts that $\langle \mathscr{R}_\alpha : \alpha < \mathfrak{H}\rangle$ is shattering. ⊣

Lemma V.3.5. $dsb(\mathbb{W}) \leq \mathfrak{H}$.

Proof. In the proof of Lemma V.3.4 we constructed a sequence $\langle \mathscr{D}_\alpha : \alpha < \mathfrak{H}\rangle$ of open and dense sets with an empty intersection. Therefore $\bigcap_{\alpha<\mathfrak{H}} \mathscr{D}_\alpha$ is not dense. ⊣

Corollary V.3.6. $cov(\boldsymbol{R}^\flat_0) \leq \mathfrak{H}$.

Proof. In the proof of Lemma V.3.4, we proved in fact that $cov(\boldsymbol{R}^\flat_0) \leq \mathfrak{H}$. ⊣

Corollary V.3.7. $add(\boldsymbol{R}^\flat_0) = cov(\boldsymbol{R}^\flat_0) = dsb(\mathbb{W}) = \mathfrak{H}$.

Proof. It is clear that $add(\boldsymbol{R}^\flat_0) \leq cov(\boldsymbol{R}^\flat_0)$. By the Lemmas V.3.3 and V.3.5 we know that $\mathfrak{H} = dsb(\mathbb{W})$. Further by the Lemma V.3.2 and the Corollary V.3.6 it follows that $\mathfrak{H} \leq add(\boldsymbol{R}^\flat_0) \leq cov(\boldsymbol{R}^\flat_0) \leq \mathfrak{H}$. Hence we have $add(\boldsymbol{R}^\flat_0) = cov(\boldsymbol{R}^\flat_0) = dsb(\mathbb{W}) = \mathfrak{H}$. ⊣

COROLLARY V.3.8. The union of less than $\mathfrak{H}$ completely dual Ramsey sets is dual Ramsey, but the union of $\mathfrak{H}$ completely dual Ramsey sets can be a set which does not have the dual Ramsey property.

PROOF. Follows from Remark 1 and Corollary V.3.7. $\dashv$

4. On the consistency of $\mathfrak{H} > \omega_1$

First, let us give some facts concerning the dual Mathias forcing: The conditions of **dual Mathias forcing** $\mathbb{M}^\flat$ are pairs $\langle s, X\rangle$ such that $s \in (\mathbb{N})$, $X \in (\omega)^\omega$ and $s \sqsubseteq X$, stipulating

$$\langle s, X\rangle \leq \langle t, Y\rangle \text{ if and only if } (s, X)^\omega \subseteq (t, Y)^\omega$$

(see also Chapter VII.5).

It will be shown in Chapter VI.2 that dual Mathias forcing can be decomposed as $\mathbb{W} * \mathbb{M}^\flat_{\mathscr{U}}$, where $\mathbb{W} = \big\langle (\omega)^\omega, \sqsubseteq^* \big\rangle$ and $\mathbb{M}^\flat_{\mathscr{U}}$ denotes restricted dual Mathias forcing, *i.e.*, conditions must have their second coordinate in $\mathscr{U}$, where $\mathscr{U}$ is a $\mathbb{W}$-generic partition-ultrafilter (see again Chapter VII.5).

Because dual Mathias forcing has pure decision (see Chapter VI.2), it is proper and has the Laver property and therefore adds no Cohen reals. (For the definition of properness and the Laver property we refer the reader to [**18**].)

After an ω_2-iteration of dual Mathias forcing with countable support, starting from a model in which the continuum hypothesis holds, we get a model in which the dual-shattering cardinal $\mathfrak{H}$ is equal to ω_2.

Let V be a model of CH and let $\mathbb{P}_{\omega_2} := \langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ be a countable support iteration of dual Mathias forcing, *i.e.*, for all $\alpha < \omega_2$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{Q}_\alpha \approx \mathbb{M}^\flat$".

In the sequel we will not distinguish between a member of $\mathbb{W}$ and its representative. In the proof of the following theorem, a set $C \subseteq \omega_2$ is called an ω_1-club if C is unbounded in ω_2 and closed under increasing sequences of length ω_1.

THEOREM V.4.1. If G is $\mathbb{P}_{\omega_2}$-generic over V, where $V \models \mathsf{CH}$, then $V[G] \models \mathfrak{H} = \omega_2$.

PROOF. In $V[G]$ let $\langle D_\nu : \nu < \omega_1\rangle$ be a family of open dense subsets of $\mathbb{W}$. Because dual Mathias forcing is proper and by a standard Löwenheim-Skolem argument, we find a ω_1-club $C \subseteq \omega_2$ such that for each $\alpha \in C$ and every $\nu < \omega_1$ the set $D_\nu \cap V[G_\alpha]$ belongs to $V[G_\alpha]$ and is open dense in $\mathbb{W}^{V[G_\alpha]}$. Let $A \in \mathbb{W}^{V[G]}$ be arbitrary. By properness and genericity and because $\mathbb{P}_{\omega_2}$ has countable support, we may assume that $A \in G(\alpha)'$ for an $\alpha \in C$, where $G(\alpha)'$ is the first component according to the decomposition of Mathias forcing of the $\dot{Q}_\alpha[G_\alpha]$-generic object determined by G. As $\alpha \in C$, $G(\alpha)'$ clearly meets every D_ν ($\nu < \omega_1$). But now X_α, the $\dot{Q}_\alpha$-generic partition (determined by $G(\alpha)''$) is below each member of $G(\alpha)'$, hence below A and in $\bigcap_{\nu<\omega_1} D_\nu$. Because A was arbitrary, this proves that $\bigcap_{\nu<\omega_1} D_\nu$ is dense in $\mathbb{W}$ and therefore $dsb(\mathbb{W}) > \omega_1$. Again by properness of dual Mathias forcing $V[G] \models 2^{\omega_0} = \omega_2$ and we finally have $V[G] \models \mathfrak{H} = \omega_2$. $\dashv$

In the model constructed in the proof of Theorem V.4.1 we have $\mathfrak{H} > \mathfrak{t}$, where $\mathfrak{t}$ is the well-known tower number (for a definition of $\mathfrak{t}$ cf. [**65**]). Moreover, we can show the following:

Corollary V.4.2. The statement $\mathfrak{H} > cov(\boldsymbol{B}_0)$ is relatively consistent with ZFC.

Proof. Because dual Mathias forcing is proper and does not add Cohen reals, $\mathbb{P}_{\omega_2}$ does also not add Cohen reals. Further it is known that $\mathfrak{t} \leq cov(\boldsymbol{B}_0)$ (*cf.* [**53**] or [**3**]). Now because forcing with $\mathbb{P}_{\omega_2}$ does not add Cohen reals, in $V[G]$, the covering number $cov(\boldsymbol{B}_0)$ is still ω_1 (because each real in $V[G]$ is in a meager set with code in V). This completes the proof. $\dashv$

Remark 2. In [**65**] Theorem 3.1.(c) it is shown that $\omega \leq \kappa < \mathfrak{t}$ implies that $2^\kappa = 2^{\omega_0}$. We do not have a similar result for the dual-shattering cardinal $\mathfrak{H}$. If we start our forcing construction $\mathbb{P}_{\omega_2}$ with a model $V \models \mathsf{CH} + 2^{\omega_1} = \omega_3$, then again by properness of dual Mathias forcing we have $V[G] \models \mathfrak{H} = \omega_2 = 2^{\omega_0} < 2^{\omega_1} = \omega_3$, where G is $\mathbb{P}_{\omega_2}$-generic over V.

Remark 3. By iterating just Mathias forcing, Spinas showed in [**62**] that $\mathfrak{H} < \mathfrak{h}$ is consistent with ZFC. Further, Jörg Brendle has proved in [**10**] that also $\mathsf{MA} + \omega_1 = \mathfrak{H} < \mathfrak{h} = \omega_2 = \mathfrak{c}$ is consistent with ZFC.

5. The diagram of the results

In ZFC it is provable that $\mathfrak{H} \leq \mathfrak{h}$ and $\mathfrak{H} \leq \mathfrak{S}$, where $\mathfrak{S}$ is the dual-splitting cardinal (*cf.* [**12**] or see Chapter II). Thus, if we summarize the results which are known about $\mathfrak{H}$, we get the following diagram:

$$\begin{tikzcd}
\mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{S}' \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
\mathfrak{h} \arrow[d, dash] & & \\
\mathfrak{H} \arrow[r, dash] \arrow[d, dash] & \mathfrak{S} \arrow[d, dash] & \\
\omega_1 \arrow[r, dash] & cov(\boldsymbol{B}_0) &
\end{tikzcd}$$

In the diagram, the cardinal characteristics grow larger as one moves up or to the right.

Consistency results:

- $cov(\boldsymbol{B}_0) < \mathfrak{H}$
- $\mathfrak{H} < cov(\boldsymbol{B}_0)$ (this is because $\mathfrak{h} < cov(\boldsymbol{B}_0)$ is consistent with ZFC)
- $\mathsf{MA} + \omega_1 = \mathfrak{H} < \mathfrak{h} = \omega_2 = \mathfrak{c}$

CHAPTER VI

Symmetries between two Ramsey properties

In this chapter we compare the Ramsey property with the dual Ramsey property, which was introducted in Chapter V. Even though the two properties are different, it can be shown that all classical results known for the Ramsey property also hold for the dual Ramsey property. In particular we will see that the dual Ramsey property is closed under a generalized Suslin operation (the similar result for the Ramsey property was proved by Matet). Further we compare two notions of forcing, the Mathias forcing and a dual form of it, and will give some symmetries between them.

1. Two Ramsey properties and two notions of forcing

First we define a topology on $[\omega]^\omega$. Let $x \in [\omega]^\omega$ and $a \in [\omega]^{<\omega}$ such that $\max(a) < \min(x)$; then $[a,x]^\omega := \{y \in [\omega]^\omega : y \subseteq (a \cup x) \wedge a \subseteq y\}$. Now let the basic open sets on $[\omega]^\omega$ be the sets $[a,x]^\omega$. These sets are called the **Ellentuck neighborhoods**. The topology induced by the Ellentuck neighborhoods is called the **Ellentuck topology**.

Related to the Ellentuck topology we get the **Mathias forcing** $\mathbb{M}$, which is defined as follows:

$$\langle a,x\rangle \in \mathbb{M} \quad\Leftrightarrow\quad a \in [\omega]^{<\omega} \ \wedge\ x \in [\omega]^\omega \ \wedge\ \max(a){<}\min(x),$$

$$\langle a,x\rangle \leq \langle b,y\rangle \quad\Leftrightarrow\quad b \subseteq a \ \wedge\ x \subseteq y \ \wedge\ (a \setminus b) \subseteq y.$$

If $\langle a,x\rangle$ is an $\mathbb{M}$-condition, then we call a the **stem** of the condition. The Mathias forcing $\mathbb{M}$ has a lot of combinatorial properties (see [**49**], [**39**], or [**25**]). Note that we can consider an $\mathbb{M}$-condition $\langle a,x\rangle$ as an Ellentuck neighborhood $[a,x]^\omega$ and $\langle a,x\rangle \leq \langle b,y\rangle$ if and only if $[a,x]^\omega \subseteq [b,y]^\omega$.

The classical Ramsey property is a property of sets of infinite subsets of ω (of sets of reals). A set $A \subseteq [\omega]^\omega$ has the **Ramsey property** (or is **Ramsey**) if $\exists x \in [\omega]^\omega([x]^\omega \subseteq A \vee [x]^\omega \cap A = \emptyset)$. If there exists an x such that $[x]^\omega \cap A = \emptyset$ we call A a **Ramsey null** set. A set $A \subseteq [\omega]^\omega$ is **completely Ramsey** if for every Ellentuck neighborhood $[s,y]^\omega$ there is an $x \in [s,y]^\omega$ such that $[s,x]^\omega \subseteq A$ or $[s,x]^\omega \cap A = \emptyset$. If we are always in the latter case, then we call A **completely Ramsey null**.

The **dual Ramsey property**, which is a property of sets of partitions, was already introduced in Chapter V.1, where one can find also the definition of the **dual Ellentuck topology**.

Related to the dual Ellentuck topology we get the **dual Mathias forcing** $\mathbb{M}^\flat$, which was already defined in Chapter V. Dual Mathias forcing is similarly to Mathias forcing, but uses the dual Ellentuck neighborhoods instead of the Ellentuck neighborhoods. So,

$$\langle s, X\rangle \in \mathbb{M}^\flat \;\Leftrightarrow\; (s,X)^\omega \text{ is a dual Ellentuck neighborhood}$$

and

$$\langle s, X\rangle \leq \langle t, Y\rangle \;\Leftrightarrow\; (s,X)^\omega \subseteq (t,Y)^\omega.$$

If $\langle s, X\rangle$ is an $\mathbb{M}^\flat$-condition, then we call s again the **stem** of the condition. Because dual Mathias forcing is very close to Mathias forcing, it also has some nice properties similar to those of $\mathbb{M}$.

Now we can start to give some symmetries between the two Ramsey properties and between the two Mathias forcings.

2. Basic facts

In this section we give the tools to consider sets of partitions as sets of reals and to compare the two Ramsey properties. We will give also some basic facts and well-known results concerning the dual Ramsey property and dual Mathias forcing. Further we give some symmetries between Mathias forcing and the dual Mathias forcing.

To compare the two Ramsey properties we first show that we can consider each $A \subseteq [\omega]^\omega$ as a set of infinite partitions of ω and vice versa. For this we define some arithmetical relations and functions.

Let $n, m \in \omega$, then $\operatorname{div}(n,m) := \max\{k \in \omega : k \cdot m \leq n\}$ and

$$\varsigma\{n,m\} := \frac{1}{2}\big((\max\{n,m\})^2 - \max\{n,m\}\big) + \min\{n,m\}\,,$$

where we consider $\varsigma\{n,m\}$ as undefined if $n = m$.

Let $x \in [\omega]^\omega$; then $\mathit{trans}(x) \subseteq \omega$ is such that $n \notin \mathit{trans}(x)$ iff there is a finite sequence s of natural numbers of length $l+1$ such that

$$n = \varsigma\{s(0), s(l)\} \;\textit{and}\; \forall k \in \{1,\dots,l\}\big(\varsigma\{s(k-1), s(k)\} \notin x\big).$$

Note that $\mathit{trans}(x) \subseteq x$. If $x \in [\omega]^\omega$, then we can consider x as a partition with

$$\natural_x(n,m) \iff n = m \;\textit{or}\; \varsigma\{n,m\} \notin \mathit{trans}(x).$$

The **corresponding partition** of a real $x \in [\omega]^\omega$ is denoted by $cp(x)$. Note that $cp(x) \in (\omega)^\omega$ iff $\forall k\, \exists n > k\, \forall m < n\, \big(\neg\natural_x(n,m)\big)$, and further, if $y \subseteq x$, then $cp(y) \sqsubseteq cp(x)$.

A partition X of ω we encode by a real $pc(X)$ (the **partition code** of X) as follows.

$$pc(X) := \big\{k \in \omega : \exists n\, \exists m\, \big(k = \varsigma\{n,m\} \wedge \neg\natural_X(n,m)\big)\big\}.$$

Note that if $X_1 \sqsubseteq X_2$ then $pc(X_1) \subseteq pc(X_2)$. With these definitions we get the following.

Fact VI.2.1. The dual Ellentuck topology is finer than the topology of the Baire space.

PROOF. Let $s \in \omega^{<\omega}$ and $U_s = \{f \in \omega^\omega : s \subset f\}$ be a basic open set in the Baire space ω^ω. Because there is a bijection between ω^ω and $[\omega]^\omega$, we can write U_s as a set

$$V_{s'} = \{r \in [\omega]^\omega : s' \subset r \wedge \min(r \setminus s) > \max(s)\}.$$

Now $cp[V_{s'}] \cap (\omega)^\omega$ (where $cp[V_{s'}] := \{cp(r) : r \in V_{s'}\}$) is open with respect to the dual Ellentuck topology. Therefore, the dual Ellentuck topology is finer than the topology of the Baire space. ⊣

REMARK 1. A similar result is true for the Ellentuck topology (*cf.* [**15**]).

FACT VI.2.2. A set $C \subseteq (\omega)^\omega$ is completely dual Ramsey if and only if C has the Baire property with respect to the dual Ellentuck topology and it is completely dual Ramsey null if and only if it is meager with respect to the dual Ellentuck topology.

PROOF. This is proved in [**11**]. ⊣

REMARK 2. The analogous result is known for the Ramsey property with respect to the Ellentuck topology (*cf.* [**15**]).

Let us now give some symmetries between the two Mathias forcings: If X_G is $\mathbb{M}^\flat$-generic over $\mathbf{V}$ and $Y \in (X_G)^\omega$, then also Y is $\mathbb{M}^\flat$-generic over $\mathbf{V}$ (*cf.* [**11**, Theorem 5.5]). From this it follows immediately that $\mathbb{M}^\flat$ is proper and therefore does not collapse ω_1.

Further, for any $\mathbb{M}^\flat$-condition $\langle s, X\rangle$ and any sentence Φ of the forcing language $\mathbb{M}^\flat$, there is an $\mathbb{M}^\flat$-condition $\langle s, Y\rangle \leq \langle s, X\rangle$ such that $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \Phi$ or $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \neg\Phi$ (*cf.* [**11**, Theorem 5.2]). This property is called **pure decision**.

REMARK 3. The similar results for Mathias forcing $\mathbb{M}$ can be found in [**49**] (or in [**37**]).

We can write dual Mathias forcing as a two step iteration where one first forces with $\mathbb{W} = \big\langle (\omega)^\omega, \sqsubseteq^* \big\rangle$ (defined in Chapter III).

Also Mathias forcing can be written as a two step iteration, where the first step is the forcing notion $\mathbb{U} = \langle \mathcal{P}(\omega)/fin, \subseteq^* \rangle$, where $x \subseteq^* y$ if $|x \setminus y| < \omega$.

