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MULTIDIMENSIONAL VALUATION

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Introduction

The first part of the text is devoted to explaining the nature of insurance losses – technical as well as financial losses – in the classical framework. The second part discusses the financial losses based on the technique of Multidimensional Valuation. The principle tool derived in this part is the Valuation Portfolio. The financial risk is then treated as the difference between the Portfolio of Assets and the Valuation Portfolio. One possibility to assess the financial risk is by option pricing (Margrabe option to switch the Asset Portfolio for the Valuation Portfolio). The standard approach to cope with financial risk – Capital at Risk – is also discussed.

1. The individual insurance contract

The individual insurance contract can always be seen as a Stochastic Flow of Payments

$$X = (X_0, X_1, \dots X_k, \dots X_N) \sim \text{ random vector } X$$

where the random variable X_k is understood as the stochastic payment made at time k.

The insurance contract starts at $k = 0$ and ends at $k = N$. N may be either deterministic or stochastic.

Typically, the time is measured in years, but of course other units of time can be used (e.g. months, yearly quarters). To fix the ideas let us think in units of years. **It is important to note that with the idea of a Stochastic Flow of Payments we can model all types of insurance contracts.**

Example A The Life Insurance Policy

$$k = 0, 1, \dots, N\,.$$

Here X_k is understood as the **claim payments** made in $[k-1, k]$ minus the **premium** paid at time k (Swiss convention is to place the claim payment **at the end** of the period when they occur, other conventions (e.g. French) put them **in the middle** of the interval).

For the Life Insurance Policy N becomes the **duration of the contract** and is contractually defined.

Example B The Automobile Insurance Contract

This is typically an annual contract for which **a single premium** is paid at inception. The annual contract **covers** the insured for events **occurring** within

[1]This article refers to the lectures held at the Kolmogorov Centennial Conference at Moscow State University and at the Colloquium Lyon–Lausanne at HEC Lausanne.

the one year period. However **claim payments** due to such events may very well extend to **years beyond the one year period of coverage**. In practice automobile insurance contracts are of course often renewed after one year, but it is essential to understand the **annual nature** of Automobile Insurance. In this case our payment stream X is interpreted as

$$X_0 \sim \text{ minus premium paid by the insured in the beginning}$$

for $k \geq 1$

$$X_k \sim \text{ claim payment made by the insurer in interval } [k-1, k].$$

Standard terminology is to call k the **development year** k. N is then the **final development year** which is typically stochastic.

To have a complete mathematical description of our stochastic vector $X = (X_0, X_1, \dots, X_N)$ we need the filtration $\mathcal{F} = (\mathcal{F}_k)_{k=0}^N$ where $\mathcal{F}_k$ is the class (σ-algebra) of those events known at time k. Obviously the sequence is increasing and X_k is known at time k, which translates into mathematical terminology that X_k is $\mathcal{F}_k$-measurable.

2. Life and Non-life Insurance

In **non-life insurance** the contracts are typically of the type described by **Example B**, let us call them of type B. The essential technical difference among non-life insurance policies derives from the nature of N (i.e. from the length of the development period). N is always stochastic, but if N tends to be **big** we speak of **long tail business** (e.g. automobile insurance, third party liability insurance), if N tends to be **small** we speak of **short term business** (e.g. fire insurance).

The important notion of non-life insurance is the **final claim amount** (ultimate claim amount)

$$C = \sum_{k=1}^{N} X_k$$

or the **underwriting loss** (**ultimate loss**)

$$U = \sum_{k=0}^{N} X_k.$$

(Observe $X_0 = U - C$ stands for the premium with a minus sign, which is paid in the beginning.)

In the following I shall work with the **underwriting loss** U because this is again the quantity which can also be used in life insurance (as we shall see). There is an obvious methodological question to be raised in connection with the just defined quantity U: Should U not depend also on the **time points** at which the payments X_k are made? The answer is - of course - yes, it would make sense, but reality is that in practice the quantity called **underwriting loss** typically does not take this consideration into account. Insurers seem to be used to live with this, which means that the economic meaning of the underwriting loss has to be interpreted with care. In defense of this attitude one may raise the rather insecure time pattern of development in Non-life Insurance.

