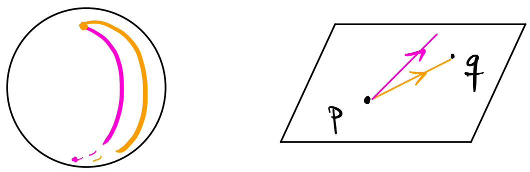
5. Curvature

Throughout, (M,g) will denote a Riemannian manifold.

Motivations: (i) generalize (intrinsic!) Gauss curvature of surfaces

(ii) do geodesics refocus?



(iii) measure path-dependence of parallel transport

(iv) $O(x^2)$ terms of g_{ij} in normal coordinates

All of these turn out to lead to the same object — the Riemann curvature tensor.

@ (i). Riemann 1854: let $\Pi \subset T_pM$ be a plane, and for $\varepsilon > 0$ small consider the C^∞ surface

$$\Sigma := \{ \exp_p(v) : v \in \Pi, |v| < \varepsilon \}$$

with its induced metric g_Σ. Then

$$K(\Pi) := \text{Gauss curvature of } (\Sigma, g_\Sigma) \text{ at } p$$

(This uses the Theorema egregium: $K(p,\Pi)$ can be computed from g_Σ only.) How to compute?

@ (ii). Let $p \in M$, $v : (-1,1) \to T_pM$, $\tilde{\gamma}(s,t) = \exp_p(t\, v(s))$,

$\gamma(t) = \tilde{\gamma}(0,t)$.

Let $V(t) = \frac{d}{ds}\tilde{\gamma}(s,t)\big|_{s=0}$ (variation vector field).

Then $V(0) = \frac{d}{ds} p \big|_{s=0} = 0$,

$$\nabla_{\partial_t} V(0) = \nabla_{\partial_t}(\partial_s \tilde{\gamma}(s,t))\big|_{s=0} = \nabla_{\partial_s}(\partial_t \tilde{\gamma}(s,t))\big|_{s=0} = \nabla_{\partial_s}(v(s)) = v'(0),$$

and

$$\nabla^2_{\partial_t} V = \nabla_{\partial_t} \nabla_{\partial_s} \partial_t \tilde{\gamma}$$

$$= \nabla_{\partial_s}(\underbrace{\nabla_{\partial_t} \partial_t \tilde{\gamma}}_{\substack{=0 \text{ since} \\ \tilde{\gamma}(s,\cdot) \text{ is a geodesic}}}) + [\nabla_{\partial_t}, \nabla_{\partial_s}]\, \partial_t \tilde{\gamma}.$$

This computation suggests defining (for X, Y with $[X,Y]=0$)

$$R(X,Y)Z = [\nabla_X, \nabla_Y] Z.$$

$$\Longrightarrow \boxed{V'' := \nabla^2_{\partial_t} V = R(\gamma', V)\gamma'.} \quad \text{(Jacobi equation.)}$$

(Rough interpretation: if $R(\gamma', V)\gamma'$ points in the direction opposite to V, $V(t)$ tends to "decrease", so neighboring geodesics tend to refocus.)

@ (iii), (iv): Exercises.

5.1. Riemann curvature tensor

Definition 5.1. The Riemann curvature tensor is the map

$$R : \Gamma(TM)^3 \longrightarrow \Gamma(TM),$$
$$(X,Y,Z) \longmapsto R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Without any calculation, only the $\mathbb{R}$-multilinearity of R is clear.

Proposition 5.2. R is tensorial ($C^\infty(M)$-linear) in each of its arguments.

Corollary 5.3. The map

$$\Gamma(TM)^4 \longrightarrow C^\infty(M), \quad (X,Y,Z,W) \longmapsto R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$$

defines $R \in \Gamma(T_{(0,4)}M)$.

Proof of Proposition 5.2.

$$\begin{aligned} \cdot\; R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - (\nabla_{f[X,Y]} Z - \nabla_{(Yf)X} Z) \\ &= f R(X,Y)Z - \cancel{(Yf)\nabla_X Z} + \cancel{(Yf)\nabla_X Z} \\ &= f R(X,Y)Z. \end{aligned}$$

$\cdot\; R(Y,X)Z = -R(X,Y)Z$, so tensoriality in Y follows immediately.

$$\begin{aligned} \cdot\; R(X,Y)(fZ) &= \nabla_X\big((Yf)Z + f\nabla_Y Z\big) - \nabla_Y\big((Xf)Z + f\nabla_X Z\big) \\ &\quad - ([X,Y]f)Z - f\nabla_{[X,Y]}Z \\ &= \cancel{(X(Yf))Z} + \cancel{(Yf)\nabla_X Z} + \cancel{(Xf)\nabla_Y Z} + f\nabla_X\nabla_Y Z \\ &\quad - \cancel{(Y(Xf))Z} - \cancel{(Xf)\nabla_Y Z} - \cancel{(Yf)\nabla_X Z} - f\nabla_Y\nabla_X Z \\ &\quad - \cancel{([X,Y]f)Z} \qquad - f\nabla_{[X,Y]}Z \end{aligned}$$

Here, the reason for including $\nabla_{[X,Y]}Z$ in the def. of R becomes clear!

$$= f\,R(X,Y)Z.$$ ∎

The tensor $R(X,Y,Z,W)$ has many symmetries:

Proposition 5.3. (i) $R(X,Y,Z,W) = -R(Y,X,Z,W)$; $\quad \dagger$

$\qquad R(X,Y,Z,W) = -R(X,Y,W,Z)$. $\quad \ddagger$

(ii) (First Bianchi identity.)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \qquad \circledast$$

(iii) $R(X,Y,Z,W) = R(Z,W,X,Y)$

Proof: (i) $\dagger$ is obvious.

$\ddagger$ follows from $R(X,Y,Z,Z)=0$, which we now show:

$$\begin{aligned}
R(X,Y,Z,Z) &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, Z\rangle \\
&= X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle \\
&\quad - Y\langle \nabla_X Z, Z\rangle + \langle \nabla_X Z, \nabla_Y Z\rangle \\
&\quad - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= \tfrac{1}{2} X(Y\langle Z,Z\rangle) - \tfrac{1}{2} Y(X\langle Z,Z\rangle) - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle \\
&= 0.
\end{aligned}$$

(ii) L.H.S. of $\circledast$ is

$$\begin{aligned}
&\nabla_X \nabla_Y Z + \nabla_Y \nabla_Z X + \nabla_Z \nabla_X Y \\
&- \nabla_Y \nabla_X Z - \nabla_Z \nabla_Y X - \nabla_X \nabla_Z Y \\
&- \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
= {}& \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
= {}& [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \text{(Jacobi identity).}
\end{aligned}$$

