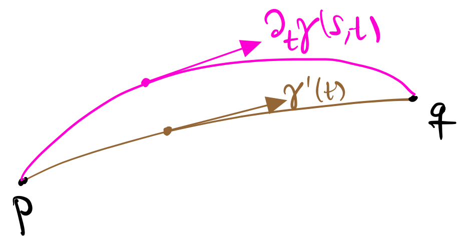
One motivation (among many): which curves $\gamma(t)$ ($t\in[0,1]$, $|\gamma'(t)|=\text{const.}$) from p to q in (M,g), have minimal length?

Strategy: consider variation $\gamma(s,t)$ ($s\in(-1,1)$) with $\gamma(s,0)=p$, $\gamma(s,1)=q$, $\gamma(0,t)=\gamma(t)$, and require

$$\frac{d}{ds}\Big|_{s=0} L(\gamma(s,\cdot)) = \frac{d}{ds}\Big|_{s=0}\int_0^1 g_{\gamma(s,t)}\left(\partial_t\gamma(s,t),\partial_t\gamma(s,t)\right)^{\frac{1}{2}}dt$$

$$= \int_0^1 \frac{1}{|\gamma'(t)|}\,\frac{\partial}{\partial s}\, g_{\gamma(s,t)}\left(\partial_t\gamma(s,t),\partial_t\gamma(s,t)\right)\Big|_{s=0} dt$$

$$\overset{!}{=} 0.$$



$\Longrightarrow$ Need to make sense of "$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\gamma(s,t)\right)$" (how does $\partial_t\gamma(s,t)$ vary with s?), among other things. But $\frac{\partial}{\partial t}\gamma(s,t)\in T_{\gamma(s,t)}M$ lives in different tangent spaces as s varies!

One can, more generally, consider differentiation of sections of vector bundles.

Definition 3.1. Let M be a C^∞ manifold, $E \xrightarrow{\pi} M$ a C^∞ rank k vector bundle. A **connection** is an $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(TM)\times\Gamma(E)\longrightarrow\Gamma(E), \qquad \nabla(V,s) =: \nabla_V s,$$

satisfying: (i) Leibniz rule: $\nabla_V(fs) = (Vf)s + f\nabla_V s$ $\forall\, V\in\Gamma(TM)$, $f\in C^\infty(M)$, $s\in\Gamma(E)$;

(ii) tensorial in V: $\nabla_{fV}s = f\nabla_V s$ $\forall\, V, f, s$.

• Interpretation: $\nabla_X s$ is the directional derivative of s in the direction X.

• Consider an open set U on M on which TM, E are trivial.

Fix $V_1, \dots, V_m \in \Gamma(TU)$, $s_1, \dots, s_k \in \Gamma(E|_U)$ s.t.

$$T_pM = \mathrm{span}\{V_i(p)\}, \quad E_p = \mathrm{span}\{s_\alpha(p)\} \quad \forall p \in U.$$

Write $\nabla_{V_i} s_\alpha = \Gamma^\beta_{i\alpha} s_\beta$: this defines the connection coefficients

$\Gamma^\beta_{i\alpha} \in C^\infty(U)$, $1 \le i \le m$, $1 \le \alpha, \beta \le k$.

Given $X \in \Gamma(TU)$, $\sigma \in \Gamma(E|_U)$, write

$$X = X^i V_i, \quad \sigma = \sigma^\alpha s_\alpha \qquad (X^i, \sigma^\alpha \in C^\infty(U)),$$

then
$$\nabla_X \sigma = \nabla_{X^i V_i}(\sigma^\alpha s_\alpha) = X^i (V_i \sigma^\alpha) s_\alpha + X^i \sigma^\beta \nabla_{V_i} s_\beta$$
$$= \left(X\sigma^\alpha + X^i \sigma^\beta \Gamma^\alpha_{i\beta} \right) s_\alpha .$$

Remark 3.2. To compute $(\nabla_X \sigma)(p)$, only need to know $X(p)$ and $\sigma|_{\mathrm{im}(\gamma)}$ where $\gamma: (-\varepsilon, \varepsilon) \to M$ is any curve with $\gamma(0) = p$, $\gamma'(0) = X(p)$.

Remark 3.3. (i) Every vector bundle admits a connection. (Exercise.)

(ii) The space of all connections is an affine space modeled on $\Gamma(T^*M \otimes E \otimes E^*)$. (Exercise.) Thus, it is an ∞-dimensional space.