In **life insurance** the time pattern of the insurance contract (see Example A) is by far easier to catch and has traditionally always been recognized. The standard approach fixes one interest rate i (called technical interest rate) and discounts with the powers of $v = \frac{1}{1+i}$. Hence in life insurance the **ultimate loss** is defined as $U = \sum_{k=0}^{N} v^k X_k$. At this point a life actuary working in practice may question my terminology. But he will find out soon, that I have only slightly twisted his terminology. I do this on purpose to have a unified approach both for life and non-life insurance.

3. Valuation – the Main Task of the Actuary

We are now prepared to do mathematics. Our object is the random vector

$$X = (X_0, X_1, \dots, X_N) \sim \text{ Stochastic Flow of Payments}$$

adapted to $\mathcal{F} = (\mathcal{F}_k)_{k=0}$ and we look at

$$U = \sum_{k=0}^{B} v^k X_k \sim \text{ ultimate (underwriting) loss discounted to time 0.}$$

Possibly we have $v = 1$ (non-life insurance)

The task of the actuary is to predict U at time $k = 0, 1, 2, \dots N-1$.

The standard and obvious way is to predict U by **its conditional expectation**, hence the prediction at time k is

$$U_k = E[U/\mathcal{F}_k] \quad k = 0, 1, 2, \dots, N\,.$$

Writing out the definition of U and observing that for $j \leq k$ X_j is known at time k (X_j is $\mathcal{F}_k$ measurable) we obtain

$$U_k = \sum_{j=0}^{k} v^j X_j + E[\sum_{j=k+1}^{N} v^j X_j/\mathcal{F}_k] \tag{1}$$

or

$$U_k(1+i)^k = \sum_{j=0}^{k} (1+i)^{k-j} X_j + E[\sum_{j=k+1}^{N} v^{j-k} X_j/\mathcal{F}_k].$$

The left hand side of the last equation stands for the **present value** of the best prediction **at time** k, namely for the best prediction of U transported to time k with yearly interest i. It is worthwhile to use the notation

$$V_k := U_k(1+i)^k \sim \text{ Present Value of } U \text{ at time } k.$$

Hence we have

$$V_k = \underbrace{\sum_{j=0}^{k} (1+i)^{k-j} X_j}_{\text{payments made accumulated with interest}} + \underbrace{E[\sum_{j=k+1}^{N} v^{j-k} X_j/\mathcal{F}_k]}_{\text{prospective reserve for future payments: } R_k} \tag{2}$$

The life actuary uses typically the expression "**valuation**" for his task to evaluate R_k. It is indeed true that the part R_k needs to be calculated by the actuary (the person who masters the probability laws of the random variables $X_{k+1}, X_{k+2}, \dots X_N$), whereas $\sum_{j=0}^{k} (1+i)^{k-j} X_j$ is observed and needs no modelling. The life actuary as a risk manager should however always also have a close look at V_k, as it measures the accumulated loss made on the policy up till time k.

It is remarkable that - although usually with interest zero - the non-life-actuary calculates the reserve R_k the other way around. He/she produces first an estimate for the **final claim amount** C and subtracts the **paid losses** to get the reserve.

4. Annual losses

The company who holds the reserves as described in the previous section and who earns annual interest i on these reserves makes the **annual loss** L_k at time k on the payment stream X, which is defined as

$$\begin{aligned} L_k &= X_k + R_k - (1+i)R_{k-1} \text{ for } k = 1, 2, \ldots N \\ L_0 &= X_0 + R_0 \quad \textbf{is the initial loss}. \end{aligned}$$

In the so called **net calculation** one should have $L_0 = 0$ (equivalence principle). In practice one should have $L_0 < 0$ (in favor of insurer) to compensate for costs of administration and capital.

