A SET-THEORETIC APPROACH
TO COMPLETE MINIMAL SYSTEMS
IN BANACH SPACES OF BOUNDED FUNCTIONS

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ABSTRACT
Using independent families from combinatorial set theory, it is shown that for every infinite cardinal $\kappa$, $\ell_\infty(\kappa)^*$ contains a subspace which is isomorphic to a Hilbert space of dimension $2^\kappa$. This provides a new proof for the first step in the construction of complete minimal systems in Banach spaces of bounded functions.

1. Introduction

Let $X$ be a Banach space and let $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ be an arbitrary set of vectors of $X$. Let $[x_\lambda : \lambda \in \Lambda]$ denote the closure of the linear span of $\{x_\lambda : \lambda \in \Lambda\}$. A set $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ is called a complete system if $[x_\lambda : \lambda \in \Lambda] = X$, and it is called a minimal system if for every $\lambda' \in \Lambda$, $x_{\lambda'} \notin [x_\lambda : \lambda \in \Lambda \setminus \{\lambda'\}]$. A complete minimal system, abbreviated c.m.s., is a complete system which is also minimal.

Using functionals, we can characterize minimal systems (and consequently c.m.s.) also in the following way: Let $X$ be a Banach space. A pair of sequences $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ and $\{f_\lambda : \lambda \in \Lambda\} \subseteq X^*$ is called a biorthogonal system if $f_\lambda(x_{\lambda'}) = \delta_{\lambda,\lambda'}$. Now, a sequence $\{x_\lambda : \lambda \in \Lambda\} \subseteq X$ is minimal if and only if there is a sequence $\{f_\lambda : \lambda \in \Lambda\} \subseteq X^*$, such that the pair $\{\{x_\lambda : \lambda \in \Lambda\}, \{f_\lambda : \lambda \in \Lambda\}\}$ is a biorthogonal

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system. A biorthogonal system which corresponds to a complete minimal system is called complete biorthogonal system.

Even though not every Banach space has a c.m.s. (see e.g., [Pl80] or [GK80]), it is known that \( \ell_\infty \) has a c.m.s. The first (not completely correct) proof for the existence of a c.m.s. in \( \ell_\infty \) was given by William Davis and William Johnson in [DJ73]. Later, Borys Godun gave a correct (and slightly easier) proof in [Go83]. However, the crucial point in both proofs is the following result due to Haskel Rosenthal (cf. [Ro69, Proposition 3.4]):

**Proposition 1.** The space \( \ell_\infty^* \) contains a subspace isomorphic to a Hilbert space of dimension the continuum.

Let us briefly sketch why Proposition 1 implies the existence of a c.m.s. in \( \ell_\infty \):
Let \( Y \subseteq \ell_\infty \) be isomorphic to a Hilbert space of dimension the continuum. Since \( Y \) is reflexive, \( Y \) is weakly\(^*\) closed (cf. e.g., [Ro69, Proposition 1.2]), and therefore, \( (\perp Y)^\perp = Y \), where \( \perp Y = \{ x \in \ell_\infty : \forall y \in Y (y(x) = 0) \} \) and \( (\perp Y)^\perp := \{ x^* \in \ell_\infty^* : \forall x \in \perp Y (x^*(x) = 0) \} \). Thus, \( (\ell_\infty/\perp Y)^* \) is isomorphic to the Hilbert space \( Y \), which implies that also \( \ell_\infty/\perp Y \) is isomorphic to \( Y \). Now, following [Go83], with the orthonormal basis in \( Y \) we can easily construct a c.m.s. in \( \ell_\infty \). At this point we like to mention that starting with generalized version of Proposition 1 (cf. [Ro69, p. 203, Remark 2]), a similar construction yields a c.m.s. in \( \ell_\infty(\kappa) \) for any infinite cardinal \( \kappa \).

Rosenthal’s proof of Proposition 1 involves some deep results from functional analysis. On the other hand, from a set-theoretical point of view a c.m.s. in \( \ell_\infty \) is just a set of bounded real-valued sequences, and therefore, it was natural to seek a more combinatorial or set-theoretical proof of Proposition 1 and the aim of this paper is to provide such a proof.

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2. **Some set theory**

2.1. **Set-theoretic terminology.** Our set-theoretical axioms are the axioms of Zermelo and Fraenkel including the Axiom of Choice. All our set-theoretical notations and definitions are standard and can be found in textbooks like [Ku83].