Without further structure, there is no "preferred" connection on E.
We are mainly interested in $E = TM$ (cf. the motivation above).

3.1. Levi-Civita connection

· **More motivation.** Consider a surface $\Sigma \subset \mathbb{R}^3$. Given $X, Y \in \Gamma(T\Sigma)$, $p \in \Sigma$, consider an open neighborhood $U \subset \mathbb{R}^3$ of p and extensions $\tilde{X}, \tilde{Y} : U \to \mathbb{R}^3$ of X, Y, i.e. $\tilde{X}|_{\Sigma \cap U} = X$, $\tilde{Y}|_{\Sigma \cap U} = Y$.

Then $(D_{\tilde{X}}\tilde{Y})_p := (\tilde{X}_p(\tilde{Y}^1), \tilde{X}_p(\tilde{Y}^2), \tilde{X}_p(\tilde{Y}^3)) \in \mathbb{R}^3$ (componentwise directional derivative of $\tilde{Y}$ along $\tilde{X}$) typically does not lie in $T_p\Sigma$. Define

$$\nabla_X Y := \pi(D_{\tilde{X}}\tilde{Y}), \qquad \pi : \mathbb{R}^3 \to T_p\Sigma \text{ orthogonal projection.}$$

Lemma 3.4. (i) $\nabla_X Y$ defines a connection on $T\Sigma$.

(ii) $\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \Gamma(T\Sigma)$.

(iii) $X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \Gamma(T\Sigma)$.

Proof (i) Only the well-definedness is not obvious, specifically the independence of $(\nabla_X Y)_p$ from the choice $\tilde{X}, \tilde{Y}$ of extensions.

To show: · if $\tilde{X}|_\Sigma = 0$, then $(\nabla_{\tilde{X}}\tilde{Y})_p = 0$. Obvious!

· if $\tilde{Y}|_\Sigma = 0$, then $(\nabla_{\tilde{X}}\tilde{Y})_p = 0$. True, since we can write such $\tilde{Y}$ as $f\tilde{Z}$ where $f \in C^\infty(U)$ vanishes at Σ, $\tilde{Z} \in C^\infty(U; \mathbb{R}^3)$

$$\Rightarrow (\nabla_{\tilde{X}}\tilde{Y})_p = \underbrace{(\tilde{X}_p f)}_{\substack{=0 \text{ since} \\ \tilde{X}_p \in T_p\Sigma}} \tilde{Z}_p + \underbrace{f(p)}_{=0}(\nabla_{\tilde{X}}\tilde{Z})_p = 0.$$

(ii) $\nabla_X Y - \nabla_Y X = \pi(D_{\tilde{X}}\tilde{Y} - D_{\tilde{Y}}\tilde{X}) = \pi([\tilde{X}, \tilde{Y}])$. But $[\tilde{X}, \tilde{Y}]|_{\Sigma \cap U} = [X, Y]$ is tangent to Σ, so $\pi([\tilde{X}, \tilde{Y}]) = [X, Y]$.

(iii) $X\, g(Y,Z) = \tilde{X}(\tilde{Y}\cdot\tilde{Z}) = (D_{\tilde{X}}\tilde{Y})\cdot\tilde{Z} + \tilde{Y}\cdot(D_{\tilde{X}}\tilde{Z})$;

but since for $p \in \Sigma \cap U$, $\tilde{Z}_p \in T_p\Sigma$, we have

$$(D_{\tilde{X}}\tilde{Y})\cdot\tilde{Z} = \pi(D_{\tilde{X}}\tilde{Y})\cdot\tilde{Z} = \nabla_X Y\cdot Z = g(\nabla_X Y, Z).$$

Likewise

$$\tilde{Y}\cdot(D_{\tilde{X}}\tilde{Z}) = g(Y, \nabla_X Z). \qquad \square$$

Definition 3.5 Let M be a smooth manifold, and let ∇ be a connection on TM.

(i) The *torsion* of ∇ is the map

$$T : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If $T=0$, ∇ is *torsion-free*.

(ii) Let g be a Riemannian metric on M. Then ∇ is *compatible with g* (or *a metric connection*) if $\forall\, X,Y,Z \in \Gamma(TM)$

$$X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Remark 3.6. $T \in \Gamma(T_{(1,2)}M)$ (via $T(X,Y,\xi) = \xi(T(X,Y))$),

i.e. $T(fX,Y) = fT(X,Y) = T(X,fY)$. (Cf Lemma 1.17.)