Multiplying L_k by v^k and adding up till time m one finds the crucial relation

$$\sum_{k=0}^{m} v^k L_k = \sum_{k=0}^{m} v^k X_k + v^m R_m \text{ which by (1) and (2) equals } U_m.$$

Hence U_m, i.e. the best estimate of ultimate (discounted) loss at time m, equals the (discounted) sum of annual losses up till time m.

Another interesting interpretation is the following. One sees immediately (either by martingale convergence or more simply by the fact that $R_{N+1} = 0$) that

$$\sum_{k=0}^{N} v^k L_k = \sum_{k=0}^{N} v^k X_k = U.$$

The left side is another decomposition of U which has the crucial property that the **partial sums** till time m are the **best estimates** of U at time m.

This last property could have been another starting point for defining the role of the reserves $R_k, k = 0, 1, \ldots, m$. For didactical reasons I have chosen a different route. Still I want to emphasize that

$$U_m = \sum_{k=0}^{m} v_k L_k, m = 0, 1, \ldots, N, \quad \text{is a } \mathcal{F}\text{-martingale}.$$

Hence the **annual losses** $\{L_k\}_{k=0}^{N}$ are uncorrelated and $E[L_k/F_j] = 0$ for $k > 0$ and $j < k$.

This statement is usually quoted as Theorem of Hattendorf [1] and means that annual losses can be **diversified over time**. In addition they can be **diversified over** the **mass of policies** provided **the policies are independent**.

This diversification both over mass and time explains why insurance works. Of course one has also to watch, that no single random variable subject to this diversification effect is dominating. Such dominating random variables need to be controlled by reinsurance protection in the form of risk transfers. Such risk transfers are quite common in the insurance sector and we may therefore say that the losses $\{L_k\}$ can be managed.

5. Technical and Financial Annual Losses

The losses L_k, $k = 1, 2, \ldots N$, defined in Section 4, are usually called **technical losses** as they occur **within** the standard insurance model. They disappear (with the exception of L_0) if the random variables X_k are replaced by $E[X_k/\mathcal{F}_{k-1}]$ for $k \geq 1$. In other words they are due to the differences of X_k to their predicted mean based on the actuarial probability distributions (e.g. for mortality, claims settlement in automobile insurance etc.).

It is of utmost importance in practice to understand that the source called **technical loss** is in most cases (in particular in life insurance) of minor importance than the so called **financial loss**.

The financial loss occurs due to the fact that in the standard model the assumption is made that the reserve R_{k-1} at time $k-1$ is transported (by proper investment) to become $(1+i)R_{k-1}$ at time k.

In reality the insurer will earn

$$i_{\text{real}} R_{k-1} \text{ on the invested reserve } R_{k-1}.$$

Hence we define

$$(i - i_{\text{real}})R_{k-1} = F_k \quad \textbf{financial loss} \text{ in } (k-1, k].$$

As said above the financial loss is due to the fact that the standard insurance model assumes a deterministic constant interest rate for all investments. Hence the financial loss occurs **outside** the model.

One can extend the standard model to a model with stochastic interest rates (see e.g. Life Insurance with Stochastic Interest Rates [2]), where financial losses occur **inside the model**. The idea is the following:

For the time interval $(k-1, k]$, i_{real} is described by the $\mathcal{F}_k$-measurable random variable δ_k. Assuming Markov structure for δ-variables (and independence between δ-variables and X-variables) we write

$$R_k(\delta_k, X^{(k)}) \text{ for the reserve at time } k\,,$$

thus highlighting the dependence of the reserve on the conditional expectations $X^{(k)} = \{E[X_m/\mathcal{F}_k]; m > k\}$ (as before) and on δ_k.

