Sets of Range Uniqueness for Multivariate Polynomials and Linear Functions with Rank k

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Abstract

Let $\mathcal{F}$ be a set of functions with common domain X and common range Y. A set $S \subseteq X$ is called a set of range uniqueness (SRU) for $\mathcal{F}$, if for all $f, g \in \mathcal{F}$

$$f[S] = g[S] \Rightarrow f = g.$$

Let $\mathcal{P}_{n,k}$ be the set of all real polynomials in n variables of degree at most k and let $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all linear functions $f : \mathbb{R}^n \to \mathbb{R}^n$ with rank k. We show that there are SRU's for $\mathcal{P}_{n,k}$ of cardinality $2\binom{n+k}{k} - 1$, but there are no such SRU's of size $2\binom{n+k}{k} - 2$ or less. Moreover, we show that there are SRU's for $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$ of size

$$\begin{cases} 2n-1, & \text{if } k = 1, \\ 2n-k+1, & \text{if } k > 1, \end{cases}$$

but there are no such SRU's of smaller size.

Key words: sets of range uniqueness, polynomials, magic sets, unique range, Vandermonde

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1 Introduction

Let $\mathcal{F}$ be a set of functions with a common domain X and a common range Y. A set $S \subseteq X$ is called a *set of range uniqueness (SRU) for* $\mathcal{F}$, if for all $f, g \in \mathcal{F}$,

$$f[S] = g[S] \implies f = g,$$

where $f[S] := \{f(s) \in Y \mid s \in S\}$. The set $S \subseteq X$ is called a *magic set for* $\mathcal{F}$ if for all $f, g \in \mathcal{F}$,

$$f[S] \subseteq g[S] \implies f = g.$$

Clearly, if S is an SRU or a magic set for $\mathcal{F}$, then S is also an SRU or a magic set for any subset $\mathcal{G} \subseteq \mathcal{F}$.

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Burke and Ciesielski have shown in [2] that SRU's always exist (*i.e.*, provable in ZFC) for the set of all Lebesgue-measurable real functions on $\mathbb{R}$. In [4] Diamond, Pomerance, and Rubel construct SRU's for the set $C^{\omega}(\mathbb{C})$ of entire functions. The continuum hypothesis implies the existence of an SRU for the class $C^{\mathrm{nwc}}(\mathbb{R})$ of continuous nowhere constant functions from $\mathbb{R}$ to $\mathbb{R}$. This has been shown by Berarducci and Dikranjan in [1]. Halbeisen, Lischka and Schumacher have replaced the continuum hypothesis by a weaker condition (see [7]). However, the existence of such a set is not provable in ZFC as Ciesielski and Shelah showed in [3].

At the other end of the regularity spectrum of functions lies the following result: For every $s \geq 2n+1$ there exist SRU's of cardinality s for the set of all real polynomials on $\mathbb{R}$ of degree n. But there is no SRU of cardinality $2n$ for this set (see [5]). Magic sets for polynomials are constructed in [6]: For polynomials of degree n there are magic sets of cardinality $s \geq 2n+1$ and no SRU's of smaller size exist. And there are SRU's for polynomials which are not magic.

In this paper we study SRU's for the set of multivariate real polynomials in n variables of degree at most k, for linear and affine functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and for linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ with given rank k.

2 Linear and affine functions

We will begin this section by introducing the following vector spaces

$$\begin{aligned}\mathcal{A}(\mathbb{R}^n, \mathbb{R}) &:= \{f : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \langle x, a \rangle + b \mid a \in \mathbb{R}^n, b \in \mathbb{R}\} \\ \mathcal{P}_n &:= \{p \in \mathbb{R}[x] \mid \deg(p) \leq n\}\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Note that both $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$ and $\mathcal{P}_n$ are $n+1$ dimensional vector spaces.

Notice that

$$\Phi : \mathcal{A}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_n, \quad f \mapsto f \circ h,$$

where $h : \mathbb{R} \to \mathbb{R}^n, x \mapsto (x, x^2, x^3, \ldots, x^n)$, is an isomorphism of vector spaces. Then we have:

Theorem 1. *If S is an SRU for $\mathcal{P}_n$, then $\overline{S} := h[S] \subseteq \mathbb{R}^n$ is an SRU for $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$.*

Proof. Let $f, g \in \mathcal{A}(\mathbb{R}^n, \mathbb{R})$ be such that $f\left[\overline{S}\right] = g\left[\overline{S}\right]$. Then consider the polynomials $F := \Phi(f)$ and $G = \Phi(g)$ in $\mathcal{P}_n$. We have

$$F[S] = (f \circ h)[S] = f[h[S]] = f\left[\overline{S}\right] = g\left[\overline{S}\right] = g[h[S]] = (g \circ h)[S] = G[S]$$

and hence $F = G$, since S is an SRU for $\mathcal{P}_n$. As Φ is an isomorphism of the vector spaces $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$ and $\mathcal{P}_n$ it follows $f = g$. ■

We now conclude immediately by the main result in [5, Section 3]:

Corollary 2. *For every $s \geq 2n+1$ there exists an SRU of size s for the set $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$.*

We transfer this finding to the n-dimensional vector space

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}) := \{f : \mathbb{R}^n \to \mathbb{R}, x \mapsto \langle x, a\rangle \mid a \in \mathbb{R}^n\}$$

of linear functionals on $\mathbb{R}^n$. Observe that

$$\Psi : \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{A}(\mathbb{R}^{n-1}, \mathbb{R}), \quad f \mapsto f \circ k,$$

where $k : \mathbb{R}^{n-1} \to \mathbb{R}^n, x \mapsto (x, 1)$, is an isomorphism of vector spaces. We find:

Theorem 3. *If S is an SRU for $\mathcal{A}(\mathbb{R}^{n-1}, \mathbb{R})$, then $\overline{S} := k[S]$ is an SRU for $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.*

Proof. Let $f, g \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ be such that $f\left[\overline{S}\right] = g\left[\overline{S}\right]$. Then consider $F := \Psi(f)$ and $G = \Psi(g)$ in $\mathcal{A}(\mathbb{R}^{n-1}, \mathbb{R})$. We have

$$F[S] = (f \circ k)[S] = f[k[S]] = f\left[\overline{S}\right] = g\left[\overline{S}\right] = g[k[S]] = (g \circ k)[S] = G[S]$$

and hence $F = G$, because S is an SRU for $\mathcal{A}(\mathbb{R}^{n-1}, \mathbb{R})$. Since Ψ is an isomorphism of the vector spaces $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ and $\mathcal{A}(\mathbb{R}^{n-1}, \mathbb{R})$ it follows $f = g$. ■

Thus, by Corollary 2 we obtain:

Corollary 4. *For every $s \geq 2n - 1$ there exists an SRU of size s for the set $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.*

Now we show, that Corollary 4 is optimal.