FACT VI.2.3. The forcing notion $\mathbb{W}$ is σ-closed and if $\mathscr{D}$ is $\mathbb{W}$-generic over $\mathbf{V}$, then $\mathrm{Min}(\mathscr{D})$ is a Ramsey ultrafilter in $\mathbf{V}[\mathscr{D}]$.

PROOF. Let $X_1 \geq X_2 \geq \dots$ be a decreasing sequence $\mathbb{W}$-conditions. Choose a sequence f_i $(i \in \omega)$ of finite sets of natural numbers, such that $X_{i+1} \sqcap \{f_i\} \sqsubseteq X_i$. Define $y_0 := X_0(0)$ and $y_n := X_n(k)$ where $k := 3 + \bigcup_{i<n}(\bigcup f_i)$. Now

$$Y := \{y_i : i \in \omega\} \cup (\omega \setminus \bigcup_{i\in\omega} y_i)$$

is coarser* than each X_i $(i \in \omega)$ and therefore $\mathbb{W}$ is σ-closed.

Now we claim that the set $\{\mathrm{Min}(X) : X \in \mathscr{D}\}$ is a Ramsey ultrafilter in $\mathbf{V}[\mathscr{D}]$. Remember that a forcing notion which is σ-closed adds no new reals to $\mathbf{V}$ (*cf.* [**36**, Lemma 19.6]). Take a $\pi \in 2^{[\omega]^2}$ and a $Y \in (\omega)^\omega$, then by Ramsey's Theorem, for $\mathrm{Min}(Y) \in [\omega]^\omega$ there exists an infinite $r \subseteq \mathrm{Min}(Y)$ such that π is constant on $[r]^2$. Finally let

$$X := \{b : b \in Y \wedge b \cap r \neq \emptyset\} \cup \bigcup \{b : b \in Y \wedge b \cap r = \emptyset\},$$

then $X \sqsubseteq Y$ and $\mathrm{Min}(X) = r$. Thus, $\mathscr{H}_\pi := \{X \in (\omega)^\omega : \pi|_{[\mathrm{Min}(X)]^2} \text{ is constant}\}$ is dense in $\mathbb{W}$, and therefore $\mathscr{H}_\pi \cap \mathscr{D} \neq \emptyset$. ⊣

Remark 4. It is easy to see that the forcing notion $\mathbb{U}$ is σ-closed. Further we have that if $\mathcal{U}$ is $\mathbb{U}$-generic over $\mathbf{V}$, then $\mathcal{U}$ is a Ramsey ultrafilter in $\mathbf{V}[\mathcal{U}]$.

The forcing notion $\mathbb{W}$ is stronger than the forcing notion $\mathbb{U}$.

Fact VI.2.4. If $\mathscr{U}$ is $\mathbb{W}$-generic, then the set $\{\mathrm{Min}(X) : X \in \mathscr{U}\}$ is $\mathbb{U}$-generic.

Proof. Let $\mathcal{A} \subseteq [\omega]^\omega$ be a maximal anti-chain in $\mathbb{U}$, *i.e.*, $\mathcal{A}$ is a maximal almost disjoint family. Then the set $\mathscr{D}_\mathcal{A} := \{X \in \mathbb{W} : \exists a \in \mathcal{A}(\mathrm{Min}(X) \subseteq^* a)\}$ is dense in $\mathbb{W}$. ⊣

We define now the second step of the two step iteration: Let $\mathscr{F} \subseteq (\omega)^\omega$, then the partial order $\mathbb{P}_\mathscr{F}$ is defined as follows.

$$\langle s, X\rangle \in \mathbb{P}_\mathscr{F} \quad\Leftrightarrow\quad X \in \mathscr{F} \;\wedge\; (s,X)^\omega \text{ is a dual Ellentuck neighborhood,}$$

$$\langle s, X\rangle \leq \langle t, Y\rangle \quad\Leftrightarrow\quad (s,X)^\omega \subseteq (t,Y)^\omega\,.$$

Remark 5. For $\mathcal{F} \subseteq [\omega]^\omega$ we can define the partial order $\mathbb{P}_\mathcal{F}$ similarly.

Fact VI.2.5. Let $\dot{\mathscr{U}}$ be the canonical $\mathbb{W}$-name for the $\mathbb{W}$-generic object, then

$$\mathbb{W} * \mathbb{P}_{\dot{\mathscr{U}}} \approx \mathbb{M}^\flat.$$

Proof.

$$\begin{aligned} \mathbb{W} * \mathbb{P}_{\dot{\mathscr{U}}} &= \{\langle p, \langle\tilde{s}, \tilde{X}\rangle\rangle : p \in \mathbb{W} \wedge p \Vdash_\mathbb{W} \langle\tilde{s}, \tilde{X}\rangle \in \mathbb{P}_{\dot{\mathscr{U}}}\} \\ &= \{\langle p, \langle\tilde{s}, \tilde{X}\rangle\rangle : p \in (\omega)^\omega \wedge p \Vdash_\mathbb{W} (\tilde{X} \in \dot{\mathscr{U}} \wedge \tilde{s} \sqsubseteq \tilde{X})\}. \end{aligned}$$

Now the embedding

$$\begin{array}{rccc} h: & \mathbb{M}^\flat & \longrightarrow & \mathbb{W} * \mathbb{P}_{\dot{\mathscr{U}}} \\ & \langle s, X\rangle & \longmapsto & \langle X, \langle\check{s}, \check{X}\rangle\rangle \end{array}$$

is a dense embedding (see **[18]** Definition 0.8):

1. It is easy to see that h preserves the order relation $\leq$.
2. Let $\langle p, \langle\tilde{s}, \tilde{X}\rangle\rangle \in \mathbb{W} * \mathbb{P}_{\dot{\mathscr{U}}}$. Because $\mathbb{W}$ is σ-closed, there is a condition $q \leq p$, a segment $s \in (\mathbb{N})$ and a partition $X \in (\omega)^\omega$ such that $q \Vdash_\mathbb{W} \check{s} = \tilde{s} \wedge \check{X} = \tilde{X}$. Evidently, $\langle q, \langle\check{s}, \check{X}\rangle\rangle \in \mathbb{W} * \mathbb{P}_{\dot{\mathscr{U}}}$ is stronger than $\langle p, \langle\tilde{s}, \tilde{X}\rangle\rangle$. Let $Z := q \sqcap X$ and let $Z' \sqsubseteq^* Z$ be such that $s \sqsubseteq Z'$, and we have $h(\langle s, Z'\rangle) \leq \langle p, \langle\tilde{s}, \tilde{X}\rangle\rangle$.

⊣

REMARK 6. Let $\dot{\mathcal{U}}$ be the canonical $\mathbb{U}$-name for the $\mathbb{U}$-generic object, then $\mathbb{U}*\mathbb{P}_{\dot{\mathcal{U}}} \approx \mathbb{M}$.

The dual Mathias forcing is stronger than the Mathias forcing.

FACT VI.2.6. The dual Mathias forcing adds Mathias reals.

PROOF. Let $\mathscr{U}$ be $\mathbb{W}$-generic over $\mathbf{V}$; then by Fact VI.2.4, $\mathcal{U} := \{\mathrm{Min}(X) : X \in \mathscr{U}\}$ is $\mathbb{U}$-generic over $\mathbf{V}$. Now we define $h : \mathbb{P}_{\mathscr{U}} \to \mathbb{P}_{\mathcal{U}}$ as follows.

$$\begin{array}{rccc} h: & \mathbb{P}_{\mathscr{U}} & \longrightarrow & \mathbb{P}_{\mathcal{U}} \\ & \langle s, X\rangle & \longmapsto & \big\langle \mathrm{Min}(s), \mathrm{Min}(X) \setminus \mathrm{Min}(s)\big\rangle \end{array}$$

For h, the following is true:

(i) If $q_1, q_2 \in \mathbb{P}_{\mathscr{U}}$, $q_1 \leq q_2$, then $h(q_1) \leq h(q_2)$.

(ii) For all $q \in \mathbb{P}_{\mathscr{U}}$ and for all $p' \leq h(q)$, there is a $q' \in \mathbb{P}_{\mathscr{U}}$ such that q and q' are compatible and $h(q') \leq p'$.

Therefore, with [**37**] Part I, Lemma 2.7 we finally get $\mathbf{V}^{\mathbb{M}} \subseteq \mathbf{V}^{\mathbb{M}^\flat}$. ⊣

3. The dual Ramsey property and Suslin's operation

In this section we will show that the dual Ramsey property is closed under a generalized Suslin operation. As a corollary we will get the already known result that analytic sets are completely dual Ramsey.

Following Chapter V, let $\boldsymbol{R}_0^\flat \subseteq \mathcal{P}\big((\omega)^\omega\big)$ be the ideal of all completely dual Ramsey null sets. Recall that $add(\boldsymbol{R}_0^\flat)$ is the smallest cardinal κ such that there exists a family $\mathscr{J} = \{\mathscr{J}_\alpha \in \boldsymbol{R}_0^\flat : \alpha < \kappa\}$ with $\bigcup\mathscr{J} \notin \boldsymbol{R}_0^\flat$, and that $cov(\boldsymbol{R}_0^\flat)$ is the smallest cardinal κ such that there exists a family $\mathscr{J} = \{\mathscr{J}_\alpha \in \boldsymbol{R}_0^\flat : \alpha < \kappa\}$ with $\bigcup\mathscr{J} = (\omega)^\omega$. In Chapter V (see also [**22**]) it is shown that $cov(\boldsymbol{R}_0^\flat) = add(\boldsymbol{R}_0^\flat) = \mathfrak{H}$ (where $\mathfrak{H}$ is the dual-shattering cardinal) and that $\mathfrak{H} > \omega_1$ is relatively consistent with ZFC.

Let $\mathrm{Seq}(\kappa) := \kappa^{<\omega}$ and for $f \in \kappa^\omega$ and $n \in \omega$, let $\bar{f}(n)$ denote the finite sequence $\big\langle f(0), f(1), \ldots, f(n-1)\big\rangle$. The **generalized Suslin operation** $\boldsymbol{\mathcal{A}}_\kappa$ (for a cardinal κ) is defined as follows:

$$\boldsymbol{\mathcal{A}}_\kappa\big\{\mathscr{Q}_s : s \in \mathrm{Seq}(\kappa)\big\} := \bigcup_{f\in\kappa^\omega}\bigcap_{n\in\omega}\mathscr{Q}_{\bar{f}(n)}\,,$$

where $\mathscr{Q}_s \subseteq (\omega)^\omega$ for all $s \in \mathrm{Seq}(\kappa)$. In Theorem VI.3.5 below we will show that for each cardinal $\kappa < \mathfrak{H}$, the completely dual Ramsey sets are closed under the operation $\boldsymbol{\mathcal{A}}_\kappa$. But first we give some other results.

A set $\mathscr{R} \subseteq (\omega)^\omega$ is **dual Ellentuck meager** if $\mathscr{R}$ is meager with respect to the dual Ellentuck topology. Remember that a set is dual Ellentuck meager if and only if it is completely dual Ramsey null and a set is completely dual Ramsey if and only if it has the Baire property with respect to the dual Ellentuck topology.

If $(s,X)^\omega$ is a dual Ellentuck neighborhood, then we say that $\mathscr{R}$ is dual Ellentuck meager in $(s,X)^\omega$ if $\mathscr{R} \cap (s,X)^\omega$ is dual Ellentuck meager. By [**11**, Theorem 4.1], $\mathscr{R}$

is dual Ellentuck meager in $(s,X)^\omega$ if for all $(t,Y)^\omega \subseteq (s,X)^\omega$ there exists a partition $Z \in (t,Y)^\omega$ such that $(t,Z)^\omega \cap \mathscr{R} = \emptyset$.

Fix a set $\mathscr{R} \subseteq (\omega)^\omega$ and let

$$M := \bigcup \left\{ (s,X)^\omega : \mathscr{R} \text{ is dual Ellentuck meager in } (s,X)^\omega \right\}.$$

Further let $M(\mathscr{R}) := M \cap \mathscr{R}$. We first show the following.

LEMMA VI.3.1. If $(s,X)^\omega$ is a dual Ellentuck neighborhood such that $(s,X)^\omega \subseteq M$, then $\mathscr{R}$ is dual Ellentuck meager in $(s,X)^\omega$.

PROOF. If $(s,X)^\omega \subseteq M$, then $(s,X)^\omega = \bigcup \left\{ (t,Y)^\omega \subseteq (s,X)^\omega : R \text{ is dual Ellentuck meager in } (t,Y)^\omega \right\}$. Let $N := \bigcup \left\{ (u,Z)^\omega \subseteq (s,X)^\omega : R \cap (u,Z)^\omega = \emptyset \right\}$. Because N is an Ellentuck open set, N is completely dual Ramsey. Therefore, for any $(t,Y)^\omega \subseteq (s,X)^\omega$ there exists a partition $Y' \in (t,Y)^\omega$ such that $(t,Y')^\omega \subseteq N$ or $(t,Y')^\omega \cap N = \emptyset$. If we are in the latter case, then because $(t,Y')^\omega \subseteq (s,X)^\omega$, we find a $(u,Y'')^\omega \subseteq (t,Y')^\omega$ such that $\mathscr{R}$ is dual Ellentuck meager in $(u,Y'')^\omega$. Hence, there exists a $(u,Z)^\omega \subseteq (u,Y'')^\omega$ such that $(u,Z)^\omega \cap \mathscr{R} = \emptyset$, which contradicts $(t,Y')^\omega \cap N = \emptyset$. So we are always in the former case, which implies that $\mathscr{R}$ is dual Ellentuck meager in $(s,X)^\omega$. ⊣

With this result, we can easily prove the following

LEMMA VI.3.2. The set $M(\mathscr{R})$ is dual Ellentuck meager.

PROOF. Take a dual Ellentuck neighborhood $(s,X)^\omega$ and let

$$S := \bigcup \left\{ (t,Z)^\omega \subseteq (s,X)^\omega : \mathscr{R} \text{ is dual Ellentuck meager in } (t,Z)^\omega \right\}.$$

Then S, as the union of open sets, is open and a subset of $(s,X)^\omega$. Because $(s,X)^\omega$ is also closed (in the dual Ellentuck topology), the set $C := (s,X)^\omega \setminus S$ is closed. By [**11**, Theorem 4.1], the sets C and S both are completely dual Ramsey. Therefore we find for every $(s',X')^\omega \subseteq (s,X)^\omega$ a partition $Y \in (s',X')^\omega$ such that $(s',Y)^\omega \subseteq S$ or $(s',Y)^\omega \subseteq C$. Now if $(s',Y)^\omega \subseteq S$, then by Lemma VI.3.1, $\mathscr{R}$ is dual Ellentuck meager in $(s',Y)^\omega$ and if $(s',Y)^\omega \subseteq C$, then $(s',Y)^\omega \cap M(\mathscr{R}) = \emptyset$. To see this, assume there is an $H \in M(\mathscr{R}) \cap (s',Y)^\omega$. Because $H \in M(\mathscr{R})$ there exists a dual Ellentuck neighborhood $(t,Z)^\omega$ such that $H \in (t,Z)^\omega$ and $\mathscr{R}$ is dual Ellentuck meager in $(t,Z)^\omega$. Because $H \in (t,Z)^\omega$ and $H \in (s',Y)^\omega$ there is a dual Ellentuck neighborhood $(u,U)^\omega \subseteq (t,Z)^\omega \cap (s',Y)^\omega$. But with $(u,U)^\omega \subseteq (t,Z)^\omega$ it follows that $\mathscr{R}$ is dual Ellentuck meager in $(u,U)^\omega$ and therefore $(u,U)^\omega \subseteq S$, a contradiction to $(u,U)^\omega \subseteq (s',Y)^\omega \subseteq C$.

Therefore, in both cases $M(\mathscr{R})$ is dual Ellentuck meager in $(s',Y)^\omega \subseteq (s',X')^\omega$ and because $(s,X)^\omega$ and $(s',X')^\omega \subseteq (s,X)^\omega$ were arbitrary, the set $M(\mathscr{R})$ is dual Ellentuck meager in each dual Ellentuck neighborhood. Hence, the set $M(\mathscr{R})$ is dual Ellentuck meager. ⊣

COROLLARY VI.3.3. The set $\mathscr{R} \cup \big((\omega)^\omega \setminus M\big)$ has the dual Ellentuck Baire property.

PROOF. Because M is open, $(\omega)^\omega \setminus M$ is closed and $\mathscr{R} \cup \big((\omega)^\omega \setminus M\big) = (\mathscr{R} \cap M) \cup \big((\omega)^\omega \setminus M\big) = M(\mathscr{R}) \cup \big((\omega)^\omega \setminus M\big)$ which is the union of a meager set and a closed set and therefore has the dual Ellentuck Baire property. ⊣

THEOREM VI.3.4. If $\mathscr{R} \subseteq (\omega)^\omega$, then we can construct a set $\mathscr{B} \supseteq \mathscr{R}$ which has the dual Ellentuck Baire property and whenever $\mathscr{C} \subseteq \mathscr{B} \setminus \mathscr{R}$ has the dual Ellentuck Baire property, then $\mathscr{C}$ is dual Ellentuck meager.

PROOF. Let $\mathscr{B} := \mathscr{R} \cup ((\omega)^\omega \setminus M)$ where M is as above. By Lemma VI.3.2 and Corollary VI.3.3 we know that $\mathscr{B}$ has the dual Ellentuck Baire property. Now let $\mathscr{C} \subseteq \mathscr{B} \setminus \mathscr{R}$ with the dual Ellentuck Baire property. If $\mathscr{C}$ is not dual Ellentuck meager, then there exists a dual Ellentuck neighborhood $(u,U)^\omega$, such that $(u,U)^\omega \setminus \mathscr{C}$ and therefore $(u,U)^\omega \cap \mathscr{R}$ are dual Ellentuck meager. Hence, $\mathscr{R}$ is dual Ellentuck meager in $(u,U)^\omega$ and therefore $(u,U)^\omega \subseteq M$. Since $(u,U)^\omega \cap \mathscr{C} \neq \emptyset$ and $\mathscr{C} \cap M = \emptyset$, there is a $Y \in (u,U)^\omega$ such that $Y \notin M$, a contradiction to $\mathscr{R}$ is dual Ellentuck meager in $(u,U)^\omega$. ⊣

Now we can prove the following.