The **financial loss** in $(k-1, k]$ is then

$$F_k = E[X_k/\mathcal{F}_{k-1}] + R_k(\delta_k, X^{(k-1)}) - (1+\delta_k)R_{k-1}(\delta_{k-1}, X^{(k-1)}).$$

The **technical loss** in $(k-1, k]$ amounts to

$$L_k = X_k - E[X_k/\mathcal{F}_{k-1}] + R_k(\delta_k, X^{(k)}) - R_k(\delta_k, X^{(k-1)}).$$

For details see [2]. It is remarkable that **all losses** defined in this way (i.e. all L_k, $k = 1, 2, \ldots, N$; F_k, $k = 1, 2, \ldots N$) are uncorrelated and have, conditioned on the past, expectation zero. Hence financial losses (properly defined) may also be diversified over time. The point is that they can typically not be diversified over the mass, as individual insurance contracts are financed by (related to the earnings of) either one single investment portfolio or, in the case where the earnings result from separated portfolios, such portfolios are mostly highly correlated.

The point I want to make is the following: financial losses need to be managed differently than technical losses. The basic instrument for this purpose is a new concept of actuarial valuation. This new concept is called **Multidimensional Actuarial Valuation**.

6. Multidimensional Valuation of a Life Insurance Policy - Example of a standard non participating Endowment Policy

The idea is to measure the liability of the insurance carrier as a portfolio of financial instruments. We call this portfolio **Valuation Portfolio** (VaPo).

Observe that at this point the financial instruments involved may or may not be traded on an existing market. Still these financial instruments **do** exist in economic reality by the fact that the insurance carrier sells the insurance contract.

6.1. The example.

To illustrate the idea we take a non participating Endowment Policy for the amount 1 written at age x for a period of $n = 5$ years against a yearly premium of P. We calculate with the deterministic model of life insurance, i.e. with the mortability table

$$\left.\begin{array}{l} l_x \\ l_{x+1} \\ l_{x+2} \\ l_{x+3} \\ l_{x+4} \\ l_{x+5} \end{array}\right\} l_{x+t} \sim \text{ number of persons alive at age } x+t.$$

In this model

$$d_{x+t} = l_{x+t} - l_{x+t+1}$$

stands for the deterministic number of persons dying in the age interval $[x+t, x+t+1]$.

Following the convention made in Section 1 we assume that death benefits are paid at the end of the age interval, whereas the premiums are paid at the beginning of each interval.

As financial instruments for this policy we need Zero Coupon Bounds $Z^{(t)}$ maturing after $t = 0, 1, 2, 3, 4, 5$ years. The **Valuation Portfolio** (VaPo) for this policy at inception we obtain as below. We calculate for l_{50} persons (for a single person divide by l_{50}).

$$(\mathrm{VaPo})_0 = \begin{cases} \text{unit} & \text{number of units} \\ Z^{(0)} & -Pl_{50} \\ Z^{(1)} & -Pl_{51} + d_{50} \\ Z^{(2)} & -Pl_{52} + d_{51} \\ Z^{(3)} & -Pl_{53} + d_{52} \\ Z^{(4)} & -Pl_{54} + d_{53} \\ Z^{(5)} & \quad l_{55} + d_{54}. \end{cases}$$

For convenience $(\mathrm{VaPo})_0$ is calculated at time $t = 0$ **before** the first premium P is received. In some practical applications one might prefer to make the valuation **after** the first premium is received. The difference of the two portfolios amounts to $l_{50}PZ^{(0)}$, which is equal to the cash amount Pl_{50} at time zero.

We emphasize at this point the **multidimensional** character of the Valuation Portfolio. In our example the dimension is 6. We have used 6 **units** to make the valuation and, when constructing the Valuation Portfolio, we consciously have **refrained from assigning monetary values to these units**.