Theorem 5. *Let $S = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{R}^n$ with $k \leq 2n - 2$. Then there are two functionals $f, g \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ with $f[S] = g[S]$ but $f \neq g$. In other words: There is no SRU for $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ of cardinality at most $2n - 2$.*

Proof. We are looking for functions $f(x) = \langle a, x\rangle$, $g(x) = \langle b, x\rangle$ with $a, b \in \mathbb{R}^n$, $a \neq b$. If $U := \operatorname{span}\{x_1, \ldots, x_k\} \neq \mathbb{R}^n$, we can choose $f \equiv 0$ and extend $g = 0$ on U nontrivially to $\mathbb{R}^n$, and we are done with $f[S] = g[S] = \{0\}$. So, we may assume without loss of generality that $U = \mathbb{R}^n$.

Consider the homogeneous linear system of equations

$$\langle a, x_i\rangle = \langle b, x_{i+1}\rangle \quad \text{for } i = 1, 2, \ldots, k, \tag{1}$$

where we take indices cyclically; $(a, b) \in \mathbb{R}^{2n}$ are the unknowns. This system has rank at most $k \leq 2n-2$ and therefore the null space is at least two-dimensional. Hence there exist non-trivial solutions $(a, b) \in \mathbb{R}^{2n}$ of (1). Observe that solutions for which $a = b$ span at most a one dimensional subspace, since $a = b$ implies $\langle a, x_i\rangle = \langle b, x_i\rangle = c$ for all i for some constant c. ■

It now follows from Theorem 3 and Theorem 5 that also Corollary 2 is optimal:

Corollary 6. *There is no SRU for $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$ of cardinality $\leq 2n$.*

Remark 7. It is clear that $S \subseteq \mathbb{R}^n$ is an SRU for $\mathcal{A}(\mathbb{R}^n, \mathbb{R}^m)$ of vector valued affine functions if and only if S is an SRU for $\mathcal{A}(\mathbb{R}^n, \mathbb{R})$. And the analogous statement holds for vector valued linear maps $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Hence the results of this section hold mutatis mutandis for $\mathcal{A}(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, respectively.

3 Generalized Vandermonde matrices

Let $n \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N}$ be fixed, arbitrary natural numbers. Let us denote by $s \in \mathbb{N}$, the number of all possible monomials in n variables of degree at most k, and for $0 \le p \le s$, we denote s_p the number of monomials of degree exactly p. The latter, s_p, is a standard combinatorial result call "Stars and Bars" problem with k stars and $n-1$ bars[1], and hence,

$$\text{for all } p \text{ with } 0 \le p \le k, \quad s_p = \binom{n+p-1}{p}.$$

Using the combinatorial identity $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$, we also see that

$$\binom{n+0-1}{0} + \binom{n+1-1}{1} + \ldots + \binom{n+k-1}{k} = \binom{n+k}{k}.$$

In other words, the identity $s = \sum_{p=0}^{k} s_p$ holds as a direct consequence. For all $1 \le i \le s$, let

$$x_i := (x_{i,1}, x_{i,2}, \ldots, x_{i,n}) \in \mathbb{R}^n \quad \text{and} \quad \alpha_i \in \mathbb{R} \setminus \{0\}.$$

In short, the x_i are real vectors and the α_i are non-zero real scalars for all $1 \le i \le s$.

For every $0 \le i \le k$, let γ_i be the following collection of n-tuples:

$$\gamma_i := \Big\{ (m_1, \ldots, m_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} m_j = i \Big\}.$$

Intuitively, each n-tuple in γ_i represents the exponents of a monomial of degree exactly i. Note that the sets γ_i are pairwise disjoint. Without loss of generality, we will order the elements in γ_i lexicographically. For every $1 \le j \le k$ and every $\beta = (\beta_1, \ldots, \beta_n) \in \gamma_j$, we define

$$x_i^{\beta} := \prod_{l=1}^{n} x_{il}^{\beta_l},$$

where x_{il} is the l^{th} component of x_i. In particular, x_i^{β} represents the monomial whose exponents are given by β, and the value of the l^{th} variable is determined by the l^{th} component of x_i. We will also adopt the following notation to simplify writing: For all $1 \le i, j \le$ let

$$x_i^{\gamma_j} := \big(x_i^{\gamma_{j,1}}, x_i^{\gamma_{j,2}}, \ldots, x_i^{\gamma_{j,s_j}} \big).$$

The generalized Vandermonde matrix $A_{n,k}(x_1, \ldots, x_s)$ is given by the following $s \times s$-matrix:

$$A_{n,k}(x_1, \ldots, x_s) = \begin{pmatrix} \alpha_1 & x_1^{\gamma_1} & x_1^{\gamma_2} & \ldots & x_1^{\gamma_k} \\ \alpha_2 & x_2^{\gamma_1} & x_2^{\gamma_2} & \ldots & x_2^{\gamma_k} \\ \vdots & \vdots & \ddots & \vdots & \\ \alpha_s & x_s^{\gamma_1} & x_s^{\gamma_2} & \ldots & x_s^{\gamma_k} \end{pmatrix}$$

[1] This is also the "Bins and Balls" problem with n bins and k balls, but they are equivalent.

Example 8. For $n = k = 2$ and $\alpha_i = i$ for all $0 \leq i \leq 6$ we have that

$$A_{2,2}(x_1, x_2) = \begin{pmatrix} 1 & x_{11} & x_{12} & x_{11}^2 & x_{11}x_{12} & x_{12}^2 \\ 2 & x_{21} & x_{22} & x_{21}^2 & x_{21}x_{22} & x_{22}^2 \\ 3 & x_{31} & x_{32} & x_{31}^2 & x_{31}x_{32} & x_{32}^2 \\ 4 & x_{41} & x_{42} & x_{41}^2 & x_{41}x_{42} & x_{42}^2 \\ 5 & x_{51} & x_{52} & x_{51}^2 & x_{51}x_{52} & x_{52}^2 \\ 6 & x_{61} & x_{62} & x_{61}^2 & x_{61}x_{62} & x_{62}^2 \end{pmatrix}$$

Lemma 9. *For all $n \in \mathbb{N} \setminus \{0\}$ and all $k \in \mathbb{N}$ we have that* $\det(A_{n,k}) \not\equiv 0$.

Proof. We prove this Lemma by induction on k. If $k = 0$, then $s = \binom{n}{0} = 1$ and

$$\det(A_{n,0}) = \alpha_1 \neq 0.$$

Now let $k \geq 1$. Consider the Laplace expansion of $A_{n,k}$ along the first row. We have that

$$\det(A_{n,k}) = (-1)^{s+1} x_1^{\gamma_{k,s_k}} \det\left(A_{n,k}^1(x_2, \ldots, x_s)\right) + \delta_1,$$

where $A_{n,k}^1$ is the submatrix obtained by deleting the first row and the last column from $A_{n,k}$, and δ_1 is the remaining polynomial term due to the Laplace expansion; note that no summand in δ_1 is divisible by $x_1^{\gamma_{k,s_k}}$. If $A_{n,k}^1 \neq A_{n,k-1}$ we do a Laplace expansion of $A_{n,k}^1$ along the first row and we get that

$$\det(A_{n,k}) = (-1)^{2s+1} x_1^{\gamma_{k,s_k}} x_2^{\gamma_{k,s_k-1}} \det\left(A_{n,k}^2(x_3, \ldots, x_s)\right) + \delta_2,$$

As before $A_{n,k}^2$ is the submatrix obtained by deleting the first row and the last column from $A_{n,k}^1$, and δ_2 represent the remaining polynomial in which no summand is divisible by $x_1^{\gamma_{k,s_k}} x_2^{\gamma_{k,s_k-1}}$.

