RESEARCH STATEMENT

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1. Research summary

I am mainly interested in the study of local and non-local, non-linear elliptic partial differential equations arising from natural sciences and differential geometry. I study qualitative properties including existence and symmetry, as well as concentration phenomena.

My research results are summarized below. More elaborated discussions for each problem will be done in the later subsections. At the end I will also state my proposal for future research directions.

(1) (with Y. Liu and J. Wei) For $1/2 < s < 1$, we developed a fractional gluing scheme and constructed solutions of the fractional Allen–Cahn equation $(-\Delta)^s u - u + u^3 = 0$ which vanish on a deformed catenoid. (pdf)

(with J. Dávila, M. del Pino, Y. Liu and J. Wei) Then we construct monotone solutions which are not one-dimensional, giving counterexamples to the fractional De Giorgi conjecture in dimensions $n \geq 9$. Furthermore we also construct a nontrivial global minimizer in dimension $n = 8$. (In progress)

(2) I solved the multiplier problem in the calculus of variations for scalar ordinary differential equations by giving a necessary and sufficient condition for a given differential equation to be variational. [pdf]

(3) (with L.F.O Faria, N. Ghoussoub, S. Mazumdar and S. Shakerian) We defined a notion of hyperbolic mass and obtained sufficient conditions for the best constant of a hyperbolic Hardy–Sobolev inequality to be attained. Hence we asserted the existence of solutions of a non-linear Hardy–Schrödinger equation. [pdf]

(4) (with W. Ao, M.d.M. González and J. Wei) We constructed entire fast decaying singular solutions to the fractional Lane–Emden equation $(-\Delta)^s u = u^p$ in $\mathbb{R}^n \setminus \{0\}$. Using these we showed existence of positive weak solutions with a prescribed singular set. For the fractional Yamabe problem, we obtain solutions singular in the whole space. [pdf]

(5) (with W. Ao, A. DeLaTorre, M.d.M. González and J. Wei) We construct similar building blocks as in (4). Combined with non-local ODE Hamiltonian and gluing technique we construct singular solutions to fractional Yamabe problem with higher dimensional singular set. (In progress)

(6) (with J. Wei) We constructed traveling wave solutions of the fractional Allen–Cahn equation $(-\Delta)^s u - cu x_3 - f(u) = 0$ in $\mathbb{R}^3$, whose wavefront is asymptotically of pyramidal shape. [pdf]

(7) (with W. Ao and J. Wei) We exhibited a new concentration phenomenon of the Lin–Ni–Takagi problem, or the singularly perturbed non-linear Schrödinger equation $\varepsilon^2 \Delta u - u + u^p = 0$ with Neumann boundary condition, by constructing solutions in $\mathbb{R}^2$ concentrating on a segment of the boundary. [pdf]

(8) (with L.F.O. Faria and S. Shakerian) We prove a Caffarelli–Kohn–Nirenberg inequality in the hyperbolic space and discuss issues of subcritical existence and supercritical non-existence. [pdf]

1.1. The De Giorgi conjecture. A generalized version of the De Giorgi conjecture can be stated as follows.

CONJECTURE 1.1.1. Let $0 < s \leq 1$. At least for $n \leq 8$, all solutions of

$(-\Delta)^s u - u + u^3 = 0 \quad \text{in} \quad \mathbb{R}^n$

satisfying $\frac{\partial u}{\partial x_n} > 0$ are one-dimensional.

In the classical case $s = 1$, Conjecture 1.1.1 is almost completely solved: $n = 2$ by Ghoussoub–Gui [28], $n = 3$ by Ambrosio–Cabré [2] and $4 \leq n \leq 8$ by Savin [53], under an extra condition $\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1$: on the other hand counterexamples are constructed for $n \geq 9$ by del Pino–Kowalczyk–Wei [20]. When the monotone condition is replaced by energy minimality, Savin [53] proved that all global minimizers are one dimensional when $n \leq 7$, while Liu–Wang–Wei [40] constructed a counterexample in dimension $n = 8$, which is based on the stable solutions of Pacard–Wei [51].
Recently there have been intensive interests in the De Giorgi conjecture for the fractional Allen–Cahn equation 1. For $s \geq 1/2$, positive results have been obtained: $n = 2$ by Sire–Valdinoci 56, $n = 3$ by Cabre–Cinti 8, $n = 4$ and $s = 1/2$ by Figalli–Serra 27, and the remaining cases for $n \leq 8$ by Savin 54 under the additional limiting condition as for $s = 1$.