THEOREM VI.3.5. Let $\kappa < \mathfrak{H}$ be a cardinal number and for each $s \in \mathrm{Seq}(\kappa)$ let $\mathscr{Q}_s \subseteq (\omega)^\omega$. If all the sets $\mathscr{Q}_s$ are completely dual Ramsey, then the set

$$\boldsymbol{\mathcal{A}}_\kappa\{\mathscr{Q}_s : s \in \mathrm{Seq}(\kappa)\}$$

is completely dual Ramsey, too.

PROOF. Let $\{\mathscr{Q}_s : s \in \mathrm{Seq}(\kappa)\}$ be a set of completely dual Ramsey sets and let $\mathscr{A} := \boldsymbol{\mathcal{A}}_\kappa\{\mathscr{Q}_s : s \in \mathrm{Seq}(\kappa)\}$. For two sequences s and f in $\kappa^{\leq\omega}$ we write $s \subseteq f$ if s is an initial segment of f. If $s \in \kappa^{<\omega}$ is a finite sequence, then $|s|$ denotes the length of s. Without loss of generality we may assume that $\mathscr{Q}_s \supseteq \mathscr{Q}_t$ whenever $s \subseteq t$.
For $s \in \mathrm{Seq}(\kappa)$ let

$$\mathscr{A}_s := \bigcup_{\substack{f \in \kappa^\omega \\ s \subseteq f}} \bigcap_{\substack{n \in \omega \\ n \geq |s|}} \mathscr{Q}_{\bar{f}(n)} .$$

In addition we have $\mathscr{A}_s \subseteq \mathscr{Q}_s$, $\mathscr{A}_s = \bigcup_{\alpha<\kappa} \mathscr{A}_{s^\frown\alpha}$ and $\mathscr{A} = \mathscr{A}_\emptyset$. By Theorem VI.3.4, for each $s \in \mathrm{Seq}(\kappa)$ we find a $\mathscr{B}_s \supseteq \mathscr{A}_s$ which is completely dual Ramsey and if $\mathscr{C} \subseteq \mathscr{B}_s \setminus \mathscr{A}_s$ has the dual Ramsey property, then $\mathscr{C}$ is dual Ramsey null. Because $\mathscr{Q}_s \supseteq \mathscr{A}_s$ is completely dual Ramsey, we may assume that $\mathscr{B}_s \subseteq \mathscr{Q}_s$ and therefore

$$\mathscr{A} = \boldsymbol{\mathcal{A}}_\kappa\{\mathscr{B}_s : s \in \mathrm{Seq}(\kappa)\} .$$

Let $\mathscr{B} := \mathscr{B}_\emptyset$. Note that $\mathscr{A} = \bigcup_{\alpha<\kappa} \mathscr{A}_{\langle\alpha\rangle} \subseteq \bigcup_{\alpha<\kappa} \mathscr{B}_{\langle\alpha\rangle}$, and therefore $\mathscr{B} \subseteq \bigcup_{\alpha<\kappa} \mathscr{B}_{\langle\alpha\rangle}$. Now we show that

$$\mathscr{B} \setminus \mathscr{A} \subseteq \bigcup_{\alpha<\kappa} \mathscr{B}_{\langle\alpha\rangle} \subseteq \bigcup_{f \in \kappa^\omega} \bigcap_{n \in \omega} \mathscr{B}_{\bar{f}(n)} \subseteq \bigcup_{s \in \mathrm{Seq}(\kappa)} \Big(\mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha}\Big) .$$

Assume $x \notin \bigcup_s \left(\mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha}\right)$. If we have for all $\alpha < \kappa$, that $x \notin \mathscr{B}_{\langle\alpha\rangle}$, then $x \notin \mathscr{B}$. And if there exists an $\alpha_0 < \kappa$ such that $x \in \mathscr{B}_{\langle\alpha_0\rangle}$, because $x \notin \bigcup_s \left(\mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha}\right)$ we find an α_1 such that $x \in \mathscr{B}_{\langle\alpha_0,\alpha_1\rangle}$ and finally we find an $f \in \kappa^\omega$ such that for all $n \leq \omega$: $x \in \mathscr{B}_{\bar{f}(n)}$. But this implies that $x \in \mathscr{A}$. Now, because

$$\mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha} \subseteq \mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{A}_{s^\frown\alpha} = \mathscr{B}_s \setminus \mathscr{A}_s$$

and because $\bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha}$ is the union of less than $\mathfrak{H}$ completely dual Ramsey sets, $\mathscr{B}_s \setminus \bigcup_{\alpha<\kappa} \mathscr{B}_{s^\frown\alpha}$ is completely dual Ramsey and as a subset of $\mathscr{B}_s \setminus \mathscr{A}_s$, it is completely dual Ramsey null. Therefore, $\mathscr{B} \setminus \mathscr{A}$ as a subset of the union of less than $\mathfrak{H}$ completely dual Ramsey null sets is completely dual Ramsey null, and because $\mathscr{B}$ is completely dual Ramsey, $\mathscr{A}$ is completely dual Ramsey too. ⊣

Remark 7. A similar result holds also for the Ramsey property and is proved by Matet in [**47**].

As a corollary we get a result which was first proved by Carlson and Simpson (*cf.* [**11**]).

Corollary VI.3.6. Every analytic set is completely dual Ramsey.

Proof. This follows from Theorem VI.3.5 and because each analytic set $\mathscr{A} \subseteq [\omega]^\omega$ can be written as

$$\mathscr{A} = \boldsymbol{\mathcal{A}}\{Q_s : s \in \mathrm{Seq}(\omega)\}$$

where each $Q_s \subseteq [\omega]^\omega$ is a closed set in the Baire space. ⊣

Remark 8. For a similar result see [**15**] or [**61**].

4. Game-families and the forcing notion $\mathbb{P}_{\mathscr{F}}$

Firstly we define a game and the corresponding game-families. Secondly we show that for game-families $\mathscr{F}$, the forcing notion $\mathbb{P}_{\mathscr{F}}$ has pure decision and if X is $\mathbb{P}_{\mathscr{F}}$-generic and $Y \in (X)^\omega$, then Y is $\mathbb{P}_{\mathscr{F}}$-generic, too.

We call a family $\mathscr{F} \subseteq (\omega)^\omega$ **non-principal** if for all $X \in \mathscr{F}$ there is a $Y \in \mathscr{F}$ such that $Y \sqsubseteq X$ and $\neg(Y \stackrel{*}{=} X)$. A family $\mathscr{F}$ is **closed under refinement** if $X \sqsubseteq Y$ and $X \in \mathscr{F}$ implies that $Y \in \mathscr{F}$. Further, it is **closed under finite changes** if for all $s \in (\mathbb{N})$ and $X \in \mathscr{F}$, $s \sqcap X \in \mathscr{F}$.

In the sequel, $\mathscr{F}$ is always a non-principal family which is closed under refinement and finite changes.

If $s \in (\mathbb{N})$ and $s \sqsubseteq X \in \mathscr{F}$, then we call the dual Ellentuck neighborhood $(s,X)^\omega$ an **$\mathscr{F}$-dual Ellentuck neighborhood** and write $(s,X)^\omega_{\mathscr{F}}$ to emphasize that $X \in \mathscr{F}$. A set $\mathscr{O} \subseteq (\omega)^\omega$ is called $\mathscr{F}$-open if $\mathscr{O}$ can be written as the union of $\mathscr{F}$-dual Ellentuck neighborhoods.

Fix a family $\mathscr{F} \subseteq (\omega)^\omega$ which is non-principal and closed under refinement and finite changes. Let $X \in \mathscr{F}$ and $s \in (\mathbb{N})$ be such that $s \sqsubseteq X$. We associate with

$(s,X)^\omega_{\mathscr{F}}$ the following game. This type of game was introduced first by Kastanas in [**40**].

$$\begin{array}{llllllll} \mathrm{I} & \langle X_0\rangle & & \langle X_1\rangle & & \langle X_2\rangle & & \\ & & & & & & & \dots \\ \mathrm{II} & & \langle t_0, Y_0\rangle & & \langle t_1, Y_1\rangle & & \langle t_2, Y_2\rangle & \end{array}$$

All the X_i of player I and the Y_i of player II must be elements of the family $\mathscr{F}$. Player I plays $\langle X_0\rangle$ such that $X_0 \in (s,X)^\omega_{\mathscr{F}}$, then II responds with $\langle t_0, Y_0\rangle$, where $Y_0 \in (s,X_0)^\omega_{\mathscr{F}}$, $s \prec t_0^* \prec Y_0$ and $|t_0| = |s|$. (Recall that for $s \in (\mathbb{N})$, $s^* = s \cup \{\{\mathrm{dom}(s)\}\}$.) For $n \geq 1$, the n^{th} move of player I is $\langle X_n\rangle$ such that $X_n \in (t^*_{n-1},Y_{n-1})^\omega_{\mathscr{F}}$ and then player II responds with $\langle t_n, Y_n\rangle$ where $Y_n \in (t^*_{n-1},X_n)^\omega_{\mathscr{F}}$, $t^*_{n-1} \prec t^*_n \prec Y_n$ and $|t_n| = |t_{n-1}| + 1$. Player I wins iff the only Y with $t_n \prec Y$ (for all n) is in $\mathscr{F}$. We denote this game by $\mathcal{G}(\mathscr{F})$ *starting with* $\langle s, X\rangle$.

A non-principal family $\mathscr{F}$ which is closed under refinement and finite changes is a **game-family** if player II has no winning strategy in the game $\mathcal{G}(\mathscr{F})$.

A family $\mathscr{F} \subseteq (\omega)^\omega$ is called a **filter** if for any $X, Y \in \mathscr{F}$, also $X \sqcap Y \in \mathscr{F}$. A filter which is also a game-family is called a **game-filter**. Note that $(\omega)^\omega$ is a game-family but not a game-filter. It is not known if game-filters exist under CH, but as we will see in Theorem VI.5.1, the existence of game-filters is consistent with ZFC.

Let $\mathscr{O} \subseteq (\omega)^\omega$ be an $\mathscr{F}$-open set. Call $(s,X)^\omega_{\mathscr{F}}$ **good** (with respect to $\mathscr{O}$), if for some $Y \in (s,X)^\omega_{\mathscr{F}} \cap \mathscr{F}$, $(s,Y)^\omega_{\mathscr{F}} \subseteq \mathscr{O}$; otherwise call it **bad**. Note that if $(s,X)^\omega_{\mathscr{F}}$ is bad and $Y \in (s,X)^\omega_{\mathscr{F}} \cap \mathscr{F}$, then $(s,Y)^\omega_{\mathscr{F}}$ is bad, too. We call $(s,X)^\omega_{\mathscr{F}}$ **ugly** if $(t^*,X)^\omega_{\mathscr{F}}$ is bad for all $s \prec t^* \sqsubseteq X$ with $|t| = |s|$. Note that if $(s,X)^\omega_{\mathscr{F}}$ is ugly, then $(s,X)^\omega_{\mathscr{F}}$ is bad, too.

To prove the following two lemmas, we will follow in fact the proof of Lemma 19.15 in [**41**].

Lemma VI.4.1. Let $\mathscr{F}$ be a game-family and $\mathscr{O} \subseteq (\omega)^\omega$ an $\mathscr{F}$-open set. If $(s,X)^\omega_{\mathscr{F}}$ is bad (with respect to $\mathscr{O}$), then there exists a $Z \in (s,X)^\omega_{\mathscr{F}}$ such that $(s,Z)^\omega_{\mathscr{F}}$ is ugly.

Proof. We begin by describing a strategy for player II in the game $\mathcal{G}(\mathscr{F})$ starting with $\langle s, X\rangle$. Let $\langle X_n\rangle$ be the n^{th} move of player I and t_n be such that $s \prec t_n$, $|t_n| = |s| + n$ and $t_n^* \prec X_n$. Let $\{t_n^i : i \leq m\}$ be an enumeration of all t such that $s \prec t \sqsubseteq t_n$, $|t| = |s|$ and $\mathrm{dom}(t) = \mathrm{dom}(t_n)$. Further let $Y^{-1} := X_n$. Now choose for each $i \leq m$ a partition $Y^i \in \mathscr{F}$ such that $Y^i \sqsubseteq Y^{i-1}$, $t_n^* \prec Y^i$ and $((t_n^i)^*,Y^i)^\omega_{\mathscr{F}}$ is bad or $((t_n^i)^*,Y^i)^\omega_{\mathscr{F}} \subseteq \mathscr{O}$. Finally, let $Y_n := Y^m$ and let player II play $\langle t_n, Y_n\rangle$.

Because player II has no winning strategy, player I can play so that the only Y with $t_n \prec Y$ (for all n) belongs to $\mathscr{F}$. Let $S_Y := \{t^* \sqsubseteq Y : s \prec t \wedge |t| = |s|\}$; then, because of the strategy of player II, for all $t \in S_Y$ we have either $(t^*,Y)^\omega_{\mathscr{F}}$ is bad or $(t^*,Y)^\omega_{\mathscr{F}} \subseteq \mathscr{O}$. Now let $C_0 := \{t \in S_Y : (t,Y)^\omega_{\mathscr{F}}$ is bad$\}$ and $C_1 := \{t \in S_Y : (t^*,Y)^\omega_{\mathscr{F}} \subseteq \mathscr{O}\} = S_Y \setminus C_0$. By a result of [**29**] (see also [**26**, Section 7]), there exists a partition $Z \in (s,Y)^\omega_{\mathscr{F}} \cap \mathscr{F}$, such that $S_Z \subseteq C_0$ or $S_Z \subseteq C_1$. If we are in the latter case, we have $(s,Z)^{\omega}_{\mathscr{F}} \subseteq \mathscr{O}$, which contradicts that $(s,X)^{\omega}_{\mathscr{F}}$ is bad. So we must have $S_Z \subseteq C_0$, which implies that $(s,Z)^{\omega}_{\mathscr{F}}$ is ugly and completes the proof of the Lemma. $\dashv$

LEMMA VI.4.2. If $\mathscr{F}$ is a game-family and $\mathscr{O} \subseteq (\omega)^{\omega}$ is an $\mathscr{F}$-open set, then for every $\mathscr{F}$-dual Ellentuck neighborhood $(s,X)^{\omega}_{\mathscr{F}}$ there exists a $Y \in (s,X)^{\omega}_{\mathscr{F}} \cap \mathscr{F}$ such that $(s,Y)^{\omega}_{\mathscr{F}} \subseteq \mathscr{O}$ or $(s,Y)^{\omega}_{\mathscr{F}} \cap \mathscr{O} \cap \mathscr{F} = \emptyset$.

PROOF. If $(s,X)^{\omega}_{\mathscr{F}}$ is good, then we are done. Otherwise we consider the game $\mathcal{G}(\mathscr{F})$ starting with $\langle s, X\rangle$. Let $\langle X_0\rangle$ be the first move of player I. Because $(s,X_0)^{\omega}_{\mathscr{F}}$ is bad, by Lemma VI.4.1 we can choose $Y' \in (s,X_0)^{\omega}_{\mathscr{F}} \cap \mathscr{F}$ such that $(s,Y')^{\omega}_{\mathscr{F}}$ is ugly. Let t_0 be such that $s \prec t_0^* \prec Y'$ and $|t_0| = |s|$. Now we choose $Y_0 \in (t_0^*,Y')^{\omega}_{\mathscr{F}} \cap \mathscr{F}$ such that $(t_0^*,Y_0)^{\omega}_{\mathscr{F}}$ is ugly, which is is possible because $(t_0,Y')^{\omega}_{\mathscr{F}}$ is ugly and therefore $(t_0^*,Y')^{\omega}_{\mathscr{F}}$ is bad. Note that for all t with $s \prec t \sqsubseteq t_0$ and $\mathrm{dom}(t) = \mathrm{dom}(t_0)$ we have $(t^*,Y_0)^{\omega}_{\mathscr{F}}$ is ugly. Now player II plays $\langle t_0, Y_0\rangle$.

Let $\langle X_{n+1}\rangle$ be the $(n+1)^{th}$ move of player I. By the strategy of player II we have $(t^*,X_{n+1})^{\omega}_{\mathscr{F}}$ is ugly for all t with $s \prec t \sqsubseteq t_n$ and $\mathrm{dom}(t) = \mathrm{dom}(t_n)$. Let t_{n+1} be such that $|t_{n+1}| = |t_n| + 1 = |s| + n$ and $t_n^* \prec t_{n+1}^* \prec X_{n+1}$. Let $\{t_{n+1}^i : i \leq m\}$ be an enumeration of all t such that $s \prec t \sqsubseteq t_{n+1}$ and $\mathrm{dom}(t) = \mathrm{dom}(t_{n+1})$. Further let $Y^{-1} := X_{n+1}$. Now choose for each $i \leq m$ a partition $Y^i \in \mathscr{F}$ such that $Y^i \sqsubseteq Y^{i-1}$, $t_{n+1}^* \prec Y^i$ and $((t_{n+1}^i)^*,Y^i)^{\omega}_{\mathscr{F}}$ is ugly. (This is possible because we know that $(t^*,X_k)^{\omega}_{\mathscr{F}}$ is ugly for all $k \leq n$ and t with $s \prec t \sqsubseteq t_k$ and $\mathrm{dom}(t) = \mathrm{dom}(t_k)$, which implies that $((t_{n+1}^i)^*,X_{n+1})^{\omega}_{\mathscr{F}}$ is bad.) Finally, let $Y_{n+1} := Y^m$ and let player II play $\langle t_{n+1}, Y_{n+1}\rangle$.

