A RESULT IN DUAL RAMSEY THEORY

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Abstract

We present a result which is obtained by combining a result of Carlson with the Finitary Dual Ramsey Theorem of Graham-Rothschild.

We start by introducing some notation.

We conform to the usual practice of identifying the least infinite ordinal ω with the set of non-negative integers.

Given $\alpha, \beta \leq \omega$, a **partition of $\boldsymbol{\alpha}$ into $\boldsymbol{\beta}$ blocks** is an onto function $X : \alpha \to \beta$ such that $\min\big(X^{-1}(\{n\})\big) < \min\big(X^{-1}(\{m\})\big)$ whenever $n < m < \beta$. Thus, the blocks of X are ordered as their **leaders** (*i.e.*, their least elements).

The **leader function** $\ell : (\alpha)^\beta \times \beta \to \alpha$ is defined by $\ell(X, m) := \min\big(X^{-1}(\{m\})\big)$. Hence, the function $m \mapsto \ell(X, m)$ enumerates the leaders of X in increasing order.

Given $X \in (\alpha)^\beta$ and $Y \in (\alpha)^\gamma$, where $\alpha, \beta, \gamma \leq \omega$, we let $Y \leqslant X$ if Y is **coarser** than X, *i.e.*, each block of Y is a union of blocks of X.

Given $\alpha, \beta, \gamma \leq \omega$ and $X \in (\alpha)^\beta$, $(X)^\gamma := \{Y \in (\alpha)^\gamma : Y \leqslant X\}$.

Given $\alpha, \beta \leq \omega$ and $k < \omega$, $(\alpha)^\beta_k$ denotes the set of all $X \in (\alpha)^\beta$ such that
(a) $X^{-1}(\{n\})$ is finite if $k \leq n < \beta$, and
(b) $\max\big(X^{-1}(\{n\})\big) < \ell(X, n+1)$ if $k \leq n$ and $n + 1 < \beta$.

Given $\alpha, \beta, \gamma \leq \omega$, $X \in (\alpha)^\beta$ and $k, m < \omega$ such that $k \leq \gamma$ and $m \leq \beta$, $(k, m, X)^\gamma$ is the set of all $Y \in (X)^\gamma$ such that

$$\big\{\ell(Y, i) : i < k\big\} \subseteq \big\{\ell(X, j) : j < m\big\}.$$

Note that $(0, m, X)^\gamma = (1, m, X)^\gamma = (X)^\gamma$ for all $m \leq \beta$.

The authors thank the referee for helpful comments.
*The author wishes to thank the *Swiss National Science Foundation* for supporting him.

2000 Mathematics Subject Classification: 03E05 05D10
Key-words and phrases: partition, Dual Ellentuck Theorem, Finitary Dual Ramsey Theorem

The **amalgamation function** $\mathcal{A}$ is defined as follows: Given $X \in (\omega)^\omega$ and $t \in (p)^m$, where $0 < m \leq p < \omega$, $\mathcal{A}(t, X)$ is the partition of ω whose blocks are

$$\bigcup_{i \in t^{-1}(\{0\})} X^{-1}(\{i\})\,, \ldots \bigcup_{i \in t^{-1}(\{m-1\})} X^{-1}(\{i\})\,, X^{-1}(\{p\})\,, X^{-1}(\{p+1\})\,, \ldots$$

For $t \in (p)^m$, where $m \leq p < \omega$, let $O_t := \{X \in (\omega)^\omega : X \restriction p = t\}$.

We topologize $(\omega)^\omega$ by taking as basic open sets $\emptyset$ and O_t for $t \in \bigcup_{m \leq p < \omega} (p)^m$.

A function $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, is **clopen** if $F^{-1}(\{i\})$ is a clopen subset of $(\omega)^\omega$ for each $i < r$.

Our starting point is the following immediate consequence of the Dual Ellentuck Theorem (Theorem 4.1 in [1]) of Carlson-Simpson.

Proposition 1. Given $X \in (\omega)^\omega$ and a clopen $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, there is $Y \in (X)^\omega$ such that F is constant on $(Y)^\omega$.

Even if every block of X is finite, there may not be any homogeneous Y having infinitely many finite blocks.

Proposition 2. There is a clopen $F : (\omega)^\omega \to 2$ with the property that there is no $Y \in (\omega)^\omega$ such that F is constant on $(Y)^\omega$ and Y has infinitely many finite blocks.

Proof. Define $F : (\omega)^\omega \to 2$ by stipulating that $F(X) = 0$ if and only if $X^{-1}(\{1\}) \cap \ell(X, 3) \subseteq \ell(X, 2)$. Obviously, F is clopen. Now suppose that there is $Y \in (\omega)^\omega$ such that Y has infinitely many finite blocks and F is constant on $(Y)^\omega$. Pick $Z \in (\omega)^\omega_1$ with $Z \leqslant Y$. Then F is constant on $(Z)^\omega$, which is clearly impossible. $\dashv$

Carlson established a "specialized" version (Theorem 6.9 of [1], which follows from Theorem 2 of [2]) of the Dual Ellentuck Theorem that deals with partitions of ω having finitely many infinite blocks. Carlson's result immediately implies the following.

Proposition 3. Given $k < \omega$, $X \in (\omega)^\omega_k$ and a clopen $F : (\omega)^\omega \to r$, where $1 \leq r < \omega$, there is $Y \in (\omega)^\omega_k \cap (k, k, X)^\omega$ such that F is constant on $(k, k, Y)^\omega$.

The purpose of this paper is to present the combinatorial result which is obtained by combining Proposition 3 with the Finitary Dual Ramsey theorem of Graham-Rothschild [3]. This last reads as follows.