Inductively apply the Laplace expansion until we get $A_{n,k}^l = A_{n,k-1}$ for some $l \in \mathbb{N} \setminus \{0\}$. Thus we can write

$$\det(A_{n,k}) = \epsilon \det\left(A_{n,k-1}(x_{l+1}, x_{l+2}, \ldots, x_s)\right) + \delta_l$$

with a monomial $\epsilon := \epsilon(x_1, \ldots, x_l)$ that does not divide any summand in δ_l. By the induction hypothesis $\det(A_{n,k-1}) \not\equiv 0$, it follows that $\det(A_{n,k}) \not\equiv 0$. ■

4 SRU's for multivariate polynomials

Let $n \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N}$ and define

$$\mathcal{P}_{n,k} := \left\{ f \in \mathbb{R}[x_1, x_2, \ldots, x_n] \;\middle|\; \deg(f) \leq k \right\}.$$

We will prove that for all $l \geq 2 \cdot \binom{n+k}{k} - 1$ there is an SRU of size l for $\mathcal{P}_{n,k}$. To do this, it will be necessary to generalize the proof of [5, Theorem 8] considerably. Let $\{x_1, x_2, \ldots, x_l\} \subseteq \mathbb{R}^n$. The following family $\mathcal{G}$ of directed graphs will play a crucial role in the construction of SRU's of size l for the set $\mathcal{P}_{n,k}$:

Definition 10. $\mathcal{G}$ is the family of all directed graphs $G = (V, E)$ with vertex set $V = \{x_1, x_2, \dots, x_l\}$ and a set E of directed edges (x_i, x_j) such that for each $x \in V$ we have

$$\text{indegree}_G(x) \geq 1 \text{ and } \text{outdegree}_G(x) \geq 1.$$

Definition 11. Let $G = (V, E)$ be a directed graph.

- A *cycle* of G is a subgraph $C = (V_C, E_C)$ of G with $V_C = \{c_0, c_1, \dots, c_{m-1}\}$ and $E_C = \{(c_i, c_{(i+1) \bmod m}) \mid i \in \mathbb{N}\}$ for an $m \geq 2$.
- A *loop* is a subgraph $L = (V_L, E_L)$ of G with $V_L = \{w\}$ and $E_L = \{(w, w)\}$.
- A *path* is a subgraph $P = (V_P, E_P)$ of G with $V_P = \{p_0, p_1, \dots, p_{m-1}\}$ and $E_C = \{(p_i, p_{i+1}) \mid 0 \leq i \leq m-2\}$ for an $m \geq 2$.

Definition 12. Let $s \in \mathbb{N}$. Cycles and loops $C_1 = (V_{C_0}, E_{C_0}), \dots, C_s = (V_{C_l}, E_{C_l})$ are called *obviously different* if for every $1 \leq i \leq s$ there is a $x_i \in V_{C_i}$ with

$$x_i \in V_{C_i} \setminus \left(\bigcup_{j=1, j \neq i}^{s} V_{C_j} \right).$$

Definition 13. A graph $G = (V, E)$ is of type $1_{n,k}$ iff there are at most $\binom{n+k}{k} - 1$ obviously different cycles and loops in G. Otherwise G is of type $2_{n,k}$.

4.1 Graphs and matrices of type $1_{n,k}$

Definition 14. Let $n \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N}$ and let $G = (V, E)$ be a graph of type $1_{n,k}$ with $|V| \geq 2 \cdot \binom{n+k}{k} - 1$. A *nice sequence of length* s is a sequence of graphs

$$G_0 = (V_0, E_0) \subseteq G_1 = (V_1, E_1) \subseteq \dots \subseteq G_s = (V_s, E_s) \subseteq G$$

with the following properties: For all $0 \leq i \leq s$

1. we have that $|E_i| \in \{2i, 2i+1\}$;
2. there are at most i obviously different loops and cycles in G_i,
3. we have that $E_{i+1} \setminus E_i$ has one of the following forms:
   - $E_{i+1} \setminus E_i = \{(x_j, x_j), (x_m, x_p)\}$ with $\deg_{G_i}(x_j) = 0$, and $\det_{G_i}(x_m) = 0$ or $\deg_{G_i}(x_p) = 0$;
   - $E_{i+1} \setminus E_i = \{(x_j, x_m), (x_p, x_j)\}$ with $\deg_{G_i}(x_j) = 0$.

Theorem 15. *Let $n \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N}$ and $s := \binom{n+k}{k}$. Every graph $G = (V, E)$ of type $1_{n,k}$ with $|V| \geq 2 \cdot \binom{n+k}{k} - 1$ has a nice sequence*

$$G_0 \subseteq G_1 \subseteq \dots \subseteq G_{s-1} = (V_{s-1}, E_{s-1})$$

with $|E_{s-1}| = 2 \cdot \binom{n+k}{k} - 1$.

Proof. The proof is the analogue of the proofs of [5, Lemma 12] and [5, Corollary 15]. ■

For all $0 \leq i \leq k$ let $s_i := \binom{n+i}{i}$. Define $M_{G_0} := \begin{pmatrix} 1 \end{pmatrix}$. And for all $0 \leq i < k$ and all $1 \leq l \leq s_{i+1} - s_i$ we define

$$M_{G_{s_i+l}} = \begin{pmatrix} x_{j_1}^{\bigcup_{p=0}^{i} \gamma_p} & x_{j_1}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} & -x_{l_1}^{\bigcup_{p=1}^{i} \gamma_p} & -x_{l_1}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} \\ x_{j_2}^{\bigcup_{p=0}^{i} \gamma_p} & x_{j_2}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} & -x_{l_2}^{\bigcup_{p=1}^{i} \gamma_p} & -x_{l_2}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j_{2(s_i+l)+1}}^{\bigcup_{p=0}^{i} \gamma_p} & x_{j_{2(s_i+l)+1}}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} & -x_{l_{2(s_i+l)+1}}^{\bigcup_{p=1}^{i} \gamma_p} & -x_{l_{2(s_i+l)+1}}^{\bigcup_{q=1}^{l} \gamma_{i+1,q}} \end{pmatrix}$$

where for all $1 \leq m \leq s_i + l$ we have that $(x_{j_m}, x_{l_m}) \in E_{G_{s_i+l}}$.

Lemma 16. *For all $1 \leq i \leq s_k$ we have that* $\det(M_{G_i}) \not\equiv 0$.

Proof. We prove this Lemma by induction over i. For $i = 0$ we have that

$$\det(M_{G_1}) = 1 \not\equiv 0.$$

Now let $0 \leq i \leq k$ and $0 < l \leq s_{i+1} - s_i$ such that $s_i + l \geq 1$. We want to show that $\det(M_{G_{s_i+l}}) \not\equiv 0$. There are three cases:

Case 1: $E_{s_i+l} \setminus E_{s_i+l-1} = \{(x_j, x_m), (x_p, x_j)\}$ with $\deg_{G_{s_i+l-1}}(x_j) = 0$.