(iii) We write down 4 versions of the 1st Bianchi identity and sum:

$$
\begin{aligned}
0 = {} & \cancel{R(X,Y,Z,W)} + \cancel{R(Y,Z,X,W)} + \underline{R(Z,X,Y,W)} \\
& + \cancel{R(Y,Z,W,X)} + \cancel{R(Z,W,Y,X)} + \underline{R(W,Y,Z,X)} \\
& + \cancel{R(Z,W,X,Y)} + \cancel{R(W,X,Z,Y)} + \underline{R(X,Z,W,Y)} \\
& + \cancel{R(W,X,Y,Z)} + \cancel{R(X,Y,W,Z)} + \underline{R(Y,W,X,Z)} \\
= {} & 2\big(\underline{R(X,Z,W,Y)} - \underline{R(W,Y,X,Z)} \big).
\end{aligned}
$$

$\square$

- We next discuss this in local coordinates $x^1, \dots, x^m$ on M: then

$$
\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k \\
&= \nabla_i (\Gamma_{jk}^q \partial_q) - \nabla_j (\Gamma_{ik}^q \partial_q) \\
&= \underbrace{\left(\partial_i \Gamma_{jk}^q - \partial_j \Gamma_{ik}^q + \Gamma_{jk}^r \Gamma_{ir}^q - \Gamma_{ik}^r \Gamma_{jr}^q \right)}_{=: R_{ijk}{}^q} \partial_q \\
&= R_{ijk}{}^q \partial_q .
\end{aligned}
$$

$[\partial_i, \partial_j] = 0$

Moreover, $R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = g_{lq} R_{kij}{}^q$; and Proposition 5.3 reads:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij},$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Example 5.4. ($\mathbb{R}^m$, Euclidean metric). $R \equiv 0$. (Either directly from Def. 5.1; or in standard coordinates from $\Gamma_{ij}^k = 0$.)

Before giving further examples, we shall prove that one can recover $R(X,Y,Z,W)$ entirely from $R(X,Y,X,Y)$ $\forall X, Y$; see the next section.

5.2. Sectional curvature

Definition 5.5. Let $\Pi \subset T_pM$ be a plane (2-dimensional linear subspace). Then the *sectional curvature of Π at p* is

$$K(\Pi) := -\frac{R(X,Y,X,Y)}{|X\wedge Y|^2},$$

where $X,Y \in T_pM$ span Π, and $|X\wedge Y|^2 := |X|^2|Y|^2 - \langle X,Y\rangle^2$ (area^2 of parallelogram in Π spanned by X,Y).

Lemma 5.6. This is well-defined, i.e. independent of the choice of basis X,Y of Π.

Proof. The expression $-\frac{R(X,Y,X,Y)}{|X\wedge Y|^2}$ is invariant under:

$$\{X,Y\} \longrightarrow \{Y,X\}$$
$$\{X,Y\} \longrightarrow \{\lambda X,Y\} \qquad (\lambda \in \mathbb{R})$$
$$\{X,Y\} \longrightarrow \{X+Y,Y\}.$$

Using these 3 transformations, one can pass from any basis $\{X,Y\}$ of Π to any other basis. $\square$

Example 5.7: *Theorema egregium* revisited. Let $\Sigma \subset \mathbb{R}^3$ be an embedded surface, $p \in \Sigma$. After a rigid motion, we may assume $p=0$, $T_p\Sigma = \mathbb{R}^2\times\{0\}$ and locally near p, Σ is the graph

$$\{(x,y,f(x,y))\}$$

of a function $f:\mathbb{R}^2\to\mathbb{R}$ with $f(0)=\nabla f(0)=0$, $\mathrm{Hess}_f(0)=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix}$. Thus, κ_1,κ_2 are the *principal curvatures* of Σ at p, and

$K = \varkappa_1\varkappa_2$ is the Gauss curvature of Σ at p.

To compute the sectional curvature $K(T_p\Sigma)$, we note that for
$F(x,y) := (x, y, f(x,y))$,

we have $F_* \partial_x = \partial_1 + (\partial_1 f)\partial_3 = \partial_1 + \varkappa_1 x\, \partial_3 + O(|x|^2+|y|^2)$

$F_* \partial_y = \partial_2 + (\partial_2 f)\partial_3 = \partial_2 + \varkappa_2 y\, \partial_3 + O(|x|^2+|y|^2)$

$$\Rightarrow (g_{ij}) = \begin{pmatrix} 1+\varkappa_1^2 x^2 & \varkappa_1\varkappa_2 xy \\ \varkappa_1\varkappa_2 xy & 1+\varkappa_2^2 y^2 \end{pmatrix} + O(|x|^3+|y|^3).$$

$$\Rightarrow \Gamma_{ij}^k(0) = 0$$

$$\Rightarrow K(T_p\Sigma) = -R(F_*\partial_x, F_*\partial_y, F_*\partial_x, F_*\partial_y)|_p = -R_{xyxy}(0) = R_{yxx}{}^y(0)$$

$$= \underbrace{\partial_y \Gamma_{xx}^y(0)}_{\substack{= \partial_x g_{12} \\ + O(|x|^2+|y|^2)}} - \underbrace{\partial_x \Gamma_{yx}^y(0)}_{=0}$$

$$= \varkappa_1\varkappa_2 .$$

Since $K(T_p\Sigma)$ is an intrinsic quantity (only depending on (Σ, g_Σ)), this reproves Gauss' Theorema egregium.

That Riemann's original proposal contains all the information in the curvature tensor is a consequence of the following result, applied to $V = T_pM$:

Lemma 5.8. Let V be a vector space, $\dim V \geq 2$, equipped with a positive definite inner product $\langle \cdot,\cdot \rangle$. Suppose

$$R, R' : V \times V \times V \to V$$

are trilinear mappings s.t. $\tilde{R}(X,Y,Z,W) = \langle R(X,Y,Z), W\rangle$, $\tilde{R}'$ satisfy the symmetries of Prop 5.3. If $\forall$ planes $\Sigma \subset V$, $\Sigma = \operatorname{span}\{X,Y\}$,

$$K(\Sigma) := \frac{\tilde{R}(X,Y,X,Y)}{|X\wedge Y|^2} = K'(\Sigma),$$

then $R = R'$.

(Thus, $R|_p$ is uniquely determined by $\{K(\Pi) : \Pi \subset T_pM \text{ plane}\}$.)

<u>Proof of Lemma 5.8.</u> We must show that $\tilde{R}(X,Y,X,Y) = \tilde{R}'(X,Y,X,Y)$ $\forall X,Y$ implies $\tilde{R} = \tilde{R}'$, or equivalently: $Q := \tilde{R} - \tilde{R}' = 0$.