6.2. **The recursion.** Suppose that we have to make the valuation of the same policy one year later. This implies - of course - that the insured is among the l_{51} persons still alive. Calculating for l_{51} persons we obtain

$$(\mathrm{VaPo})_1 = \begin{cases} \text{unit} & \text{number of units} \\ Z^{(1)} & -Pl_{51} \\ Z^{(2)} & -Pl_{52} + d_{51} \\ Z^{(3)} & -Pl_{53} + d_{52} \\ Z^{(4)} & -Pl_{54} + d_{53} \\ Z^{(5)} & \quad l_{55} + d_{54}. \end{cases}$$

(VaPo)$_1$ is a part of (VaPo)$_0$: All we need to do to go from $(\mathrm{VaPo})_0$ to $(\mathrm{VaPo})_1$ is **adding the cash stream obtained** from and to the l_{50} policies thus compensating the disappearing number of units:

$$(\mathrm{VaPo})_1 = (\mathrm{VaPo})_0 + \underbrace{l_{50}PZ^{(0)} - d_{50}Z^{(1)}}_{\text{premium minus benefits in age interval [50, 51]}} .$$

In more finance oriented language we may also summarize this finding as follows.

Remark: (Self-financing property)

"The step from $(\mathrm{VaPo})_t$ to $(\mathrm{VaPo})_{t+1}$ is exactly financed from the cash flow generated by the policies to which the Valuation Portfolio is referring."

The statement in this exact form is true in the deterministic model. In reality there will be deviations from expected values as defined by the deterministic model. But these deviations are - in the terminology of Section 5 - technical losses and hence, as discussed there, they can be handled by traditional methods (effect of Law of Large Numbers plus reinsurance). Observe that these technical losses are now expressed in units rather than in money amount, but otherwise the arguments made in Section 5 are still valid.

6.3. **The General Case.** Example 6.1 shows how to construct the Valuation Portfolio for a very special case of a Life Insurance Contract. Such a construction of a Valuation Portfolio can in principle be made for all insurance contracts. In the area of life insurance the construction is obvious. All that needs to be added is a richer ensemble of financial instruments including participations in funds and European options. In this paper we omit the details of such constructions. On the non-life side where “claim amounts” are event driven it takes more sophistication to arrive at Valuation Portfolios. But also in these insurance branches one can define reasonable Valuation Portfolios.

It is important to note, that also in all these cases the “self-financing property” remains - up to technical losses - a valid proposition.

7. Asset Liability Management

With the construction of the Valuation Portfolio for each insurance contract we have laid the basis for sound Asset Liability Management.

Take a well defined part of the business written by the insurance carrier, e.g. a branch or a regional collective of risks. By adding up all Valuation Portfolios of the individual contracts we arrive at the

Valuation Portfolio of the total business under consideration.

This total VaPo (which for simplicity we call VaPo from here on) needs to be compared with the

Portfolio of Assets S covering the liabilities of the total portfolio under consideration.

Observe that we have now a situation where assets and liabilities are assessed in a commensurate way, contrary to the situation encountered in a standard balance sheet where assets and liabilities are expressed in a one dimensional figure (e.g. in Euros). The reduction of assets and liabilities to a simple money amount is lacking transparency and may have occurred on the basis of non uniform accounting principles.

8. Solvency

By solvency we mean in this presentation protection against the **financial risk**. For didactical reasons we omit possible **technical losses** which we assume to be controlled a priori (e.g. by reinsurance or by the Law of Large Numbers).

An insurer is solvent if he can fulfill his obligations for the whole time period of the contracts which he has written. Using the terminology of section 7 and our restriction of solvency to the aspect of financial risk we define

(Solvency)$_1$ “**The insurer is solvent at time t_0 if at any $t \geq t_0$ he could switch S_t for (VaPo)$_t$.**”

This possibility of switching is usually expressed by an **accounting principle** A which assigns a monetary value to a given portfolio

$$A\colon \text{ Portfolio in units } \longmapsto \text{ Money amount.}$$

Mathematically A is a linear mapping and the interpretation is

$$\text{If } A[S_t] \geq A[\tilde{S}_t] \text{ the portfolio } S \text{ can be switched into } \tilde{S} \text{ at time } t.$$

We assume hence that our **accounting principle** is such that it guarantees the **switching property**.

Remark: One obvious choice for A is Market Value Accounting if markets exist. We have taken care to define A in a more general way such that the existence of markets is not a precondition for our reasoning.