First of all we do a Laplace expansion along the row containing x_j and x_m. We get

$$\det(M_{G_{s_i+l}}) = \epsilon_0 x_j^{\gamma_{i+1,l}} \det(\overline{M_{G_{s_i+l}}}) + \delta_0, \tag{2}$$

where $\overline{M_{G_{s_i+l}}}$ is the matrix we obtain from $M_{G_{s_i+l}}$ by deleting the row containing x_j and x_m and the column containing $x_j^{\gamma_{i+1,l}}$. Moreover, we have that $\epsilon_0 \in \{-1, 1\}$ and that $\left(x_j^{\gamma_{i+1,l}}\right)^2$ does not divide any summand in δ_0 because from $\deg_{G_{s_i+l-1}}(x_j) = 0$ it follows that x_j is only contained in two rows of $M_{G_{s_i+l}}$. Now we do a second Laplace expansion along the row containing x_j and x_p. We get that

$$\det(\overline{M_{G_{s_i+l}}}) = \epsilon_1 x_j^{\gamma_{i+1,l}} \det(M_{G_{s_i+l-1}}) + \delta_1, \tag{3}$$

where $\epsilon_1 \in \{-1, 1\}$ and $x_j^{\gamma_{i+1,l}}$ does not divide any summand in δ_1. Combining (2) and (3) we get that

$$\det(M_{G_{s_i+l}}) = \epsilon_0 \epsilon_1 \left(x_j^{\gamma_{i+1,l}}\right)^2 \det(M_{G_{s_i+l-1}}) + \epsilon_0 x_j^{\gamma_{i+1,l}} \delta_1 + \delta_0,$$

where $\epsilon_0 x_j^{\gamma_{i+1,l}} \delta_1 + \delta_0$ does not contain a summand that can be divided by $\left(x_j^{\gamma_{i+1,l}}\right)^2$. Since by the induction hypothesis $\det(M_{G_{s_i+l-1}}) \not\equiv 0$ it follows that

$$\det(M_{G_{s_i+l}}) \not\equiv 0.$$

Case 2: $E_{s_i+l} \setminus E_{s_i+l-1} = \{(x_j, x_j), (x_m, x_p)\}$ with $\deg_{G_{s_i+l-1}}(x_m) = 0$ and $\deg_{G_{s_i+l-1}}(x_j) = 0$.

As in Case 1 we do two Laplace expansions. First we do one along the row containing x_j and then we do one along the row containing x_m and x_p. We get that

$$\det(M_{G_{s_i+l}}) = \epsilon_0 \epsilon_1 x_j^{\gamma_{i+1,l}} x_m^{\gamma_{i+1,l}} \det(M_{G_{s_i+l-1}}) + \epsilon_0 x_j^{\gamma_{i+1,l}} \delta_1 + \delta_0,$$

where $\epsilon_0, \epsilon_1 \in \{-1, 1\}$ and $x_j^{\gamma_{i+1,l}} x_m^{\gamma_{i+1,l}}$ does not divide any summand in $\epsilon_0 x_j^{\gamma_{i+1,l}} \delta_1 + \delta_0$. Since by the induction hypothesis $\det(M_{G_{s_i+l-1}}) \not\equiv 0$ it follows that

$$\det(M_{G_{s_i+l}}) \not\equiv 0.$$

Case 3: $E_{s_i+l} \setminus E_{s_i+l-1} = \{(x_j, x_j), (x_m, x_p)\}$ with $\deg_{G_{s_i+l-1}}(x_p) = 0$ and $\deg_{G_{s_i+l-1}}(x_j) = 0$.

This case is similar to Case 2. ■

4.2 Graphs and matrices of type $2_{n,k}$

Let $G = (V, E)$ be a graph of type $2_{n,k}$. So G contains at least $s = \binom{n+k}{k}$ obviously different loops and cycles

$$C_1 = (V_{C_1}, E_{C_1}), C_2 = (V_{C_2}, E_{C_2}), \ldots, C_s = (V_{C_s}, E_{C_s}).$$

Without loss of generality we can assume that for all $1 \leq i \leq s$ we have that

$$x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n}) \in V_{C_i} \setminus \left(\bigcup_{j=1, j \neq i}^{s} V_{C_j} \right).$$

Let $t := \binom{n+k-1}{k}$ and define

$$N_G(x_1, x_2, \ldots, x_l) := \begin{pmatrix} |V_{C_1}| & \sum_{x \in V_{C_1}} x^{\gamma_{1,1}} & \sum_{x \in V_{C_1}} x^{\gamma_{1,2}} & \cdots & \sum_{x \in V_{C_1}} x^{\gamma_{k,t}} \\ |V_{C_2}| & \sum_{x \in V_{C_2}} x^{\gamma_{1,1}} & \sum_{x \in V_{C_2}} x^{\gamma_{1,2}} & \cdots & \sum_{x \in V_{C_2}} x^{\gamma_{k,t}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_s}| & \sum_{x \in V_{C_s}} x^{\gamma_{1,1}} & \sum_{x \in V_{C_s}} x^{\gamma_{1,2}} & \cdots & \sum_{x \in V_{C_s}} x^{\gamma_{k,t}} \end{pmatrix}$$

Now we want to show that $\det(N_G(x_1, \ldots, x_l)) \not\equiv 0$. It suffices to prove that the determinant of

$$N_G(x_1, \ldots, x_s, 0, \ldots, 0) = \begin{pmatrix} |V_{C_1}| & x_1^{\gamma_1} & x_1^{\gamma_2} & \cdots & x_1^{\gamma_k} \\ |V_{C_2}| & x_2^{\gamma_1} & x_2^{\gamma_2} & \cdots & x_2^{\gamma_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_s}| & x_s^{\gamma_1} & x_s^{\gamma_2} & \cdots & x_s^{\gamma_k} \end{pmatrix}$$

is not identically equal to zero. Since $|V_{C_i}| \neq 0$ for all $1 \leq i \leq s$ this follows from Lemma 9.

4.3 An SRU for $\mathcal{P}_{n,k}$

Lemma 17. *For every $l \geq 2 \cdot \binom{n+k}{k} - 1$ there is an SRU of size l for $\mathcal{P}_{n,k}$.*

Proof. This Lemma can be proven as in [5, Theorem 8]. ■

5 Minimal cardinality of SRU's for $\mathcal{P}_{n,k}$

In this section we will prove that there are no SRU's of size at most $2 \cdot \binom{n+k}{k} - 2$ for $\mathcal{P}_{n,k}$. To do this, we will generalize the proof of [8, Theorem 2.7].

In this section let $n \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N}$. For all $0 \leq p \leq k$ define $s_p := \binom{n+p-1}{p}$.

Remark 18. For all $0 \leq p \leq k$ there are $\binom{n+p-1}{p}$ monomials of degree exactly p in n variables and there are $\binom{n+p}{p}$ monomials of degree at most p in n variables.