(i) $$0 = Q(X+Z, Y, X+Z, Y)$$
$$= \underbrace{Q(X,Y,X,Y)}_{=0} + Q(X,Y,Z,Y) + \underbrace{Q(Z,Y,X,Y)}_{=Q(X,Y,Z,Y)} + \underbrace{Q(Z,Y,Z,Y)}_{=0}$$

$\Rightarrow$ $Q(X,Y,Z,Y) = 0$ $\forall X,Y,Z$.

(ii) $$0 = Q(X, Y+W, Z, Y+W)$$
$$= \underbrace{Q(X,Y,Z,Y)}_{=0} + Q(X,Y,Z,W) + Q(X,W,Z,Y) + \underbrace{Q(X,W,Z,W)}_{=0}$$

$\Rightarrow$ $Q(X,Y,Z,W) = -Q(X,W,Z,Y) = -Q(Z,Y,X,W) = Q(Y,Z,X,W)$,

i.e. Q is invariant under cyclic permutations of its first three arguments. But

$$0 = Q(X,Y,Z,W) + \underbrace{Q(Y,Z,X,W)}_{=Q(X,Y,Z,W)} + \underbrace{Q(Z,X,Y,W)}_{=Q(X,Y,Z,W)}$$
$$= 3\,Q(X,Y,Z,W).$$

$\square$

This Lemma can also be used to determine the form of the Riemann curvature tensor in the case that all sectional curvatures at a point p are equal. (The corresponding metrics turn out to be those on $\mathbb{R}^m, \mathbb{H}^m, \mathbb{S}^m$ — later.)

Proposition 5.9. Let (M,g) be a Riemannian manifold, $p \in M$. Then $K(\Sigma) = k_0$ $\forall$ planes $\Sigma \subset T_pM$ if and only if

$$R = k_0 R', \qquad \langle R'(X,Y,Z),W\rangle = \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle.$$

Proof. R' satisfies the symmetries of Prop. 5.3. By Lemma 5.8, we only need to verify that $K'(\Sigma)=1$ for all Σ. But if X, Y are orthonormal, then

$$\begin{aligned} K'(\operatorname{span}\{X,Y\}) &= -\langle R'(X,Y)X, Y\rangle \\ &= \langle X,X\rangle\langle Y,Y\rangle = 1 \end{aligned}$$

indeed. $\square$

Example 5.10. Let $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ be the unit sphere. Then $K(\Pi) = 1$ $\forall \Pi$. (You can check this already now by convincing yourself that the image of a plane $\Pi \subset T_p\mathbb{S}^m$ under $\exp_p$ is a standard 2-sphere (i.e. isometric to the unit 2-sphere in $\mathbb{R}^3$).)

$$\Rightarrow R(X,Y,Z,W) = \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle.$$

5.3. Operations on tensors I: contractions

& Ricci and scalar curvature.

The Riemann curvature tensor contains rather much information. To define simpler (and less precise) notions of curvature, we shall take its various "traces".

- Consider a $(1,1)$-tensor T, so $T \in \Gamma(T_{(1,1)}M) = \Gamma(TM \otimes T^*M)$. We shall construct a function on M out of it.
- Recall that $V \otimes W^* \cong \mathrm{Hom}(W, V)$ for (finite-dimensional) vector spaces; in particular $V \otimes V^* \cong \mathrm{End}(V)$. The isomorphism is given by linearity from $v \otimes \lambda \mapsto (V \ni u \mapsto v\lambda(u))$.

– If $\{e_1, \dots, e_k\}$ is a basis of V,

$\{\varepsilon^1, \dots, \varepsilon^k\}$ the dual basis of V^*,

then $A^i{}_j e_i \otimes \varepsilon^j \longmapsto A = (u^j e_j \mapsto a^i{}_j u^j e_i)$,

so the matrix of A in the basis $\{e_1, \dots, e_k\}$ is

$$(A^i{}_j) = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \cdots & A^1{}_k \\ \vdots & & \ddots & \vdots \\ A^k{}_1 & A^k{}_2 & \cdots & A^k{}_k \end{pmatrix}.$$

– The trace of $A \in \mathrm{End}(V)$ is $\mathrm{Tr}(A) = \sum_{i=1}^{k} A^i{}_i$; it is independent of the choice of basis. Identifying $A \in V \otimes V^*$: $\mathrm{Tr}(A) = \sum_{i=1}^{k} A(\varepsilon^i, e_i)$.

Invariantly: $\mathrm{Tr} \in (\mathrm{End}(V))^*$ is the image of the identity map $\mathrm{Id}_V \in \mathrm{End}(V)$ under the sequence of isomorphisms

$$\mathrm{End}(V) = V\otimes V^* \underset{V=V^{**}}{=} (V^*\otimes V)^* = (V\otimes V^*)^* = (\mathrm{End}(V))^*.$$

- Returning to our tensor $T\in\Gamma(TM\otimes T^*M)$: for each $p\in M$, $T_p\in T_pM\otimes T_p^*M \cong \mathrm{End}(T_pM)$ has a well-defined trace $\mathrm{Tr}(T_p)$, and we write $(C_1^1 T)(p) := \mathrm{Tr}(T_p) = \sum_{i=1}^m T_p(\varepsilon^i, e_i)$
  (e_i basis of T_pM, ε^i dual basis of T_p^*M)

- In local coordinates $x^1,\dots,x^m$ on $U\subset M$:

$$T|_U = T^i{}_j\, \frac{\partial}{\partial x^i}\otimes dx^j, \quad \mathrm{Tr}(T) = T^i{}_i.$$

Definition 5.11. For $T\in\Gamma(T_{(r,s)}M)$, $C^{r'}_{s'}T\in\Gamma(T_{(r-1,s-1)}M)$ is defined by

$$(C^{r'}_{s'}T)(\omega_1,\dots,\omega_{r-1}, V_1,\dots,V_{s-1}) :=$$
$$\sum_{k=1}^m T(\omega_1,\dots\omega_{r'-1}, \varepsilon^k, \omega_{r'},\dots,\omega_{r-1}, V_1,\dots,V_{s'-1}, e_k, V_{s'},\dots,V_{s-1}).$$

In components: $T^{i_1\cdots i_r}_{j_1\cdots j_s} = T(dx^{i_1},\dots,dx^{i_r}, \frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_s}})$

$$\Rightarrow (C^{r'}_{s'}T)^{i_1\cdots i_{r-1}}_{j_1\cdots j_{s-1}} = T^{i_1\cdots i_{r'-1}\,k\,i_{r'}\cdots i_{r-1}}_{j_1\cdots j_{s'-1}\,k\,j_{s'}\cdots j_{s-1}} \quad \text{(sum over } k\text{)}.$$

Definition 5.12. $\tilde{R}(\omega,X,Y,Z) := \omega\big(R(X,Y)Z\big)$ defines $\tilde{R}\in\Gamma(T_{(1,3)}M)$.
The Ricci tensor is $\mathrm{Ric} := C^1_1\tilde{R}\in\Gamma(T_{(0,2)}M)$.