For a set \( x \), the **cardinality** of \( x \), denoted by \( |x| \), is the least ordinal number \( \alpha \) for which there exists a bijection \( f : \alpha \to x \); such an ordinal number \( \alpha \) is called a **cardinal number** (or just a **cardinal**). The least infinite ordinal number, which is also a cardinal, is denoted by \( \omega \), thus, \( |\omega| = \omega \). In particular, \( \omega = \{ 0, 1, 2, \ldots \} \) is the set of natural numbers. A set \( x \) is called finite, if \( |x| \in \omega \), otherwise it is called infinite. Further it is called countable, if \( |x| \leq \omega \). For a set \( x \), \( P(x) \) denotes the power
set of $x$ and $[x]^{<\omega}$ denotes the set of all finite subsets of $x$. For a cardinal $\kappa$, $|\mathcal{P}(\kappa)|$ is denoted by $2^\kappa$. For example there exists a bijection between the reals $\mathbb{R}$ and $\mathcal{P}(\omega)$, hence $|\mathbb{R}| = |\mathcal{P}(\omega)| = 2^{\omega}$. For every infinite cardinal we have $2^\kappa > \kappa$ and $|[\kappa]^{<\omega}| = \kappa$.

2.2. **Independent families.** Let $\kappa$ be an infinite cardinal and let $\mathcal{I} \subseteq \mathcal{P}(\kappa)$, then $\mathcal{I}$ is called an **independent family** (on $\kappa$), if whenever $m$ and $n - 1$ belong to $\omega$, and $x_0, \ldots, x_m, \ldots, x_{m+n}$ are distinct members of $\mathcal{I}$, then

$$\left| \bigcap_{0 \leq i \leq m} x_i \setminus \bigcup_{1 \leq j \leq n} x_{m+j} \right| = \kappa.$$ 

To make this paper self-contained, let us prove the following result due to Felix Hausdorff (cf. [Ha36]):

**Proposition 2.** For any infinite cardinal $\kappa$, there is an independent family on $\kappa$ of cardinality $2^\kappa$.

**Proof.** We just follow Exercise (A6) on p. 288 of [Ku83]. Let

$$J = \{ (s, A) : s \subseteq \kappa \text{ and } |s| < \omega \text{ and } A \subseteq \mathcal{P}(s) \}.$$ 

Notice that $|J| = \kappa$, so, it is enough to construct an independent family of cardinality $2^\kappa$ on $J$. For $x' \subseteq \kappa$, let $x := \{ (s, A) : x' \cap s \in A \}$. Then $\mathcal{I} = \{ x : x' \in \mathcal{P}(\kappa) \}$ is an independent family on $J$ of cardinality $2^\kappa$. Indeed, let $x_0', \ldots, x_m', \ldots, x_{m+n}'$ be distinct members of $\mathcal{P}(\kappa)$ (for some $m$ and $n - 1$ in $\omega$). Then there is a finite set $s' \subseteq \kappa$ such that for all $i, j$ with $0 \leq i < j \leq m + n$ we have $x_i' \cap s' \neq x_j' \cap s'$. Let $A = \{ s \cap x_i' : 0 \leq i \leq m \} \subseteq \mathcal{P}(s)$, and for every $\alpha \in \kappa \setminus s$, let $s_\alpha = s \cup \{ \alpha \}$ and $A_\alpha = A \cup \{ t \cup \{ \alpha \} : t \in A \}$. Then

$$\{ (s_\alpha, A_\alpha) : \alpha \in \kappa \setminus s \} \subseteq \bigcap_{0 \leq i \leq m} x_i \setminus \bigcup_{1 \leq j \leq n} x_{m+j},$$

which implies that $\left| \bigcap_{0 \leq i \leq m} x_i \setminus \bigcup_{1 \leq j \leq n} x_j \right| = \kappa$, and therefore, $\mathcal{I}$ is an independent family on $J$ of cardinality $2^\kappa$.

As an easy consequence we get the following

**Fact.** If $\mathcal{I} = \{ x_\alpha : \alpha \in 2^\kappa \}$ is an independent family on $\kappa$ and $\alpha_1, \ldots, \alpha_n$ are finitely many distinct elements of $2^\kappa$, then $\left| \bigcap_{1 \leq i \leq n} y_{\alpha_i} \right| = \kappa$, where for every $1 \leq i \leq n$, the set $y_{\alpha_i}$ is either equal to the set $x_{\alpha_i}$, or to its complement $\kappa \setminus x_{\alpha_i}$.