Moreover, $T(Y,X) = -T(X,Y)$. (Exercise.)

Theorem/Definition 3.7. Let (M,g) be a Riemannian manifold. Then $\exists!$ connection ∇ on TM which is torsion-free and compatible with g. It is called the *Levi-Civita connection* of (M,g).

It is characterized by the Koszul formula (writing $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$)

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\ & - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle . \end{aligned} \qquad \circledast$$

Thus, $\oplus$ above agrees with the Levi-Civita connection of (Σ, g).

Proof (i) The formula $\circledast$. Given a torsion-free metric connection ∇, we compute

$$\begin{aligned} & X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\ & = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ & = \langle X, [Y,Z]\rangle + \langle Y, [X,Z]\rangle + 2\langle \nabla_X Y, Z\rangle + \langle Z, [Y,X]\rangle , \end{aligned}$$

which is $\circledast$.

(ii) Uniqueness. Since the right hand side of $\circledast$ is defined solely in terms of the data (M,g), and since $\langle \cdot,\cdot \rangle$ is non-degenerate, $\circledast$ uniquely determines $\nabla_X Y$ $\forall X, Y \in \Gamma(TM)$.

(iii) Existence. (iii.1) We first show that, for fixed $X, Y \in \Gamma(TM)$, there exists a $W \in \Gamma(TM)$ s.t. $\forall Z \in \Gamma(TM)$,

$$\langle W, Z\rangle = \text{R.H.S. of } \circledast =: F(Z).$$

(Then $\nabla_X Y := W$; W is unique by (ii).)

· First, we note that $F(Z)$ is tensorial in Z: if $f \in C^\infty(M)$, then

$$\begin{aligned} F(fZ) = {} & X\langle Y, fZ\rangle + Y\langle X, fZ\rangle - fZ\langle X,Y\rangle \\ & - \langle X, [Y, fZ]\rangle - \langle Y, [X, fZ]\rangle + \langle fZ, [X,Y]\rangle \end{aligned}$$

$\langle Y, fZ\rangle = f\langle Y,Z\rangle$

$[Y, fZ] = f[Y,Z] + (Yf)Z$

$$= f\,\mathcal{F}(Z) + (Xf)\langle Y,Z\rangle + (Yf)\langle X,Z\rangle$$
$$- \langle X, (Yf)Z\rangle - \langle Y, (Xf)Z\rangle$$
$$= f\,\mathcal{F}(Z).$$

- Then: let (φ, U) be a chart, and fix $V_1, \ldots, V_m \in \Gamma(TU)$ s.t. $\{V_i(p)\} \subset T_pM$ is an orthonormal basis $\forall p \in U$.

  Set $W_U := \sum_{i=1}^{m} \mathcal{F}(V_i)V_i \in \Gamma(TU)$.

  Then $\forall Z \in \Gamma(TU)$ with $\operatorname{supp} Z \subset U$, write $Z = f^i V_i$, $f^i \in C_c^\infty(U)$, and note that

$$\langle W_U, Z\rangle = \mathcal{F}(V_i)f^i = \mathcal{F}(f^i V_i) = \mathcal{F}(Z).$$

- Finally, $W_U = W_V$ on $U \cap V$ by uniqueness; thus, $\exists\, W \in \Gamma(TM)$ s.t. $W|_U = W_U$ $\forall$ charts U; and $\langle W, Z\rangle = \mathcal{F}(Z)$ $\forall Z \in \Gamma(TM)$ follows from a partition of unity argument.

(iii.2) Finally, we need to check that $\nabla_X Y$ is a torsion-free metric connection; this follows from calculations similar to those in (iii.1) above. (Exercise.) $\square$

- In local coordinates $x^1, \ldots, x^m$: Christoffel symbols Γ_{ij}^k = connection coefficients, $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$.

Lemma 3.8 $\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) = \Gamma_{ji}^k$.