An open question is whether or not Savin’s result is optimal. There are three main difficulties in constructing counterexamples. Firstly, the basic profile has algebraic hence slow decay. Secondly, there are interactions of non-locality and higher dimensional concentrations. Moreover, no local Fermi coordinates expressions for the fractional counterexamples. Firstly, the basic profile has algebraic hence slow decay. Secondly, there are interactions of non-locality and higher dimensional concentrations. Moreover, no local Fermi coordinates expressions for the fractional counterexamples. Firstly, the basic profile has algebraic hence slow decay. Secondly, there are interactions of non-locality and higher dimensional concentrations. Moreover, no local Fermi coordinates expressions for the fractional counterexamples.

**THEOREM 1.1.2** (Chan–Liu–Wei 10). For $1/2 < s < 1$ and $n = 3$, there exist solutions of 1 vanishing on a rotationally symmetric surface, which locally resembles a catenoid and grows sub-linearly at infinity.

**THEOREM 1.1.3** (Chan–Dávila–del Pino–Liu–Wei 13). Let $1/2 < s < 1$. There exist solutions of
\[
(-\Delta)^s u - u + u^3 = 0 \quad \text{in } \mathbb{R}^n
\]
with $\frac{\partial u}{\partial x_0} > 0$ and $u$ is not one-dimensional.

Using a fractional infinite dimensional Lyapunov–Schmidt reduction procedure, we also construct a global minimizer of the energy functional over the Simons cone in dimension $n = 8$. Then a monotone solution can be obtained via a modification of the Jerison–Monneau program 35. This extends 10 to the fractional case.

We believe that our approach also holds for $s = \frac{1}{2}$ and we are currently working on this.

1.2. The multiplier problem. The calculus of variations is a very powerful tool in the study of differential equations. Let $L(x, p_0, p_1, \ldots, p_n)$ be a given smooth function. The Euler–Lagrange equation for the Lagrangian $L(x, u, u', \ldots, u^{(n)})$ is given by

\[
LE[u](x) := \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} L_{p_k}(x, u(x), \ldots, u^{(n)}(x)) = 0.
\]

This is an ordinary differential equation with leading term $L_{p_n} p_n(x, u, \ldots, u^{(n)}) u^{(2n)}$. The multiplier problem in the calculus of variations asks a converse question:

**PROBLEM 1.2.1.** Given a differential equation, say

\[
u^{(2n)} - f(x, u(x), \ldots, u^{(2n-1)}(x)) = 0,
\]
when is there a Lagrangian $L(x, p_0, p_1, \ldots, p_n)$ of order $n$ and some positive multiplier $\rho(x, p_0, \ldots, p_n)$ such that

\[
LE[u](x) = \rho(u)(u^{(2n)} - f[u](x))
\]
holds for all smooth functions $u$?

This is a long-standing problem and can be dated back to 1979; see 55 for a historical account. Darboux’s classical result 19 states that a multiplier always exists when the order is two. Fels 25 proved that for fourth order equations the existence of a variational multiplier is equivalent to the vanishing of two quantities. In the same spirit, Juráš 36 solved the problem for sixth and eighth order equations using a tool called variational bicomplexes and considered more general cases ($3 \leq n \leq 100$) using symbolic computations 37.

Despite the power of such differential-algebraic tools, it is difficult to write down a clean condition in terms of $f$, for (3) to admit a variational multiplier, for an arbitrary $n$. In 12 I attacked the problem with a completely different and more elementary approach. By comparing the coefficients of the highest order term on both sides of (4), I obtained an overdetermined system of linear partial differential equations. A careful analysis of this system reveals a deeper structure which yields the exact differential-integral forms of all involved functions $f$, $\rho$ and $L$. In this way I proposed a complete solution to the multiplier problem, for all $n \geq 2$. As an illustration I will state the theorem for $n = 2$.

