Zähringer Logics
new approaches to multi-valued modal logics

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Abstract

We provide two new approaches to multi-valued modal logic with truth-values between $-2$ and 2, where both approaches are based on the interpretation of the possibility operator as $\Box \varphi \iff \varphi \lor \varphi$.

1 Introduction

The aim of this article is to semantically better grasp the meaning of the natural language logical connectives, in particular the propositional extensional as well as the modal ones. The purpose is to grasp them better or more accurately than in the standard truth-functional and modal semantics. This task is carried out by introducing two different multi-valued modal logics. The point of departure is that all atomic statements are either 1 (true) or $-1$ (false). The question then is: How do the just mentioned logical operators affect the truth-values of compound statements? Most of the ideas for these logics were developed during several discussions in the Café Zähringer (at Zähringerplatz in Zürich). This is why they are called Zähringer-Logics or just Z-logics.

To be more specific, a main motivation to develop a multi-valued modal logic was the following observation: Assume you are sitting in a room without windows and you say “it is raining outside” (without knowing whether this statement is true or false). For simplicity, we denote this statement by $A$ and give $A$ the truth-value 1 if $A$ is true, and $-1$ if $A$ is false. The truth-value of $A$ is denoted by $\|A\|$. Notice that the truth-value of $A$ satisfies $\|\neg A\| = -\|A\|$, which will always be the case. Let us now compare the statement $A$ with the statements “it is necessarily raining outside” (denoted $\Box A$) and “it is possibly raining outside” (denoted $\Diamond A$). Now, even in the case when $A$ is true, i.e., “it is raining outside”, it might not be the case that “it is necessarily raining outside”, i.e., there might be no necessity or necessary reason for the rain. So, the truth-value of $\Box A$ should be strictly less than the truth-value of $A$, i.e., $\|\Box A\| < \|A\|$. On the other hand, if $A$ is false, i.e., “it is not raining outside”, it can still be possible that “it is raining outside”. So, the truth-value of $\Diamond A$ should be strictly greater than the truth-value of $A$, i.e., $\|\Diamond A\| > \|A\|$. Since the statements $\neg \Box \neg A$ and $\Diamond A$, as well as the statements $\neg \Diamond \neg A$ and $\Box A$, will be logically equivalent, apart from the truth-values $-1$ and 1 we need truth-values which are bigger than 1, others which are less than $-1$, and yet others which are between $-1$ and 1.

In a first step we extended the scale of the truth-values of formulae $\varphi$ from the two-element set $\{-1, 1\}$ to a proper interval. In fact, the extended truth-values will be strictly between $-2$
and 2. But we assume that $\|\varphi\| \neq \|\neg\varphi\|$ and $\|\neg\varphi\| = -\|\varphi\|$ for all formulae $\varphi$ and therefore, we have that $\|\varphi\| \neq 0$. This intuitively is important: since 1 represents true, all positive truth-values represent more or less true, and since $-1$ represents false, all negative truth-values represent more or less false. 0 would represent something like a genuine third truth-value which is undesirable. In a second step, we define in two different ways truth-functions, which assign, for arbitrary formulae $\varphi$ and $\psi$, a truth-value to $\varphi \rightarrow \psi$, $\varphi \land \psi$, etc. But let us first consider the construction of formulae in the two Z-logics.

We start with **atomic statements**, denoted by $A, B, \ldots$ with truth-value either $-1$ or 1, and concatenate them with logical operators. The logical operators are defined in two different ways, depending on which modal logic we are in. However, in both logics they satisfy the following laws for all statements $\varphi$ and $\psi$ and the operators $\neg$ (negation), $\land$ (and), $\lor$ (or), $\rightarrow$ (implication), $\Box$ (necessary) and $\Diamond$ (possible):

1. $\neg\neg\varphi : \iff \varphi$;
2. $\Box\varphi : \iff \neg\Diamond\neg\varphi$;
3. $\varphi \rightarrow \psi : \iff \psi \lor \neg\varphi$;
4. $\varphi \land \psi : \iff \neg(\neg\varphi \lor \neg\psi)$ (de Morgan’s law);
5. $\Diamond\varphi : \iff \varphi \lor \varphi$ (Zahringer law).