So: $\mathrm{Ric}(Y,Z) = \sum_{i=1}^m \varepsilon^i(R(e_i,Y)Z)$ (#)

In local coordinates, $\mathrm{Ric}_{jk} = R_{ijk}{}^i$ (from (#)).

- If e_i is an ONB, then $\varepsilon^i := g_p(e_i,\cdot) = \langle e_i,\cdot\rangle$ is the dual basis;

thus $$\mathrm{Ric}(Y,Z) = \sum_{i=1}^{m} \langle e_i, R(e_i,Y)Z\rangle = \sum_{i=1}^{m} \langle R(e_i,Y)Z, e_i\rangle \qquad (\circledast)$$
$$= \sum_{i=1}^{m} R(e_i,Y,Z,e_i).$$

Corollary 5.13. Ric is symmetric: $\mathrm{Ric}(Y,Z)=\mathrm{Ric}(Z,Y)$.

Proof. $$R(e_i,Y,Z,e_i) = R(Z,e_i,e_i,Y) = R(e_i,Z,Y,e_i).$$

The claim now follows from $\circledast$. $\square$

Remark 5.14. Directly from the definition, for $Y\in T_pM$, $|Y|=1$,
$$\mathrm{Ric}(Y,Y) = \sum_{i=1}^{m-1} K(\mathrm{span}\{Y,X_i\})$$ where $\{X_1,\dots,X_{m-1},Y\}$ is an orthonormal basis of T_pM. One can also write
$$\mathrm{Ric}(Y,Y) = (m-1)\int_{\substack{\text{planes } \Pi\subset T_pM\\ Y\in\Pi}} K(\Pi)\, d\Pi$$
for a suitable measure $d\Pi$ on the space $\{\Pi\subset T_pM : \dim\Pi=2, \Pi\ni Y\}$. (with $\int d\Pi = 1$). So $\mathrm{Ric}(Y,Y)$ is (up to a factor $(m-1)$) the average of the sectional curvatures of all planes containing Y.

Example 5.15. $S^m\subset\mathbb{R}^{m+1}$: by Example 5.10, $\forall$ unit $Y\in T_pS^m$,
$$\mathrm{Ric}(Y,Y) = m-1 = (m-1)\,g(Y,Y)$$
$$\Rightarrow \mathrm{Ric} = (m-1)\,g.$$

If we started with the Riemann curvature tensor as an element of $\Gamma(T_{(0,4)}M)$, it appears that we could not define any contraction of it. But note that in the passage between $\Gamma(T_{(1,3)}M)$ and $\Gamma(T_{(0,4)}M)$ we used the metric tensor (in components, $R_{ijkl} = g_{lq}R_{ijk}{}^{q}$).

Definition 5.16. · Let (M,g) be a Riemannian manifold. For a tensor $T \in \Gamma(T_{(0,2)}M)$, we define its **metric contraction** by

$$C_{12}T := C^1_1 \tilde{T}, \quad \text{where} \quad \tilde{T}(\omega, X) = T(\omega^\#, X).$$

· Alternative notation: $\mathrm{tr}_g T = C_{12} T$.

· Similarly, $C_{r_1 r_2} : \Gamma(T_{(r,s)}M) \to \Gamma(T_{(r-2,s)}M)$, etc.

· In local coordinates, $T = T_{ij}\, dx^i \otimes dx^j$, $\tilde{T} = g^{ki} T_{ij} \frac{\partial}{\partial x^k} \otimes dx^j$

(indeed, $\tilde{T}(dx^k, \frac{\partial}{\partial x^j}) = T(g^{ki}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g^{ki}T_{ij}$), so

$$C_{12}T = \tilde{T}^k{}_k = g^{ki}T_{ik} = g^{ik}T_{ik}.$$

Example 5.17. $\mathrm{tr}_g(g) = C_{12}\, g = C^2_1 \tilde{g}$, where $\tilde{g}^k{}_i = g^{kj}g_{ji} = \delta^k_i$,

so $\mathrm{Tr}_g(g) = \delta^k_k = m = \dim M$.

Definition 5.18. The **scalar curvature** of (M,g) is the metric contraction of Ric, so $\mathrm{scal} = \mathrm{tr}_g \mathrm{Ric}$. (So $\mathrm{scal} \in C^\infty(M)$.)

In coordinates, $\mathrm{scal} = \mathrm{Ric}_j{}^j = R_{ij}{}^{ji} = g^{jk} R_{ijk}{}^i$.

Remark 5.19. Analogously to Remark 5.14, $\mathrm{scal}_p = \sum_{i=1}^m \mathrm{Ric}_p(E_i, E_i)$ (ONB of T_pM)

$= m \fint_{\{v \in T_pM : |v|=1\}} \mathrm{Ric}_p(v,v)\, dv$ is m times the average of $\mathrm{Ric}_p(v,v)$ $\forall$ unit vectors $v \in T_pM$, and then

$$\mathrm{scal} = m(m-1) \fint_{\substack{\Pi \subset T_pM \\ \text{plane}}} K(\Pi)\, d\Pi$$

is $m(m-1)$ times the average sectional curvature at p.

Example 5.20. For $S^m \subset \mathbb{R}^{m+1}$, $\mathrm{scal} = m(m-1)$ (constant function on S^m).

5.4. Operations on tensors II: covariant differentiation & the second Bianchi identity.

We shall elevate the Levi-Civita connection to a connection on all tensor bundles. This is an instance of a general construction for connections on vector bundles.

Proposition 5.21. Let $E, F \to M$ be vector bundles (of ranks k_E, k_F), and let ∇^E, ∇^F be connections on them.

(i) There exists a unique connection ∇^{E^*} on E^* such that, for all $X \in \Gamma(TM)$, $\varepsilon \in \Gamma(E^*)$, and $e \in \Gamma(E)$,

$$X(\underbrace{\varepsilon(e)}_{\substack{M \ni p \mapsto \varepsilon_p(e_p) \in \mathbb{R},\\ \text{i.e. a } C^\infty \text{ function on } M}}) = (\underbrace{\nabla^{E^*}_X \varepsilon}_{\in \Gamma(E^*)})(e) + \varepsilon(\underbrace{\nabla^E_X e}_{\in \Gamma(E)}). \qquad (*)$$

(ii) There exists a unique connection ∇ on $E \otimes F$ such that, for all $X \in \Gamma(TM)$, $e \in \Gamma(E)$, $f \in \Gamma(F)$,

$$\nabla_X (e \otimes f) = (\nabla^E_X e) \otimes f + e \otimes (\nabla^F_X f). \qquad (**)$$

The requirements $(*)$ and $(**)$ are the obviously desirable Leibniz rules.