Proof Koszul formula: $[\partial_i, \partial_j] = 0$ etc., so

$$2\langle \nabla_{\partial_i}\partial_j, \partial_q\rangle = 2\Gamma_{ij}^k g_{kq}$$

$$= \partial_i \underbrace{\langle \partial_j, \partial_q\rangle}_{=g_{jq}} + \partial_j \underbrace{\langle \partial_i, \partial_q\rangle}_{=g_{iq}} - \partial_q \underbrace{\langle \partial_i, \partial_j\rangle}_{=g_{ij}}. \qquad \square$$

- As before, for $V = V^i\partial_i$, $W = W^i\partial_i$,

$$\nabla_V W = (V^i\partial_i W^k + \Gamma_{ij}^k V^i W^j)\partial_k$$

depends at $p \in M$ only on $W|_{\mathrm{im}(\gamma)}$ for any $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, $\gamma'(0) = V$.

- Towards differentiating vector fields along curves (cf. the initial motivation), we now introduce:

Definition 3.8 Let $F : N \xrightarrow{C^\infty} M$, where N, M are smooth manifolds. Then a (smooth) *vector field along F* is a (smooth) map $W : N \to TM$ s.t. $W(p) \in T_{F(p)}M$ $\forall p \in N$.

Example 3.9. $\gamma : [0,1] \xrightarrow{C^\infty} M$. Then $(-1,1) \ni t \mapsto \gamma'(t) \in T_{\gamma(t)}M$ is a smooth vector field along γ.

Definition 3.10. Let $W : N \to TM$ be a vector field along $F : N \to M$, with (M,g) Riemannian. Let $V \in \Gamma(TN)$. Then $\nabla_V W$ is the vector field along F defined as follows: for $p \in N$, coordinates $x^1, \dots, x^m$ around $F(p) \in M$, write $W(q) = \sum_{i=1}^m W^i(q)\,\partial_i|_{F(q)} \in T_{F(q)}M$ $(q \in N)$, then

$$(\nabla_V W)(p) := \left(V_p(W^k) + d_pF^i(V)\,W^j(p)\,\Gamma_{ij}^k(F(p))\right)\partial_k|_{F(p)} \in T_{F(p)}M. \quad \oplus$$

Example 3.11. For $\gamma : [0,1] \to M$, $\nabla_{\partial_t} \gamma'(t) = \nabla_{\gamma'(t)} \gamma'(t) =: \gamma''(t)$ (acceleration of γ).

Lemma 3.12. The definition $\#$ of $(\nabla_V W)(p)$ is independent of the choice of coordinates around $F(p)$.

Proof. If $V_p = 0$, $(\nabla_V W)(p) = 0$. So assume $V_p \neq 0$.

Case 1. $d_pF(V) = 0$. Consider another coordinate system $y = y(x)$. Then

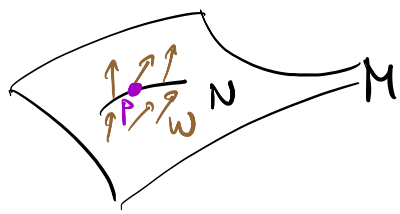
$$W^i(q) \frac{\partial}{\partial x^i}\Big|_{F(q)} = W^i(q) \frac{\partial y^j}{\partial x^i}(F(q)) \frac{\partial}{\partial y^j}\Big|_{F(q)} =: \tilde{W}^j(q) \frac{\partial}{\partial y^j}\Big|_{F(q)}$$

we have

$$V_p(\tilde{W}^j) \frac{\partial}{\partial y^j}\Big|_{F(p)} = V_p(W^i) \frac{\partial y^j}{\partial x^i}(F(p)) \frac{\partial}{\partial y^j}\Big|_{F(p)}$$
$$+ W^i(p) \underbrace{d_pF(V)\left(\frac{\partial y^j}{\partial x^i}\right)}_{=0} \frac{\partial}{\partial y^j}\Big|_{F(p)}$$
$$= V_p(W^i) \frac{\partial}{\partial x^i}\Big|_{F(p)}.$$

(The idea is that W, on a curve segment through p with direction V, is "essentially" valued in the fixed vector space $T_{F(p)}M$; and $\nabla_V W$ is the derivative of this vector-valued function.)

Case 2. $d_pF(V) \neq 0$. We may then replace N by an embedded curve $N \subset M$, with $F : N \hookrightarrow M$ the inclusion map and $V = d_pF(V) \in T_pM$. Since only $W|_N$ enters in $\#$, we may replace W by a C^∞ vector field defined near p. Then $\#$ is the local coordinate expression for $(\nabla_V W)(p)$.