Using the accounting priciple A as defined we arrive at a new definition of solvency:

(Solvency)$_2$ **"The insurer is solvent at time** t_0 if

$$A[S_t] \geq A[(\text{VaPo})_t] \text{ for any } t \geq t_0.''$$

9. Strategies of Investment to be solvent

Assume that at all time points the insurance company knows the VaPo of its liabilities. In order to control financial risk the insurance company is then confronted with the question:

How to choose the investment portfolio S such that solvency is guaranteed?

The choice of S is hence a strategic investment decision based on the liabilities as expressed by the VaPo. In order not to overload the complexity of this presentation let us assume from here on that the VaPo can be bought and sold on a market and that $A[(\text{VaPo})_t]$ stands for its market price at time t.

Let us also assume that the same holds for all portfolios S under consideration. (This means that we consider only possible investments in liquid markets.) This restriction on the VaPo and on S can be considered as a reasonable working hypothesis for life insurance. In the non–life area as well as in social insurance one encounters more frequently situations where the VaPo can not be bought on a market. In this case one needs to find the Minimum Risk Portfolio in a given market which approximates the VaPo in an optimal way. In todays presentation I assume however that the Minimum Risk Portfolio and the VaPo coincide.

9.1. The prudent investment strategy.

Suppose that we are at time t_0. If we choose at this time point to invest in the VaPo we have

at time t_0: portfolio $(\text{VaPo})_{t_0}$ with value $A[(\text{VaPo})_{t_0}]$ and by the self financing property (see 6.2)

at time $t > t_0$: portfolio $(\text{VaPo})_t$ with value $A[(\text{VaPo})_t]$.

Observe that with this investment strategy the insurance company is - up to technical losses - solvent for **any accounting principle**.

With this prudent investment strategy all that is needed to be solvent is **enough initial investment volume** $A[(\mathbf{VaPo})_{t_0}]$.

9.2. **The courageous investment strategy.** The insurance company may want to invest in a more profitable way than under the prudent strategy. We call such an investment strategy courageous because it is exposed to additional financial risk.

It is important to understand that under the courageous investment strategy solvency can be achieved **recursively**: As in 9.1 we start with an initial investment volume which is sufficient and we select at time t_0 a portfolio S which satisfies

Condition C_{t_0}**:** i) $A[S_{t_0}] \geq A[(\text{VaPo})_{t_0}]$
ii) $A[S_{t_0+1}] \geq A[(\text{VaPo})_{t_0+1}]$

At time t_0+1 select a **new portfolio** $\tilde{S}$ which satisfies

Condition C_{t_0+1}**:** i) $A[\tilde{S}_{t_0+1}] \geq A[(\text{VaPo})_{t_0+1}]$
ii) $A[\tilde{S}_{t_0+2}] \geq A[(\text{VaPo})_{t_0+2}]$

Observe that such portfolios S and $\tilde{S}$ exist. We can always choose them to be equal to the VaPo.

Hence if at time t_0 the insurance company chooses S_{t_0} to satisfy condition C_{t_0} (and if the company in later years reshuffles the portfolio to S_t satisfying $C_t(t > t_0)$), the company is solvent according to definition $(\text{Solvency})_2$.

Observe the liquidity assumption inherent in the procedure to be able to change the portfolio at each time point t.

The question to be answered is then:

"How at time t_0 can we choose a portfolio to satisfy C_{t_0}?"

Let us discuss two possible solutions:

I) Solution by Margrabe Option

$S_{t_0} = \tilde{S}_{t_0} + M_{t_0}^{(t_0+1)}$ where $A[\tilde{S}_{t_0}] = [A(\text{VaPo})_{t_0}]$

$M_t^{(t_0+1)} \sim$ Margrabe option for switching $\tilde{S}$ into (VaPo) at time t_0+1.

It is obvious that C_{t_0} is satisfied by this construction.