Because player II has no winning strategy, player I can play so that the only Y with $t_n \prec Y$ (for all n) belongs to $\mathscr{F}$. We claim that $(s,Y)^{\omega}_{\mathscr{F}} \cap \mathscr{O} \cap \mathscr{F} = \emptyset$. Let $Z \in (s,Y)^{\omega}_{\mathscr{F}} \cap \mathscr{O} \cap \mathscr{F}$. Because $\mathscr{O}$ is $\mathscr{F}$-open we find a $t \prec Z$ such that $(t^*,Z)^{\omega}_{\mathscr{F}} \subseteq \mathscr{O}$. Because $t^* \sqsubseteq Y$ we know by the strategy of player II that $(t^*,Y)^{\omega}_{\mathscr{F}}$ is bad. Hence, there is no $Z \in (t^*,Y)^{\omega}_{\mathscr{F}}$ such that $(t^*,Z)^{\omega}_{\mathscr{F}} \subseteq \mathscr{O}$. This completes the proof. $\dashv$

Now we give two properties of the forcing notion $\mathbb{P}_{\mathscr{F}}$, where $\mathbb{P}_{\mathscr{F}}$ is defined as in Section 2 and $\mathscr{F}$ is a game-family. Note that for $\mathscr{F} = (\omega)^{\omega}$ (which is obviously a game-family) the forcing notion $\mathbb{P}_{\mathscr{F}}$ is the same as dual Mathias forcing. First we show that the forcing notion $\mathbb{P}_{\mathscr{F}}$ has pure decision.

THEOREM VI.4.3. Let $\mathscr{F}$ be a game-family and let Φ be a sentence of the forcing language $\mathbb{P}_{\mathscr{F}}$. For any $\mathbb{P}_{\mathscr{F}}$-condition $(s,X)^{\omega}_{\mathscr{F}}$ there exists a $\mathbb{P}_{\mathscr{F}}$-condition $(s,Y)^{\omega}_{\mathscr{F}} \leq (s,X)^{\omega}_{\mathscr{F}}$ such that $(s,Y)^{\omega}_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \Phi$ or $(s,Y)^{\omega}_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \neg\Phi$.

PROOF. With respect to Φ we define

$$\mathscr{O}_1 := \{Y : (t,Y)^{\omega}_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \Phi \text{ for some } t \prec Y \in \mathscr{F}\}$$

and

$$\mathscr{O}_2 := \{Y : (t,Y)^{\omega}_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \neg\Phi \text{ for some } t \prec Y \in \mathscr{F}\}.$$

Clearly $\mathscr{O}_1$ and $\mathscr{O}_2$ are both $\mathscr{F}$-open and $\mathscr{O}_1 \cup \mathscr{O}_2$ is even dense (with respect to the partial order $\mathbb{P}_{\mathscr{F}}$). Because $\mathscr{F}$ is a game-family, by Lemma VI.4.2 we know that for any $(s,X)^\omega_{\mathscr{F}} \in \mathbb{P}_{\mathscr{F}}$ there exists $Y \in (s,X)^\omega_{\mathscr{F}} \cap \mathscr{F}$ such that either $(s,Y)^\omega_{\mathscr{F}} \subseteq \mathscr{O}_1$ or $(s,Y)^\omega_{\mathscr{F}} \cap \mathscr{O}_1 \cap \mathscr{F} = \emptyset$. In the former case we have $(s,Y)^\omega_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \Phi$ and we are done. In the latter case we find $Y' \in (s,Y)^\omega_{\mathscr{F}} \cap \mathscr{F}$ such that $(s,Y')^\omega_{\mathscr{F}} \subseteq \mathscr{O}_2$. (Otherwise we would have $(s,Y')^\omega_{\mathscr{F}} \cap (\mathscr{O}_2 \cup \mathscr{O}_1) \cap \mathscr{F} = \emptyset$, which is impossible by the density of $\mathscr{O}_1 \cup \mathscr{O}_2$.) Hence, $(s,Y')^\omega_{\mathscr{F}} \Vdash_{\mathbb{P}_{\mathscr{F}}} \neg\Phi$. ⊣

Let $\mathscr{F}$ be a game-family. If G is $\mathbb{P}_{\mathscr{F}}$-generic, then let $X_G := \bigcap G$. Now X_G is an infinite partition and $G = \{(s,Z)^\omega_{\mathscr{F}} : s \prec X_G \sqsubseteq Z\}$. Therefore we can consider the partition $X_G \in (\omega)^\omega$ as a $\mathbb{P}_{\mathscr{F}}$-generic object. Further we have $G \subseteq \mathbb{P}_{\mathscr{F}}$ is $\mathbb{P}_{\mathscr{F}}$-generic if and only if $X_G \in \bigcup D$ for all $D \subseteq \mathbb{P}_{\mathscr{F}}$ which are dense in $\mathbb{P}_{\mathscr{F}}$. Note that if D is dense in $\mathbb{P}_{\mathscr{F}}$, then $\bigcup D$ is $\mathscr{F}$-open.

The next theorem shows in fact that if $\mathscr{F}$ is a game-family, then $\mathbb{P}_{\mathscr{F}}$ is proper.

Theorem VI.4.4. Let $\mathscr{F} \subseteq (\omega)^\omega$ be a game-family. If $X_0 \in (\omega)^\omega$ is $\mathbb{P}_{\mathscr{F}}$-generic over $\mathbf{V}$ and $Y_0 \in (X_0)^\omega \cap \mathbf{V}[X_0]$, then Y_0 is also $\mathbb{P}_{\mathscr{F}}$-generic over $\mathbf{V}$.

Proof. Take an arbitrary dense set $D \subseteq \mathbb{P}_{\mathscr{F}}$, *i.e.*, for all $(s,X)^\omega_{\mathscr{F}}$ there exists a $(t,Y)^\omega_{\mathscr{F}} \subseteq (s,X)^\omega_{\mathscr{F}}$ such that $(t,Y)^\omega_{\mathscr{F}} \in D$. Let D' be the set of all $(s,Z)^\omega_{\mathscr{F}}$ such that $(t,Z)^\omega_{\mathscr{F}} \subseteq \bigcup D$ for all $t \sqsubseteq s$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$.

First we show that D' is dense in $\mathbb{P}_{\mathscr{F}}$. For this take an arbitrary $(s,W)^\omega_{\mathscr{F}}$ and let $\{t_i : 0 \leq i \leq m\}$ be an enumeration of all $t \in (\mathbb{N})$ such that $t \sqsubseteq s$ and $\mathrm{dom}(t) = \mathrm{dom}(s)$. Because D is dense in $\mathbb{P}_{\mathscr{F}}$ and $\bigcup D$ is $\mathscr{F}$-open, we find for every t_i a $W' \in \mathscr{F}$ such that $t_i \sqsubseteq W'$ and $(t_i,W')^\omega_{\mathscr{F}} \subseteq \bigcup D$. Moreover, if we define $W_{-1} := W$, we can choose for every $i \leq m$ a partition $W_i \in \mathscr{F}$ such that $W_i \sqsubseteq W_{i-1}$, $s \prec W_i$ and $(t_i,W_i)^\omega_{\mathscr{F}} \subseteq \bigcup D$. Now $(s,W_m)^\omega_{\mathscr{F}} \in D'$ and because $(s,W_m)^\omega_{\mathscr{F}} \subseteq (s,W)^\omega_{\mathscr{F}}$, D' is dense in $\mathbb{P}_{\mathscr{F}}$.

Since D' is dense and $X_0 \in (\omega)^\omega$ is $\mathbb{P}_{\mathscr{F}}$-generic, there exists a $(s,Z)^\omega_{\mathscr{F}} \in D'$ such that $s \prec X_0 \sqsubseteq Z$. Because $Y_0 \in (X_0)^\omega$ we have $t \prec Y_0 \sqsubseteq Z$ for some $t \sqsubseteq s$ and because $(t,Z)^\omega_{\mathscr{F}} \subseteq \bigcup D$, we get $Y_0 \in \bigcup D$. Hence, $Y_0 \in \bigcup D$ for every dense $D \subseteq \mathbb{P}_{\mathscr{F}}$, which completes the proof. ⊣

Remark 9. Similar results are proved in [**49**] and [**48**].

5. On dual Mathias forcing and game-filters

In this section we show that it is consistent with ZFC that game-filters exist. Further we show that the dual Mathias forcing $\mathbb{M}^\flat$ is flexible and with this result we can prove that if $\mathbf{V}$ is $\mathbf{\Sigma}^1_4$-$\mathbb{M}^\flat$-absolute, then $\omega_1^{\mathbf{V}}$ is inaccessible in $\mathbf{L}$, where $\mathbf{L}$ denotes Gödel's constructible universe.

In the sequel, let $\mathbb{W}$ be the forcing notion we defined in section 2.

Theorem VI.5.1. If $\mathscr{U}$ is $\mathbb{W}$-generic over $\mathbf{V}$, then $\mathscr{U}$ is a game-filter in $\mathbf{V}[\mathscr{U}]$ with respect to the game $\mathcal{G}(\mathscr{U})$.

Proof. Because $\mathscr{U}$ is $\mathbb{W}$-generic over $\mathbf{V}$, we know that $\mathscr{U} \subseteq (\omega)^\omega$ is a non-principal family in $\mathbf{V}[\mathscr{U}]$ which is closed under refinement and finite changes, and for $X, Y \in \mathscr{U}$

we also have $X \sqcap Y \in \mathscr{U}$. It remains to show that player II has no winning strategy in the game $\mathcal{G}(\mathscr{U})$.

Let $\tilde{\sigma}$ be a $\mathbb{W}$-name for a strategy for player II in the game $\mathcal{G}(\dot{\mathscr{U}})$, where $\dot{\mathscr{U}}$ is the canonical $\mathbb{W}$-name for the $\mathbb{W}$-generic object. Let us assume that player II will follow this strategy. We may assume that

$$\mathbb{1} \Vdash_{\mathbb{W}} \text{“}\tilde{\sigma} \text{ is a strategy for II in the game } \mathcal{G}(\dot{\mathscr{U}})\text{”}.$$

If

$$Z \Vdash_{\mathbb{W}} \tilde{\sigma}(\langle \tilde{X}_0 \rangle, \langle \tilde{t}_0, \tilde{Y}_0 \rangle, \dots, \langle \tilde{X}_n \rangle) = \langle \tilde{t}_n, \tilde{Y}_n \rangle,$$

then for $n \geq 1$ we get

$$Z \Vdash_{\mathbb{W}} (|\tilde{t}_n| = |\tilde{t}_{n-1}| + 1 \wedge \tilde{t}^*_{n-1} \prec \tilde{t}^*_n \prec \tilde{Y}_n \sqsubseteq \tilde{X}_n \wedge \tilde{Y}_n \in \dot{\mathscr{U}})$$

and for $n = 0$ we have

$$Z \Vdash_{\mathbb{W}} (|\tilde{t}_0| = |\tilde{s}| \wedge \tilde{s} \prec \tilde{t}^*_0 \prec \tilde{Y}_0 \sqsubseteq \tilde{X}_0 \sqsubseteq \tilde{X} \wedge \tilde{Y}_0 \in \dot{\mathscr{U}}) \,,$$

where $\langle \tilde{s}, \tilde{X} \rangle$ is the starting point of $\mathcal{G}(\dot{\mathscr{U}})$.

Now let $\langle \tilde{s}, \tilde{X} \rangle$ (the starting point of the game $\mathcal{G}(\dot{\mathscr{U}})$) be such that $(\tilde{s},\tilde{X})^\omega$ is a $\mathbb{W}$-name for a dual Ellentuck neighborhood and let $Z_0 \in (\omega)^\omega \cap \mathbf{V}$ be a $\mathbb{W}$-condition in $\mathbf{V}$ such that $Z_0 \Vdash_{\mathbb{W}} \tilde{X} \in \dot{\mathscr{U}}$. Therefore, $Z_0 \Vdash_{\mathbb{W}}$ “$(\tilde{s},\tilde{X})^\omega$ is a $\dot{\mathscr{U}}$-dual Ellentuck neighborhood". By Fact VI.2.3 we know that the forcing notion $\mathbb{W}$ adds no new reals (and therefore no new partitions) to $\mathbf{V}$. So, we find a $Z_0' \sqsubseteq^* Z_0$ and a dual Ellentuck neighborhood $(s,X)^\omega$ in $\mathbf{V}$ such that

$$Z_0' \Vdash_{\mathbb{W}} \langle \tilde{s}, \tilde{X} \rangle = \langle \check{s}, \check{X} \rangle \,,$$

where $\check{s}$ and $\check{X}$ are the canonical $\mathbb{W}$-names for s and X. Because $Z_0' \Vdash_{\mathbb{W}} \check{X} \in \dot{\mathscr{U}}$, we must have $Z_0' \leq X$, which is the same as $Z_0' \sqsubseteq^* X$. Finally put $X_0 \in (\omega)^\omega$ such that $X_0 \stackrel{*}{=} Z_0'$ and $X_0 \in (s,X)^\omega$. Player I plays now $\langle \check{X}_0 \rangle$. Since player II follows the strategy $\tilde{\sigma}$, player II plays now $\tilde{\sigma}(\langle \check{X}_0 \rangle) =: \langle \tilde{t}_0, \tilde{Y}_0 \rangle$. Again by Fact VI.2.3 there exists a $Z_1 \sqsubseteq^* X_0$ and a dual Ellentuck neighborhood $(t_0,Y_0)^\omega$ in $\mathbf{V}$ such that

$$Z_1 \Vdash_{\mathbb{W}} \langle \tilde{t}_0, \tilde{Y}_0 \rangle = \langle \check{t}_0, \check{Y}_0 \rangle \,.$$

And again by $Z_1 \Vdash_{\mathbb{W}} \check{Y}_0 \in \dot{\mathscr{U}}$ we find $X_1 \stackrel{*}{=} Z_1$ such that $t_0^* \prec X_1 \sqsubseteq Y_0$. Player I plays now $\langle \check{X}_1 \rangle$.

In general, if $\tilde{\sigma}(\langle \tilde{X}_0 \rangle, \langle \tilde{t}_0, \tilde{Y}_0 \rangle, \dots, \langle \tilde{X}_n \rangle) = \langle \tilde{t}_n, \tilde{Y}_n \rangle$, then player I can play $\check{X}_{n+1}$ such that $X_n \Vdash_{\mathbb{W}} \langle \tilde{t}_n, \tilde{Y}_n \rangle = \langle \check{t}_n, \check{Y}_n \rangle$ and $t_n^* \prec X_{n+1} \sqsubseteq Y_n$. For $n \geq m$ we also have $X_n \sqsubseteq X_m$. Let $Y \in (\omega)^\omega$ be the such that $t_n \prec Y$ (for all n), then

$$Y \Vdash_{\mathbb{W}} \text{“the only } \tilde{Y} \text{ such that } \tilde{t}_n \prec \tilde{Y} \text{ (for all } n) \text{ is in } \dot{\mathscr{U}}\text{”}.$$

Hence, the strategy $\tilde{\sigma}$ is not a winning strategy for player II and because $\tilde{\sigma}$ was an arbitrary strategy, player II has no winning strategy at all. $\dashv$

REMARK 10. A similar result is proved in [**49**] (see also [**47**]).

As a corollary we get that the forcing notion $\mathbb{P}_{\mathscr{U}}$, where $\mathscr{U}$ is $\mathbb{W}$-generic over $\mathbf{V}$, has pure decision in $\mathbf{V}[\mathscr{U}]$.

COROLLARY VI.5.2. Let $\mathscr{U}$ be $\mathbb{W}$-generic over $\mathbf{V}$. Then the forcing notion $\mathbb{P}_{\mathscr{U}}$ has pure decision in $\mathbf{V}[\mathscr{U}]$.

PROOF. This follows from Theorem VI.4.3 and Theorem VI.5.1. $\dashv$

Corollary VI.5.2 follows also from the facts that the dual Mathias forcing has pure decision (*cf.* [**11**]) and that it can be written as a two step iteration as in section 2.

REMARK 11. If $\mathcal{U}$ is $\mathbb{U}$-generic over $\mathbf{V}$, then $\mathbb{P}_{\mathcal{U}}$ has pure decision in $\mathbf{V}[\mathcal{U}]$ (*cf.* [**49**]).

6. More properties of $\mathbb{M}^\flat$

Let $\mathbb{P}$ be a notion of forcing in the model $\mathbf{V}$. We say that $\mathbf{V}$ is $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-**absolute** if for every $\mathbf{\Sigma}^1_n$-sentences Φ with parameters in $\mathbf{V}$ the following holds for any G which is $\mathbb{P}$-generic over $\mathbf{V}$:

$$\mathbf{V} \models \Phi \textit{ if and only if } \mathbf{V}[G] \models \Phi\,.$$

Now we will show that if $\mathbf{V}$ is $\mathbf{\Sigma}^1_4$-$\mathbb{M}^\flat$-absolute, then $\omega_1^{\mathbf{V}}$ is inaccessible in $\mathbf{L}$. For this we first will translate the dual Mathias forcing in a tree forcing notion.

If $s \in (\mathbb{N})$, then s is a partition of some natural number $n \in \omega$ and therefore s is a finite set of finite sets of natural numbers. Let t be a finite set of natural numbers, then $\sharp t$ is such that for all $k \in \omega$ we have $\mathrm{div}(\sharp t, 2^k)$ *is odd* $\Leftrightarrow k \in s$. Remember that $\mathrm{div}(n,m) := \max\{k \in \omega : k \cdot m \leq n\}$. Now, let $\sharp s$ be such that for all $k \in \omega$:

$$\mathrm{div}(\sharp s, 2^k) \textit{ is odd } \Leftrightarrow k = \sharp t \textit{ for some } t \in s\,.$$

In fact, $\sharp s$ is defined for any finite set of finite sets of natural numbers. If $s \in (\mathbb{N})$, then $|s|$ denotes the cardinality of s, which is the number of blocks of s.

For $s \in (\mathbb{N})$ with $|s| = k$ let $\bar{s}$ be the finite sequence $\langle n_1, \dots, n_k\rangle$ where $n_i := \sharp s_i$ and $s_i \in (\mathbb{N})$ is such that $|s_i| = i$ and $s_i^* \prec s^*$.