As a consequence of (2), (5), and (4) we obtain

$$\Box\varphi \iff \neg\Diamond\neg\varphi \iff \neg(\neg\varphi \land \neg\varphi) \iff \varphi \land \varphi.$$ 

Notice that the definitions (1)–(4) are as in classical logic. So, in Z-logics, just the Zahringer law (5), which defines the operator “$\Diamond$” with the operator “$\lor$” is new. In particular, if we require, for example, $\|\varphi\| < \|\Diamond\varphi\|$, then (5) implies $\|\varphi\| < \|\varphi \lor \varphi\|$, which shows one of the main differences between Z-logics and classical modal logics.

In the next two sections we will define in two different ways truth-values for arbitrary compound formulae $\varphi$, but first, let us define our range of truth-values: The truth value $\|\varphi\|$ of a compound formula $\varphi$ will always be a rational number of the form

$$\frac{a}{2^n},$$

where $n$ is a non-negative integer and $a$ is an odd integer with

$$-2^{n+1} + 1 \leq a \leq 2^{n+1} - 1.$$ 

Thus, for each formula $\varphi$ we have $-2 < \|\varphi\| < 2$, where $\|\varphi\| \neq 0$ (as mentioned above). As a matter of fact we would like to mention that there is no particular reason to restrict the possible truth-values to the interval $(-2, 2)$. We could replace $(-2, 2)$ by any other open interval. It does not even have to be a bounded interval.
The following table shows how we can define (in a classical way) truth-values of compound formulae such that the conditions (1) – (5) are satisfied.

\[
\begin{align*}
\|\neg \varphi\| &= -\|\varphi\|,
\|\varphi \land \psi\| &= \min\{\|\varphi\|, \|\psi\|\}
\|\varphi \lor \psi\| &= \max\{\|\varphi\|, \|\psi\|\}
\|\varphi \rightarrow \psi\| &= \|\psi \lor \neg \varphi\|
\|\Box \varphi\| &= \|\varphi \land \varphi\| = \|\varphi\|
\|\Diamond \varphi\| &= \|\varphi \lor \varphi\| = \|\varphi\|
\end{align*}
\]

Notice, that with this classical definition of truth-values, the requirement that, for example, \(\|\varphi\| < \|\Box \varphi\|\), is not satisfied. Furthermore notice, that since formulae are built from atomic statements, and since the truth-value of atomic statements is either 1 or \(-1\), the truth-value defined in this way is just one of the values of either 1 or \(-1\).

Let us now turn back to the Z-logics. A truth-function for a Z-logic is a function which assigns to each formula \(\varphi\) a value \(\|\varphi\|\) such that \(-2 < \|\varphi\| < 2\), \(\|\varphi\| \neq 0\) and

(E) if \(\varphi \iff \psi\), then \(\|\varphi\| = \|\psi\|\),

(N) \(\|\neg \varphi\| = -\|\varphi\|\) (negation rule),

(Z) \(\|\varphi\| < \|\Diamond \varphi\|\) (Zähringer rule).

Rule (E) makes sure that any two logically equivalent formulae have the same truth value. In particular, by (1), for any formula \(\varphi\) we must have \(\|\varphi\| = \|\neg \varphi\|\), and as a consequence of (Z) we obtain \(\|\Box \varphi\| < \|\varphi\|\).