Proof of Prop. 5.21. (i) We define ∇^{E^*} via $(*)$, i.e.

$$(\nabla^{E^*}_X \varepsilon)(e) := X(\varepsilon(e)) - \varepsilon(\nabla^E_X e).$$

- Claim: the R.H.S. is tensorial in e, and thus defines $\nabla^{E^*}_X \varepsilon \in \Gamma(E^*)$.
  (Using: $\underbrace{C^\infty(M)\text{-linear maps } \Gamma(E) \to C^\infty(M)}_{\substack{\text{such as}\\ e \mapsto X(\varepsilon(e)) - \varepsilon(\nabla^E_X e)}}$ are the same as elements of $\Gamma(E^*)$.)

Proof of claim: for $u \in C^\infty(M)$,

$$X(\varepsilon(ue)) - \varepsilon(\nabla^E_X(ue))$$
$$= X(u \cdot \varepsilon(e)) - \varepsilon(u \cdot \nabla^E_X e + (Xu)e)$$
$$= \cancel{(Xu)\cdot\varepsilon(e)} + u\, X(\varepsilon(e)) \quad - \quad u\cdot\varepsilon(\nabla^E_X e) - \cancel{(Xu)\varepsilon(e)}$$
$$= u\cdot\left(X(\varepsilon(e)) - \varepsilon(\nabla^E_X e)\right).$$

- Claim: ∇^{E^*}, thus defined, is a connection.

Proof of claim: $C^\infty(M)$-linearity in X follows from

$$(\nabla^{E^*}_{uX}\varepsilon)(e) = uX(\varepsilon(e)) - \varepsilon(\underbrace{\nabla^E_{uX} e}_{= u\cdot\nabla^E_X e})$$
$$= u\left(X(\varepsilon(e)) - \varepsilon(\nabla^E_X e)\right)$$
$$= u\,(\nabla^{E^*}_X\varepsilon)(e);$$

the Leibniz rule from

$$(\nabla^{E^*}_X(u\varepsilon))(e) = X(u\cdot\varepsilon(e)) - u\cdot\varepsilon(\nabla^E_X e)$$
$$= u\cdot(\nabla^{E^*}_X\varepsilon)(e) + (Xu)\,\varepsilon(e).$$

(ii) The proof is completely analogous to that of part (i).

Exercise. $\square$

Corollary 5.22. The Levi-Civita connection ∇ on (M,g) induces a unique connection $\nabla^{(r,s)}$ on $T_{(r,s)}M$ for all $r,s\in\mathbb{N}_0$ which

(i) for $(r,s)=(0,0)$ (i.e. on $T_{(0,0)}M = M\times\mathbb{R}$) is given by

$$\nabla^{(0,0)}_X u = Xu \qquad (X\in\Gamma(TM),\ u\in C^\infty(M))$$

(ii) for $(r,s)=(1,0)$ (i.e. on $T_{(1,0)}M = TM$) equals ∇,

(iii) satisfies the Leibniz rule
$$\nabla_X^{(r+r',s+s')}(T\otimes T') = (\nabla_X^{(r,s)}T)\otimes T' + T\otimes(\nabla_X^{(r',s')}T').$$

Moreover,

(iv) these connections "commute with contractions C", in that
$$C(\nabla_X^{r,s}T) = \nabla_X^{r-1,s-1}(CT).$$

Proof. Only part (iv) does not follow from Proposition 5.21.

We only verify it in the case $(r,s)=(1,1)$.

(a) For a simple $(1,1)$-tensor $T=V\otimes\omega$, $V\in\Gamma(TM)$, $\omega\in\Gamma(T^*M)$,

we have $CT=\omega(V)$ and
$$\nabla_X^{1,1}T = \nabla_X V\otimes\omega + V\otimes\nabla_X\omega,$$
so
$$\begin{aligned} C(\nabla_X^{1,1}T) &= \omega(\nabla_X V) + (\nabla_X\omega)(V) \\ &\overset{(iii)}{=} X(\omega(V)) \\ &= X(CT) \end{aligned}$$
indeed.

(b) Given a general tensor $T\in\Gamma(T_{(1,1)}M)$, we verify
$$C(\nabla_X^{1,1}T) = X(CT) \quad \text{in a coordinate chart } \mathcal{U};$$
the point is that on $\mathcal{U}$, $T=\sum T^i{}_j\,\frac{\partial}{\partial x^i}\otimes dx^j$ is a finite sum of simple tensors, so (a) gives the result. $\square$

One also often writes $\nabla^{r,s}T\in\Gamma(T_{(r,s+1)}M)$ for the tensor
$$(\omega_1,\dots,\omega_r,X,V_1,\dots,V_s)\longmapsto(\nabla_X^{r,s}T)(\omega_1,\dots,\omega_r,V_1,\dots,V_s).$$

Exception: for $u \in C^\infty(M)$, $\quad du \in \Gamma(T_{(0,1)}M) = \Gamma(T^*M)$.

Example 5.23. (i) We compute the covariant derivative of the metric tensor itself: for $X, Y, Z \in \Gamma(TM)$,

$$\underbrace{(\nabla_X g)}_{:= \nabla^{(0,2)}_X g}(Y,Z) = \underbrace{X\big(g(Y,Z)\big) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)}_{=0}$$

since the Levi-Civita connection is compatible with the metric.

I.e. the metric compatibility is equivalent to $\nabla g = 0$!

(ii) Covariant derivative of 1-forms in local coordinates: let $\omega = \omega_i \, dx^i$, then

$$\begin{aligned}(\nabla_{\partial_i}\omega)(\partial_j) &= \partial_i\big(\omega(\partial_j)\big) - \omega(\nabla_{\partial_i}\partial_j) \\ &= \partial_i \omega_j - \omega(\Gamma^k_{ij}\partial_k) \\ &= \partial_i \omega_j - \Gamma^k_{ij}\omega_k .\end{aligned}$$

(Note the minus sign compared to the computation, for $V = V^k \partial_k$, of

$$\begin{aligned}(\nabla_{\partial_i} V)^j = dx^j(\nabla_{\partial_i} V) &= dx^j\big((\partial_i V^k)\partial_k + \Gamma^\ell_{ik} V^k \partial_\ell\big) \\ &= \partial_i V^j + \Gamma^j_{ik} V^k .)\end{aligned}$$

Equipped with these notions, we can now define various "natural" differential operators on Riemannian manifold (M,g).

Definition 5.24. (i) The gradient of $u \in C^\infty(M)$ is $\nabla u = (du)^\# \in \Gamma(TM)$.