Now let $p = (s,X)^\omega$ be an $\mathbb{M}^\flat$-condition. Without loss of generality we may assume that $s^* \sqsubseteq X$. The tree $T_p \subseteq \omega^{<\omega}$ is defined as follows.

$$\sigma \in T_p \;\Leftrightarrow\; \exists t \in (\mathbb{N})\big((t^* \prec s^* \vee s \prec t) \wedge t^* \sqsubseteq X \wedge \sigma = \bar{t}\big)\,.$$

FACT VI.6.1. Let p, q be two $\mathbb{M}^\flat$-conditions. Then T_p is a subtree of T_q if and only if $p \leq q$.

Finally let $\boldsymbol{T}_{\mathbb{M}^\flat} := \{T_p : p \in \mathbb{M}^\flat\}$; then $\boldsymbol{T}_{\mathbb{M}^\flat}$ is a set of trees. We stipulate that $T_p \leq T_q$ if T_p is a subtree of T_q. Then (by Fact VI.6.1) forcing with $\mathbb{T}_{\mathbb{M}^\flat} := \langle \boldsymbol{T}_{\mathbb{M}^\flat}, \leq\rangle$ is the same as forcing with $\mathbb{M}^\flat$.

Now we will give the definition of a flexible forcing notion $\mathbb{P}$. But first we have to give some other definitions.

A set $T \subseteq \omega^{<\omega}$ is called a **Laver-tree** if

$$T \text{ is a tree and } \exists \tau \in T\ \forall \sigma \in T\big(\sigma \subseteq \tau \vee (\tau \subseteq \sigma \;\wedge\; |\{n : \sigma^\frown n \in T\}| = \omega)\big)\,.$$

We call τ the **stem** of T. For $\sigma \in T$ we let $\mathrm{succ}_T(\sigma) := \{n : \sigma^\frown n \in T\}$, the **successors** of σ in T, and $T_\varrho := \{\sigma \in T : \sigma \subseteq \varrho \ \wedge \ \varrho \subseteq \sigma\}$.

For a Laver-tree T, we say $A \subseteq T$ is a **front** if $\sigma \neq \tau$ in A implies $\sigma \not\subseteq \tau$ and for all $f \in [T]$ there is an $n \in \omega$ such that $f|_n \in A$.

The meaning of $p \leq [\![\Phi]\!]$ and $p \cap [\![\Phi]\!]$ are $U_p \subseteq [\![\Phi]\!]$ and $U_p \cap [\![\Phi]\!]$, respectively.

(i) We say a forcing notion $\mathbb{P}$ is **Laver-like** if there is a $\mathbb{P}$-name $\tilde{r}$ for a dominating real such that

(i) the complete Boolean algebra generated by the family $\big\{[\![\tilde{r}(i) = n]\!] : i, n \in \omega\big\}$ equals r.o.$(\mathbb{P})$, and

(ii) for each condition $p \in \mathbb{P}$ there exists a Laver-tree $T \subseteq \omega^{<\omega}$ so that for all $\sigma \in T$ we have:

$$p(T_\sigma) := \prod_{n\in\omega} \sum_{\tau \in T_\sigma} \big\{p \cap [\![\tilde{r}|_{\mathrm{lg}(\tau)} = \tau]\!] : \mathrm{lg}(\tau) = n\big\} \in \mathrm{r.o.}(\mathbb{P}) \setminus \{\mathbf{0}\}\,.$$

We express this by saying $p(T) \neq \emptyset$, where $p(T) := p(T_{stem(T)})$.

(ii) If $\tilde{r}$ is a $\mathbb{P}$-name that witnesses that $\mathbb{P}$ is Laver-like, we say that $\mathbb{P}$ has **strong fusion** if for countably many open dense sets $D_n \subseteq \mathbb{P}$ and for $p \in \mathbb{P}$, there is a Laver-tree T such that $p(T) \neq \emptyset$ and for each n the set

$$\big\{\sigma \in T : p(T) \cap [\![\tilde{r}|_{\mathrm{lg}(\sigma)} = \sigma]\!] \in D_n\big\}$$

contains a front.

(iii) A Laver-like $\mathbb{P}$ is **closed under finite changes** if given $p \in \mathbb{P}$ and Laver trees T and T' so that for all $\sigma \in T'$, if $p(T) \neq \emptyset$ then $|\mathrm{succ}_T(\sigma) \setminus \mathrm{succ}_{T'}(\sigma)| < \omega$, then $p(T') \neq \emptyset$, too.

We call a forcing notion $\mathbb{P}$ **flexible**, if $\mathbb{P}$ is Laver-like, has strong fusion and is closed under finite changes.

With this definition we can show – as a further symmetry between the forcing notions $\mathbb{M}$ and $\mathbb{M}^\flat$ – that dual Mathias forcing $\mathbb{M}^\flat$ is flexible.

LEMMA VI.6.2. The dual Mathias forcing $\mathbb{M}^\flat$ is flexible.

PROOF. Since $\mathbb{M}^\flat \approx \boldsymbol{T}_{\mathbb{M}^\flat}$, it is enough to prove that the forcing notion $\boldsymbol{T}_{\mathbb{M}^\flat}$ is flexible. Let $\tilde{r}$ be the canonical $\boldsymbol{T}_{\mathbb{M}^\flat}$-name for the $\boldsymbol{T}_{\mathbb{M}^\flat}$-generic object. By the definition of the function "$\sharp$" and the construction of $\boldsymbol{T}_{\mathbb{M}^\flat}$, $\tilde{r}$ is a name for a dominating real. The rest of the proof is similar to the proof that Mathias forcing is flexible, which is given in **[25]**. $\dashv$

Let $\mathbf{W}$ be a submodel of $\mathbf{V}$. If all $\boldsymbol{\Sigma}^1_n$-sets in $\mathbf{V}$ with parameters in $\mathbf{V} \cap \mathbf{W}$ have the Ramsey property $\boldsymbol{R}$ or the dual Ramsey property $\boldsymbol{R}^\flat$, then we write $\mathbf{V} \models \boldsymbol{\Sigma}^1_n(\boldsymbol{R})_{\mathbf{W}}$ and $\mathbf{V} \models \boldsymbol{\Sigma}^1_n(\boldsymbol{R}^\flat)_{\mathbf{W}}$, respectively. If $\mathbf{V} = \mathbf{W}$, then we omit the index $\mathbf{W}$. The notations for $\boldsymbol{\Delta}^1_n$-sets and $\boldsymbol{\Pi}^1_n$-sets are similar. Further, $\boldsymbol{B}$ stands for the Baire property and $\boldsymbol{L}$ stands for Lebesgue measurability.

Now we can prove the following

THEOREM VI.6.3. If $\mathbf{V}$ is $\mathbf{\Sigma}^1_4$-$\mathbb{M}^\flat$-absolute, then $\omega_1^{\mathbf{V}}$ is inaccessible in $\mathbf{L}$.

PROOF. To prove the corresponding result for Mathias forcing (*cf.* [**25**]), one uses only that $\mathbb{M}$ is flexible and that, if $\mathbf{V}$ is $\mathbf{\Sigma}^1_4$-$\mathbb{M}$-absolute, then $\mathbf{V} \models \mathbf{\Sigma}^1_2(\boldsymbol{R})$, which is the same as $\mathbf{\Sigma}^1_3$-$\mathbb{M}$-absoluteness (*cf.* [**25**, Theorem 4.1]). Therefore, it is enough to prove that $\mathbf{\Sigma}^1_3$-$\mathbb{M}^\flat$-absoluteness implies $\mathbf{\Sigma}^1_3$-$\mathbb{M}$-absoluteness. It follows immediately from Fact VI.2.6 that $\mathbf{V} \subseteq \mathbf{V}^{\mathbb{M}} \subseteq \mathbf{V}^{\mathbb{M}^\flat}$, and since $\mathbf{\Sigma}^1_3$-formulas are upwards absolute, this completes the proof. ⊣

7. Iteration of dual Mathias forcing

In this section we will build two models in which every $\mathbf{\Sigma}^1_2$-set is dual Ramsey. In the first model $\mathfrak{c} = \omega_1$ and in the second model $\mathfrak{c} = \omega_2$. With the result that dual Mathias forcing has the Laver property we further can show that $\mathbf{\Sigma}^1_2(\boldsymbol{R}^\flat)$ implies neither $\mathbf{\Sigma}^1_2(\boldsymbol{L})$ nor $\mathbf{\Sigma}^1_2(\boldsymbol{B})$, but first we give a result similar to Theorem 1.15 of [**39**].

LEMMA VI.7.1. Let $\mathscr{U}$ be $\mathbb{W}$-generic over $\mathbf{V}$. If X_G is $\mathbb{P}_{\mathscr{U}}$-generic over $\mathbf{V}[\mathscr{U}]$, then $\mathbf{V}[\mathscr{U}][X_G] \models \mathbf{\Sigma}^1_2(\boldsymbol{R}^\flat)_{\mathbf{V}}$.

PROOF. Let $\dot{X}_G$ be the canonical name for the $\mathbb{P}_{\mathscr{U}}$-generic object X_G over $\mathbf{V}[\mathscr{U}]$ and let $\varphi(Y)$ be a $\mathbf{\Sigma}^1_2$-formula with parameters in $\mathbf{V}$. By Theorem VI.5.1 and Corollary VI.5.2, the forcing notion $\mathbb{P}_{\mathscr{U}}$ has pure decision. So, there exists a $\mathbb{P}_{\mathscr{U}}$-condition $p \in \mathbf{V}[\mathscr{U}]$ with empty stem, or in other words, there is a $p \in \mathscr{U}$ so that $\mathbf{V}[\mathscr{U}] \models$ "$p \Vdash_{\mathbb{P}_{\mathscr{U}}} \varphi(\dot{X}_G)$" or $\mathbf{V}[\mathscr{U}] \models$ "$p \Vdash_{\mathbb{P}_{\mathscr{U}}} \neg\varphi(\dot{X}_G)$". Assume the former case holds. Because $X_G \sqsubseteq^* q$ for all $q \in \mathscr{U}$, there is an $f \in [\omega]^{<\omega}$ such that $X_G \sqcap \{f\} \sqsubseteq p$. By Theorem VI.5.1 and Theorem VI.4.4 we know that if X is $\mathbb{P}_{\mathscr{U}}$-generic over $\mathbf{V}[\mathscr{U}]$ and $X' \in (X)^\omega \cap \mathbf{V}[\mathscr{U}][X_G]$, then X' is also $\mathbb{P}_{\mathscr{U}}$-generic over $\mathbf{V}[\mathscr{U}]$. Hence, every $X'_G \sqsubseteq X_G \sqcap \{f\} \sqsubseteq p$ is $\mathbb{P}_{\mathscr{U}}$-generic over $\mathbf{V}[\mathscr{U}]$ and therefore $\mathbf{V}[\mathscr{U}][X'_G] \models \varphi(X'_G)$. Because $\mathbf{\Sigma}^1_2$-formulas are absolute we get $\mathbf{V}[\mathscr{U}][X_G] \models \varphi(X'_G)$. Thus, $\mathbf{V}[\mathscr{U}][X_G] \models \exists X\, \forall Y \in (X)^\omega \big(\varphi(Y)\big)$. The case when $\mathbf{V}[\mathscr{U}] \models$ "$p \Vdash_{\mathbb{P}_{\mathscr{U}}} \neg\varphi(\dot{X}_G)$" is similar. Hence, we finally have $\mathbf{V}[\mathscr{U}][X_G] \models \mathbf{\Sigma}^1_2(\boldsymbol{R}^\flat)_{\mathbf{V}}$. ⊣

REMARK 12. The proof of the analogous result can be found in [**39**].

Because Gödel's constructible universe $\mathbf{L}$ has a $\mathbf{\Delta}^1_2$-well-ordering of the reals, $\mathbf{L}$ is neither a model for $\mathbf{\Delta}^1_2(\boldsymbol{R}^\flat)$ nor a model for $\mathbf{\Delta}^1_2(\boldsymbol{R})$. But we can build a model in which $\mathfrak{c} = \omega_1$ and all $\mathbf{\Sigma}^1_2$-sets are dual Ramsey.

THEOREM VI.7.2. After an ω_1-iteration of dual Mathias forcing with countable support starting from $\mathbf{L}$, we get a model in which every $\mathbf{\Sigma}^1_2$-set of reals is dual Ramsey and $\mathfrak{c} = \omega_1$.

PROOF. The proof follows immediately from Fact VI.2.5, Lemma VI.7.1 and the fact that dual Mathias forcing is proper. ⊣

REMARK 13. The proof of a similar result can be found in [**38**].

We can build also a model in which all $\mathbf{\Sigma}^1_2$-sets are dual Ramsey and in which $\mathfrak{c} = \omega_2$.

THEOREM VI.7.3. After an ω_2-iteration of dual Mathias forcing with countable support starting from $\mathbf{L}$, we get a model in which every $\mathbf{\Sigma}^1_2$-set of reals is dual Ramsey and $\mathfrak{c} = \omega_2$.

PROOF. In Chapter V (see also [**22**]) it was shown that an ω_2-iteration of dual Mathias forcing with countable support starting from $\mathbf{L}$ yields a model in which $\mathfrak{c} = \omega_2$ and the union of fewer than ω_2 completely dual Ramsey sets is completely dual Ramsey. Now because each $\mathbf{\Sigma}^1_2$-set can be written as the union of ω_1 analytic sets, and because analytic sets are completely dual Ramsey, all $\mathbf{\Sigma}^1_2$-sets are dual Ramsey in that model. $\dashv$

REMARK 14. A similar result is true because an ω_2-iteration of Mathias forcing with countable support starting from $\mathbf{L}$ yields a model in which $\mathfrak{h} = \omega_2$ (*cf.* [**63**]), and $\mathfrak{h}$ can be considered as the additivity of the ideal of completely Ramsey null sets (*cf.* [**54**]).

For the next result we have to give first the definition of the Laver property: A **cone** $\bar{A}$ is a sequence $\langle A_k : k \in \omega\rangle$ of finite subsets of ω with $|A_k| < 2^k$. We say that $\bar{A}$ **covers** a function $f \in \omega^\omega$ if for all positive $k \in \omega$ we have $f(k) \in A_k$. For a function $H \in {}^\omega\omega$, we write ΠH for the set $\big\{f \in \omega^\omega : \forall k > 0\, \big(f(k) < H(k)\big)\big\}$. Now, a forcing notion $\mathbb{P}$ is said to have the **Laver property** iff for every $H \in {}^\omega\omega$ in $\mathbf{V}$,

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}\forall f \in \Pi H\, \exists \bar{A} \in \mathbf{V}(\bar{A} \text{ is a cone covering } f)\text{”}.$$

Like Mathias forcing, dual Mathias forcing has the Laver property and therefore adds no Cohen reals (*cf.* [**18**] or [**3**]).

LEMMA VI.7.4. The forcing notion $\mathbb{M}^\flat$ has the Laver property.

PROOF. Given $f, H \in {}^\omega\omega$ such that for all $k > 0$, $f(k) < H(k)$. Let $\langle s, X\rangle$ be any $\mathbb{M}^\flat$-condition. Because $\mathbb{M}^\flat$ has pure decision and $f(1) < H(1)$, we find a $Y_0 \in (s,X)^\omega$ such that $\langle s, Y_0\rangle$ decides $f(1)$. Set $s_0 := s$. Suppose we have already constructed $s_n \in (\mathbb{N})$ and $Y_n \in (\omega)^\omega$ such that $s \prec s_n$, $|s_n| = |s| + n$ and $(s_n,Y_n)^\omega$ is a dual Ellentuck neighborhood. Choose $Y_{n+1} \in (s_n,Y_n)^\omega$ such that for all $h \in (\mathbb{N})$ with $s \prec h \sqsubseteq s_n$ and $\operatorname{dom}(h) = \operatorname{dom}(s_n)$, $\langle h, Y_{n+1}\rangle$ decides $f(k)$ for all $k < 2^{n+1}$. Further, let $s_{n+1} \in (\mathbb{N})$ be such that $s_n \prec s_{n+1}$, $|s_{n+1}| = |s_n| + 1 = |s| + n + 1$ and $s_{n+1} \prec Y_{n+1}$. Finally, let Y be the unique partition such that for all $n \in \omega$, $s_n \prec Y$. Evidently, the $\mathbb{M}^\flat$-condition $\langle s, Y\rangle$ is stronger than the given $\mathbb{M}^\flat$-condition $\langle s, X\rangle$ (or equal). Now, if $k, n \in \omega$ such that $2^n \le k < 2^{n+1}$, then let $\{h_j : j \le m\}$ be an enumeration of all $s \prec h \sqsubseteq s_n$ with $\operatorname{dom}(h) = \operatorname{dom}(s_n)$. It is clear that $m < 2^{2^n}$. Further, let

$$A_k := \big\{l \in \omega : \exists j \le m\big(\langle h_j, Y\rangle \Vdash_{\mathbb{M}^\flat} f(k) = l\big)\big\}\,,$$

then $|A_k| \le m < 2^{2^n}$, and because $2^n \le k$ we have $|A_k| < 2^k$. If we define $A_0 := \big\{l \in \omega : \langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} f(0) = l\big\}$, then the $\mathbb{M}^\flat$-condition $\langle s, Y\rangle$ forces that $\bar{A} := \langle A_k : k \in \omega\rangle$ is a cone for f. $\dashv$

Using these results we can prove the following

THEOREM VI.7.5. $\mathbf{\Sigma}^1_2(\boldsymbol{R}^\flat)$ implies neither $\mathbf{\Sigma}^1_2(\boldsymbol{L})$ nor $\mathbf{\Sigma}^1_2(\boldsymbol{B})$.