II) Solution by Risk Based Capital (Standard Solution)

$$S_{t_0} = (1+\lambda)\tilde{S}_{t_0} \text{ with } A[\tilde{S}_{t_0}] = A[(\text{VaPo})_{t_0}]$$

and with λ such as to have

$$P[A[(1+\lambda)\tilde{S}_{t_0+1}] \geq A[(\text{VaPo})_{t_0+1}] = 1-\varepsilon.$$

Observe that by this construction C_{t_0} is satisfied only with high probability. ε plays the role of a yearly default probability. (Default is not meant in a legal sense since the situation may always be remedied if the insurance company can raise additional capital.)

10. How to calculate the price for the Margabe Option and the amount of additional capital under the Risk Based Capital approach

For convenience we use the notation

$$A[(\text{VaPo})_t] =: V_t \quad \{V_t; t \geq t_0\} \sim \text{process of liabilities in money units}$$
$$A[\tilde{S}_t] =: Z_t \quad \{Z_t; t \geq t_0\} \sim \text{process of assets in money units}$$

It is convenient to use V as numeraire, hence we end up by doing our calculations for the process

$$\{X_t; t \geq t_0\} \text{ with } X_t := \frac{Z_t}{V_t}$$

X_t can be interpreted as growth of Z expressed in units of V.

I) Price of Margrabe Option (option to switch Z for V at time $t_0 + 1$)

$$\text{(MA)} \qquad \begin{cases} \text{Price } [M_{t_0}^{(t_0+1}] = & E_{t_0}^*[(1 - X_{t_0+1})^+] \text{ in } V\text{-units} \\ & E_{t_0}^*[(1 - X_{t_0+1})^+]V_{t_0} \text{ in money-units} \end{cases}$$

where $P^* \sim$ risk neutral measure for pricing $X_t(t \geq t_0)$ at time t_0

$E_{t_0}^* \sim$ conditional expectation with respect to P^* at time t_0.

In general (MA) needs to be calculated by simulation. In order to get an idea about the order of magnitude for the price of the Margrabe Option we calculate with the

Standard Probability Law

$(X_t)_{t \geq t_0}$ has P**-measure of motion**

$$X_t = e^{Y_t}$$

where $(Y_t)_{t \geq t_0}$ Brownian Motion

$$Y_t \sim \mathcal{N}(\mu(t - t_0), \sigma^2(t - t_0))$$

hence **equivalent risk neutral** P^*-measure

$$X_t = e^{Y_t}$$

where $(Y_t)_{t \geq t_0}$ Brownian Motion

$$Y_t \sim \mathcal{N}(-\frac{\sigma^2}{2}(t - t_0), \sigma^2(t - t_0)).$$

Under these assumptions the formula of Margrabe (see [3]; [4]) is as follows

$$\text{(MAS)} \qquad \text{Price } [M_{t_0}^{(t_0+1)}] = E_{t_0}^*[(1 - X_{t_0+1})^+] = 2\Phi(\frac{\sigma}{2}) - 1$$

where $\Phi \sim$ cumulative distribution function of standardized Normal distribution.

Numerical Values-Formula (MAS)

	Price $[M_{t_0}^{(t_0+1)}]$
$\sigma = 0.05$	2.00 % (in percent of $V_{t_0} = Z_{t_0}$ (initial reserves at time t_0))
$\sigma = 0.1$	3.99 %
$\sigma = 0.2$	7.97 %
$\sigma = 0.3$	11.92 %
$\sigma = 0.4$	15.85 %

Roughly speaking the price is almost linear for values of σ not too far from zero.

Interpretation: The solution to construct solvency via option pricing is theoretically quite attractive.

i) The insurance company has the **obligation** to hold the VaPo.

ii) If the shareholders of the company want to invest differently

– they have to pay for the option M

– for this price they have the right to obtain the extra yield from the investment strategy

The solvency construction via option pricing clearly separates the interests of the insured and those of the shareholders and requests a price from the shareholders for the right to use the insurance reserves to make a profit. It is important to realize that in this view the participating policy holder takes the same role as the shareholders for the part of the extra yield which is passed on to him by the insurance company.

The practical disadvantage of this approach is the rather substantial price of such a Margrabe option.

