RESEARCH STATEMENT

HARDY CHAN

1. Summary

I am interested in nonlinear partial differential equations arising from geometry and natural sciences. A large part of my research stems from two classical equations — the Allen–Cahn equation in phase transition,

\[ -\Delta u = u(1 - u^2) \quad \text{in } \mathbb{R}^n, \]

which is deeply connected to minimal surfaces and pattern formation,\(^1\) as well as the Yamabe equation in conformal geometry,

\[ -\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \text{ or } (M^n, g). \]

I study also the corresponding nonlocal models that have been recently prevalent. They have important applications not only in probability and finances, but also physics and geometry, as described below.

2. Phase transition

2.1. Overview. Phase Transitions (PTs) are changes in the molecular/atomic structure or electronic properties under unstable thermodynamic conditions. Often manifested by an abrupt modification in material properties, PTs are observed between solid-liquid-gaseous states, ferromagnetic-paramagnetic phases, altering crystallographic structures, and in superconductivity, etc. The condensation of water droplets in clouds is an evident example in the nature. A better understanding of PTs will provide a wide range of applications in countless industrial and technical processes.

Mathematical models in PTs are often described by partial differential equations (PDEs). Indeed, the analysis of the interface, in particular its location and regularity, has attracted a lot of attention in the past few decades and remains as a central subject to the present day. One well-accepted model is the Allen–Cahn equation (1).

2.2. De Giorgi conjecture. The profound relationship between (1) and minimal surfaces can be seen by the variational convergence of the energy for the singularly perturbed Allen–Cahn equation to the perimeter functional. In this way, the interface between the stable phases \( u = -1 \) and \( u = +1 \) is expected to approach a minimal surface. From the resolution of the Bernstein problem, entire minimal graphs in \( \mathbb{R}^n \) are hyperplanes when \( n \leq 8 \), and the dimension 8 is sharp. Thus De Giorgi conjectured the classification of monotone entire solutions of (1). (For more background see our survey [21].) A generalized version can be stated as follows.

\textbf{Conjecture 2.1} (De Giorgi conjecture). \textit{Let } \( s \in (0,1] \). \textit{At least for } \( n \leq 8 \), \textit{all solutions of}

\[ (-\Delta)^s u = u(1 - u^2) \quad \text{in } \mathbb{R}^n, \]

\textit{satisfying } \( \frac{\partial u}{\partial x_n} > 0 \), \textit{are one-dimensional.}

In the classical case \( s = 1 \), Conjecture 2.1 is almost completely solved: \( n = 2 \) by Ghoussoub–Gui [30], \( n = 3 \) by Ambrosio–Cabré [1] and \( 4 \leq n \leq 8 \) by Savin [39], under an extra condition \( \lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1 \); on the other hand counterexamples are built for \( n \geq 9 \) by Del Pino–Kowalczyk–Wei [26] upon the entire non-affine minimal graph of Bombieri–De Giorgi–Giusti [8]. When the monotone condition is replaced by energy minimality, Savin [39] proved that all global minimizers are one dimensional when

\(^{1}\text{Pattern formation is not the main focus in what follows; see, for instance, our article [17].}\)
n ≤ 7, while Liu–Wang–Wei [33] constructed a counterexample in dimension n = 8, which is based on the foliating stable solutions of Pacard–Wei [38].

Recently there have been intensive interests in the De Giorgi Conjecture 2.1 for the fractional Allen–Cahn equation \((2)\). For \(s \in [1/2, 1)\), positive results have been obtained: \(n = 3\) by Cabré–Sire [13] and Siré–Valdinoci [43], \(n = 3\) by Cabré–Cinti [10], \(n = 4\) and \(s = 1/2\) by Figalli–Serra [27], and the remaining cases for \(n ≤ 8\) by Savin [40, 41] under the additional limiting condition as for \(s = 1\). A very challenging and important open question is whether or not Savin’s result is optimal when \(s ∈ (1/2, 1)\).

2.3. **Counterexamples in high dimensions.** There are three main difficulties in constructing counterexamples. First, the basic profile has algebraic hence slow decay. Second, there are interactions due to nonlocality and high dimensional transitions. Besides, no local Fermi coordinates expressions for the fractional Laplacian exist due to its coupling.

Nonetheless, with Liu and Wei we have developed a **fractional gluing scheme** [18] (preprint) to construct entire solutions of the fractional Allen–Cahn equation when \(s ∈ (1/2, 1)\), vanishing near a minimal surface. One crucial ingredient is a fractional computation in Fermi coordinates that we have done in [20] (JDE 2017). This scheme enables one to construct counterexamples of Conjecture 2.1. With most ingredients prepared in [18], the key is, by exploiting symmetry, to improve the decay of the error with very high precision, in the spirit of [33]. Together with Dávila, del Pino, Liu and Wei, our goal is to establish the existence of a minimizing solution of \((2)\).