$\square$

Lemma 3.13. $F: N \to (M,g)$ C^∞, V, W vector fields along F, $Z \in T(TN)$

$$\Rightarrow \quad \underbrace{Z\langle V, W\rangle}_{\text{at } p\in N:\ Z_p f,\ f(q) = g_{F(q)}(V_q, W_q)} = \langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle$$

Proof. Exercise. $\square$

Remark 3.14. Definition 3.10 and Lemma 3.12 can be generalized.

(i) Given $F: N \to M$ and a vector bundle $E \xrightarrow{\pi} M$, one can define the pullback bundle $F^*E \to N$ by

$$F^*E = \{(n,e) \in N \times E : F(n) = \pi(e)\}$$

(so $(F^*E)_p = E_{F(p)}$). Example: a vector field along F is the same as a smooth section of $F^*(TM) \to N$.

(ii) Given a connection ∇ on E, there exists a unique connection $F^*\nabla$ on F^*E (called the pullback connection) for which, moreover,

$$(F^*\nabla)_V (F^*s) = F^*(\nabla_{F_*V} s) \quad \forall\, s \in T(E),\ V \in T(TN),$$

(Here $(F^*s)(n) = s(F(n))$) (The Leibniz rule for $F^*\nabla$ can be used to show that there exists at most one such connection — since every section of F^*E is, locally on N, a finite sum of products of $C^\infty(N)$ functions and pullback sections F^*s, $s \in T(E)$. The existence of $F^*\nabla$ can be proved analogously to Lemma 3.12 above.)

3.2. Parallel transport

A connection ∇ on a vector bundle $E \xrightarrow{\pi} M$ "connects" different fibers of E.

Proposition 3.15. Given a smooth curve $\gamma : [a,b] \to M$ and $e_0 \in E_{\gamma(a)}$, there exists a unique section $e : [a,b] \to E$ along γ (i.e. $e(t) \in E_{\gamma(t)}$) s.t.

$$e(a) = e_0, \quad \nabla_{\gamma'} e = 0, \qquad (\circledast)$$

called the parallel transport of e_0 along γ.

Proof In local coordinates and trivializations, write $e(t) = e^\alpha(t) s_\alpha$.
We have $\nabla_{\partial_t} s_\alpha = \partial_t \gamma^j(t)\, \Gamma^\beta_{\alpha j}(\gamma(t))\, s_\beta$. We thus need to solve

$$\begin{cases} \partial_t e^\alpha(t) + e^\beta(t)\, \partial_t \gamma^j(t)\, \Gamma^\alpha_{\beta j}(\gamma(t)) = 0 \\ e^\alpha(a) = e^\alpha_0(a). \end{cases}$$

This is a linear ODE with a unique smooth solution. To construct the global solution of $\circledast$ along γ, use the compactness of $[a,b]$. $\square$

Remark 3.16. Parallel transport $E_{\gamma(a)} \to E_{\gamma(b)}$ depends on γ! The dependence is quantified by the curvature of ∇ (for $E = TM$: curvature of (M,g)).

Lemma 3.17. Let (M,g) be a Riemannian manifold, $\gamma : [a,b] \xrightarrow{C^\infty} M$, $V_0, W_0 \in T_{\gamma(a)}M$, and V, W the parallel transports of V_0, W_0 along γ. Then $\langle V(t), W(t) \rangle = \text{const}$.

Proof $\partial_t \langle V(t), W(t) \rangle = \langle \nabla_{\partial_t} V, W \rangle + \langle V, \nabla_{\partial_t} W \rangle = 0.$ $\square$

For $V = W$, we conclude that parallel transport preserves lengths.

3.3. First variation of length; geodesics.

We can now carry out the plan from the beginning of this section. Let $\gamma : (-1,1)_s \times [0,1]_t \xrightarrow{C^\infty} (M,g)$, $\gamma(t) := \gamma(0,t)$, with $|\partial_t \gamma(t)| = \text{const.} = \lambda$.