II Amount of Capital at Risk needed to cover the Financial Risk

We construct a portfolio S_{t_0} satisfying C_{t_0} with probability $1-\varepsilon$ by using an increased capital basis. Hence we invest in a portfolio

$$S_{t_0} = (1+\lambda)\tilde{S}_{t_0} \text{ with } A[\tilde{S}_{t_0}] = A[(\text{VaPo})_{t_0}].$$

Using the same notation as for the calculation of the Margrabe Option price we find λ from the equation

$$P_{t_0}[(1+\lambda)X_{t_0+1} \geq 1] = 1-\varepsilon. \tag{3}$$

Under the Standard Probability Law (Observe that one must calculate with P **not** with P^*) we obtain

$$\begin{aligned} P_{t_0}[(1+\lambda)X_{t_0+1} \geq 1] &= 1-\varepsilon \\ P_{t_0}[Y_{t_0+1} \geq -\ln(1+\lambda)] &= 1-\varepsilon \\ P_{t_0}\left[\frac{Y_1-\mu}{\sigma} \geq -\frac{\ln(1+\lambda)+\mu}{\sigma}\right] &= 1-\varepsilon \end{aligned}$$

which leads to

(CAPS) $$\ln(1+\lambda) = \sigma z_\varepsilon - \mu$$

the quantile z_ε is defined by $\Phi(z_\varepsilon) = 1-\varepsilon$.

Numerical Values

Formula (CAPS): see Table 1: Amount of Capital at Risk in percentages of Z_{t_0}.

	$\sigma = 0.05$ $\mu = 0.012$	$\sigma = 0.1$ $\mu = 0.024$	$\sigma = 0.15$ $\mu = 0.036$	$\sigma = 0.2$ $\mu = 0.048$	$\sigma = 0.25$ $\mu = 0.06$	$\sigma = 0.3$ $\mu = 0.072$
$\varepsilon = .01$ % $Z_\varepsilon = 3.71$	18.9 %	41.5 %	68.3 %	101.7 %	138.1 %	183.2 %
$\varepsilon = .03$ % $Z_\varepsilon = 3.43$	17.3 %	37.6 %	61.4 %	89.3 %	122.0 %	160.4 %
$\varepsilon = .07$ % $Z_\varepsilon = 3.20$	16.0 %	34.4 %	55.9 %	80.8 %	109.6 %	143.0 %
$\varepsilon = .18$ % $Z_\varepsilon = 2.91$	14.3 %	30.6 %	49.3 %	70.6 %	94.9 %	122.8 %
$\varepsilon = 1.08$ % $Z_\varepsilon = 2.30$	10.8 %	22.9 %	36.2 %	51.0 %	67.4 %	85.5 %
$\varepsilon = 6.41$ % $Z_\varepsilon = 1.52$	6.6 %	13.7 %	21.2 %	29.2 %	37.7 %	46.8 %
$\varepsilon = 11.61$ % $Z_\varepsilon = 1.20$	4.9 %	10.1 %	15.5 %	21.2 %	27.1 %	33.4 %

Table 1. Amount of Capital at Risk in percentages of Z_{t_0}.

The different columns represent scenarios S of the form $S = \alpha F + (1-\alpha)$ VaPo where F is a stock fund with expected extra return 0.06 and volatility $\sigma = 0.25$ (expressed in units of liability). One sees that the Margrabe Option leads to lower figures than the extra capital needed with the Risk Based Capital approach. This is of course well understandable as the price for the Margrabe Option is spent in all cases, whereas the additional capital is only immobilized i.e. only used if needed.

11. Final Remarks

The problems addressed in this presentation are the basic risk management issues of an insurance company. Actuaries may substantially contribute to make the risk management process more transparent by a new understanding of the valuation process: Valuation must be seen as a multidimensional task leading to a Valuation Portfolio (rather than only to a onedimensional figure expressed e.g. in Euros). Only by such an understanding assets and liabilities are on a commensurate basis.

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