2 First Zähringer-Logic

The first Zähringer-Logic, denoted \(\hat{Z}\)-Logic, is based on the basic intuition and subsequently the requirement that for all formulae \(\varphi\), the truth-value of \(\varphi \rightarrow \varphi\) (\(\hat{Z}\)-implication) is equal to 1. More formally, the first Zähringer-Logic is characterised by the requirement, that for all formulae \(\varphi\), we have

\[
\|\varphi \rightarrow \varphi\| = 1. \tag{Z1}
\]

Thus, by the conditions above, in order to define a truth-function which satisfies \(\|\varphi \rightarrow \varphi\| = 1\) (for all formulae \(\varphi\)), we have to find a suitable truth-function for \(\varphi \lor \psi\) (\(\hat{Z}\)-OR). With such a truth-function we can then define truth-values for \(\varphi \land \psi\) (\(\hat{Z}\)-AND), \(\Diamond \varphi\) (\(\hat{Z}\)-possible), and \(\Box \varphi\) (\(\hat{Z}\)-necessary).

For every \(s \in (-2, 2)\), let

\[
\mu(s) := 1 + \frac{s}{2} - \frac{s^2}{16},
\]
and for all formulae $\varphi$ and $\psi$ let

$$\| \varphi \land \psi \| := \mu (\| \varphi \| + \| \psi \|).$$

With this definition it is easy to verify that for all formulae $\varphi$ and $\psi$ we have

$$\| \varphi \rightarrow \varphi \| = 1,$$
$$\| \varphi \rightarrow \psi \| = \mu (\| \psi \| - \| \varphi \|),$$
$$\| \varphi \lor \psi \| = \mu (\| \varphi \| + \| \psi \|),$$
$$\| \varphi \land \psi \| = -\mu (-\| \varphi \| - \| \psi \|),$$
$$\| \lozenge \varphi \| = -\mu (-2\| \varphi \|),$$
$$\| \lozenge \varphi \| = \mu (2\| \varphi \|).$$

Thus, the truth-function $\| \cdot \|$, based on the function $\mu$, satisfies (Z1). On the one hand, there are also other functions $(-2, 2) \rightarrow (-2, 2)$ such that the corresponding truth-function satisfies (Z1). On the other hand, the function $\mu$ has some nice properties. For example, if $\| \varphi \| = 1$, then $\lim_{k \rightarrow \infty} \| \lozenge^k \varphi \| = 2$ and $\lim_{k \rightarrow \infty} \| \Box^k \varphi \| = 0$. Moreover, in contrast to the classical operators “$\lor$” (OR) and “$\land$” (AND), the operators “$\lor$” and “$\land$” are not associative. For example, if $\| \varphi \| = -1$ and $\| \psi \| = 1$, then

$$\| (\varphi \lor \psi) \lor \psi \| = \frac{7}{4},$$

whereas

$$\| \varphi \lor (\psi \lor \psi) \| = \frac{343}{256}.$$ 

In particular, there are formulae $\varphi$ and $\psi$ (for example, if $\| \varphi \| = 1$ and $\| \psi \| = -1$), such that

$$\| \lozenge (\varphi \lor \psi) \| \neq \| \lozenge \varphi \lor \lozenge \psi \|.$$ 

Furthermore, note that for $\| \varphi \| > 2(1 - \sqrt{2})$ we have that $\| \lozenge \varphi \| > 0$. So, even in the case when $\| \varphi \|$ is negative, $\| \lozenge \varphi \|$ can be positive. This also corresponds to our intuition about the meaning of possible and necessary. However, in the second Zährringer-Logic, which we will introduce in the next section, this will not be the case.

3 Second Zährringer-Logic

The second Zährringer-Logic, denoted $\tilde{Z}$-Logic, is based on the intuition and the subsequent requirement that the truth-value of $\varphi \lor \psi$ is in general not the same as the truth-value of $\psi \lor \varphi$, but the formulae $\tilde{\lor} (\varphi \lor \psi)$ and $\tilde{\lor} \varphi \lor \tilde{\lor} \psi$ are still logically equivalent. This logic purports to account for the fact that according to their meaning in a natural language the connectives AND and OR often are neither commutative nor associative (cf. section 6). More formally, the second Zähringer-Logic is characterised by the requirement that there are formulae $\varphi$ and $\psi$ such that

$$\| \varphi \lor \psi \| \neq \| \psi \lor \varphi \|,$$

(Z2.a)
but for all formulae \( \varphi \) and \( \psi \) we have

\[
\| \odot (\varphi \land \psi) \| = \| \varphi \land \odot \psi \|. 
\] (Z2.b)