Denoting the highest order dependence of the derivatives by a subscript, we can rewrite (4) as

\[
E_4^2 L_2 = \rho_2 (p_4 - f_3),
\]
where $E_4^2$ is the Euler–Lagrange operator. For fourth order equations we have for $D_2 = \partial_x + p_1 \partial_{p_0} + p_2 \partial_{p_1}$ and certain lower order Euler–Lagrange type operators $E_4^2$ and $E_1^4$, the following
THEOREM 1.2.2 (Chan [12]). Let \( n = 2 \). Given \( f_3 \), suppose that (5) is solved for some \( L_2 \) and \( \rho_2 \). Then there are functions \( R_2, f_0, f_1 \) and \( N_1 \) such that

\[
\begin{align*}
f_3 &= \partial p_2 R_2 \cdot p_2^2 + 2D_2 R_2 \cdot p_3 - e^{R_2} \left( E_2^2 \int \int e^{-R_2} + f_1 p_2 - E_1^1 \int \int f_1 + f_0 \right) , \\
\rho_2 &= e^{-R_2} , \\
L_2 &= \int \int e^{-R_2} - \int \int f_1 + \int \int f_0 + D_2 N_1 .
\end{align*}
\]

Conversely, given any functions \( R_2, f_0, f_1 \) and \( N_1 \), the functions \( f_3, L_2, \) and \( \rho_2 \) defined in (6) solves (5).

For \( n \geq 3 \), the forms of the functions have a similar pattern but \( f_{2n-1} \) is linear in its highest derivative.

1.3. Mass and extremals of a Hardy–Schrödinger operator on hyperbolic space. In the Euclidean space, the loss of compactness and the resulting concentration phenomena are seen with the Hardy–Schrödinger operator \(-\Delta - \gamma |x|^{-2} \) and the non-linearity \(|x|^{-s} |u|^2(s)\) which are critical. Here \( \gamma \) is less than the best constant \( \frac{6(n-2)}{n-4} \) of the Hardy’s inequality, \( 2^*(s) = \frac{2(n-s)}{n-2} \) and \( 0 \leq s < 2 \). As an extension to Brezis–Nirenberg’s idea of restoring compactness with a linear perturbation [7], the attainability of extremal in

\[
\mu_{\gamma,s,\lambda}^{B^n}(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int \Omega \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|} - \lambda u^2 \right) dx}{\left( \int \Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{n}{2}}} ,
\]

has received a lot of attention (26, 34 and many more) and is summarized in the survey [30]. While solutions always exist in higher dimensional case provided that \( \lambda > 0 \) is small, the positivity of a “Hardy-singular mass” is necessary when \( \gamma \) is close to \( \frac{(n-2)^2}{4} \) in order that the infimum is attained. Indeed, it allows for a fine test function argument to go through, see [29].

It will be interesting to see such phenomena in manifolds. Recently in [52] the authors came up with several integral inequalities involving weights \( V_p \) on the hyperbolic ball model \( B^n \) that are invariant under scaling, once restricted to radial functions. In fact, one has \( V_2 \sim |x|^{-2} \) and \( V_2(s) \sim |x|^{-s} \) near the origin as well as the validity of the Hardy–Sobolev inequality

\[
C \left( \int_{B^n} V_2^*(s) |u|^{2^*(s)} dv_{g_B^n} \right)^{\frac{2}{2^*(s)}} \leq \int_{\Omega} \left( |\nabla_{B^n} u|^2 - \gamma V_2 u^2 \right) dv_{g_B^n}
\]

when \( \gamma < \frac{(n-2)^2}{4} \). Therefore one can consider the Hardy–Schrödinger operator \( L_\gamma := -\Delta_{B^n} - \gamma V_2 \) and pose the

PROBLEM 1.3.1. Let \( n \geq 3 \) and \( 0 \in \Omega \subseteq B^n \). Under what conditions is the best constant

\[
\mu_{\gamma,s}(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int \Omega \left( |\nabla_{B^n} u|^2 - \gamma V_2 u^2 \right) dv_{g_B^n}}{\left( \int \Omega V_2(s) |u|^{2^*(s)} dv_{g_B^n} \right)^{\frac{n}{2}}} ,
\]

attained?

Such generalization is not trivial. As opposed to the case \( \gamma = s = 0 \) studied in [43], even the radial ordinary differential equation does not have constant coefficients. Worse than that, there is no Kelvin’s transform in the hyperbolic space, making the asymptotic analysis hard.