2.3. The Banach spaces $\ell_2(\kappa)$ and $\ell_\infty(\kappa)$. Let $\kappa$ be an infinite cardinal. The Banach space $\ell_\infty(\kappa)$ is the set of all bounded functions from $\kappa$ to $\mathbb{R}$, where for $x \in \ell_\infty(\kappa)$, $\| x \| = \sup \{ x(\alpha) : \alpha \in \kappa \}$. The Banach space $\ell_2(\kappa)$ is the set of all functions $x$ from $\kappa$ to $\mathbb{R}$ such that $\sum_{\alpha \in \kappa} x(\alpha)^2 = \| x \|^2 < \infty$. It is common to write $\ell_2$ and $\ell_\infty$ instead of $\ell_2(\omega)$ and $\ell_\infty(\omega)$ respectively. Like for $\ell_2$ and $\ell_\infty$, one can show that $\ell_2(\kappa)^* = \ell_2(\kappa)$ and that $\ell_\infty(\kappa)^*$ is isometric to the space of all finitely additive signed
measures \( \mu \) of bounded variation on \( \mathcal{P}(\kappa) \), supplied with the norm \( \| \mu \| = |\mu|((\kappa)) \), where \( |\mu| \) is the total variation of \( \mu \).

For \( \alpha, \beta \in \kappa \), let
\[
\delta_{\alpha}^{\beta} = \begin{cases} 
1 & \text{if } \alpha = \beta, \\
0 & \text{otherwise},
\end{cases}
\]
and let \( e_\alpha : \kappa \to \{0, 1\} \) be such that \( e_\alpha(\beta) = \delta_{\alpha}^{\beta} \). It is easy to see that the set of vectors \( \{e_\alpha : \alpha \in \kappa\} \) is a c.m.s. of \( \ell_2(\kappa) \). On the other hand, the set \( \{e_\alpha : \alpha \in \kappa\} \) is much too small to be a c.m.s. of \( \ell_\infty(\kappa) \). In general, the cardinality of a complete minimal system \( S \) of an infinite dimensional real Banach space \( X \) is always equal to the density character of \( X \). Indeed, on the one hand, the set of all finite linear combinations of \( S \) with rational coefficients is dense in \( X \), and on the other hand, \( S \) is discrete in \( X \). In particular, the density character of \( \ell_\infty(\kappa) \) is \( 2^\kappa \), so, any c.m.s. of \( \ell_\infty(\kappa) \) must have cardinality \( 2^\kappa \).

3. \( \ell_\infty(\kappa)^* \) Contains an Isomorphic Copy of \( \ell_2(2^\kappa) \)

Now we are ready to prove the main result.

**Theorem.** Let \( \kappa \) be an infinite cardinal. Then any independent family on \( \kappa \) of cardinality \( 2^\kappa \) induces a subspace of \( \ell_\infty(\kappa)^* \) which is isomorphic to the Hilbert space \( \ell_2(2^\kappa) \).

**Proof.** Let \( \mathcal{I} = \{x_\alpha : \alpha \in 2^\kappa\} \) be an independent family on \( \kappa \) of cardinality \( 2^\kappa \) (which exists by Proposition 2). Define a measure \( \hat{\mu} \) on the set \( B \) of all Boolean combinations of elements of \( \mathcal{I} \) by stipulating

- \( \hat{\mu}(x_\alpha) = \hat{\mu}(\kappa \setminus x_\alpha) = 1/2 \) (for all \( x_\alpha \in \mathcal{I} \)),
- \( \hat{\mu}(x_\alpha \cap x_\beta) = \hat{\mu}(x_\alpha \cap (\kappa \setminus x_\beta)) = 1/4 \) (for all distinct \( x_\alpha, x_\beta \in \mathcal{I} \)),

and in general, if \( \alpha_1, \ldots, \alpha_n \) are finitely many distinct elements of \( 2^\kappa \) and \( 0 \leq j \leq n \), then
\[
\hat{\mu}\left(\bigcap_{1 \leq i \leq j} x_{\alpha_i} \cap \bigcap_{j < i \leq n} (\kappa \setminus x_{\alpha_i})\right) = 2^{-n}.
\]

The measure \( \hat{\mu} \) induces a normalized linear functional \( \varphi_{\hat{\mu}} \) on a subspace of \( \ell_\infty(\kappa) \). Thus, by the normed space version of the Hahn-Banach Extension Theorem, there is a normalized functional on all of \( \ell_\infty(\kappa) \) which extends the functional \( \varphi_{\hat{\mu}} \). In particular, there is a measure \( \mu \) on \( \mathcal{P}(\kappa) \) with \( \| \mu \| = 1 \), such that \( \mu|_B \equiv \hat{\mu} \). For every \( \alpha \in 2^\kappa \) let \( f_\alpha : \kappa \to \{1, -1\} \) such that
\[
f_\alpha(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in x_\alpha, \\
-1 & \text{otherwise}.
\end{cases}
\]
Now, for every \( \alpha \in 2^\kappa \), let the measure \( \mu_\alpha \) on \( \mathcal{P}(\kappa) \) be defined by
\[
\mu_\alpha(E) = \mu(E \cap x_\alpha) - \mu(E \cap (\kappa \setminus x_\alpha)),
\]
and let \( \varphi_\alpha \) be the linear functional on \( \ell_\infty(\kappa) \) induced by the measure \( \mu_\alpha \). It is not hard to see that for all \( \alpha, \beta \in 2^\kappa \), \( \varphi_\alpha(f_\beta) = \delta^\beta_\alpha \) and that \( \|\varphi_\alpha\|_{\ell_\infty(\kappa)^*} = 1 \). Let \( Y = [\varphi_\alpha : \alpha \in 2^\kappa] \subseteq \ell_\infty(\kappa)^* \), and let \( \sum_{i=1}^n a_i \varphi_{\alpha_i} \in Y \).