PROOF. Because a forcing notion with the Laver property adds no Cohen reals and because the Laver property is preserved under countable support iterations of proper forcings (with the Laver property), in the model constructed in Theorem VI.7.2 no real is Cohen over $\mathbf{L}$. Therefore, in that model $\mathbf{\Delta}^1_2(\boldsymbol{B})$ fails, and because $\mathbf{\Sigma}^1_2(\boldsymbol{L})$ implies $\mathbf{\Sigma}^1_2(\boldsymbol{B})$ (*cf.* [**38**]), also $\mathbf{\Sigma}^1_2(\boldsymbol{L})$ must fail in that model. ⊣

REMARK 15. For the analogous result see [**39**].

8. Appendix: On the dual Ramsey property of projective sets

Although the Ramsey property and the dual Ramsey property are very similar, one can show that the two Ramsey properties are different.

THEOREM VI.8.1. Using the axiom of choice one can construct a set which is Ramsey but not dual Ramsey.

PROOF. We will construct a set $\mathcal{R} \subseteq [\omega]^\omega$ which is Ramsey but not dual Ramsey.

Remember that the relation " $\overset{*}{=}$ " is an equivalence-relation on $(\omega)^\omega$, where $X \overset{*}{=} Y$ if and only if there are $f, g \in [\omega]^{<\omega}$ such that $X \sqcap \{f\} \sqsubseteq Y$ and $Y \sqcap \{g\} \sqsubseteq X$. For $X \in (\omega)^\omega$, let $\{X^\sim$ denote the equivalence class of X. Now, choose from each equivalence class $X^\sim$ an element A_X and for $X \in (\omega)^\omega$ let

$$h_X := \min\big\{|f| + |g| : f, g \in [\omega]^{<\omega} \textit{ and } X \sqcap \{f\} \sqsubseteq A_X \textit{ and } A_X \sqcap \{g\} \sqsubseteq X\big\}.$$

Further, define a function $F : (\omega)^\omega \to \{0, 1\}$ by stipulating

$$F(X) := \begin{cases} 1 & \text{if } h_X \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the set $\{X \in (\omega)^\omega : F(X) = 1\}$ is obviously not dual Ramsey and therefore, the set $\mathcal{R} := \big\{x \in [\omega]^\omega : \exists X \in (\omega)^\omega\big(x = pc(X) \wedge F(X) = 1\big)\big\}$ is not dual Ramsey as well.

Now, define $r := \{\varsigma\{k, k+1\} : k \in \omega\}$, where "$\varsigma$" as in Section 2, then $cp(r) = \big\{\{\omega\}\big\} \notin (\omega)^\omega$ and hence, $[r]^\omega \cap \mathcal{R} = \emptyset$. So, the set $\mathcal{R}$ is Ramsey. ⊣

On the other hand, for projective sets one can show that the dual Ramsey property is stronger than the Ramsey property.

LEMMA VI.8.2. If $\mathbf{V} \models \mathbf{\Sigma}^1_n(\boldsymbol{R}^\flat)$, then $\mathbf{V} \models \mathbf{\Sigma}^1_n(\boldsymbol{R})$.

PROOF. Given a $\mathbf{\Sigma}^1_n$-formula $\varphi(x)$ with parameters in $\mathbf{V}$. Let $\psi(y)$ be defined as follows:

$$\psi(y) \iff \exists x\big(x = \operatorname{Min}\big(cp(y)\big) \wedge \varphi(x)\big).$$

It is easy to see that $\psi(y)$ is also a $\mathbf{\Sigma}^1_n$-formula (even with the same parameters as φ). Now, if there is an $X \in (\omega)^\omega$ such that for all $Y \in (X)^\omega$, $\psi\big(pc(Y)\big)$ holds, then for all $y \in [x]^\omega$ where $x = \mathrm{Min}(X)$, $\varphi(y)$ holds. The case where for all $Y \in (X)^\omega$, $\neg\psi\big(pc(Y)\big)$ holds, is similar. $\dashv$

In [**11**, Section 5], Carlson and Simpson prove that in the Solovay model, constructed by collapsing an inaccessible cardinal to ω_1, every projective set is dual Ramsey (it is unknown whether the inaccessible cardinal is necessary for that).

Another question connected to the dual Ramsey property of projective sets is the following. As with the standard Ramsey property we can ask whether an appropriate amount of determinacy implies the dual Ramsey property. As usually with regularity properties of sets of reals we would expect that $\mathrm{Det}(\mathbf{\Pi}^1_n)$ implies the dual Ramsey property for all $\mathbf{\Sigma}^1_{n+1}$ sets. But a direct implication using determinacy is not as easy as with the more prominent regularity properties (as Lebesgue measurability and the Baire property) since the games connected to the dual Ramsey property (the Banach-Mazur games in the dual Ellentuck topology) cannot be played using natural numbers.

The same problem had been encountered with the classical Ramsey property and had been solved by Leo Harrington and Alexander Kechris in [**33**] by making use of the scale property and the periodicity theorems. They showed the following.

Proposition VI.8.3. If $\mathrm{Det}(\mathbf{\Delta}^1_{2n+2})$, then every $\mathbf{\Pi}^1_{2n+2}$-set is Ramsey.

Using the techniques of Harrington and Kechris, Benedikt Löwe could strengthen their result and prove the following (see [**26**, Section 6]).

Proposition VI.8.4. If $\mathrm{Det}(\mathbf{\Delta}^1_{2n+2})$, then every $\mathbf{\Sigma}^1_{2n+2}$-set is dual Ramsey.

CHAPTER VII

Ramseyan Ultrafilters and Dual Mathias Forcing

In this chapter we investigate families of partitions which are related to special coideals, so-called happy families, and give a dual form of Ramsey ultrafilters in terms of partitions. The combinatorial properties of these partition-ultrafilters, which we call Ramseyan ultrafilters, are similar to those of Ramsey ultrafilters. For example it will be shown that dual Mathias forcing restricted to a Ramseyan ultrafilter has the same features as Mathias forcing restricted to a Ramsey ultrafilter. Further we introduce an ordering on the set of partition-filters and consider the dual form of some cardinal characteristics of the continuum.

1. Introduction

The Stone-Čech compactification $\beta\omega$ of the natural numbers, or equivalently, the ultrafilters over ω, is a well-studied space (*cf.* *e.g.* [**66**] and [**14**]) which has a lot of interesting topological and combinatorial features (*cf.* [**34**] and [**64**]). In the late 1960's, a partial ordering on the non-principal ultrafilters $\beta\omega\setminus\omega$, the so-called Rudin-Keisler ordering, was established and "small" points with respect to this ordering were investigated rigorously (*cf.* [**8**], [**5**], [**6**] and [**45**]). The minimal points have a nice combinatorial characterization which is related to Ramsey's Theorem (*cf.* [**57**, Theorem A]) and so, the ultrafilters which are minimal with respect to the Rudin-Keisler ordering are also called Ramsey ultrafilters (for further characterizations of Ramsey ultrafilters see [**3**, Chapter 4.5]). Families, not necessarily filters, having similar combinatorial properties as Ramsey ultrafilters, are the so-called happy families (*cf.* [**49**]), which are very important in the investigation of Mathias forcing (*cf.* [**49**]).

In the sequel we will introduce an ordering on the set of partition-filters which is similar to the Rudin-Keisler ordering on $\beta\omega\setminus\omega$ and introduce a partition form of Ramsey ultrafilters, so-called Ramseyan ultrafilters. Further we will investigate dual Mathias forcing restricted to Ramseyan ultrafilters and consider the dual form of some cardinal characteristics of the continuum which are related to Ramseyan ultrafilters.

2. An ordering on the set of partition-filters

Following Chapter III, let $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ denote the set of all partition-filters. We define a partial ordering on $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ which has some similarities with the Rudin-Keisler ordering on $\beta\omega\setminus\omega$.

To keep the notation short, for $\mathscr{H} \subseteq \mathcal{P}\big(\mathcal{P}(\omega)\big)$ and a function $f : \omega \to \omega$ we define

$$f^{-1}(\mathscr{H}) := \{f^{-1}(X) : X \in \mathscr{H}\},$$

where for $X \in \mathscr{H}$ we define

$$f^{-1}(X) := \{f^{-1}(b) : b \in X\},$$

where for $b \subseteq \omega$, $f^{-1}(b) := \{n : f(n) \in b\}$.

Let $f : \omega \twoheadrightarrow \omega$ be any surjection from ω onto ω and let $X \in (\omega)^{\leq\omega}$ be any partition. Then $f(X)$ denotes the finest partition such that whenever n and m lie in the same block of X, then $f(n)$ and $f(m)$ lie in the same block of $f(X)$.
For any partition-filter $\mathscr{F} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$ define

$$f(\mathscr{F}) := \big\{Y \in (\omega)^{\leq\omega} : \exists X \in \mathscr{F}\big(f(X) \sqsubseteq Y\big)\big\}.$$

We define the ordering "$\lesssim$" on $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$ as follows:

$$\mathscr{F} \lesssim \mathscr{G} \quad \text{if and only if} \quad \mathscr{F} = f(\mathscr{G}) \text{ for some surjection } f : \omega \twoheadrightarrow \omega\,.$$

Since the identity map is a surjection and the composition of two surjections is again a surjection, the partial ordering "$\lesssim$" is reflexive and transitive.

Fact VII.2.0.1. Let $\mathscr{F}$, $\mathscr{G} \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$ and assume $f(\mathscr{G}) = \mathscr{F}$ for some surjection $f : \omega \twoheadrightarrow \omega$. Then $\mathscr{G} \subseteq f^{-1}(\mathscr{F})$ and $f^{-1}(\mathscr{F}) \in \mathtt{PF}\big((\omega)^{\leq\omega}\big)$.

Proof. Let $\mathscr{H} = f^{-1}(\mathscr{F})$, where $f : \omega \twoheadrightarrow \omega$ is such that $f(\mathscr{G}) = \mathscr{F}$. Since $\mathscr{F}$ is a partition-filter and f is a function, for any $X_1, X_2 \in \mathscr{F}$ we have $X_1 \sqcap X_2 \in \mathscr{F}$ and $f^{-1}(X_1 \sqcap X_2) = f^{-1}(X_1) \sqcap f^{-1}(X_2)$, and therefore, $\mathscr{H}$ is a partition-filter. Further, for any $Y \in \mathscr{G}$ we get $f(Y) \in \mathscr{F}$ and $f^{-1}(f(Y)) \sqsubseteq Y$, which implies $\mathscr{G} \subseteq \mathscr{H}$. $\dashv$

The ordering "$\lesssim$" induces in a natural way an equivalence relation "$\simeq$" on the set of partition-filters $\mathtt{PF}\big((\omega)^{\leq\omega}\big)$:

$$\mathscr{F} \simeq \mathscr{G} \quad \text{if and only if} \quad \mathscr{F} \lesssim \mathscr{G} \text{ and } \mathscr{G} \lesssim \mathscr{F}\,.$$

So, the ordering "$\lesssim$" induces a partial ordering of the set of equivalence classes of partition-filters. Concerning partition-ultrafilters, we get the following.

Fact VII.2.0.2. Let $\mathscr{U}, \mathscr{V} \in \mathtt{PUF}\big((\omega)^{\leq\omega}\big)$ and assume that $\mathscr{U}$ is principal or contains a partition, all of whose blocks are infinite. If $\mathscr{U} \simeq \mathscr{V}$, then there is a permutation h of ω such that $h(\mathscr{U}) = \mathscr{V}$.

Proof. Because $\mathscr{U} \lesssim \mathscr{V}$ and $\mathscr{V} \lesssim \mathscr{U}$, there are surjections f and g from ω onto ω such that $\mathscr{V} = f(\mathscr{U})$ and $\mathscr{U} = g(\mathscr{V})$, and because $\mathscr{U}$ and $\mathscr{V}$ are both partition-ultrafilters, by Fact VII.2.0.1 we get $\mathscr{U} = f^{-1}(\mathscr{V})$ and $\mathscr{V} = g^{-1}(\mathscr{U})$.
First assume that $\mathscr{U}$ is principal and therefore contains a 2-block partition $X = \{b_0, b_1\}$. Because $g^{-1}(X) \in \mathscr{V}$, the partition-ultrafilter $\mathscr{V}$ is also principal and we get

$$\mathscr{V} = \big\{Y \in (\omega)^{\leq\omega} : g^{-1}(X) \sqsubseteq Y\big\},$$

where

$$g^{-1}(X) = \{g^{-1}(b_0), g^{-1}(b_1)\} =: \{c_0, c_1\}.$$

Now, because $\mathscr{U} = f^{-1}(\mathscr{V})$, we must have $f^{-1}(g^{-1}(X)) = X$, which implies that $f^{-1}(g^{-1}(b_i)) \in \{b_0, b_1\}$ (for $i \in \{0,1\}$). If one of the blocks of X is finite, say b_0, then $f|_{b_0}$ as well as $g|_{f(b_0)}$ must be one-to-one, and therefore, b_0 has the same cardinality as c_0. Hence, no matter if one of the blocks of X is finite or not, we can define a permutation h of ω such that $h(b_0) = c_0$ and $h(b_1) = c_1$, which implies $h(\mathscr{U}) = \mathscr{V}$. Now assume that $\mathscr{U}$ contains a partition $X = \{b_i : i \in \omega\}$, all of whose blocks b_i are infinite. Because g is a surjection, $g^{-1}(X)$, which is a member of $\mathscr{V}$, is a partition, all of whose blocks are infinite. Let h be a permutation of ω such that $h(b_i) = g^{-1}(b_i)$. Take any $Y \in \mathscr{V}$ with $Y \sqsubseteq g^{-1}(X)$. By the definition of h we have $h^{-1}(Y) = g(Y)$ and since $\mathscr{U} = g(\mathscr{V})$ there is a $Z \in \mathscr{U}$ such that $g(Y) = Z$, which implies $h(Z) = Y$, hence, $h(\mathscr{U}) = \mathscr{V}$. ⊣

The following proposition shows that "$\lesssim$" is directed upward (for a similar result concerning the Rudin-Keisler ordering see [**5**, p. 147]).

FACT VII.2.0.3. For any partition-filters $\mathscr{D}, \mathscr{E} \in \mathtt{PF}((\omega)^{\leq\omega})$, there is a partition-filter $\mathscr{F} \in \mathtt{PF}((\omega)^{\leq\omega})$, such that $\mathscr{D} \lesssim \mathscr{F}$ and $\mathscr{E} \lesssim \mathscr{F}$.

PROOF. Let ϱ_1 and ϱ_2 be two functions from ω into ω defined by $\varrho_1(n) := 2n$ and $\varrho_2(n) := 2n+1$. For a partition X and $i \in \{0,1\}$, let $\varrho_i(X) := \{\varrho_i(b) : b \in X\}$, where $\varrho_i(b) := \{\varrho_i(n) : n \in b\}$. Now, take any two partition-filters $\mathscr{D}, \mathscr{E} \in \mathtt{PF}((\omega)^{\leq\omega})$ and define $\mathscr{F}$ by

$$\mathscr{F} := \{\varrho_1(X) \cup \varrho_2(Y) : X \in \mathscr{D} \wedge Y \in \mathscr{E}\}.$$

Clearly, this defines a partition-filter. Define two surjections f and g from ω onto ω as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

$$g(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $f(\mathscr{F}) = \mathscr{D}$ and $g(\mathscr{F}) = \mathscr{E}$, which implies $\mathscr{D} \lesssim \mathscr{F}$ and $\mathscr{E} \lesssim \mathscr{F}$. ⊣

3. Ramseyan ultrafilters

3.1. Coloring segments. For the reader's convenience, let us recall some definitions: For $n \in \omega$, $(\omega)^{n*}$ denotes the set of all $u \in (\mathbb{N})$ such that $|u| = n$. Further, for $n \in \omega$ and $X \in (\omega)^{\omega}$ let

$$(X)^{n*} := \{u \in (\mathbb{N}) : |u| = n \wedge u^* \sqsubseteq X\};$$

and if $s \in (\mathbb{N})$ is such that $|s| \leq n$ and $s \sqsubseteq X$, let

$$(s, X)^{n*} := \{u \in (\mathbb{N}) : |u| = n \wedge s \prec u \wedge u^* \sqsubseteq X\}.$$

Let us state again Theorem IV.2.1:

PROPOSITION VII.3.1.1. For any coloring of $(\omega)^{(n+1)*}$ with $r+1$ colors, where $r, n \in \omega$, and for any $Z \in (\omega)^\omega$, there is an infinite partition $X \in (Z)^\omega$ such that $(X)^{(n+1)*}$ is monochromatic.

As we have seen in Chapter IV, this combinatorial result – which can also be derived from [**11**, Theorem 1.2] – is the partition form of Ramsey's Theorem.

We say that a surjection $f : \omega \twoheadrightarrow \omega$ **respects** the partition $X \in (\omega)^\omega$, if we have $f^{-1}(f(X)) = X$, otherwise, we say that it **disregards** the partition X. If $f^{-1}(f(X)) = \{\omega\}$, then we say that f **completely disregards** the partition X.

LEMMA VII.3.1.2. For any surjection $f : \omega \twoheadrightarrow \omega$ and for any $Z \in (\omega)^\omega$, there is an $X \in (Z)^\omega$ such that f either respects or completely disregards the partition X.

PROOF. For a surjection $f : \omega \twoheadrightarrow \omega$, define the coloring $\pi : (\omega)^{2*} \to \{0, 1\}$ as follows. $\pi(s) := 0$ if and only if $f(s(0)) \cap f(s(1)) = \emptyset$. By Proposition VII.3.1.1, there is a partition $X \in (Z)^\omega$ such that $(X)^{2*}$ is monochromatic with respect to π, which implies that f respects X in case of $\pi|_{(X)^{2*}} = \{0\}$, and f completely disregards X is case of $\pi|_{(X)^{2*}} = \{1\}$. $\dashv$

In the sequel we will use a slightly stronger version of Proposition VII.3.1.1, which is given in the following two corollaries.