As above, in order to define a truth-function which satisfies (Z2.a) and (Z2.b), it is enough to find a suitable truth-function for \( \varphi \lor \psi \) (\( \mathcal{Z} \)-OR). With such a truth-function we can then define truth-values for \( \varphi \land \psi \) (\( \mathcal{Z} \)-AND), \( \varphi \rightarrow \psi \) (\( \mathcal{Z} \)-implication), \( \oplus \varphi \) (\( \mathcal{Z} \)-possible), and \( \Box \varphi \) (\( \mathcal{Z} \)-necessary).

Let \( \varphi \) and \( \psi \) be formulae with \( \| \varphi \| = \frac{a}{2^n} \) and \( \| \psi \| = \frac{b}{2^n} \), where \( a \) and \( b \) are both odd. Then we define

\[
\| \varphi \lor \psi \| := \frac{3a + b + 3}{2\max\{n,m\} + 2}.
\]

Notice that by the rules (3) and (E) we get that \( \| \varphi \rightarrow \psi \| = \| \varphi \lor \neg \varphi \| \), which is in general not the same as \( \| \neg \varphi \lor \psi \| \). Notice also that this operator is not associative. For \( \| \varphi \| = 1 \) and \( \| \psi \| = -1 \) we have that

\[
\| \varphi \lor (\varphi \lor \psi) \| = \frac{11}{16} \neq \frac{23}{16} = \| (\varphi \lor \varphi) \lor \psi \|
\]

First we have to verify that this truth-function satisfies (Z2.a) and (Z2.b). For (Z2.b), notice that for \( \| \varphi \| = \frac{a}{2^n} \) and \( \| \psi \| = \frac{b}{2^n} \), where \( a \) and \( b \) are both odd, we have

\[
\| \odot (\varphi \lor \psi) \| = \frac{12a + 4b + 15}{2\max\{n,m\} + 4} = \| \varphi \lor \odot \psi \|.
\]

To see that also (Z2.a) is satisfied, let \( \varphi \) and \( \psi \) be formulae with \( \| \varphi \| = \frac{a}{2^n} \) and \( \| \psi \| = \frac{b}{2^n} \), where \( a \) and \( b \) are both odd and \( a \neq b \). Then

\[
\| \varphi \land \psi \| = \frac{3a + b + 3}{2^{n+2}} \neq \frac{3b + a + 3}{2^{n+2}} = \| \psi \land \varphi \|.
\]

Concerning the operators “\( \odot \)” and “\( \boxdot \)”, it is easy to verify that if \( \| \varphi \| = \frac{a}{2^n} \) with \( a \) odd, then

\[
\| \odot \varphi \| = \frac{4a + 3}{2^{n+2}} \quad \text{and} \quad \| \boxdot \varphi \| = \frac{4a - 3}{2^{n+2}}.
\]

In particular, if \( A \) is an atomic statement and \( \| A \| = 1 \), then, for example, we get \( \| \odot A \| = \frac{7}{4} \), \( \| \odot \odot A \| = \frac{31}{16} \), \( \| \boxdot \odot A \| = \frac{25}{16} \), \( \| \boxdot A \| = \frac{1}{4} \), \( \| \boxdot \boxdot A \| = \frac{1}{16} \) and \( \| \boxdot \boxdot \Box A \| = \frac{7}{16} \). Moreover, we have that for all positive integers \( k \) and for all formulae \( \varphi \) with \( \| \varphi \| = \frac{a}{2^n} \)

\[
\| \odot^k \varphi \| = \frac{4^k(a + 1) - 1}{2^{n+2k}} \quad \text{and} \quad \| \boxdot^k \varphi \| = \frac{4^k(a - 1) + 1}{2^{n+2k}}
\]

Thus, if \( \| \varphi \| = 1 \), then, similar as in \( \hat{Z} \)-Logic we have \( \lim_{k \to \infty} \| \odot^k \varphi \| = 2 \) and \( \lim_{k \to \infty} \| \boxdot^k \varphi \| = 0 \).