Lemma 19. *Let $f, g \in \mathcal{P}_{n,k}$ and let x_i, x_j be such that $f(x_i) = g(x_j)$ and $f(x_j) = g(x_i)$. Then we have that*

$$\sum_{p=0}^{k} \sum_{q=1}^{s_p} (a_{pq} - b_{pq})(x_i^{\gamma_{p,q}} + x_j^{\gamma_{p,q}}) = 0$$

and

$$\sum_{p=1}^{k} \sum_{q=1}^{s_p} (a_{pq} + b_{pq})(x_i^{\gamma_{p,q}} - x_j^{\gamma_{p,q}}) = 0,$$

where $f(x) = \sum_{p=0}^{k} \sum_{q=1}^{s_p} a_{pq} x^{\gamma_{p,q}}$ and $g(x) = \sum_{p=0}^{k} \sum_{q=1}^{s_p} b_{pq} x^{\gamma_{p,q}}$.

Proof. Since $f(x_i) = g(x_j)$ and $f(x_j) = g(x_i)$ we have that

$$\begin{aligned}\sum_{p=0}^{k} \sum_{q=1}^{s_p} a_{pq}(x_i^{\gamma_{p,q}} + x_j^{\gamma_{p,q}}) = f(x_i) + f(x_j) &= g(x_i) + g(x_j) \\ &= \sum_{p=0}^{k} \sum_{q=1}^{s_p} b_{pq}(x_i^{\gamma_{p,q}} + x_j^{\gamma_{p,q}}).\end{aligned}$$

By rearranging this equation we get that

$$\sum_{p=0}^{k} \sum_{q=1}^{s_p} (a_{pq} - b_{pq})(x_i^{\gamma_{p,q}} + x_j^{\gamma_{p,q}}) = 0.$$

Moreover, we have that

$$
\begin{aligned}
\sum_{p=1}^{k}\sum_{q=1}^{s_p} a_{pq}(x_i^{\gamma_{p,q}} - x_j^{\gamma_{p,q}}) &= \sum_{p=0}^{k}\sum_{q=1}^{s_p} a_{pq}(x_i^{\gamma_{p,q}} - x_j^{\gamma_{p,q}}) = f(x_i) - f(x_j) \\
&= g(x_j) - g(x_i) = \sum_{p=0}^{k}\sum_{q=1}^{s_p} b_{pq}(x_j^{\gamma_{p,q}} - x_i^{\gamma_{p,q}}) \\
&= \sum_{p=1}^{k}\sum_{q=1}^{s_p} b_{pq}(x_j^{\gamma_{p,q}} - x_i^{\gamma_{p,q}}).
\end{aligned}
$$

By rearranging this equation we get that

$$\sum_{p=1}^{k}\sum_{q=1}^{s_p} (a_{pq} + b_{pq})(x_i^{\gamma_{p,q}} - x_j^{\gamma_{p,q}}) = 0. \qquad \blacksquare$$

Lemma 20. *Let $S = \{x_1, x_2, \dots, x_r\} \subseteq \mathbb{R}^n$ with $1 \le r \le 2 \cdot \binom{n+k}{k} - 2$. Choose an $l \in \mathbb{N}$ such that $r = 2l$ if r is even and $r = 2l+1$ if r is odd. Then there exist $c_{p,q} \in \mathbb{R}$ and $d_{pq} \in \mathbb{R}$ with*

$$\sum_{p=0}^{k}\sum_{q=1}^{s_p} c_{pq}(x_{2i-1}^{\gamma_{p,q}} + x_{2i}^{\gamma_{p,q}}) = 0 = \sum_{p=0}^{k}\sum_{q=1}^{s_p} d_{p,q}(x_{2i-1}^{\gamma_{p,q}} - x_{2i}^{\gamma_{p,q}}) \tag{4}$$

for all $1 \le i \le l$. Moreover, if r is odd,

$$\sum_{p=0}^{k}\sum_{q=1}^{s_p} c_{pq} x_r^{\gamma_{p,q}} = 0.$$

Proof. First of all we assume that $r = 2l \le 2 \cdot \binom{n+k}{k} - 2$ is even. We will look at the other case later. For all $1 \le i \le l$ we want that

$$\sum_{p=0}^{k}\sum_{q=1}^{s_p} c_{pq}(x_{2i-1}^{\gamma_{p,q}} + x_{2i}^{\gamma_{p,q}}) = 0 \quad \text{and} \quad \sum_{p=0}^{k}\sum_{q=1}^{s_p} d_{pq}(x_{2i-1}^{\gamma_{p,q}} - x_{2i}^{\gamma_{p,q}}) = 0.$$

Define the following two $l \times \binom{n+k}{k}$-matrices X_1 and X_2:

$$X_1 := \begin{pmatrix} 1 & x_1^{\gamma_1} & x_1^{\gamma_2} & \dots & x_1^{\gamma_k} \\ 1 & x_3^{\gamma_1} & x_3^{\gamma_2} & \dots & x_3^{\gamma_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2l-1}^{\gamma_1} & x_{2l-1}^{\gamma_2} & \dots & x_{2l-1}^{\gamma_k} \end{pmatrix} \quad X_2 := \begin{pmatrix} 1 & x_2^{\gamma_1} & x_2^{\gamma_2} & \dots & x_2^{\gamma_k} \\ 1 & x_4^{\gamma_1} & x_4^{\gamma_2} & \dots & x_4^{\gamma_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2l}^{\gamma_1} & x_{2l}^{\gamma_2} & \dots & x_{2l}^{\gamma_k} \end{pmatrix}.$$

Since $l < \binom{n+k}{k}$ there are non-trivial vectors

$$(c_{pq}) = c \in \ker(X_1 + X_2) \setminus \{0\} \text{ and } (d_{pq}) = d \in \ker(X_1 - X_2) \setminus \{0\}.$$

So in case r is even we are done. Assume now that $r = 2l + 1 \leq 2 \cdot \binom{n+k}{k} - 2$ is odd. Consider the following two $(l+1) \times \binom{n+k}{k}$-matrices

$$X_3 := \begin{pmatrix} 1 & x_1^{\gamma_1} & x_1^{\gamma_2} & \dots & x_1^{\gamma_k} \\ 1 & x_3^{\gamma_1} & x_3^{\gamma_2} & \dots & x_3^{\gamma_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2l-1}^{\gamma_1} & x_{2l-1}^{\gamma_2} & \dots & x_{2l-1}^{\gamma_k} \\ 1 & x_{2l+1}^{\gamma_1} & x_{2l+1}^{\gamma_2} & \dots & x_{2l+1}^{\gamma_k} \end{pmatrix}$$

and

$$X_4 := \begin{pmatrix} 1 & x_2^{\gamma_1} & x_2^{\gamma_2} & \dots & x_2^{\gamma_k} \\ 1 & x_4^{\gamma_1} & x_4^{\gamma_2} & \dots & x_4^{\gamma_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2l}^{\gamma_1} & x_{2l}^{\gamma_2} & \dots & x_{2l}^{\gamma_k} \\ 1 & x_{2l+1}^{\gamma_1} & x_{2l+1}^{\gamma_2} & \dots & x_{2l+1}^{\gamma_k} \end{pmatrix}.$$

Thus $X_3 + X_4$ and $X_3 - X_4$ are matrices of dimension $(l+1) \times \binom{n+k}{k}$ and since $l + 1 < \binom{n+k}{k}$ there are

$$(c_{pq}) = c \in \ker(X_3 + X_4) \setminus \{0\} \text{ and } (d_{pq}) = d \in \ker(X_3 - X_4) \setminus \{0\}.$$