COROLLARY VII.3.1.3. For any coloring of $(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r, n, k \in \omega$, and for any dual Ellentuck neighborhood $(s, Y)^\omega$, where $|s| = n + 1$, there is an infinite partition $X \in (s, Y)^\omega$ such that $(s, X)^{(n+k+1)*}$ is monochromatic.

PROOF. Let $(s, Y)^\omega$ be any dual Ellentuck neighborhood, with $|s| = n + 1 \geq 1$. Set $Y' := s \sqcap Y$, $R := \bigcup_{i<n+1} Y'(i)$ and $Y_R := Y' \setminus \{Y'(i) : i < n+1\}$, and take any order-preserving bijection $f : \omega \setminus R \to \omega$. Then $Z := f(Y_R)$ is an infinite partition of ω. For $u \in (Z)^{n+k+1*}$ we define $\xi(u) \in (s, Y)^{n+k+1*}$ as follows. $\mathrm{dom}(\xi(u)) := f^{-1}(\mathrm{dom}(u))$ and for $i < n + k + 1$,

$$\xi(u)(i) := \begin{cases} \big(Y'(i) \cap \mathrm{dom}(u)\big) \cup f^{-1}(u(i)) & \text{for } i < n+1, \\ f^{-1}(u(i)) & \text{otherwise.} \end{cases}$$

Let $\pi : (\omega)^{(n+k+1)*} \to r + 1$ be any coloring. Define $\tau : (\omega)^{(n+k+1)*} \to r + 1$ by stipulating $\tau(u) := \pi(\xi(u))$. By Proposition VII.3.1.1 there is an infinite partition $X' \in (Z)^\omega$ such that $(X')^{n+k+1*}$ is monochromatic with respect to the coloring τ. Now let $X \in (\omega)^\omega$ be such that

$$X(i) := \begin{cases} Y'(i) \cup f^{-1}(X'(i)) & \text{for } i < n+1 \\ f^{-1}(X'(i)) & \text{otherwise.} \end{cases}$$

Then, by definition of τ and X', $X \in (s,Y)^\omega$ and $(s,X)^{(n+k+1)*}$ is monochromatic with respect to π. $\dashv$

COROLLARY VII.3.1.4. For any coloring of $\bigcup_{n\in\omega}(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r,k \in \omega$, and for any $Z \in (\omega)^\omega$, there is an infinite partition $X \in (Z)^\omega$ such that for any $n \in \omega$ and for any $s \prec X$ with $|s| = n+1$, $(s,X)^{(n+k+1)*}$ is monochromatic.

PROOF. Using Corollary VII.3.1.3 repeatedly, we can construct the partition $X \in (\omega)^\omega$ straight forward by induction on n. $\dashv$

We say that a family $\mathscr{C} \subseteq (\omega)^\omega$ has the **segment-coloring-property**, if for every coloring of $\bigcup_{n\in\omega}(\omega)^{(n+k+1)*}$ with $r+1$ colors, where $r,k \in \omega$, and for any $Z \in \mathscr{C}$, there is an infinite partition $X \in (Z)^\omega \cap \mathscr{C}$, such that for any $n \in \omega$ and for any $s \prec X$ with $|s| = n+1$, $(s,X)^{(n+k+1)*}$ is monochromatic.

If a partition-ultrafilter $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ has the segment-coloring-property, then it is called a **Ramseyan ultrafilter**.

The next lemma shows that every partition-filter $\mathscr{F} \in \mathtt{PF}\big((\omega)^\omega\big)$ which has the segment-coloring-property is a partition-ultrafilter. A similar result we have for Ramsey filters over ω, since every Ramsey filter is an ultrafilter.

LEMMA VII.3.1.5. If $\mathscr{F} \subseteq (\omega)^\omega$ is a partition-filter which has the segment-coloring-property, then $\mathscr{F} \subseteq (\omega)^\omega$ is a partition-ultrafilter.

PROOF. Take any $Z \in (\omega)^\omega$ such that for any $X \in \mathscr{F}$, $Z \sqcap X \in (\omega)^\omega$. Define the coloring $\pi : (\omega)^{2*} \to \{0,1\}$ by stipulating $\pi(u) = 0$ if and only if $u \in (Z)^{2*}$. Because $\mathscr{F}$ has the segment-coloring-property, there is a partition $X \in \mathscr{F}$ such that $(X)^{2*}$ is monochromatic with respect to π, which implies that $X \sqsubseteq Z$ in case of $\pi|_{(X)^{2*}} = \{0\}$, and $X \sqcap Z = \{\omega\}$ in case of $\pi|_{(X)^{2*}} = \{1\}$. By the choice of Z we must have $X \sqsubseteq Z$, thus, since $\mathscr{F}$ is a partition-filter, $Z \in \mathscr{F}$. $\dashv$

The following lemma gives a relation between Ramseyan and Ramsey ultrafilters.

LEMMA VII.3.1.6. If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\{\mathrm{Min}(X) \setminus \{0\} : X \in \mathscr{U}\}$ is a Ramsey ultrafilter over ω (to be pedantic, one should say "over $\omega \setminus \{0\}$").

PROOF. Let $\tau : [\omega]^n \to r$ be any coloring of the n-element subsets of ω with r colors, where n and r are positive natural numbers. Define $\pi : (\omega)^{n*} \to r$ by stipulating $\pi(s) := \tau(\mathrm{Min}(s^*) \setminus \{0\})$. Take $X \in \mathscr{U}$ such that $(X)^{n*}$ is monochromatic with respect to π, then, by the definition of π, the set $[\mathrm{Min}(X) \setminus \{0\}]^n$ is monochromatic with respect to τ. $\dashv$

Ramsey ultrafilters over ω build the minimal points of the Rudin-Keisler ordering on $\beta\omega \setminus \omega$. This fact can also be expressed by saying that a non-principal ultrafilter $\mathcal{U}$ is a Ramsey ultrafilter if and only if any function $g : \omega \to \omega$ is either constant or one-to-one on some set of $\mathcal{U}$. By Lemma VII.3.1.2, we get a similar result for Ramseyan ultrafilters with respect to the ordering "$\lesssim$".

THEOREM VII.3.1.7. If $\mathscr{U}$ is a Ramseyan ultrafilter, then for any surjection $f : \omega \twoheadrightarrow \omega$ there is an $X \in \mathscr{U}$ such that f either respects or completely disregards X.

PROOF. The proof is the same as the proof of Lemma VII.3.1.2, but restricted to the partition-ultrafilter $\mathscr{U}$. ⊣

3.2. On the existence of Ramseyan ultrafilters. As we have seen above, every Ramseyan ultrafilter induces a Ramsey ultrafilter over ω. It is not clear if the converse holds as well. However, Ramseyan ultrafilters are always forceable: Let $\mathbb{W}$ be the forcing notion consisting of infinite partitions, stipulating $X \leq Y \Leftrightarrow X \sqsubseteq^* Y$. $\mathbb{W}$ is the natural dualization of the forcing notion $\mathbb{U} = \langle \mathcal{P}(\omega)/\text{fin}, \subseteq^* \rangle$, which was defined in Chapter VI, and it is not hard to see that if $\mathscr{G}$ is $\mathbb{W}$-generic over $\mathbf{V}$, then $\mathscr{G}$ is a Ramseyan ultrafilter in $\mathbf{V}[\mathscr{G}]$. Since $\mathbb{W}$ is σ-closed, as a consequence we get that Ramseyan ultrafilters exist if we assume CH. On the other hand we know by Lemma VII.3.1.6 that Ramseyan ultrafilters cannot exist if there are no Ramsey ultrafilters. Kenneth Kunen proved (*cf.* [**36**, Theorem 91]) that it is consistent with ZFC that Ramsey ultrafilters don't exist. We like to mention that Saharon Shelah showed that even p-points, which are weaker ultrafilters than Ramsey ultrafilters, may not exist (see [**58**, VI §4]). He also proved that it is possible that – up to isomorphisms – there exists a unique Ramsey ultrafilter (see [**58**, VI §5]).

In the following, $\mathfrak{c}$ denotes the cardinality of the continuum and $2^{\mathfrak{c}}$ denotes the cardinality of its power-set.

Andreas Blass proved that MA implies the existence of $2^{\mathfrak{c}}$ Ramsey ultrafilters (see [**5**, Theorem 2]). He mentions in this paper that with CH in place of MA, this result is due to Keisler and with 1 in place of $2^{\mathfrak{c}}$, it is due to Booth (*cf.* [**8**, Theorem 4.14]). Further he mentions that his proof is essentially the union of Keisler's and Booth's proof. However, Blass' proof uses at a crucial point that MA implies that the tower number is equal to $\mathfrak{c}$. Such a result we don't have for partitions, because Timothy Carlson proved that the dual-tower number is equal to ω_1 (see [**46**, Proposition 4.3]). So, concerning the existence of Ramseyan ultrafilters under MA, we cannot simply translate the proof of Blass, and it seems that MA and sets of partitions are quite unrelated. But as mentioned above, if one assumes CH, then Ramseyan ultrafilters exist. Moreover, with respect to the equivalence relation "$\simeq$" (defined in Section 2) we get the following (for a similar result w.r.t. the Rudin-Keisler ordering see [**5**, p. 149]).

THEOREM VII.3.2.1. CH implies the existence of $2^{\mathfrak{c}}$ pairwise non-equivalent Ramseyan ultrafilters.

PROOF. Assume $\mathbf{V} \models \mathsf{CH}$. Let χ be large enough such that $\mathcal{P}((\omega)^\omega) \in H(\chi)$, *i.e.*, the power set of $(\omega)^\omega$ (in $\mathbf{V}$) is hereditarily of size $< \chi$. Let $\mathbf{N}$ be an elementary submodel of $\langle H(\chi), \in \rangle$ with $|\mathbf{N}| = \omega_1$, containing all reals (or equivalently, all partitions) of $\mathbf{V}$. We consider the forcing notion $\mathbb{W}$ in the model $\mathbf{N}$. Since $|\mathbf{N}| = \omega_1$, in $\mathbf{V}$ there is an enumeration $\{D_\alpha \subseteq (\omega)^\omega : \alpha < \omega_1\}$ of all dense sets of $\mathbb{W}$ which lie in $\mathbf{N}$. For any $Z \in (\omega)^\omega \cap \mathbf{V}$, let $Y_Z^{\alpha,0}, Y_Z^{\alpha,1} \in D_\alpha$ be such that $Y_Z^{\alpha,0} \sqsubseteq^* Z$, $Y_Z^{\alpha,1} \sqsubseteq^* Z$

and $Y_Z^{\alpha,0} \sqcap Y_Z^{\alpha,1} \notin (\omega)^\omega$ (since D_α is dense, such partitions exist). For any function $\zeta : \mathfrak{c} \to \{0,1\}$ we can construct a set $H_\zeta = \{X_\alpha : \alpha < \omega_1\}$ in $\mathbf{V}$ such that for all $\beta < \alpha < \omega_1$ we have $X_\alpha \sqsubseteq^* Y_{X_\beta}^{\beta,\zeta(\beta)}$. By construction, for any function ζ, the set $G_\zeta := \{X \in (\omega)^\omega : X_\alpha \sqsubseteq^* X$ for some $X_\alpha \in H_\zeta\}$ is $\mathbb{W}$-generic over $\mathbf{N}$, thus, a Ramseyan ultrafilter in $\mathbf{N}[G_\zeta]$, and since $\mathbb{W}$ is σ-closed and therefore adds no new reals, G_ζ is also a Ramseyan ultrafilter in $\mathbf{V}$. Furthermore, if $\zeta \neq \zeta'$, then the two Ramseyan ultrafilters G_ζ and $G_{\zeta'}$ are different (consider the two partitions $X_{\beta+1} \in H_\zeta$ and $X'_{\beta+1} \in H_{\zeta'}$, where $\zeta(\beta) \neq \zeta'(\beta)$). Hence, in $\mathbf{V}$, there are $2^{\mathfrak{c}}$ Ramseyan ultrafilters. Because there are only $\mathfrak{c}$ surjections from ω onto ω, no equivalence class (w.r.t. "$\simeq$") can contain more than $\mathfrak{c}$ Ramseyan ultrafilters, so, in $\mathbf{V}$, there must be $2^{\mathfrak{c}}$ pairwise non-equivalent Ramseyan ultrafilters. $\dashv$

4. The happy families' relatives

4.1. Relatively happy families. As we will see below, the partition-families which have the segment-coloring-property are related to special coideals, so-called happy families, which are introduced and rigorously investigated by Adrian Mathias in [**49**]. So, partition-families with the segment-coloring-property can be considered as "relatives of happy families".

Let us first consider the definition of Mathias' happy families. Recall that $[\omega]^\omega$ is the set of all infinite subsets of ω, and that $[\omega]^{<\omega}$ is the set of all finite subsets of ω. A set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a **free ideal**, if $\mathcal{I}$ is an ideal which contains the **Fréchet ideal** $[\omega]^{<\omega}$. A set $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **free filter**, if $\{y : \omega \setminus y \in \mathcal{F}\}$ is an ideal containing the Fréchet ideal. For $a \in [\omega]^{<\omega}$, let $a^* := \max\{n+1 : n \in a\}$, in particular, $0^* = 0$. For $x, y \in \mathcal{P}(\omega)$ we write $y \subseteq^* x$ if $(y \setminus x) \in [\omega]^{<\omega}$. For a set $\mathcal{B} \subseteq \mathcal{P}(\omega)$, let $\mathrm{fil}(\mathcal{B})$ be the free filter generated by $\mathcal{B}$, so, $x \in \mathrm{fil}(\mathcal{B})$ if and only if there is a finite set $y_0, \dots, y_n \in \mathcal{B}$ such that $(y_0 \cap \ldots \cap y_n) \subseteq^* x$.

A set $x \subseteq \omega$ is said to **diagonalize** the family $\{x_a : a \in [\omega]^{<\omega}\}$, if $x \subseteq x_0$ and for all $a \in [\omega]^{<\omega}$, if $\max(a) \in x$, then $(x \setminus a^*) \subseteq x_a$.

The family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is **happy**, if $\mathcal{P}(\omega) \setminus \mathcal{A}$ is a free ideal and whenever $\mathrm{fil}\{x_a : a \in [\omega]^{<\omega}\} \subseteq \mathcal{A}$, there is an $x \in \mathcal{A}$ which diagonalizes $\{x_a : a \in [\omega]^{<\omega}\}$.

In terms of happy families one can define Ramsey ultrafilters as follows: A Ramsey ultrafilter is an ultrafilter that is also a happy family.

Now we turn back to partitions. The Fréchet ideal corresponds to the set of finite partitions, and therefore, the notion of a free filter corresponds to partition-filters containing only infinite partitions, hence, to partition-filters $\mathscr{F} \subseteq (\omega)^\omega$. For a set $\mathscr{B} \subseteq (\omega)^\omega$, let $\mathrm{fil}(\mathscr{B})$ be the partition-filter generated by $\mathscr{B}$, so, $X \in \mathrm{fil}(\mathscr{B})$ if and only if there is a finite set of partitions $Y_0, \dots, Y_n \in \mathscr{B}$ such that $(Y_0 \sqcap \ldots \sqcap Y_n) \sqsubseteq^* X$.

A partition X is said to **diagonalize** the family $\{X_s : s \in (\mathbb{N})\}$, if $X \sqsubseteq X_\emptyset$ and for all $s \in (\mathbb{N})$, if $s^* \prec X$, then $\left(\bigcup s^* \sqcap X\right) \sqsubseteq X_s$.

The family $\mathscr{A} \subseteq (\omega)^\omega$ is **relatively happy**, if whenever $\mathrm{fil}\{X_s : s \in (\mathbb{N})\} \subseteq \mathscr{A}$, there is an $X \in \mathscr{A}$ which diagonalizes $\{X_s : s \in (\mathbb{N})\}$. An example of a relatively happy family is $(\omega)^\omega$, the set of all infinite partitions (compare with [**49**, Example 0.2]). Another example of a much smaller relatively happy family is given in the following theorem (compare with [**49**, p. 63]).

THEOREM VII.4.1.1. Every Ramseyan ultrafilter is relatively happy.

PROOF. Let $\mathscr{U} \subseteq (\omega)^\omega$ be a partition-ultrafilter which has the segment-coloring-property and let $\{X_s : s \in (\mathbb{N})\} \subseteq \mathscr{U}$ be any family. Since $\mathscr{U}$ is a partition-filter, we obviously have $\mathrm{fil}\{X_s : s \in (\mathbb{N})\} \subseteq \mathscr{U}$. For $t \in (\mathbb{N})$ with $|t| \geq 2$, let s_t be such that $s_t^* \prec t$ and $|s_t| = |t| - 2$. Define the coloring $\pi : \bigcup_{n\in\omega}(\omega)^{(n+2)*} \to \{0,1\}$ by stipulating

$$\pi(t) := \begin{cases} 0 & \text{if } \left(\bigcup s_t^* \sqcap t^*\right) \sqsubseteq X_{s_t}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $X \in (X_\emptyset)^\omega \cap \mathscr{U}$ be such that for any $n \in \omega$ and for any $s^* \prec X$ with $|s| = n$, $(s^*, X)^{(n+2)*}$ is monochromatic with respect to π. Take any $s^* \prec X$. Since $(s^*, X)^{(|s|+2)*}$ is monochromatic with respect to π, each $t^* \sqsubseteq X$ with $s^* \prec t$ and $|t| = |s|+2$ gets the same color. Hence, for all such t's we have either $\left(\bigcup s^* \sqcap t^*\right) \sqsubseteq X_s$, which implies $X \sqsubseteq^* X_s$, or $\left(\bigcup s^* \sqcap t^*\right) \not\sqsubseteq X_s$, which implies $X \sqcap X_s \notin (\omega)^\omega$. The latter is impossible, since it contradicts the assumption that $\mathscr{U}$ is a partition-filter. So, we are always in the former case, which completes the proof. ⊣

4.2. A game characterization. There is a characterization of happy ultrafilters over ω, *i.e.*, of Ramsey ultrafilters, in terms of games (*cf.* [**3**, Theorem 4.5.3]). A similar characterization we get for relatively happy partition-ultrafilter.