In contrast to the \( \hat{Z} \)-logic, \( \| \varphi \rightarrow \varphi \| \) is not constant anymore. In fact, if \( \| \varphi \| = \frac{a}{2^n} \), then

\[
\| \varphi \rightarrow \varphi \| = \| \varphi \land \neg \varphi \| = \frac{\| \varphi \|}{2} + \frac{3}{2^{n+2}}.
\]
We would like to mention that, beside the truth-function defined above, there are also other truth-functions which satisfy (Z2.a) and (Z2.b). Hence, as for the First Zähringer-Logic, the truth-function we defined is not unique. However, it has some additional properties. For example, for all formulae $\varphi$, if $\|\varphi\| < 0$, then also $\|\boxdot \varphi\| < 0$. Notice that this is not the case for the truth-function we defined for $\check{Z}$-Logic.

4 Truth tables

In this section, we compute some kind of truth tables for different formulae $\theta$. If there is just one formula $\varphi$ involved in $\theta$, then the truth value of $\theta$ just depends on the truth value of $\varphi$, and therefore, the truth value of $\theta$ is a function $t_\theta : (-2,2) \rightarrow (-2,2)$, where

$$(-2,2) := \{ x \in \mathbb{Q} : -2 < x < 2 \text{ and } x = \frac{a}{2^n} \text{ for } a \text{ odd}\}.$$  

If two formulae $\varphi$ and $\psi$ are involved in $\theta$, then the truth value of $\theta$ depends on the truth values of $\varphi$ and $\psi$, and therefore, the truth value of $\theta$ is a function $t_\theta : (-2,2) \times (-2,2) \rightarrow (-2,2)$. In general, if $n$ formulae are involved in $\theta$, then $t_\theta : (-2,2)^n \rightarrow (-2,2)$.

4.1 Truth functions of some basic formulae

\begin{tabular}{|c|c|}
\hline
$\check{Z}$-logic & $\check{Z}$-logic \\
\hline
\end{tabular}
$\mathcal{Z}$-logic

$\| \diamond \varphi \| = \| \varphi \lor \varphi \|

\| \varphi \land \psi \|

\| \Box \varphi \| = \| \varphi \land \varphi \|

\| \varphi \land \psi \|

\| \varphi \lor \psi \|

\| \Box \psi \| = \| \varphi \land \varphi \|

\| \varphi \land \psi \|

\| \varphi \lor \psi \|

\| \varphi \lor \psi \|

\| \varphi \land \psi \|
The following two figures show the differences between the formulae $\varphi \lor \psi$ and $\varphi \land \psi$, and the formulae $\varphi \lor \psi$ and $\psi \lor \varphi$, respectively.

\[ \| \varphi \lor \psi \| - \| \varphi \land \psi \| \]

\[ \| \varphi \lor \psi \| - \| \psi \lor \varphi \| \]

4.2 Truth tables of some non-modal logical axioms

<table>
<thead>
<tr>
<th>$\mathcal{Z}$-logic</th>
<th>$\hat{\mathcal{Z}}$-logic</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Truth Table 1" /></td>
<td><img src="image2" alt="Truth Table 2" /></td>
</tr>
<tr>
<td><img src="image3" alt="Truth Table 3" /></td>
<td><img src="image4" alt="Truth Table 4" /></td>
</tr>
<tr>
<td><img src="image5" alt="Truth Table 5" /></td>
<td><img src="image6" alt="Truth Table 6" /></td>
</tr>
<tr>
<td><img src="image7" alt="Truth Table 7" /></td>
<td><img src="image8" alt="Truth Table 8" /></td>
</tr>
</tbody>
</table>
\[
\| \varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi) \|
\]