So in case r is odd the equations (4) hold. In case r is odd the $(l+1)$-th row of $X_3 + X_4$ yields

$$\sum_{p=0}^{k} \sum_{q=1}^{s_p} 2c_{pq} x_r^{\gamma_{p,q}} = 0 \quad \Rightarrow \quad \sum_{p=0}^{k} \sum_{q=1}^{s_p} c_{pq} x_r^{\gamma_{p,q}} = 0. \qquad \blacksquare$$

Theorem 21. *Any set $S \subseteq \mathbb{R}^n$ with $|S| \leq 2 \cdot \binom{n+k}{k} - 2$ is not an SRU for $\mathcal{P}_{n,k}$.*

Proof. Let $1 \leq r \leq 2 \cdot \binom{n+k}{k} - 2$ be arbitrary, let $S = \{x_1, x_2, \dots, x_r\} \subseteq \mathbb{R}^n$ and let the c_{pq} and the d_{pq} be as in Lemma 20. Let

$$a_{pq} := d_{pq} + c_{pq} \text{ and } b_{pq} := d_{pq} - c_{pq}$$

for all $0 \leq p \leq k$ and all $0 \leq q \leq s_p$. Define $f(x) = \sum_{p=0}^{k} \sum_{q=1}^{s_p} a_{pq} x^{\gamma_{pq}}$ and $g(x) = \sum_{p=0}^{k} \sum_{q=1}^{s_p} b_{pq} x^{\gamma_{p,q}}$.

Claim 1: $f \neq g$

Since $c = (c_{pq}) \neq 0$ there are p, q with $c_{pq} \neq 0$. So

$$a_{pq} = d_{pq} + c_{pq} \neq d_{pq} - c_{pq} = b_{pq}$$

and therefore, $f \neq g$.

Claim 2: $f[S] = g[S]$

By Lemma 20 we have that

$$f(x_{2i-1}) = \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}+c_{pq})x_{2i-1}^{\gamma_{p,q}} = \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}-c_{pq})x_{2i}^{\gamma_{p,q}} = g(x_{2i})$$

and

$$f(x_{2i}) = \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}+c_{pq})x_{2i}^{\gamma_{p,q}} = \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}-c_{pq})x_{2i-1}^{\gamma_{p,q}} = g(x_{2i-1})$$

for all i with $2i \leq r$. If r is odd we additionally have that

$$\begin{aligned} f(x_r) &= \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}+c_{pq})x_r^{\gamma_{p,q}} = \sum_{p=0}^{k}\sum_{q=1}^{s_p} d_{pq}x_r^{\gamma_{p,q}} = \\ &= \sum_{p=0}^{k}\sum_{q=1}^{s_p}(d_{pq}-c_{pq})x_r^{\gamma_{p,q}} = g(x_r). \end{aligned}$$

■

Remark 22. In Lemma 20 we can choose $d = \big(1, 0, \ldots, 0\big)^T$. Therefore, for all sets $S \subseteq \mathbb{R}^n$ with $|S| \leq 2 \cdot \binom{n+k}{k} - 2$ there are functions $f, g \in \mathcal{P}_{n,k}$ with $f[S] = g[S]$ and $g = 2 - f$.

Example 23. Let $k = n = 2$ and

$$S := \{(0,0),(0,4),(4,0),(4,4),(1,2),(2,1),(2,3),(4,1),(2,2),(4,2)\}.$$

Then $|S| = 10 = 2 \cdot \binom{n+k}{k} - 2$. Indeed, for

$$f(x,y) = -9 + 6x + y - 2x^2 + xy + y^2$$

and

$$g(x,y) = 2 - f(x,y) = 11 - 6x - y + 2x^2 - xy - y^2$$

which are both in $\mathcal{P}_{n,k}$ we have $f[S] = g[S]$.

6 Linear maps with rank k

Let $n, k \in \mathbb{N} \setminus \{0\}$. In this section we are interested in the family of all linear endomorphisms with rank k, i.e.,

$$\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n) := \{f(x) = Ax \mid A \text{ is an } n \times n \text{ matrix with rank } k\}.$$

We will prove that the minimal size of an SRU for the family $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$ is

$$\begin{cases} 2n - 1 & k = 1 \\ 2n - k + 1 & k > 1 \end{cases}$$

6.1 An SRU for $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$

For $k = 1$ we have already constructed an SRU of size $2n-1$ for the family of all linear maps with rank at least $k = 1$, see Remark 7. So let $n \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N} \setminus \{0, 1\}$. Now we want to construct an SRU for the family of all linear maps with rank at least k of size $2n - k + 1$.

Let $S := \{x_1, x_2, \dots, x_{2n-k+1}\}$. We look at the family $\mathcal{F}$ of all graphs on S in which each vertex has indegree and outdegree at least one. Let $U \subseteq \mathbb{R}^{2n-k+1}$ be a non-empty, open set. If a graph $G \in \mathcal{F}$ has at least n obviously different loops and cycles $C_1, C_2, \dots, C_n$ we can find a matrix

$$M_G(x_1, \dots, x_{2n-k+1}) = \begin{pmatrix} |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \dots & \sum_{x \in V_{C_1}} x^{n-1} \\ |V_{C_2}| & \sum_{x \in V_{C_2}} x & \sum_{x \in V_{C_2}} x^2 & \dots & \sum_{x \in V_{C_2}} x^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_1}} x^2 & \dots & \sum_{x \in V_{C_n}} x^{n-1} \end{pmatrix}$$

and a non-empty open subset $V \subseteq U$, such that $M_G(x_1, \dots, x_{2n-k+1})$ has full rank, i.e. non-zero determinant, for all $(x_1, \dots, x_{2n-k+1}) \in V$. This matrix can be found as in [5, Section 3.3]. Now assume that $G = (V_G, E_G) \in \mathcal{F}$ has less than n obviously different loops and cycles. For all $1 \le i \le 2n-k+1$ we define

$$v_i := (x_i, x_i^2, \cdots, x_i^{n-1}).$$

We will show later that we can find a matrix

$$N_G(x_1, \dots, x_{2n-k+1}) = \begin{pmatrix} 1 & v_{i_1} & -1 & -v_{j_1} \\ 1 & v_{i_2} & -1 & -v_{j_2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & v_{i_{2n-k+1}} & -1 & -v_{j_{2n-k+1}} \end{pmatrix}$$

such that $(x_{i_l}, x_{j_l}) \in E_G$ for all $1 \le l \le 2n-k+1$ and an open subset $V \subseteq U$ such that $N_G(x_1, x_2, \dots, x_{2n-k+1})$ has full rank for all $(x_1, x_2, \dots, x_{2n-k+1}) \in V$.