Let $\mathscr{U}$ be a partition-ultrafilter. Define a game $G(\mathscr{U})$ played by players I and II as follows:

$$\begin{array}{ccccccccc} \mathrm{I} & X_1 & & X_2 & & X_3 & & & \\ & & & & & & & & \cdots \\ \mathrm{II} & & s_1 & & s_2 & & s_3 & & \end{array}$$

Player I on the n-th move plays a partition $X_n \in \mathscr{U}$. Player II responds with a segment $s_n \in (\mathbb{N})$ such that $|s_n| = n$, $s_{n-1}^* \prec s_n$ and for all $m < n$, $\left(\bigcup s_m^* \sqcap s_n^*\right) \sqsubseteq X_{m+1}$, where $s_0 := \emptyset$. Player I wins if and only if the unique partition X with $s_n \prec X$ (for all n) is not in $\mathscr{U}$.

THEOREM VII.4.2.1. Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$, then player I has a winning strategy in $G(\mathscr{U})$ if and only if $\mathscr{U}$ is not relatively happy.

PROOF. Assume first that the partition-ultrafilter $\mathscr{U}$ is relatively happy and that $\{X_s : s \in (\mathbb{N})\}$ is a strategy for player I. This means, player I begins with $X_\emptyset$ and then, if s_n is the n-th move of player II, player I plays X_{s_n}. Because $\mathscr{U}$ is relatively happy, there is a partition $X \in \mathscr{U}$ which diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$, in particular, $X \sqsubseteq X_\emptyset$. Now, by the definition of X and by the rules of the game $G(\mathscr{U})$, player II can play the segments of X. More precisely, player II plays on the

n-th move the segment s_n, so that $|s_n| = n$ and $s_n^* \prec X$. Since $X \in \mathscr{U}$, the strategy $\{X_s : s \in (\mathbb{N})\}$ was not a winning strategy for player I.

Now assume that the strategy $\sigma = \{X_s : s \in (\mathbb{N})\}$ is not a winning strategy for player I. Consider the game where player I is playing according to the strategy σ. In this game, player II can play segments s_n such that the unique partition X with $s_n \prec X$ (for all n) is in $\mathscr{U}$. We have to show that X diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$. For $n \in \omega$, let $s_n \in (\mathbb{N})$ be such that $s_n^* \prec X$ and $|s_n| = n$. Fix $m \in \omega$, then, by the rules of the game, for any $n > m$ we have $\left(\bigcup s_m^* \sqcap s_n^*\right) \sqsubseteq X_{m+1}$, which implies $\left(\bigcup s_m^* \sqcap X\right) \sqsubseteq X_{m+1}$. Since player I follows the strategy σ, $X_{m+1} = X_{s_m}$, and because m was arbitrary, for all $m \in \omega$ we get $\left(\bigcup s_m^* \sqcap X\right) \sqsubseteq X_{s_m}$. Hence, X diagonalizes the family $\{X_s : s \in (\mathbb{N})\}$. $\dashv$

5. The combinatorics of dual Mathias forcing

Let us first recall some properties of Mathias forcing and dual Mathias forcing, respectively: Mathias forcing restricted to a non-principal ultrafilter $\mathcal{U}$, denoted by $\mathbb{M}_{\mathcal{U}}$, consists of the ordered pairs $\langle a, x\rangle \in \mathbb{M}$ with $x \in \mathcal{U}$. Mathias forcing has a lot of nice combinatorial properties (some of them are mentioned below) which also hold for Mathias forcing restricted to a Ramsey ultrafilter (see [**49**]). Dual Mathias forcing restricted to a partition-ultrafilter $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$, denoted by $\mathbb{M}^\flat_{\mathscr{U}}$, consists of the ordered pairs $\langle s, X\rangle \in \mathbb{M}^\flat$ with $X \in \mathscr{U}$ (see *e.g.* [**23**] and [**26**]). As we have seen before, both, Mathias forcing as well as dual Mathias forcing, are proper forcings. Moreover, both have (i) a decomposition, (ii) pure decision and (iii) the homogeneity property (see *e.g.* [**49**], [**11**], [**23**], or Chapter VI):

(i) **Decomposition**: $\mathbb{M} \approx \mathbb{U} * \mathbb{M}_{\dot{\mathcal{U}}}$, where $\dot{\mathcal{U}}$ is the canonical $\mathbb{U}$-name for the $\mathbb{U}$-generic object ($\mathbb{U}$ as in Section 3.2).

$\mathbb{M}^\flat \approx \mathbb{W} * \mathbb{M}^\flat_{\dot{\mathscr{U}}}$, where $\dot{\mathscr{U}}$ is the canonical $\mathbb{W}$-name for the $\mathbb{W}$-generic object ($\mathbb{W}$ as in Section 3.2).

(ii) **Pure decision**: For any $\mathbb{M}$-condition $\langle a, x\rangle$ and any sentence Φ of the forcing language $\mathbb{M}$, there is an $\mathbb{M}$-condition $\langle a, y\rangle \leq \langle a, x\rangle$ such that either $\langle a, y\rangle \Vdash_{\mathbb{M}} \Phi$ or $\langle a, y\rangle \Vdash_{\mathbb{M}} \neg\Phi$.

Similarly, for any $\mathbb{M}^\flat$-condition $\langle s, X\rangle$ and any sentence Φ of the forcing language $\mathbb{M}^\flat$, there is an $\mathbb{M}^\flat$-condition $\langle s, Y\rangle \leq \langle s, X\rangle$ such that either $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \Phi$ or $\langle s, Y\rangle \Vdash_{\mathbb{M}^\flat} \neg\Phi$.

(iii) **Homogeneity property**: If x_G is $\mathbb{M}$-generic over $\mathbf{V}$ and $y \in [x_G]^\omega$, then y is also $\mathbb{M}$-generic over $\mathbf{V}$.

If X_G is $\mathbb{M}^\flat$-generic over $\mathbf{V}$ and $Y \in (X_G)^\omega$, then Y is also $\mathbb{M}^\flat$-generic over $\mathbf{V}$.

In Chapter VI (see also [**23**]) it was shown that if $\mathscr{F} \subseteq (\omega)^\omega$ is a so-called *game-family*, then $\mathbb{M}^\flat_{\mathscr{F}}$ has pure decision and the homogeneity property (see Theorem VI.4.3 and Theorem VI.4.4, respectively). Game-families have the segment-coloring-property and therefore, the so-called *game-filters*, *i.e.*, game-families which are partition-filters, are Ramseyan ultrafilters. Unlike for Ramseyan ultrafilters, it is not clear if CH implies the existence of game-filters, so, it seems that game-filters are stronger than Ramseyan ultrafilters. However, in the sequel we show that if $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has pure decision and the homogeneity property.

Recently, Stevo Todorčević gave an abstract presentation of Ellentuck's theorem by introducing the notion of a *quasi ordering with approximations* which *admits a finitization* and the notion of a *Ramsey space*. The ABSTRACT ELLENTUCK THEOREM says that a quasi ordering with approximations which admits a finitization and satisfies certain axioms is a Ramsey space.

Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ be a partition-ultrafilter and let "$\sqsubseteq$" be the quasi ordering on $\mathscr{U}$. For each $n \in \omega$, let the function $p_n : \mathscr{U} \to (\mathbb{N})$ be such that $p_n(X)$ is the unique s with $s^* \prec X$ and $|s| = n$. Let p be the sequence $(p_n)_{n\in\omega}$. It is easy to verify that the triple $(\mathscr{U}, \sqsubseteq, p)$ is a **quasi ordering with approximations**. For $n, m \in \omega$ and $X, Y \in \mathscr{U}$ define: $p_n(X) \sqsubseteq_{\text{fin}} p_m(Y)$ if and only if $\operatorname{dom}\big(p_n(X)\big) = \operatorname{dom}\big(p_m(Y)\big)$ and $p_n(X) \sqsubseteq p_m(Y)$. This definition verifies that $(\mathscr{U}, \sqsubseteq, p)$ **admits a finitization**. If $(s, X)^\omega$ is a dual Ellentuck neighborhood and $X \in \mathscr{U}$, then $(s, X)^\omega \cap \mathscr{U}$ is called a **$\mathscr{U}$-dual Ellentuck neighborhood**. The topology on $\mathscr{U}$, induced by the $\mathscr{U}$-dual Ellentuck neighborhoods, is called the **$\mathscr{U}$-dual Ellentuck topology**. With respect to the $\mathscr{U}$-dual Ellentuck topology, the topological space $\mathscr{U}$ is a **Ramsey space**, if for any subset $S \subseteq \mathscr{U}$ which has the Baire property with respect to the $\mathscr{U}$-dual Ellentuck topology, and for any $\mathscr{U}$-dual Ellentuck neighborhood $(s, Y)^\omega \cap \mathscr{U}$, there is a partition $X \in (s, Y)^\omega \cap \mathscr{U}$ such that either $(s, X)^\omega \cap \mathscr{U} \subseteq S$ or $(s, X)^\omega \cap \mathscr{U} \subseteq \mathscr{U} \setminus S$.

Let $\mathscr{U} \in \mathtt{PUF}\big((\omega)^\omega\big)$ be a Ramseyan ultrafilter. Since the triple $(\mathscr{U}, \sqsubseteq, p)$ satisfies certain axioms, by Todorčević's ABSTRACT ELLENTUCK THEOREM, the Ramseyan ultrafilter $\mathscr{U}$ with respect to the $\mathscr{U}$-dual Ellentuck topology is a Ramsey space. Moreover, we get the following two results.

THEOREM VII.5.1. If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has pure decision.

PROOF. Let Φ be any sentence of the forcing language $\mathbb{M}^\flat_{\mathscr{U}}$. With respect to Φ we define

$$D_0 := \{Y \in \mathscr{U} : \text{ for some } t \prec Y,\ \langle t, Y\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \neg\Phi\}\,,$$

and

$$D_1 := \{Y \in \mathscr{U} : \text{ for some } t \prec Y,\ \langle t, Y\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \Phi\}\,.$$

Clearly D_0 and D_1 are both open (w.r.t. the $\mathscr{U}$-dual Ellentuck topology) and $D_0 \cup D_1$ is dense (w.r.t. the partial order in $\mathbb{M}^\flat_{\mathscr{U}}$). Because $\mathscr{U}$ is a Ramsey space, for any $\mathscr{U}$-dual Ellentuck neighborhood $(s, Y)^\omega \cap \mathscr{U}$ there is an $X \in (s, Y)^\omega \cap \mathscr{U}$ such that either

$$(s, X)^\omega \cap \mathscr{U} \subseteq D_0 \ \text{ or } \ (s, X)^\omega \cap \mathscr{U} \cap D_0 = \emptyset\,.$$

In the former case we have $\langle s, X\rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \neg\Phi$ and we are done. In the latter case we find $X' \in (s, X)^\omega \cap \mathscr{U}$ such that $(s, X')^\omega \cap \mathscr{U} \subseteq D_1$. (Otherwise we would have

$(s, X')^\omega \cap \mathscr{U} \cap (D_0 \cup D_1) = \emptyset$, which is impossible by the density of $D_0 \cup D_1$.) Hence, $\langle s, X' \rangle \Vdash_{\mathbb{M}^\flat_{\mathscr{U}}} \Phi$. $\dashv$

THEOREM VII.5.2. If $\mathscr{U}$ is a Ramseyan ultrafilter, then $\mathbb{M}^\flat_{\mathscr{U}}$ has the homogeneity property.

PROOF. For a dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$, let

$$\bigcup D := \{X \in (\omega)^\omega : X \in (s, Y)^\omega \text{ for some } \langle s, Y \rangle \in D\}.$$

It is clear that a partition X_G is $\mathbb{M}^\flat_{\mathscr{U}}$-generic if and only if $X_G \in \bigcup D$ for each dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$. Let $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$ be an arbitrary dense set and let D' be the set of all $\langle s, Z \rangle \in \mathbb{M}^\flat_{\mathscr{U}}$ such that $(t, Z)^\omega \subseteq \bigcup D$ for all $t \sqsubseteq s$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$.

First we show that D' is dense in $\mathbb{M}^\flat_{\mathscr{U}}$. For this, take an arbitrary $\langle s, W \rangle \in \mathbb{M}^\flat_{\mathscr{U}}$ and let $\{t_i : 0 \leq i \leq m\}$ be an enumeration of all $t \in (\mathbb{N})$ such that $t \sqsubseteq s$ and $\mathrm{dom}(t) = \mathrm{dom}(s)$. Because D is dense in $\mathbb{M}^\flat_{\mathscr{U}}$, $\bigcup D$ is open (w.r.t. the $\mathscr{U}$-dual Ellentuck topology), and since $\mathscr{U}$ is a Ramsey space, for every t_i we find a $W' \in \mathscr{U}$ such that $t_i \sqsubseteq W'$ and $(t_i, W')^\omega \subseteq \bigcup D$. Moreover, if we define $W_{-1} := W$, for every $i \leq m$ we can choose a partition $W_i \in \mathscr{U}$ such that $W_i \sqsubseteq W_{i-1}$, $s \prec W_i$ and $(t_i, W_i)^\omega \subseteq \bigcup D$. Thus, $\langle s, W_m \rangle \in D'$, and because $\langle s, W_m \rangle \leq \langle s, W \rangle$, D' is dense in $\mathbb{M}^\flat_{\mathscr{U}}$.

Let X_G be $\mathbb{M}^\flat_{\mathscr{U}}$-generic and let $Y \in (X_G)^\omega$ be arbitrary. Since D' is dense, there is a condition $\langle s, Z \rangle \in D'$ such that $s \prec X_G \sqsubseteq Z$. Since $Y \in (X_G)^\omega$, we have $t \prec Y \sqsubseteq Z$ for some $t \sqsubseteq s$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$, and because $(t, Z)^\omega \subseteq \bigcup D$, we get $Y \in \bigcup D$. Hence, $Y \in \bigcup D$ for each dense set $D \subseteq \mathbb{M}^\flat_{\mathscr{U}}$, which completes the proof. $\dashv$

Appendix

In this section we are gathering some results concerning the dual form of some cardinal characteristics of the continuum. For the definition of the classical cardinal characteristics, as well as for the relation between them, we refer the reader to [**69**].

First we consider the shattering cardinal $\mathfrak{h}$. This cardinal was introduced in [**1**] as the minimal height of a tree π-base of $\beta\omega \setminus \omega$. Later it was shown by Szymon Plewik in ([**54**]) that $\mathfrak{h} = \mathit{add}(\boldsymbol{R}_0) = \mathit{cov}(\boldsymbol{R}_0)$, where $\boldsymbol{R}_0$ denotes the ideal of completely Ramsey null sets. It is easy to see that $\mathfrak{p} \leq \mathfrak{h}$, and therefore, MA(σ-centered) implies $\mathfrak{h} = \mathfrak{c}$.

The dual form of the classical cardinal characteristics were introduced and investigated in [**12**] and further investigated in [**22**]. Concerning the dual-shattering cardinal $\mathfrak{H}$, one easily gets $\omega_1 \leq \mathfrak{H} \leq \mathfrak{h}$, and in [**22**] it is shown that $\mathfrak{H} > \omega_1$ is consistent relative to ZFC and that $\mathfrak{H} = \mathit{add}(\boldsymbol{R}_0^\flat) = \mathit{cov}(\boldsymbol{R}_0^\flat)$, where $\boldsymbol{R}_0^\flat$ denotes the ideal of completely dual Ramsey null sets. After all these symmetries, one would not expect the following: $\mathsf{MA} + (\mathfrak{c} > \mathfrak{H})$ is consistent relative to ZFC. This was proved by Jörg Brendle in [**10**] and implies that $\mathfrak{H} < \mathfrak{p}$ is consistent relative to ZFC.

Concerning the reaping and the dual-reaping number $\mathfrak{r}$ and $\mathfrak{R}$, respectively, the situation looks different. It is shown in [**23**] that $\mathfrak{p} \leq \mathfrak{R} \leq \min\{\mathfrak{r}, \mathfrak{i}\}$, and thus we get $\mathsf{MA}(\sigma\text{-centered})$ implies $\mathfrak{R} = \mathfrak{c}$. Further, it is easy to show that $\mathfrak{R} \leq \mathfrak{U}$, where $\mathfrak{U}$ denotes the partition-ultrafilter base number, *i.e.*, the dual form of $\mathfrak{u}$, and consequently, $\mathsf{MA}(\sigma\text{-centered})$ implies $\mathfrak{U} = \mathfrak{c}$.

For a Ramsey ultrafilter $\mathcal{U}$, Brendle introduced in [**9**] the ideal $\boldsymbol{R}_{0,\mathcal{U}}$, which is the ideal of completely Ramsey null sets with respect to the ultrafilter $\mathcal{U}$. Concerning this ideal $\boldsymbol{R}_{0,\mathcal{U}}$, he showed for example that $\mathfrak{hom} \leq non(\boldsymbol{R}_{0,\mathcal{U}})$, where $\mathfrak{hom}$ is the homogeneity number investigated by Blass in [**7**, Section 6]. There, Blass also investigated the so-called partition number $\mathfrak{par}$ and showed that $\mathfrak{par} = \min\{\mathfrak{b}, \mathfrak{s}\}$. Now, replacing the Ramsey ultrafilter $\mathcal{U}$ by a Ramseyan ultrafilter $\mathscr{U}$, one obtains the ideal $\boldsymbol{R}^{\flat}_{0,\mathscr{U}}$ of completely dual Ramsey null sets with respect to $\mathscr{U}$ as the dualization of the ideal $\boldsymbol{R}_{0,\mathcal{U}}$, and replacing the colorings of $[\omega]^2$ – involved in the definition of $\mathfrak{hom}$ and $\mathfrak{par}$ – by colorings of $(\omega)^{2*}$, one obtains the cardinal characteristics $\mathfrak{Hom}$ and $\mathfrak{Par}$ and could begin to investigate them, but this is left for further research.

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List of Symbols

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