**Claim.** For each \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n \) we have
\[
\left\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \right\|_{\ell_\infty(\kappa)^*} = 2^{-n} \sum_{\varepsilon \in \{-1, 1\}^n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|.
\]

**Proof.** For each \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n \) let \( E_\varepsilon = \bigcap_{1 \leq i \leq n} y_{\alpha_i} \), where
\[
y_{\alpha_i} = \begin{cases} x_{\alpha_i} & \text{if } \varepsilon_i a_i \geq 0, \\ \kappa \setminus x_{\alpha_i} & \text{otherwise.} \end{cases}
\]
By the fact mentioned above, \( |E_\varepsilon| = \kappa \), and by the properties of the measure \( \mu \) we get \( \mu(E) = 2^{-n} \). Notice that for any distinct \( \varepsilon \) and \( \varepsilon' \) in \( \{-1, 1\}^n \) we have \( E_\varepsilon \cap E_{\varepsilon'} = \emptyset \) and that \( \kappa = \bigcup_{\varepsilon \in \{-1, 1\}^n} E_\varepsilon \). Further, for every \( \varepsilon \in \{-1, 1\}^n \) let \( f_\varepsilon : \kappa \to \{\pm 1, 0\} \) be such that
\[
f_\varepsilon(\lambda) = \begin{cases} 1 & \text{if } \lambda \in E_\varepsilon \text{ and } \sum_{i=1}^n \varepsilon_i a_i \geq 0, \\ -1 & \text{if } \lambda \in E_\varepsilon \text{ and } \sum_{i=1}^n \varepsilon_i a_i < 0, \\ 0 & \text{otherwise,} \end{cases}
\]
and let \( f = \sum_{\varepsilon \in \{-1, 1\}^n} f_\varepsilon \). It is not hard to verify that for each \( \varepsilon \in \{-1, 1\}^n \) we have
\[
(a_1 \varphi_{\alpha_1} + \ldots + a_n \varphi_{\alpha_n})(f_\varepsilon) = 2^{-n} |\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n|,
\]
and therefore,
\[
\left( \sum_{i=1}^n a_i \varphi_{\alpha_i} \right)(f) = 2^{-n} \sum_{\varepsilon \in \{-1, 1\}^n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|.
\]
Now, since the \( E_\varepsilon \)'s are pairwise disjoint, \( \|f\|_{\ell_\infty(\kappa)} = 1 \), and by the construction of \( f \) we finally get
\[
\left\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \right\|_{\ell_\infty(\kappa)^*} = 2^{-n} \sum_{\varepsilon \in \{-1, 1\}^n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|.
\]
\( ^\dagger \) **Claim**

Hence, by Khintchine’s inequality, there is a constant \( c = 1/\sqrt{2} \) such that
\[
c \cdot \sqrt{\sum_{i=1}^n a_i^2} \leq \left\| \sum_{i=1}^n a_i \varphi_{\alpha_i} \right\|_{\ell_\infty(\kappa)^*} \leq \sqrt{\sum_{i=1}^n a_i^2},
\]
which implies that the space \( Y \subseteq \ell_\infty(\kappa)^* \) is isomorphic to the Hilbert space \( \ell_2(2^\kappa) \) and completes the proof. \( ^\dagger \)
Remark. For an infinite cardinal \( \kappa \), the Banach space \( c_0(\kappa) \) is the set of all functions \( x \) from \( \kappa \) to \( \mathbb{R} \) such that for every \( \varepsilon > 0 \), the set \( \{ \alpha < \kappa : |x(\alpha)| > \varepsilon \} \) is finite. Now, the Theorem admits the following generalization: Let \( \kappa \) be an infinite cardinal. Then the space \( (\ell_\infty(\kappa)/c_0(\kappa))^* \) contains a subspace which is isomorphic to \( \ell_2(2^\kappa) \). Consequently we get: For every infinite cardinal \( \kappa \), the space \( \ell_\infty(\kappa)/c_0(\kappa) \) has a complete minimal system.

References