Proposition 24. *There is an SRU $S = \{s_1, s_2, \dots, s_{2n-k+1}\} \subseteq \mathbb{R}^n$ for the family of all linear maps with rank at least k.*

Proof. Since the family $\mathcal{F}$ is finite, we can find an open set $U \subseteq \mathbb{R}^{2n-k+1}$ such that for all $G \in \mathcal{F}$, the matrix M_G (if G contains at least n cycles) or the matrix N_G (if G contains less than n cycles) has full rank. Let $T = \{t_1, \dots, t_{2n-k+1}\} \subseteq \mathbb{R}$ such that $(t_1, \dots, t_{2n-k+1}) \in U$. For all $1 \le i \le 2n-k+1$ we define

$$s_i := (1, t_i, t_i^2, \dots, t_i^{n-1}) \quad \text{and} \quad S := \{s_1, \dots, s_{2n-k+1}\}.$$

Assume that there are functions $f(x) = Ax$ and $g(x) = Bx$ with $\operatorname{rank}(A) = \operatorname{rank}(B) \ge k$ and $f[S] = g[S]$. Our goal is to show that $f = g$. We define a graph G on S by drawing an edge pointing from s_i to s_j whenever $f(s_i) = g(s_j)$. There are two cases:

Case 1: G contains at least n cycles.
For all $1 \leq i \leq n$ let a_i and b_i be the i-th row of A or B respectively. In this case we have that

$$M_G(a_i - b_i) = 0$$

for all $1 \leq i \leq n$. However, since $\dim(\ker(M_G)) = 0$, it follows that $a_i - b_i = 0$ and therefore, $A = B = 0$ which is a contradiction.

Case 2: G contains less than n cycles.
In this case, N_G has full rank, namely, for all $(x_1, \ldots, x_{2n-k+1}) \in U$ we have $\mathrm{rank}(N_G) = 2n - k + 1$. So,

$$\dim(\ker(N_G)) = 2n - \mathrm{rank}(N_G) = 2n - 2n + k - 1 = k - 1.$$

For all $1 \leq i \leq n$ let a_i and b_i be the i-th row of A or B respectively. Note that since $f[S] = g[S]$ we have that

$$N_G \begin{pmatrix} a_i^T \\ b_i^T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for all $1 \leq i \leq n$. However, since A has rank at least k, there exists i_0 such that $1 \leq i_0 \leq n$ and

$$\begin{pmatrix} a_{i_0}^T \\ b_{i_0}^T \end{pmatrix} \notin \ker(N_G).$$

In other words, $f[S] \neq g[S]$ which is a contradiction. ■

Let $G \in \mathcal{F}$ and assume that G contains less than n obviously different loops and cycles. The matrix N_G can be found as follows: First choose a maximal nice sequence

$$G_0 = (V_{G_0}, E_{G_0}) \subseteq G_1 = (V_{G_1}, E_{G_1}) \subseteq \cdots \subseteq G_m = (V_{G_m}, E_{G_m}) \subseteq G$$

as in [5, Proof of Lemma 12]. If $|E_{G_m}| > 2n - 1$, shorten the nice sequence as in [5, Corollary 15] to a nice sequence with $|E_{G_m}| = 2n - 1$. Now look at the matrix L_G corresponding to $G_m = (V_{G_m}, E_{G_m})$. For $v_i = (x_i, x_i^2, \ldots, x_i^{n-1})$ and $v_j = (x_j, x_j^2, \ldots, x_j^{n-1})$ let $1\_v_i\_v_j := (1, x_i, x_i^2, \ldots, x_i^{n-1}, x_j, x_j^2, \ldots, x_j^{n-1})$. So $1\_v_i\_v_j$ is a row in L_G iff $(x_i, x_j) \in E_{G_m}$. If L_G is a quadratic matrix, we can show as in [5, Proposition 20] that for every open set $U \subseteq \mathbb{R}^{2n-k+1}$ there is an open subset $V \subseteq U$ with

$$\det(L_G(x_1, \ldots, x_{2n-k+1})) \neq 0$$

for all $(x_1, \ldots, x_{2n-k+1}) \in V$. So all rows of $L_G(x_1, \ldots, x_{2n-k+1})$ are linearly independent if $(x_1, \ldots, x_{2n-k+1}) \in V$. If L_G is not quadratic, let $y_1, \ldots, y_l$ with an $l \in \mathbb{N}$ be new variables in $\mathbb{R}$ and add rows $1\_w_{2i}\_w_{2i-1}$ to L_G, where

$$w_j = (y_j, y_j^2, \ldots, y_{j-1}^{n-1}) \text{ for all } 1 \leq j \leq l,$$

until we get a quadratic matrix $\overline{L_G}$. Also for this matrix we can show as in [5, Proposition 20] that for every open set $U \subseteq \mathbb{R}^{2n-k+1+l}$ there is an open subset $V \subseteq U$ such that for all $(x_1, x_2, \ldots, x_n, y_1, \ldots y_l) \in V$, $\overline{L_G}$ has non-zero determinant. In other words, all rows in $\overline{L_G}$ are linearly independent. Therefore, all rows in L_G are linearly independent. This also shows that all rows in $N_G(x_1, \ldots, x_{2n-k+1})$ are linearly independent whenever $(x_1, \ldots, x_{2n-k+1}) \in V$.

6.2 Minimal cardinality of SRU's for $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$

Lemma 25. *Any SRU S of $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$ contains a basis of $\mathbb{R}^n$.*

Proof. Suppose not, and let $\{x_1, \ldots, x_l\} \subseteq S$ be a maximum linearly independent subset. Then there exists two distinct basis transformation matrix $T_1, T_2 \in \mathbb{R}^n$ (i.e. $T_1 \neq T_2$) such that $T_1 x_i = e_i = T_2 x_i$ for $i = 1, \ldots, l$.

Let $C \in \mathbb{R}^{n \times n}$ be an arbitrary linear map of rank k. Define $f(x) := CT_1 x$ and $g(x) := CT_2 x$. We have $f[S] = g[S]$ but $f \neq g$. Moreover, CT_1 and CT_2 are rank k matrices, hence f, g are rank-k linear maps, contradicting S is an SRU of $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$. ∎

Lemma 26. *Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $k \in \{1, 2, \ldots, n\}$. We define*

$$m_k := \begin{cases} 2n - 2 & k = 1, \\ 2n - k & k > 1 \end{cases}$$

Then every set $S \subseteq \mathbb{R}^n$ containing at most m_k elements is not an SRU for the family $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. For $k = 1$ we have already seen that $2n-2$ points are not enough to form an SRU (see Remark 7). So let $k \geq 2$. Let $S \subseteq \mathbb{R}^n$ be a set with cardinality at most m_k. If S does not contain a basis, then by Lemma 25, S is not an SRU of $\mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$. So we may assume that S contains a basis of $\mathbb{R}^n$.

Consider the following system of linear equations:

$$\langle a, x_i \rangle = \langle b, x_{i+1} \rangle \text{ for } i = 1, 2, \ldots l,$$

where we take indices cyclically; $(a, b) \in \mathbb{R}^{2n}$ are the unknowns. This system has rank at most $r \leq m_k$. So the null-space is at least k-dimensional. So we can find k linearly independent vectors in the null-space, namely

$$(a_1, b_1)^T, (a_2, b_2)^T, \ldots, (a_k, b_k)^T \in \mathbb{R}^{2n}.$$

Observe that solutions for which $a = b$ span at most a one-dimensional subspace since $a = b$ implies $\langle a, x_i \rangle = \langle b, x_i \rangle = c$ for all i and some constant c. Let $1 \leq i_0 \leq k$ be such that $a_{i_0} \neq b_{i_0}$.