Proposition 4. Suppose that $1 \leq k \leq m < \omega$ and $1 \leq r < \omega$. Then there is $p < \omega$ such that $p \geq m$ and the following holds: Given $f : (p)^k \to r$, there is $s \in (p)^m$ such that f is constant on $(s)^k$.

We now state our result.

THEOREM. Given $1 < k < m < \omega$, $X \in (\omega)^\omega_k$ and a clopen $F : (\omega)^\omega \to r$, where $1 \le r < \omega$, there is $Y \in (\omega)^\omega_m \cap (X)^\omega$ such that F is constant on $(k, m, Y)^\omega$.

Proof. Using Proposition 4, select $p \ge m$ so that every $f : (p)^k \to r$ is constant on $(s)^k$ for some $s \in (p)^m$. First we define $g : \bigcup_{i \le p-k} (k - 1 + i)^{k-1} \to r$ and $Y_0, Y_1, \dots, Y_{p-k}$ so that

(0) $Y_0 \in (\omega)^\omega_k \cap (k, k, X)^\omega$ and F takes the constant value $g(u)$ on $\big(k, k, \mathcal{A}(u, Y_0)\big)^\omega$, where u is the unique element of $(k-1)^{k-1}$ (hence, $\mathcal{A}(u, Y_0) = Y_0$).
(1) $Y_1 \in (\omega)^\omega_{k+1} \cap (k+1, k+1, Y_0)^\omega$ and F takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_1)\big)^\omega$ for every $t \in (k)^{k-1}$.
(2) $Y_2 \in (\omega)^\omega_{k+2} \cap (k+2, k+2, Y_1)^\omega$ and F takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_2)\big)^\omega$ for every $t \in (k+1)^{k-1}$.

$\vdots$

$(p-k)$ $Y_{p-k} \in (\omega)^\omega_p \cap (p, p, Y_{p-k-1})^\omega$ and F takes the constant value $g(t)$ on $\big(k, k, \mathcal{A}(t, Y_{p-k})\big)^\omega$ for every $t \in (p-1)^{k-1}$.

For example, to define Y_3 and $g \restriction (k+2)^{k-1}$, proceed as follows. Let $t_0, t_1, \dots, t_q$ be an enumeration of the elements of $(k+2)^{k-1}$. Applying Proposition 3 repeatedly, define T_j, Z_j and c_j for $j \le q$ so that

(i) $T_j \in (\omega)^\omega_k$.
(ii) If $j = 0$, $T_j \in \big(k, k, \mathcal{A}(t_j, Y_2)\big)^\omega$ and $Z_j \in (k+3, k+3, Y_2)^\omega$.
(iii) If $j > 0$, $T_j \in \big(k, k, \mathcal{A}(t_j, Z_{j-1})\big)^\omega$ and $Z_j \in (k+3, k+3, Z_{j-1})^\omega$.
(iv) F takes the constant value c_j on $(k, k, T_j)^\omega$.
(v) $\mathcal{A}(t_j, Z_j) = T_j$.

Then set $Y_3 = Z_q$ and $g(t_j) = c_j$ for every $j \le q$.

Define $f : (p)^k \to r$ by $f(w) = g\big(w \restriction \ell(w, k-1)\big)$. Set $W = Y_{p-k}$. Obviously, $W \in (\omega)^\omega_p \cap (X)^\omega$. Moreover, F takes the constant value $f(w)$ on $\big(k, k, \mathcal{A}(w \restriction \ell(w, k-1), W)\big)^\omega$ for every $w \in (p)^k$. Let $s \in (p)^m$ be such that f is constant on $(s)^k$. Then $Y = \mathcal{A}(s, W)$ is as desired. $\dashv$

The referee pointed out that the theorem and similar results can be derived from Theorem 10 and Theorem 11 of [2].

The theorem is optimal in the following sense:

PROPOSITION 5. Suppose that $1 < k < m < \omega$. Then there is $F : (\omega)^\omega \to 2$ such that $F^{-1}(\{0\})$ is open and there is no $Y \in (\omega)^\omega_m$ with the property that F is constant on $(k, m, Y)^\omega$.

Proof. Let $F(Y) = 0$ exactly when $Y(m) \not\subseteq \ell(Y, m+1)$. ⊣

The theorem has the following finitary version, which is proved by arguing as for 3.2 in [1].

PROPOSITION 6. Suppose that $n \leq q \leq m < \omega$, $1 \leq k \leq m$, $n \leq k$ and $1 \leq r < \omega$. Then there is $p < \omega$ such that $p \geq m$ and the following holds: Given $f : (p)^k \to r$, there is $s \in (p)^m_q$ such that f is constant on $(n, q, s)^k$.

Proof. Assume that for every $p \geq m$ there is $f_p : (p)^k \to r$ such that for every $s \in (p)^m_q$, f_p is not constant on $(n, q, s)^k$. Define $F : (\omega)^\omega \to r$ by stipulating that $F(T) = f_{\ell(T,k)}\big(T \restriction \ell(T, k)\big)$. Using the theorem (for $1 < n < q$) or Proposition 3 (otherwise), we find $Y \in (\omega)^\omega_q$ such that F is constant on $(n, q, Y)^\omega$. Set $p = \ell(Y, m)$ and $s = Y \restriction m$. Then $p \geq m$ and $s \in (p)^m_q$. Moreover, f_p is constant on $(n, q, s)^k$. Contradiction! ⊣

When $n \in \{0, 1\}$ and $q \in \{m-1, m\}$, Proposition 6 reduces to the Finitary Dual Ramsey Theorem. When $n = k$ and $q \in \{m-1, m\}$, it reduces to the n-parameter set theorem of Graham-Rothschild [3], which generalizes the Finitary Dual Ramsey Theorem.

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