2.4. **Classification in low dimensions.** The question on the other end is the Stability Conjecture — the classification of stable solutions at least in dimension \(n = 3\), which leads to the full\(^2\) resolution of Conjecture 2.1 in dimension \(n = 4\). It is proved true for \(n = 2\) by Cabré–Sire [13] and \(n = 3\) in the asymptotic case \(s = (1/2)^−\) by Cabré–Cinti–Serra [11] and \(s = 1/2\) by Figalli–Serra [27].

Together with Figalli and Serra, we plan to derive sharp quantitative estimates leading to quadratic area growth control, in parallel to the minimal surface theory, and to prove, in the end, the following result: For \(n = 3\) and \(s ∈ (1/2, 1]\), stable solutions of \((2)\) are one-dimensional. As a consequence, for \(n = 4\) and \(s ∈ (1/2, 1]\), Conjecture 2.1 is true.

2.5. **Nonlocal minimizing cones.** When \(s ∈ (0, 1/2)\), the underlying geometric object is instead a nonlocal minimal surface, and the threshold dimension \(n\) for Conjecture 2.1, that is the maximum dimension for the validity of the nonlocal Bernstein theorem, follows from non-existence of singular nonlocal minimizing\(^3\) cones in \(\mathbb{R}^{n−1}\), as proved by Figalli–Valdinoci [28]. It is expected to be \(n = 8\) for \(s ∼ (1/2)^−\) by the regularity result of Caffarelli–Valdinoci [15], and \(n = 7\) for \(s ∼ 0^+\) by the numerical stability\(^4\) of \(s\)-Lawson cones due to Dávila–Del Pino–Wei [25]. In fact, nonlocal minimizing cones are planes by the classification results of Savin–Valdinoci [42] for \(n = 2\) and Cabré–Cinti–Serra [12] for \(n = 3\) and \(s ∼ (1/2)^−\).

As of now, no non-trivial examples of nonlocal minimizing cones are known. One major obstruction comes from the very definition of the nonlocal mean curvature which involves intricate integrals. In spite of the difficulties, together with Cabré and Wei, we intend to establish the existence of nonlocal minimizing Lawson cones by constructing foliations, in turn by developing a suitable theory for quasilinear nonlocal equations, and analyzing the singular integrals carefully.

While the full classification of \(s\)-minimizing cones up to dimension 7 or 8 appears to be out of reach, my long-term goal is to develop a gluing scheme for fractional equations in the highly nonlocal regime, \(s ∈ (0, 1/2)\), and construct counterexamples of Conjecture 2.1 in high dimensions, building on the foliation around the \(s\)-minimizing cones.

\(^2\)i.e. classification without an extra limit assumption as in Savin’s work

\(^3\)i.e. a set \(E\) that minimizes the \(s\)-perimeter \(\|1_E\|_{W^{1,2,s}}\), as introduced by Caffarelli–Roquejoffre–Savin [14].

\(^4\)a property weaker than \(s\)-perimeter-minimizing
3. YAMABE PROBLEM

3.1. Overview. In dimension two, the uniformization theorem states that every simply connected Riemann surface is conformally equivalent the open unit disk, the complex plane, or the Riemann sphere. All of the three have constant curvature. As a generalization in higher dimensions, given a compact Riemannian manifold \((M^n, g)\), the Yamabe problem asks for a conformal metric with constant scalar curvature. It is solved by the combined work of Yamabe, Schoen, Aubin and Trudinger.

Recently a notion of nonlocal curvature has been introduced by Chang–González [22], in terms of the conformal fractional Laplacian that has its root in scattering theory. The nonlocal \(Q_s\)-curvatures thus defined include the scalar curvature \((s = 1)\) and the curvatures associated to the Paneitz operator \((s = 2)\) and to the fractional Laplacian (in particular for \(s \in (0,1)\)).

A natural question is then the fractional Yamabe problem of finding a conformal metric with constant \(Q_s\)-curvature, which includes the Escobar problem (Yamabe problem on a manifold with boundary) in the case \(s = 1/2\). In the following we consider a number of aspects including singularities and their dimension, fractional curvature flow and the moduli space of metrics.