We have explored the radial aspects of the hyperbolic space \( B^n \) parallel to one in \( \mathbb{R}^n \) and overcome the aforementioned difficulties by classifying all the radial solutions of

\[
L_\gamma u = V_2(s) u^{2^*(s)-1} ,
\]

by an explicit formula in terms of the fundamental solution \( G \). Then the argument in the Euclidean case in [29] can be applied and in particular we proved

THEOREM 1.3.2 (Chan–Faria–Ghoussoub–Mazumdar–Shakerian [13]). The hyperbolic Hardy-singular mass \( m_{B^n}(\Omega) \) is well-defined.

Roughly speaking, the mass is the second-term coefficient of the expansion of the normalized nonnegative solution of \( L_\gamma H = 0 \) in \( \Omega \) near the origin.
From this point the attainability is still not straightforward because the next order of the potential $V_2$ is still singular of order $|x|^{-1}$. We tackled this difficulty by generalizing the attainability result in [29], allowing for more singular linear perturbations. Finally we obtain

**THEOREM 1.3.3** (Chan–Faria–Ghoussoub–Mazumdar–Shakerian [15]). Let $0 \in \Omega \subset \mathbb{R}^n$, $n \geq 3$ and $\gamma < \frac{(n-2)^2}{4}$. Then $\mu_{\gamma,s}(\Omega)$ is attained if either

$$0 < \gamma \leq \frac{(n-2)^2}{4} - \frac{1}{4}$$

or

$$\frac{(n-2)^2}{4} - \frac{1}{4} < \gamma < \frac{(n-2)^2}{4} \quad \text{and} \quad m_{\mathbb{R}^n}(\Omega) > 0.$$  

1.4. The fractional Lane–Emden equation. Let $0 < s < 1$, $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a smooth domain. Consider

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Owing to the recent strong interest in the study of non-local equations, we establish two related concentration results. These are a joint work with W. Ao, M.d.M. González and J. Wei.

1.4.1. Prescribed singularities. In the classical case $s = 1$, Chen–Lin [18] showed the existence of positive weak solutions with a prescribed singular set. The basic cells are positive radial solutions of $-\Delta u = u^p$ which are singular at the origin ($u \sim |x|^{-\frac{2n}{p-1}}$) and fast-decaying at infinity ($u \sim |x|^{-(n-2)}$).

Concentration in the fractional case, in contrast, is still largely open. We consider in particular the

**PROBLEM 1.4.1.** Show the existence of positive weak solutions of (7) for a given singular set.

Unlike the case $s = 1$ which can be proved by a simple phase-plane analysis, the construction of the corresponding building blocks is quite involved. First, we studied a weighted Dirichlet problem by an approximation argument, using the stability of the singular solution. Then by blowing-up we obtained an entire solution, whose Kelvin transform has the desired properties.

In what follows, $p_1$ is a threshold of stability of the singular solution $u_s = A|x|^{-\frac{2n}{p_1}}$ that is known in [24]. In fact, $p_1$ is determined by an equation involving quotients of Gamma functions and the best constant of the fractional Hardy’s inequality [11].

**THEOREM 1.4.2** (Ao–Chan–González–Wei). Suppose $\frac{n}{n-2s} < p < p_1$. Then for every $\alpha \in (0, \infty)$ there exists a positive solution $u_{\alpha}$ of

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

such that

$$u_{\alpha}(x) = \begin{cases} A|x|^{-\frac{2n}{p-1}} & \text{as } |x| \to 0, \\ \alpha|x|^{-(n-2s)} & \text{as } |x| \to \infty. \end{cases}$$

Moreover, we have $0 < u_{\alpha}(x) < u_s(x)$ for any $\alpha$ and

$$\lim_{\alpha \to 0} u_{\alpha}(x) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} u_{\alpha}(x) = u_s(x)$$

uniformly in any compact set in $\mathbb{R}^n \setminus \{0\}$.

For the same range of $p$, we have answered Problem 1.4.1 positively using the same variational machinery, but with harder non-local estimates.