\[
\| \varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi) \|
\]

\[
\| \varphi \rightarrow (\varphi \wedge \psi) \rightarrow \varphi \|
\]

\[
\| (\varphi \wedge \psi) \rightarrow \varphi \|
\]

\[
\| (\varphi \rightarrow (\varphi \vee \psi)) \|
\]

\[
\| (\varphi \rightarrow (\varphi \vee \psi)) \|
\]

\[
\| (\varphi \rightarrow (\psi \vee \varphi)) \|
\]
4.3 Truth tables of some classical modal formulae

\[ K:\quad \parallel (\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)) \parallel \]

\[ D:\quad \parallel \square \varphi \rightarrow \Diamond \varphi \parallel \]

\[ T:\quad \parallel \square \varphi \rightarrow \varphi \parallel \]

\[ \hat{K}:\quad \parallel (\Diamond (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)) \parallel \]

\[ \hat{D}:\quad \parallel \square \varphi \rightarrow \Diamond \varphi \parallel \]

\[ \hat{T}:\quad \parallel \square \varphi \rightarrow \varphi \parallel \]
\(Z\)-logic \(\Rightarrow\) \(Z\)-logic

\[
\begin{align*}
\text{B: } & \| \varphi \Rightarrow \Box \Diamond \varphi \| \\
\text{S4: } & \| \Box \varphi \Rightarrow \Box \Box \varphi \| \\
\text{S5: } & \| \Diamond \varphi \Rightarrow \Box \Diamond \varphi \|
\end{align*}
\]
5 Inference rules

An inference rule

\[
\frac{\varphi_1, \ldots, \varphi_n}{\psi}
\]

is 1-\(\hat{\mathbb{Z}}\)-valid (or 1-\(\hat{\mathbb{Z}}\)-valid) if \(\|\psi\| \geq 1\) whenever \(\|\varphi_i\| \geq 1\) for all \(1 \leq i \leq n\); and the inference rule is 0-\(\hat{\mathbb{Z}}\)-valid (or 0-\(\hat{\mathbb{Z}}\)-valid) if \(\|\psi\| > 0\) whenever \(\|\varphi_i\| > 0\) for all \(1 \leq i \leq n\).

For example, the inference rules MODUS PONENS (MP)

\[
\frac{\varphi, \varphi \rightarrow \psi}{\psi}
\]

and

\[
\frac{\varphi, \varphi \rightarrow \psi}{\psi}
\]

are 1-\(\hat{\mathbb{Z}}\)-valid and 0-\(\hat{\mathbb{Z}}\)-valid but not 0-\(\hat{\mathbb{Z}}\)-valid. Moreover, MP is also 1-\(\hat{\mathbb{Z}}\)-valid.

The Necessitation Rules (NR)

\[
\frac{\varphi}{\Box \varphi}
\]

and

\[
\frac{\varphi}{\Box \varphi}
\]

is 0-\(\hat{\mathbb{Z}}\)-valid, but not 1-\(\hat{\mathbb{Z}}\)-valid and neither 1-\(\hat{\mathbb{Z}}\)-valid nor 0-\(\hat{\mathbb{Z}}\)-valid.

Now we give a few examples of inference rules:

<table>
<thead>
<tr>
<th>Inference Rule</th>
<th>0-(\hat{\mathbb{Z}})-valid</th>
<th>1-(\hat{\mathbb{Z}})-valid</th>
<th>0-(\hat{\mathbb{Z}})-valid</th>
<th>1-(\hat{\mathbb{Z}})-valid</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi, \varphi \rightarrow \psi) (\psi)</td>
<td>(|\varphi| = 1, |\psi| = -\frac{1}{2})</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\varphi, \varphi \rightarrow \psi) (\psi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\Diamond \Box \varphi, \Box \neg \varphi) (\Box \neg \varphi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\Diamond \Box \varphi, \Box \neg \varphi) (\Box \neg \varphi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\varphi, \neg \varphi \rightarrow \psi) (\neg \varphi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\varphi, \neg \varphi \rightarrow \psi) (\neg \varphi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(\varphi \vee \psi, \neg \varphi \rightarrow \psi) (\neg \varphi)</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
6 Alternative $\hat{Z}$-Logics