3.2. Singular Yamabe problem. Writing \(g_u = u^{4\over n-2}g\), the PDE formulation on the sphere\(^5\) is

\[
(-\Delta)^s u = u^{n+2s\over n-2s} \quad \text{in } \mathbb{R}^n,
\]

and we are interested in (very) weak solutions \(u\) that blow up on a prescribed singular set of dimension \(k\). In the classical work [44], Schoen–Yau

- proved that if \(g_u\) is complete, then \(k \leq (n-2)/2\) (Schoen–Yau’s theorem); and
- conjectured that weak solutions \(u\) of (3) have a singular set of dimension \(k \leq (n-2)/2\), and the resultant metric \(g_u\) is complete (Schoen–Yau’s conjecture).

Singular solutions with prescribed singularity having a complete conformal metric are constructed by Pacard [37] for \(k = (n-2)/2\) and Mazzeo–Pacard [35] for \(k \in (0,(n-2)/2)\), while counterexamples of Schoen–Yau’s conjecture with higher dimensional singularity and incomplete resultant metrics are found by Pacard [37] for \(n = 4,6\) and Chen–Lin [23] for \(n \geq 9\).

With DelaTorre [16] (Comm. PDE 2020), we have provided an alternative construction by exploiting stability for \(k = (n-2)/2\), which is free from ODE arguments and derivative estimates. This also answers an open question posed by Aviles [7] in 1983.

3.3. Singular fractional Yamabe problem. In the fractional setting, one naturally asks about the validity of Schoen–Yau’s theorem, as well as the possibility of corresponding constructions. The building block is the fast-decay radially symmetric solution, classically obtained by phase-plane analysis.

When \(k = (n-2s)/2\), the critical exponent predicted by Schoen–Yau’s theorem, with DelaTorre we plan to construct singular fractional Yamabe metrics for \(s = 1/2\), generalizing [16].

When \(k \in (0,(n-2s)/2)\), finding a fast decay profile is a nontrivial task. We have done it with Ao, González and Wei [4] (Calc. Var. PDE 2018) in the stable regime using the theory of extremal solutions, and with Ao, DelaTorre, Fontelos, González and Wei [2] (Duke 2019) in the unstable regime using the bifurcation techniques.

The second achievement of [2] is establishing a nonlocal ODE theory: inverting the linearized operator of Hardy–Schrödinger type. Under the conformal transformation of Emden–Fowler, the linearized equation has constant coefficients and is solved explicitly using non-Euclidean harmonic analysis and complex analysis. The Green function thus obtained provides an alternative formulation, in addition to the known extension problems, where classical ODE quantities like the Wrońskián can be utilized. Thereby the nonlocal gluing scheme is established, and we obtain solutions of (3) singular on prescribed submanifolds of dimension \(k \in (0,(n-2s)/2)\).

As an application of the new theory of nonlocal ODE (see also our survey [3], to appear in J. Math. Study), we have studied radial solutions in a fractional supercritical problem [5].

\(^{5}\)under the stereographic projection
3.4. **Yamabe problem with non-smooth singularity.** The known construction of high dimensional singular metrics, whether local [16,35,37] or non-local [2,4], are based on building solutions around smooth submanifolds. With González and Wei, we wish to address the open question of existence of solutions of the Yamabe problem with prescribed non-smooth singularity. This requires an extra analysis gluing the Fermi coordinates to the cone-edge coordinates. Ultimately we hope to do this construction on any prescribed singularity of iterated cone-edge type, in the spirit of Mazzeo [34].

3.5. **Fractional Yamabe flow.** It is desirable to generate regular metrics with constant fractional curvature from the flow in the spirit of Hamilton. On the boundary $(M, g)$ of a Poincaré–Einstein manifold, a volume-preserving fractional Yamabe flow has recently been shown to exist by Daskalopoulos–Sire–Vázquez [24]. Whereas all the previous works in the fractional case are elliptic in nature, together with Sire and Sun [19] (Ann. SNS Pisa, to appear) we have developed a parabolic theory showing the convergence of the flow with small initial energy, in spite of extra challenges from the lack of a singular integral formula, and generalized the classical convergence theorem of Schwetlick–Struwe [45]. Together with Sire and Sun, we plan to prove convergence for arbitrary initial data, which requires a careful analysis of the possibly interacting bubbles, as in Brendle [9].

3.6. **Moduli space of singular Escobar metrics.** The dimension and analyticity of the moduli space of Yamabe metrics with isolated singularities are studied by Mazzeo–Pollack–Uhlenbeck [36]. Together with Sire, we plan to generalize this result to the nonlocal case using the tools developed in [2,6]. The half curvature case treats singular Escobar metrics.

References


6 unless $(M, g)$ is the round sphere, a special case treated by Jin–Xiong [32].