1.4.2. Fractional Yamabe problem. When $p$ is the critical exponent $\frac{n+2s}{n-2s}$, equation (7) can be considered as the fractional Yamabe problem. In the classical case, solutions with higher dimensional concentrations has been known to Mazzeo–Pacard [24]. We gave examples of weak solutions that are singular in the whole $\mathbb{R}^n$, generalizing another result of [18]. Again, while the main idea is similar, we completed the argument with some nontrivial estimates involving the fractional Laplacian.

**THEOREM 1.4.3** (Ao–Chan–González–Wei). Let $n \geq 9$, $s$ be sufficiently close to 1 and $p = \frac{n+2s}{n-2s}$. There exist positive weak solutions to

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^n$$

in $L^p(\mathbb{R}^n, (1 + |x|^2)^{-\frac{n-2s}{2}} \, dx)$ whose singular set is the whole $\mathbb{R}^n$. 
1.5. **Singular solution for fractional Yamabe problem.** In the same line of thought as Section 1.4 we consider the problem of constructing solutions to the fractional Yamabe problem having higher-dimensional concentration:

**PROBLEM 1.5.1.** Construct a solution to

\[
(-\Delta_{\mathbb{R}^n})^\gamma u = u^{n+2\gamma\frac{n-2}{2}} \quad \text{in } \mathbb{R}^n \setminus \Sigma
\]

that blows up exactly at a given union \( \Sigma \) of submanifolds.

The outline of our argument follows that in the local case \( \gamma = 1 \) of [44], but for our non-local equation, we encounter intrinsic difficulties in the study of the basic linearized operator. Our main contribution is to overcome such difficulties. We have succeeded in using ODE type methods for a non-local equation such as a Hamiltonian quantity and a phase portrait. An important estimate has been obtained through the construction of a homoclinic orbit. All these are done in the framework of scattering theory and rely strongly on the geometry of the problem.

**THEOREM 1.5.2** (Ao–Chan–DelaTorre–González–Wei [3], in preparation). Let \( \Sigma \) be a finite disjoint union of smooth, compact submanifolds without boundary, each of whose dimension lying in the interval \( (0, \frac{n-2}{2}) \). Then there exists a positive solution of (8) that blows up exactly on \( \Sigma \).

The last difficulty to tackle is the injectivity of the linear operator, which we are working on.

1.6. **Traveling wave solutions with non-local diffusion.** Traveling wave solutions for non-linear local diffusion equations are solutions to

\[-\Delta u - cu_{x_n} - f(u) = 0 \quad \text{in } \mathbb{R}^n.\]

Planar traveling waves are those depending only on \( x_n \), whose existence is classical [26]. In the unbalanced bistable case, namely \( f(t) = -(t-t_0)(t-1)(t+1) \) with \( t_0 \neq 0 \), the speed \( c \neq 0 \) is determined uniquely by \( f \). The study of non-planar waves began with the V-shaped one by Ninomiya–Taniguchi [49, 50] when \( n = 2 \), followed by its higher dimensional analogues [33, 57, 58].

Consider now the following problem with non-local diffusion:

**PROBLEM 1.6.1.** Construct a solution of

\[(-\Delta)^s u - cu_z - f(u) = 0 \quad \text{in } \mathbb{R}^3\]

with a non-planar wavefront.

While an idea of such construction is already contained in [57], there are some obstructions. The primary problem is that, unlike the local case, no chain rule is available.

We succeeded to deal with such difficulty by coming up with an **expansion of the fractional Laplacian in the Fermi coordinates.** On a nearly flat surface, we found out that it is closely approximated by the one-dimensional fractional Laplacian. Similar to the fractional gluing scheme mentioned in Section 1.1, we also need to be careful in the estimates because of the slow decay of the one-dimensional profile and the non-local effect.

Using sub- and super-solution method, we constructed a solution of (9) with an asymptotically pyramidal front. An example of pyramid is the square pyramid given by \( z = m^* \sqrt{2} (|x| + |y|) \) where \( m^* \) is the slope depending on the speeds.

**THEOREM 1.6.2** (Chan–Wei [17]). Let \( 1/2 < s < 1 \) and \( f \) be an unbalanced bistable non-linearity such that the speed \( k \) of the planar traveling wave given in [32] satisfies \( k > 0 \). For any \( c > k \), there exists a solution \( u \) to (9) such that, away from the edges of the pyramid, \( u \) tends uniformly to the planar traveling wave solution over the pyramid.