The reader might have noticed that the constructions of the $Z$-logics are not at all unique. First of all one could question the basic laws we introduced in section 1. For example, we could change the definition of $\|\neg \varphi\|$ in order to destroy the law that two negatives resolve to a positive (i.e., \textit{duplex negatio affirmat}). While in languages like German, \textit{duplex negatio affirmat} is common, in other languages like English, a double negation might intensify a negation.

Let us assume from now on that the laws from section 1 are all satisfied. Even with this assumption, the $Z$-logics are not unique. For example, we could redefine the $\hat{Z}$-logic as follows:

For $\|\varphi\| = \frac{2a}{2^n+1}$, we stipulate

$$\| \Diamond \varphi \| := \frac{2a + 1}{2^n+1} \quad \text{and} \quad \| \Box \varphi \| := \| \neg \Diamond \neg \varphi \| = \frac{2a - 1}{2^n+1}.$$

In particular, if $A$ is an atomic statement and $\|A\| = 1$, then, for example, we get $\| \Diamond A \| = \frac{3}{2}$, $\| \Diamond \Diamond A \| = \frac{7}{4}$, $\| \Box \Diamond A \| = \frac{5}{4}$, $\| \Box A \| = \frac{1}{2}$, $\| \Box \Box A \| = \frac{1}{4}$ and $\| \Diamond \Box A \| = \frac{3}{4}$.

Now, as in $\hat{Z}$-logic, we want that the $\hat{Z}$-OR-operator "$\lor^\ast$" satisfies

$$\| \Diamond \varphi \| = \| \varphi \lor^\ast \varphi \|.$$
If $\varphi$ and $\psi$ are formulae with $\|\varphi\| = \frac{a}{2^n}$ and $\|\psi\| = \frac{b}{2^m}$, then we can for example define

$$\| \varphi \lor \psi \| := \frac{\|\varphi\| + \|\psi\|}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{m+2}} = \frac{\|\varphi\lor\| + \|\psi\lor\|}{2}.$$ 

The corresponding operators "\&" and "\rightarrow" can be defined as follows:

$$\varphi \& \psi : \iff \neg(\neg\varphi \lor \neg\psi) \quad \text{and} \quad \varphi \rightarrow \psi : \iff \neg\varphi \lor \psi.$$ 

As in section 3, the "\lor"-operator is not associative, satisfies

$$\|\varphi \lor \psi\| \neq \|\varphi \lor \varphi\| \text{ and } \|\varphi \lor \psi\| \neq \|\psi \lor \psi\|$$

for all formulae $\varphi$ and $\psi$ with $\|\varphi\| \neq \|\psi\|$, and satisfies $\lim_{k \to \infty} \|\varphi\lor\| = 2$ and $\lim_{k \to \infty} \|\varphi\lor\| = 0$ whenever $\|\varphi\| = 1$. However, "\lor" is commutative and in general we have

$$\|\varphi \lor \psi\| \neq \|\varphi \lor \varphi\|$$

with the non-commutativity of the "\lor"-operator in section 3, we tried to introduce some temporal aspects of the natural language into the $Z$-logic. For example the meaning of the two sentences "Bob slipped and fell down" and "Bob fell down and slipped." are not the same. In the first sentence, Bob slipped at time $t_0$ and fell down at time $t_1 > t_0$. In the second sentence it is the other way round. Of course, the $\tilde{Z}$-logic is not strong enough to capture all temporal aspects of the natural language. It might be interesting to study time-dependent $Z$-logical operators in the future.