Claim: Both, $\{a_i \mid 1 \leq i \leq k\} \subseteq \mathbb{R}^n$ and $\{b_i \mid 1 \leq i \leq k\} \subseteq \mathbb{R}^n$, are sets of linearly independent vectors.

Proof of the Claim. Assume towards a contradiction that $\{a_i \mid 1 \leq i \leq k\}$ or $\{b_i \mid 1 \leq i \leq k\}$ is linearly dependent; without loss of generality we will assume that $\{a_i \mid 1 \leq i \leq k\}$ is linearly dependent, and $a_1 = \sum_{i=2}^{k} \lambda_i a_i$ for some $\lambda_i \in \mathbb{R}$. So for all $1 \leq j \leq l$ (we take the indices cyclically) we have that

$$\begin{aligned} \langle b_1, x_{j+1} \rangle = \langle a_1, x_j \rangle = \left\langle \sum_{i=2}^{k} \lambda_i a_i, x_j \right\rangle = \sum_{i=2}^{k} \lambda_i \langle a_i, x_j \rangle \\ = \sum_{i=2}^{k} \lambda_i \langle b_i, x_{j+1} \rangle = \left\langle \sum_{i=2}^{k} \lambda_i b_i, x_{j+1} \right\rangle. \end{aligned}$$

Since $S = \{x_1, \dots, x_l\}$ contains a basis of $\mathbb{R}^n$, it follows that

$$b_1 = \sum_{i=2}^{k} \lambda_i b_i$$

and hence $(a_1, b_1)^T = \sum_{i=2}^{k} \lambda_i (a_i, b_i)^T$. So $\{(a_i, b_i)^T \mid 1 \leq i \leq k\}$ is not a linearly independent set. This is a contradiction. $\blacksquare_{\text{Claim}}$

Now define the following $n \times n$-matrices

$$A := \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_k^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that $A \neq B$ because $a_{i_0} \neq b_{i_0}$ and that by the claim both matrices A and B have rank k. Moreover, A and B map S to the same set. Therefore, S is not an SRU for the family of all linear maps with rank exactly k. $\blacksquare$

Example 27. Let $n := 3$, $k := 2$ and $S := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix} \right\}$. Then $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a maximal linearly independent subfamily. There are two different basis transformations that map $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, e.g.

$$T_1 := \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \quad \text{and} \quad T_2 := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Let $C := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$A = CT_1 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = CT_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

both have rank k. For $f(x) = Ax$ and $g(x) = Bx$ we have $f[S] = g[S]$.

Example 28. Let $n := 3$, $k := 2$ and $S := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} \right\}$. Note that S forms a basis of $\mathbb{R}^3$. We consider the following system of linear equations:

$$\begin{aligned} a_1 &= b_1 + 2b_2 + b_3 \\ a_1 + 2a_2 + a_3 &= 3b_1 + 6b_2 + 4b_3 \\ 3a_1 + 6a_2 + 4a_3 &= b_1, \end{aligned}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are the unknowns. The following vectors of the form $(a_1, a_2, a_3, b_1, b_2, b_3)^T$ are in the null-space:

$$(1, 5, -8, 1, 0, 0)^T, (2, 11, -18, 0, 1, 0)^T \text{ and } (2, 15, -24, 0, 0, 2)^T.$$

So let, for example,

$$A := \begin{pmatrix} 1 & 5 & -8 \\ 2 & 11 & -18 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both matrices have rank k. For $f(x) = Ax$ and $g(x) = Bx$ we have $f[S] = g[S]$.

7 Invertible linear maps

In this section, we aim to provide a concrete SRU for the family of all invertible, linear maps

$$\mathcal{L}_n(\mathbb{R}^n, \mathbb{R}^n) = \{f(x) = Ax \mid A \in \mathbb{R}^{n \times n} \text{ is invertible}\}.$$

Theorem 29. *Let $n \in \mathbb{N} \setminus \{0\}$. For every $i \in \{1, 2, \ldots, n\}$ let e_i be the i-th standard basis vector of $\mathbb{R}^n$ and define*

$$x_i := e_i \text{ for } i \in \{1, 2, \ldots, n\} \quad \text{and} \quad x_{n+1} := \sum_{j=1}^{n} j \cdot e_j.$$

Then $S := \{x_1, x_2, \ldots, x_{n+1}\}$ is an SRU for the family $\mathcal{L}_n(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. Assume towards a contradiction that S is not an SRU for $\mathcal{L}_n(\mathbb{R}^n, \mathbb{R}^n)$. So, we can find $f, g \in \mathcal{L}_n(\mathbb{R}^n, \mathbb{R}^n)$ with $f[S] = g[S]$ but $f \neq g$. Let $A, B \in \mathbb{R}^{n \times n}$ be invertible matrices with

$$f(x) = Ax \quad \text{and} \quad g(x) = Bx$$

for all $x \in \mathbb{R}^n$. Notice that since $f \neq g$, we have $A \neq B$. There is a permutation of the set $\{1, 2, \ldots, n+1\}$ such that

$$Ax_i = Bx_{\pi(i)}$$

for all $i \in \{1, \ldots, n+1\}$. Define $C := B^{-1}A$. Then we have that

$$Cx_i = x_{\pi(i)} \text{ for all } i \in \{1, \ldots, n+1\}. \tag{5}$$

Note that $\pi \neq \mathrm{id}$ because otherwise we would have that C is the identity matrix and therefore $A = B$.

Case 1: $\pi(n+1) = n+1$.
By (5) we have that

$$Ce_i = e_{\pi(i)}$$

for all $i \in \{1, 2, \dots, n\}$. Therefore,

$$Cx_{n+1} = C\left(\sum_{j=1}^{n} je_j\right) = \sum_{j=1}^{n} jCe_j = \sum_{j=1}^{n} je_{\pi(j)}.$$

But since $\pi(n+1) = n+1$ we have that by (5)

$$\sum_{j=1}^{n} je_j = x_{n+1} = Cx_{n+1} = \sum_{j=1}^{n} je_{\pi(j)}$$

and it follows that π is the identity. This is a contradiction.

Case 2: $\pi(n+1) = i_0 \neq n+1$.
Let $j_0 := \pi^{-1}(n+1) \in \{1, 2, \dots, n\}$. So, $Ce_{j_0} = x_{\pi(j_0)} = x_{n+1}$ and we have that

$$Cx_{n+1} = C\left(\sum_{j=1}^{n} je_j\right) = \sum_{j=1}^{n} jCe_j = \left(\sum_{j=1, j\neq j_0}^{n} je_{\pi(j)}\right) + j_0\left(\sum_{j=1}^{n} je_j\right).$$

In particular, all entries of the vector Cx_{n+1} are non-zero. But by assumption $\pi(n+1) = i_0 \neq n+1$ and therefore,

$$Cx_{n+1} = x_{\pi(n+1)} = e_{i_0}.$$

So, some entries of the vector Cx_{n+1} are equal to zero. This is a contradiction.
■

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