(ii) The Laplace operator (on functions) is $\Delta u := \mathrm{tr}_g(\nabla du)$.

(iii) The Hessian of $u \in C^\infty(M)$ is $\mathrm{Hess}(u)(X,Y) = (\nabla du)(X,Y)$.

(iv) The divergence of a symmetric tensor $T \in \Gamma(T_{(0,2)}M)$ (i.e. $T(X,Y)=T(Y,X)$) is $\mathrm{div}_g T := C_{12}(\nabla T) \in \Gamma(T^*M)$.

In local coordinates: $du = (\partial_i u)\, dx^i$, so

- $\nabla u = g^{ij}(\partial_j u)\partial_i$; also often written $\mathrm{grad}\, u$;
- $\mathrm{Hess}(u)_{ij} = \partial_i \partial_j u - \Gamma_{ij}^k \partial_k u$;
- $\Delta u = g^{ij}(\partial_i \partial_j u - \Gamma_{ij}^k \partial_k u)$;
- $(\nabla T)_{ijk} = (\nabla_{\partial_i} T)(\partial_j, \partial_k) = \partial_i (T(\partial_j, \partial_k)) - T(\nabla_{\partial_i}\partial_j, \partial_k) - T(\partial_j, \nabla_{\partial_i}\partial_k)$

$$= \partial_i T_{jk} - \Gamma_{ij}^{\ell} T_{\ell k} - \Gamma_{ik}^{\ell} T_{j\ell}$$

$$\Rightarrow \mathrm{div}_g T = g^{ij} (\nabla T)_{ijk} = (\cdots).$$

Remark 5.25. (i) $\mathrm{Hess}(u) \in \Gamma(T_{(0,2)}M)$ is symmetric:

$$(\nabla du)(X,Y) = (\nabla_X (du))(Y) = X(\,(du)(Y)) - du(\nabla_X Y)$$

$$= XYu - (\nabla_X Y)u$$

∇ is torsion free

$$= YXu - (\nabla_Y X)u = (\nabla du)(Y,X)$$

(ii) $\Delta u = \mathrm{tr}_g \mathrm{Hess}(u) = \mathrm{div}\, \mathrm{grad}\, u$; here $\mathrm{div} : \Gamma(TM) \to C^\infty(M)$,
$\mathrm{div}\, X = C(\nabla X)$.

(iii) $\mathrm{div}_g\, g = 0$ (see Example 5.23 (ii)).

One naturally wonders if such operations produce interesting objects when acting on Ric and scal.

Proposition 5.26. ((Contracted) second Bianchi identity.) $\mathrm{div}_g \mathrm{Ric} = \frac{1}{2} d\, \mathrm{scal}$.

Remark 5.27. Defining the Einstein tensor $\mathrm{Ein} := \mathrm{Ric} - \frac{1}{2}\mathrm{scal} \cdot g \in \Gamma(T_{(0,2)}M)$,
Proposition 5.26 is equivalent to $\mathrm{div}\, \mathrm{Ein} = 0$. (Exercise.)

We deduce Proposition 5.26 from:

Proposition 5.28. (Second Bianchi identity.) Write $\nabla_X R : \Gamma(TM)^3 \to \Gamma(TM)$

$$(\nabla_X R)(Y,Z)W = \nabla_X\big(R(Y,Z)W\big) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W . \quad (*)$$

Then $(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$.

• Regarding $(*)$: regard the curvature tensor as $\tilde{R} \in \Gamma(T_{(0,4)}M)$, $\tilde{R}(Y,Z,W,V) = \langle R(Y,Z)W, V\rangle$, then

$$\begin{aligned}(\nabla_X \tilde{R})(Y,Z,W,V) &= X\big(\tilde{R}(Y,Z,W,V)\big) - \tilde{R}(\nabla_X Y, Z, W, V) - \tilde{R}(Y,\nabla_X Z, W, V) \\ &\qquad - \tilde{R}(Y,Z,\nabla_X W, V) - \tilde{R}(Y,Z,W,\nabla_X V) \\ &= \langle \nabla_X (R(Y,Z)W), V\rangle + \cancel{\langle R(Y,Z)W, \nabla_X V\rangle} \\ &\qquad - \langle R(\nabla_X Y, Z)W, V\rangle - \langle R(Y,\nabla_X Z)W, V\rangle \\ &\qquad - \langle R(Y,Z)\nabla_X W, V\rangle - \cancel{\langle R(Y,Z)W, \nabla_X V\rangle} \\ &= \langle (\nabla_X R)(Y,Z)W, V\rangle,\end{aligned}$$

so $\nabla_X \tilde{R} \in \Gamma(T_{(0,4)}M)$ is related to $\nabla_X R : \Gamma(TM)^3 \to \Gamma(TM)$ like $\tilde{R} \in \Gamma(T_{(0,4)}M)$ to $R : \Gamma(TM)^3 \to \Gamma(TM)$.

Proof of Proposition 5.28. Since the identity we wish to prove is tensorial in X,Y,Z,W, it suffices to check it in local coordinates and with X,Y,Z,W equal to coordinate vector fields; the point is that coordinate vector fields commute with one another: $[\partial_i, \partial_j] = 0$. Then $R(Y,Z)W = [\nabla_Y, \nabla_Z]W$ and thus

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= \nabla_X\big(R(Y,Z)W\big) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W \\ &= \nabla_X [\nabla_Y, \nabla_Z]W - [\nabla_Y, \nabla_Z]\nabla_X W - R(\nabla_X Y, Z)W + R(\nabla_X Z, Y)W\end{aligned}$$

$$= [\nabla_X, [\nabla_Y, \nabla_Z]]W + R(\nabla_X Z, Y)W - R(\nabla_X Y, Z)W.$$

Summing over cyclic permutations of X, Y, Z gives

$$[\nabla_X, [\nabla_Y, \nabla_Z]]W + \cancel{R(\nabla_X Z, Y)}W - R(\nabla_X Y, Z)W$$
$$+ [\nabla_Y, [\nabla_Z, \nabla_X]]W + R(\nabla_Y X, Z)W - R(\nabla_Y Z, X)W$$
$$+ [\nabla_Z, [\nabla_X, \nabla_Y]]W + R(\nabla_Z Y, X)W - \cancel{R(\nabla_X Z, Y)}W$$

$= 0$ (Jacobi identity)

$$= R(\underbrace{[Y, X]}_{=0}, Z)W + R(\underbrace{[Z, Y]}_{=0}, X)W$$

$$= 0. \qquad \square$$

Proof of Proposition 5.26. • We rewrite the second Bianchi identity as

$$\langle (\nabla_X R)(Y,Z)W, V\rangle - \langle (\nabla_Y R)(Z,X)V, W\rangle - \langle (\nabla_Z R)(Y,X)W, V\rangle = 0 \qquad \circledast$$

(This uses the symmetries $\langle (\nabla_Y R)(Z,X)W, V\rangle = -\langle (\nabla_Y R)(Z,X)V, W\rangle$,

$\langle (\nabla_Z R)(X,Y)W, V\rangle = -\langle (\nabla_Z R)(Y,X)W, V\rangle$,

which follow from Proposition 5.3.)