The hypothesis \( s > 1/2 \) is not merely technical; it ensures that the non-local interactions are not too strong. This is to be discussed in the research proposal. Another importance of this work is that the **Fermi coordinate expansion** of \((-\Delta)^s\) can be applied to more sophisticated methods, as I discuss in Section 1.1.

1.7. **The Lin–Ni–Takagi problem.** Consider the singularly perturbed elliptic problem

\[
\begin{align*}
\varepsilon^2 \Delta u - u + u^p &= 0 \quad \text{in } \Omega \\
u > 0 &\quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \), \( \nu \) is its normal, \( p > 2 \) and \( \varepsilon > 0 \).
The study of such equation was initiated by Lin–Ni–Takagi \[39\]. Various concentration phenomena have been exhibited, including the least energy solution with a spike at the most curved part of the boundary \[47\] \[48\], solutions having multiple interior and/or boundary spikes \[31\], for instance and those concentrating on higher dimensional sets (like \[42\]). For more background see the survey articles \[45\] \[46\] \[59\].

In all the previous work concerning higher dimensional boundary concentrations, the concentration sets were either the whole boundary or closed submanifolds of the boundary, and the solutions were constructed only for some sequence \(\varepsilon_j \to 0\) under a gap condition. A natural question is

**QUESTION 1.7.1.** Does \((10)\) have solutions concentrating on a broken segment of the boundary for all \(\varepsilon \to 0\)?

We provided an affirmative answer under certain assumption on the broken segment \(\gamma \subset \partial \Omega\), demonstrating a new concentration phenomenon. At the same time we removed a technical assumption. This was done by putting a large number of spikes over \(\gamma\). An immediate obstacle came from the reduced system with growing size. Nevertheless, we devised a new idea of approximating the system with a limiting overdetermined ODE, which is solvable under a geometric assumption on \(\gamma\).

**THEOREM 1.7.2 (Ao–Chan–Wei \[11\]).** Assume that the mean curvature function is strictly convex on \(\gamma\) and that the endpoints of \(\gamma\) have the same mean curvature. Then there is an \(\varepsilon_0 > 0\) such that for all \(\varepsilon < \varepsilon_0\), there exist boundary spike solutions to \((10)\) concentrating on \(\gamma\).

1.8. A hyperbolic Caffarelli–Kohn–Nirenberg inequality. The study of energy inequalities is a crucial part in the field of partial differential equations. Among others, Caffarelli–Kohn–Nirenberg \[9\] proved a version weighted with powers, which takes the form

\[
C \left( \int_{\mathbb{R}^n} |x|^{-b}|u|^p \, dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 \, dx, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n),
\]

where \(-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1\) and \(p = \frac{2N}{N-2+2(b-a)}\). As a first step of generalization to Cartan–Hadamard manifolds on which Hardy’s inequality is known to hold \[11\] we ask the

**QUESTION 1.8.1.** Does a version of \((11)\) hold on the model hyperbolic space \(\mathbb{H}^N\)?

A direct proof is not possible, because of the extra terms from the metric. Still, we managed to exploit the well-known change of variable \(u = d^{-a}v\) \((d = \text{dist} (\cdot, 0))\) which transforms the inequality to a Hardy type one, and absorb the remainder terms with a Poincaré inequality. We have

**THEOREM 1.8.2 (Chan–Faria–Shakerian \[14\]).** With the same \(a, b, p \in \mathbb{R}\), there holds

\[
C \left( \int_{\mathbb{B}^N} d^{-b} |u|^p \, dV \right)^{\frac{2}{p}} \leq \int_{\mathbb{B}^N} d^{-2a} |\nabla_{\mathbb{B}^N} u|^2 \, dV \quad \text{for all } u \in C_0^\infty(\mathbb{B}^N).
\]

As a consequence, we obtained variationally the existence of solutions of a weighted subcritical Hardy–Hénon equation of the form

\[
- \Delta_{\mathbb{B}^N}^\alpha u - \lambda d^{\alpha-2} u = d^\beta |u|^{q-2} u \quad \text{in } \mathbb{B}^N
\]

where \(\Delta_{\mathbb{B}^N}^\alpha\) is a weighted Laplace–Beltrami type operator.