Lemma 3.18. Define the variation vector field by

$$V(t) := \partial_s \gamma(0,t) \in T_{\gamma(t)} M.$$

(This is thus a vector field along γ.) Then

$$\frac{d}{ds} L(\gamma(s,\cdot))\Big|_{s=0} = \frac{1}{\lambda}\left(\langle V, \gamma' \rangle \Big|_0^1 - \int_0^1 \langle V(t), \gamma''(t) \rangle \, dt \right).$$

Proof $\displaystyle \frac{d}{ds} L(\gamma(s,\cdot))\Big|_{s=0} = \frac{1}{2\lambda} \int_0^1 \partial_s \left(g_{\gamma(s,t)}(\partial_t \gamma(s,t), \partial_t \gamma(s,t)) \right)\Big|_{s=0} dt$

$$= \frac{1}{\lambda} \int_0^1 \langle \nabla_{\partial_s} \partial_t \gamma, \partial_t \gamma \rangle \Big|_{s=0} dt.$$

In local coordinates on M, $\partial_t \gamma = (\partial_t \gamma^i) \partial_i$ and (by Def. 3.10)

$$\nabla_{\partial_s} \partial_t \gamma = \left((\partial_s \partial_t \gamma^k) + (\partial_s \gamma^j)(\partial_t \gamma^i) \Gamma_{ij}^k(\gamma(s,t)) \right) \partial_k$$

$$= \nabla_{\partial_t} \partial_s \gamma.$$

$$\Rightarrow \frac{d}{ds} L(\gamma(s,\cdot))\Big|_{s=0} = \frac{1}{\lambda} \int_0^1 \langle \nabla_{\partial_t} \partial_s \gamma, \partial_t \gamma \rangle \Big|_{s=0} dt$$

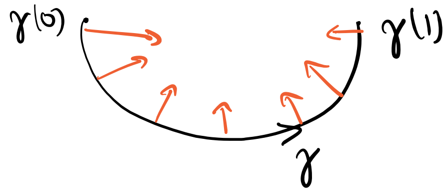
$$= \frac{1}{\lambda} \int_0^1 \langle \nabla_{\partial_t} V, \gamma' \rangle \, dt$$

$$= \frac{1}{\lambda} \int_0^1 \partial_t \langle V, \gamma' \rangle - \langle V, \nabla_{\partial_t} \gamma' \rangle \, dt$$

$$= \frac{1}{\lambda} \left(\langle V, \gamma' \rangle \Big|_0^1 - \int_0^1 \langle V, \gamma'' \rangle dt \right). \qquad \square$$

Interpretation. First term: curve gets longer if you move $\gamma(1)$ in the direction of $\gamma'(1)$ (or $\gamma(0)$ in the direction of $-\gamma'(0)$).

Second term: shifting γ towards $-\gamma''$ makes it shorter.



- For γ to be a stationary point under all endpoint-preserving variations (i.e. $\gamma(s,0)=\gamma(0)$, $\gamma(s,1)=\gamma(1)$ $\forall s$), we need $\int_0^1 \langle V, \gamma'' \rangle \, dt = 0$ $\forall$ variation vector fields V along γ with $V(0)=0$, $V(1)=0$.

Exercise. $\forall$ vector fields V along γ, $V(0)=0$, $V(1)=0$,
$\exists$ $\gamma : (-1,1) \times [0,1] \longrightarrow M$, $\gamma(0,t)=\gamma(t)$,
with fixed endpoints s.t. $V=$ variation vector field of γ.

<u>Definition 3.19.</u> A curve $\gamma : I \subseteq \mathbb{R} \to (M,g)$ (I connected) is a geodesic if $\gamma''(t) = \nabla_{\partial_t} \gamma'(t) = \nabla_{\gamma'(t)} \gamma'(t) = 0$ $\forall t \in I$.
(In other words, γ' is parallel along γ.)

- In local coordinates $x^1, \dots, x^m$ on M, $\gamma(t) = (\gamma^1(t), \dots, \gamma^m(t))$, and $\nabla_{\gamma'} \gamma' = \left(\partial_t^2 \gamma^k(t) + \partial_t \gamma^i(t) \, \partial_t \gamma^j(t) \, \Gamma_{ij}^k(\gamma(t)) \right) \partial_k = 0$ is equivalent to the coupled system of nonlinear ODEs

$$\ddot{\gamma}^k + \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = 0. \qquad (\dot{\ } := \partial_t).$$

Given initial data $\gamma^k(0)$, $\dot{\gamma}^k(0)$, there exists a unique local solution.

<u>Example 3.20.</u> (i) ($\mathbb{R}^m$, $g=$ Eucl.), standard coordinates $x^1, \dots, x^m \Rightarrow \Gamma_{ij}^k = 0$, so geodesic equation is $\ddot{\gamma}^k = 0$; solutions are straight lines.
(ii) (S^m, standard metric) $\Rightarrow$ geodesics are great circles (intersections of 2-dim. planes $\subset \mathbb{R}^{m+1}$ with S^m). (Exercise.)