• Recall that if $E_1, \dots, E_m$ is an orthonormal basis of T_pM, then

$$Ric_p(Z,W) = \sum_{i=1}^{m} \langle R(E_i, Z)W, E_i\rangle \qquad (\#)$$

• Plug $Y = V = E_i$, $Z = W = E_j$ into $\circledast$ to get

$$\sum_{i=1}^{m} \langle (\nabla_X R)(E_i, E_j)E_j, E_i\rangle = \sum_{i=1}^{m} \langle (\nabla_{E_i} R)(E_j, X)E_i, E_j\rangle + \langle (\nabla_{E_j} R)(E_i, X)E_j, E_i\rangle$$

$\sum_{j=1}^{m}$ $\Downarrow$ ∇ commutes with contractions $+ (\#)$

$$\sum_{j=1}^{m} (\nabla_X Ric)(E_j, E_j) = 2 \sum_{i,j=1}^{m} \langle (\nabla_{E_i} R)(E_j, X)E_i, E_j\rangle$$

$\|$

$C(\nabla_X Ric) = \nabla_X C Ric = \nabla_X scal$

$$\iff \nabla_X scal \overset{(\#)}{=} 2 \sum_{i=1}^{m} (\nabla_{E_i} Ric)(X, E_i)$$

$\Longleftrightarrow \quad d\,\mathrm{scal}(X) \qquad = 2\,(\mathrm{div}\,\mathrm{Ric})(X).$ $\square$

As an application of these computations, we can prove the following result of F. Schur, 1886:

Theorem 5.29. Let (M,g) be a connected Riemannian manifold, $m = \dim M \geq 3$.

(i) If $\exists f \in C^\infty(M)$ s.t. $\mathrm{Ric}(g) = fg$, then $f = $ const.

(ii) If $\forall p \in M$, $K(\Pi)$ is independent of $\Pi \subset T_pM$, then (M,g) has constant sectional curvature.

Remark 5.30. (i) $\dim M = 2 \Rightarrow$ every metric g satisfies $\mathrm{Ric}(g) = fg$ for some $f \in C^\infty(M)$.

(ii) (M,g) with $\mathrm{Ric}(g) = \mathrm{const}\cdot g$ are called Einstein manifolds; if $\mathrm{Ric}(g) = 0$: Ricci-flat.

Proof of Theorem 5.29. (i) If $\mathrm{Ric}(g) = fg$, then the contracted 2nd Bianchi identity gives

$$\begin{array}{c} \mathrm{div}\,\mathrm{Ric} = \tfrac{1}{2}\, d\,\mathrm{scal} = \tfrac{1}{2}\, d(\mathrm{tr}_g \mathrm{Ric}) = \tfrac{1}{2}\, d(\mathrm{tr}_g(fg)) = \tfrac{m}{2}\, df \\ \| \\ \mathrm{div}(fg) \overset{(\mathrm{Ex.})}{=} df \end{array}$$

$$\Rightarrow \left(\tfrac{m}{2} - 1\right) df = 0 \underset{m\geq 3}{\Longrightarrow} df = 0 \Rightarrow f = \text{const.} \quad (\text{since } M \text{ is connected})$$

(ii) $K(\Pi) = K_p \ \forall \Pi \in T_pM$ implies, by Proposition 5.9, that

$$\langle R(X,Y)Z, W\rangle = K_p\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right)$$

$\Rightarrow \mathrm{Ric}(Y,Z) = K_p(m-1)\langle Y,Z\rangle$, so the condition of part (i) is satisfied for $f = (m-1)K_p \in C^\infty(M) \Rightarrow K_p = $ const. $\square$

5.5. Curvature of submanifolds

- Suppose M is an m-dimensional submanifold of the $\tilde{m}$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. (Recall that this means: $\forall p \in M\ \exists$ chart $\varphi: U \subset \tilde{M} \to \mathbb{R}^{\tilde{m}}$ s.t. $\varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^m \times \underbrace{\{0\}}_{\subset \mathbb{R}^{\tilde{m}-m}})$. In particular, $T_pM = d_{\varphi(p)}\varphi^{-1}(\mathbb{R}^m \times \{0\}) \subset T_p\tilde{M}$.)
- $\tilde{g}$ induces a metric on M via $g_p(X,Y) = \tilde{g}_p(X,Y)$, $X, Y \in T_pM \subset T_p\tilde{M}$.

Proposition 5.31. Denote by $\tilde{\nabla}$ the Levi-Civita connection on $(\tilde{M}, \tilde{g})$, and for $p \in M$ write $\Pi_p : T_p\tilde{M} \to T_pM$ for the orthogonal projection. For $X, Y \in \Gamma(TM)$, denote by $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$ any extension of X, Y. Define $\nabla_X Y \in \Gamma(TM)$ by

$$(\nabla_X Y)(p) := \Pi_p((\tilde{\nabla}_{\tilde{X}} \tilde{Y})(p)). \qquad (*)$$

Then $\nabla : (X,Y) \mapsto \nabla_X Y$ is the Levi-Civita connection of (M, g).

Proof. · Existence of extensions $\tilde{X}, \tilde{Y}$: exercise.
· well-definedness of $\nabla_X Y$ (i.e. independence of choice of $\tilde{X}, \tilde{Y}$), ∇ is metric-compatible and torsion-free: similar to Lemma 3.4. $\square$

What about the part annihilated by Π_p in $(*)$?

Definition 5.32. The second fundamental form of $M \subset \tilde{M}$ is

$$k(X,Y) = (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^\perp \in (TM)^\perp \quad (X, Y \in \Gamma(TM),\ \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}) \text{ extensions}).$$

Lemma 5.33. k is (i) tensorial in both arguments; (ii) $k(X,Y) = k(Y,X)$.