On the other hand, we establish a non-existence result in star-shaped domains. We emphasize that the application of the Pohožaev identity requires great care because of the extra terms and this approach seems to work only under the additional supercriticality condition \(q \geq \max \left\{ \frac{2(N+\beta)}{N-2+2\alpha}, \frac{2N}{N-2} \right\} \).

2. Research proposal

There are many open problems which are natural extensions of the results I have obtained. In particular, I will work in the following directions.

1. Construct a counter-example to the De Giorgi conjecture for the half-Laplacian.

   There are two major difficulties, both arising from non-integrability: the non-local terms interacting with the mean curvature and the \(L^2\) linear theory. To these ends, I will first compute a refined expansion of the fractional Laplacian on an almost flat manifold. Next, I will need to develop a new linear theory by approximation with bounded domains.
(2) Study the multiplier problem for a system of two differential equations. Douglas [22] solved the second order case with an extensive case study. I will first examine the overdetermined system in the second order cases and check the consistency, as I did for the fourth order scalar case in [12].

(3) Solve the Hardy–Schrödinger equation with simultaneous point and boundary singularities. The Hardy–Schrödinger equation in the Euclidean space with either interior or boundary singularity is well-studied [30]. New phenomena can be seen when the singularities are combined, say in the equation

$$-\Delta v - \left(\frac{\gamma_1}{|x|^2} + \frac{\gamma_2}{(1 - |x|)^2}\right) v = \left(\frac{c_1}{|x|^s} + \frac{c_2}{(1 - |x|)^{p(s)}}\right) v^{2^*(s)-1} \quad \text{in } B_1(0) \subset \mathbb{R}^n,$$

where $s$ and $2^*(s)$ are as described in Section 1.3 and $p(s)$ is an exponent related to the behavior of the potential $V_{2^*(s)}$ near $|x| = 1$. Such an equation is already seen, up to smooth coefficients, with the conformal transformation $v = \left(\frac{2}{1 - |x|^2}\right)^{\frac{s}{2^*}} u$ where

$$-\Delta_{B^n} u - \gamma V_2 u = V_{2^*(s)} u^{2^*(s)-1},$$

because the singularity near $\partial B_1(0)$ is hidden in the metric tensor. I will solve the equation with only a singularity at the origin in the hyperbolic space, as an extension to [15].

(4)-(5) Complete the construction of solutions of the fractional Yamabe problem with multidimensional singularity. To finish the project, I will study the injectivity of the linear operator.

(6) Construct traveling wave solutions in the case $0 < s \leq 1/2$. For such $s$, in the construction of [27, 17], the mollified pyramid, or the nodal set of the super-solution, would not lie above the original pyramid (nodal set of the sub-solution). Moreover, non-local terms other than the mean curvature will be seen (this is related to (1)). For $0 < s < 1/2$ even the main term is non-local, so that the auxiliary function $S$, which is designed from the mean curvature equation with constant force, can no longer be used. I will first study the corresponding non-local mean curvature equation with force.

Construct traveling wave solutions with more general fronts. The local case is completed in [58], where the conical fronts can be generated by any convex compact set of codimension one. I will follow this idea and generalize it to the fractional case.

Construct traveling wave solutions for balanced non-linearity. With the fractional gluing approach mentioned in Section 1.1 I will extend [21] to the fractional case.

(7) Construct solutions to the Lin–Ni–Takagi problem in higher dimensions. In the subcritical case that I have studied in [4], the reduced problem resembles a local ordinary differential equation. I will work on the higher dimensional case, where the exponent $p$ can become critical and a non-local reduced equation is expected.

(8) Address existence issues in the gap interval for the exponent. The current proof of non-existence does not work when $\frac{2(N+\beta)}{N-2+\alpha} \leq q < \frac{2N}{N-2}$. It will be very interesting the see whether solutions can exist for such $q$. According to the proof, for $q$ close to but less than $\frac{2N}{N-2}$, non-existence is guaranteed in sufficiently small (star-shaped) domains.

Prove the Caffarelli–Kohn–Nirenberg inequality on Cartan–Hadamard manifolds. As mentioned in Section 1.8, I will use the existing idea on this more general setting where Hardy’s inequality hold [11, 6].

REFERENCES

8 HARDY CHAN