Proof. (i) Tensoriality in X is clear. We check tensoriality in Y

and well-definedness (indepence of $\tilde{Y}$) in one go: for $f \in C^\infty(M)$, let $\tilde{f} \in C^\infty(\tilde{M})$ denote any extension; then, at $p \in M$,

$$\tilde{\nabla}_{\tilde{X}}(\tilde{f}\tilde{Y}) = \tilde{f}\,\tilde{\nabla}_{\tilde{X}}\tilde{Y} + (\tilde{X}\tilde{f})\tilde{Y}$$
$$= f\,(\nabla_X Y + (\tilde{\nabla}_{\tilde{X}}\tilde{Y})^\perp) + (Xf)\,Y$$
$$\Rightarrow (\tilde{\nabla}_{\tilde{X}}(\tilde{f}\tilde{Y}))^\perp = f\,(\tilde{\nabla}_{\tilde{X}}\tilde{Y})^\perp.$$

In particular, if $f(p)=0$, this vanishes, so $(\tilde{\nabla}_{\tilde{X}}\tilde{Y})(p)$ only depends on $\tilde{Y}(p) = Y(p) \in T_pM$; and we get $k(X, fY) = f\,k(X,Y)$ indeed.

(ii) $k(X,Y) - k(Y,X) = (\tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X})^\perp = (\underbrace{[\tilde{X},\tilde{Y}]}_{\substack{=[X,Y] \text{ at } M, \\ \text{so tangent to } M}})^\perp = 0.$ □

Theorem 5.34. (Gauss equation.) The Riemann curvature tensors $R, \tilde{R}$ of (M,g) and $(\tilde{M},\tilde{g})$ are related as follows: for $X,Y,Z,W \in T_pM$,

$$R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + \langle k(X,W), k(Y,Z)\rangle - \langle k(X,Z), k(Y,W)\rangle.$$

Proof. • Pick extensions $\tilde{X}, \tilde{Y}, \tilde{Z}$ of X, Y, Z. By tensoriality, and taking X, Y and $\tilde{X}, \tilde{Y}$ to be coordinate vector fields, we may assume $[\tilde{X},\tilde{Y}]=0$.
Then $\tilde{R}(X,Y,Z,W) = \langle [\tilde{\nabla}_{\tilde{X}}, \tilde{\nabla}_{\tilde{Y}}]\tilde{Z}, W\rangle.$

• We have $\tilde{\nabla}_{\tilde{Y}}\tilde{Z} = \nabla_Y Z + k(Y,Z)$ on M. Since $[\tilde{\nabla}_{\tilde{X}}(\tilde{\nabla}_{\tilde{Y}}\tilde{Z})]|_M$ only depends on $(\tilde{\nabla}_{\tilde{Y}}\tilde{Z})|_M$ (since $\tilde{X}|_M = X$ is tangent to M), we may write (on M)

$$\langle \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}, W\rangle = \langle \tilde{\nabla}_{\tilde{X}}(\nabla_Y Z + k(Y,Z)), W\rangle$$
$$= \tilde{X}\,\langle \nabla_Y Z + \underbrace{k(Y,Z)}_{\perp TM}, \underbrace{W}_{\in TM}\rangle - \langle \nabla_Y Z + k(Y,Z), \tilde{\nabla}_{\tilde{X}} W\rangle$$

$$= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z + k(Y,Z), \nabla_X W + k(X,W)\rangle$$

$$= \langle \nabla_X\nabla_Y Z, W\rangle + \cancel{\langle \nabla_Y Z, \nabla_X W\rangle}$$

$$- \cancel{\langle \nabla_Y Z, \nabla_X W\rangle} - \langle k(Y,Z), k(X,W)\rangle$$

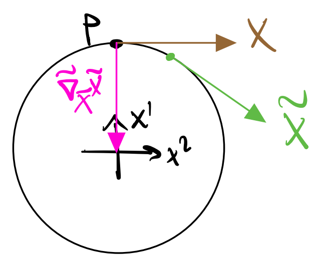
$$\Longrightarrow \tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \underbrace{\langle k(X,Z), k(Y,W)\rangle}_{\text{from } \langle \tilde\nabla_{\tilde Y}\tilde\nabla_{\tilde X}\tilde Z, W\rangle} - \underbrace{\langle k(Y,Z), k(X,W)\rangle}_{\text{from } \langle \tilde\nabla_{\tilde X}\tilde\nabla_{\tilde Y}\tilde Z, W\rangle \text{ above}}$$

$\square$

Example 5.35. $(\tilde M, \tilde g) = (\mathbb{R}^{m+1}, \text{Euclidean metric})$,
$M =$ unit sphere $S^m \subset \mathbb{R}^{m+1}$.

(i) Compute second fundamental form: up to rotation, suffices to consider
$p = (1,0,\dots,0)$, $X = (0,1,\dots,0,0) \in T_p S^m$,

$\tilde X = (-\sin t, \cos t, \dots, 0)$
$\in T_{(\cos t, \sin t, \dots, 0)} S^m$



$$\Longrightarrow \tilde\nabla_{\tilde X}\tilde X\Big|_p = \partial_t \tilde X(t)\Big|_{t=0} = (-1, 0, \dots, 0)$$

$$= \underbrace{\nabla_X \tilde X}_{=0} + k(X,X)$$

(acceleration vector of the geodesic $t \longmapsto (\cos t, \sin t, 0, \dots, 0)$ on S^m)

$\Longrightarrow k_p(X,Y) = -\langle X, Y\rangle N_p$, $N_p = p$ outward unit normal.

(ii) Get curvature of S^m from Gauss equation.

Remark 5.36. When $\dim M = \dim \tilde M - 1$, and $\exists$ C^∞ unit normal
$N: M \to (TM)^\perp \subset T_M\tilde M$ (this happens if and only if $(TM)^\perp$ is

trivial as a vector bundle over M — and then N is unique up to overall multiplication by -1), one also often calls

$$(X,Y) \longmapsto \langle k(X,Y), N\rangle$$

the second fundamental form; this is a symmetric $(0,2)$-tensor on M.

Remark 5.37. Theorema egregium, again. $M =$ surface $\subset (\tilde{M}, \tilde{g}) = (\mathbb{R}^3,$ Euclidean metric). Theorem 5.34 computes, for orthonormal $e_1, e_2 \in T_pM$,

$$K_p = R(e_1, e_2, e_2, e_1) = k(e_1, e_1)\, k(e_2, e_2) - k(e_1, e_2)^2.$$

- Comparison with $\det \mathrm{II}$, $\mathrm{II}(X,Y) = \langle X, D_Y N\rangle_{\mathbb{R}^3}$ ($N =$ unit normal of M):

$$\langle X, D_Y N\rangle = \langle \tilde{X}, \tilde{\nabla}_{\tilde{Y}} \tilde{N}\rangle = \tilde{Y}\, \underbrace{\langle \tilde{X}, \tilde{N}\rangle}_{=0\ @\ M} - \langle \tilde{\nabla}_{\tilde{Y}} \tilde{X}, \tilde{N}\rangle$$

$$= -k(Y,X) = -k(X,Y);$$

so $\underbrace{K_p}_{\text{computable from intrinsic data: metric on } M} = \underbrace{\det \mathrm{II}}_{\text{extrinsic curvature information!}}.$