8. Cartan–Hadamard manifolds

We now turn from constant curvature manifolds to those with nonpositive but possibly variable curvature.

Theorem 8.1. (Hadamard 1898, Cartan 1928) Let (M,g) be a connected complete Riemannian manifold with sectional curvature $\sec \leq 0$.

(i) Let $p \in M$. Then $\exp_p : T_pM \to M$ is a covering map.

(ii) If M is simply connected, then $\exp_p$ is a diffeomorphism.

The proof uses the following property of Jacobi fields when $\sec \leq 0$:

Lemma 8.2. $\sec \leq 0$, $\gamma : [0,\ell] \to M$ geodesic, J Jacobi field along γ $\Rightarrow$ $f(t) := |J(t)|^2_{\gamma(t)}$ is convex.

Proof $f' = 2\langle J, J'\rangle$

$$\Rightarrow f'' = 2\langle J', J'\rangle + 2\langle J, J''\rangle = 2|J'|^2 - \overbrace{2\langle R(J,\gamma')\gamma', J\rangle}^{\leq 0} \geq 0 \quad \square$$

Proof of Theorem 8.1.

(i) · Since $\sec \leq 0$, geodesics have no conjugate points. (Indeed, if $J(t)$ is a Jacobi field with $J(0)=0$ and $J(\ell)=0$, then $J \equiv 0$ by convexity of $|J(t)|^2 \geq 0$.)

· By Lemma 6.7, $\exp_p$ is a local diffeomorphism.

· Set $g_* := \exp_p^*(g)$ = Riemannian metric on T_pM.

The curves $t \mapsto tv$ $(v \in T_pM)$ are geodesics in (T_pM, g_*) since their images under $\exp_p$ are.

$\Rightarrow (T_pM, g_*)$ is complete by Hopf-Rinow.

- $\exp_p : (T_pM, g_*) \to (M,g)$ is a local isometry. The claim now follows from Proposition 7.13.

(ii) · Since T_pM is simply connected, $\pi_1(M) \cong$ group of deck transformations of $\exp_p$. If M is simply connected, there are no nontrivial deck transformations $\Rightarrow \exp_p$ is injective (cf. the proof of Proposition 7.10).

- An injective C^∞ covering map is a diffeomorphism. $\square$

Definition 8.3. A complete and simply connected Riemannian manifold with $\sec \leq 0$ is called a (Cartan-) Hadamard manifold.

Example 8.4. (i) Hyperbolic spaces $\mathbb{H}^m_\kappa$.

(ii) 2-dimensional (M,g): let $p \in M$, fix an isometry $(T_pM, g_p) \cong (\mathbb{R}^2, \langle \cdot,\cdot \rangle_{eucl.})$, normal coordinates x^1, x^2, written as

$$\begin{cases} x^1 = r\cos\theta \\ x^2 = r\sin\theta. \end{cases} \qquad (r>0,\ \theta \in [\theta_0, \theta_0 + 2\pi) \text{ are polar normal coordinates})$$

Gauss lemma $\Rightarrow g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1,\ g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0,\ g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) =: E(r,\theta).$

$$\Rightarrow (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}.$$

Have $K = -\dfrac{\partial_r^2 \sqrt{E}}{\sqrt{E}}$, so (M,g) is Hadamard iff $\partial_r^2 \sqrt{E} \geq 0$.

(Smoothness of g at $r=0$ imposes further restrictions on E.)

For instance: • $E(r,\theta) = r^2 \Rightarrow$ Euclidean plane $(K=0)$.

• $E(r,\theta) = \sinh^2 r \Rightarrow$ hyperbolic plane $(K=-1)$.

• $E(r,\theta) = \begin{cases} r^2, & r \le r_0 \\ (\text{convex})^2, & r > r_0 \end{cases}$

(iii) (M,g) = universal cover of a non-positively curved complete manifold.

We shall study isometries of Hadamard manifolds (M,g).

Definition 8.5. Let $\gamma \in \mathrm{Iso}(M,g)$.

(i) We define

• the displacement function $d_\gamma(p) = d(p, \gamma(p))$;

• $|\gamma| := \inf_{p \in M} d_\gamma(p)$;

• $\mathrm{Min}(\gamma) := \{ p \in M : d_\gamma(p) = |\gamma| \}$.

(ii) We call γ

• parabolic if $\mathrm{Min}(\gamma) = \emptyset$;

• elliptic if $\mathrm{Min}(\gamma) \neq \emptyset$, $|\gamma| = 0$ (i.e. γ has a fixed point);

• hyperbolic if $\mathrm{Min}(\gamma) \neq \emptyset$, $|\gamma| > 0$.

(iii) Let $c : \mathbb{R} \to M$ be a unit speed geodesic. Then c is an axis of γ if $\exists\, a > 0$ s.t. $\gamma(c(t)) = c(t+a)$ $\forall\, t \in \mathbb{R}$.

If γ possesses an axis, it is called axial.

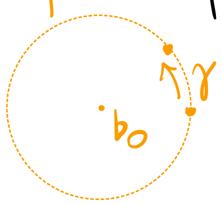
Remark 8.6. $\mathrm{Min}(\gamma)$ is closed, convex, and γ-invariant. (Exercise.)

Example 8.7. (i) $(M,g) = (\mathbb{R}^m, g_{eucl})$. Every $\gamma \in \mathrm{Iso}(\mathbb{R}^m, g_{eucl})$ is of the form $\gamma(x) = Ax + b$ $\quad (b = \gamma(0) \in \mathbb{R}^m,\ A \in O(m))$.

$\Rightarrow$ $d_\gamma(x) = |(A-I)x + b|$, which attains its minimum $(= \mathrm{dist}(\mathrm{ran}(A-I), -b))$.

$\Rightarrow$ γ is elliptic or hyperbolic.

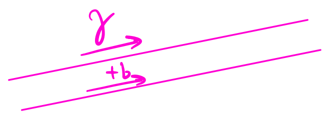
- Elliptic examples: $b = 0$, $A \in SO(n)$ (rotation around 0)



$\gamma(x) = A(x - b_0) + b_0$, i.e. $b = b_0 - Ab_0$

(rotation around $b_0 \in \mathbb{R}^m$)

- Hyperbolic examples: $b \neq 0$, $A = I$ (translation by $b \in \mathbb{R}^m$)



This is an axial isometry: every line parallel to b is an axis. $|\gamma| = b$.

(ii) $(M,g) = (\mathbb{H}^2, g_{std})$.

- Elliptic isometry: Poincaré disc model $\left(B_1(0), \frac{dx^2}{4(1-|x|^2)^2}\right)$,

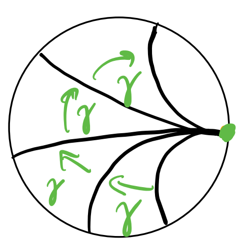
γ = rotation around 0. $\quad \mathrm{Min}(\gamma) = \{0\}$.

- Parabolic isometry: $\mathbb{H}^2 = (0,\infty)_x \times \mathbb{R}_y$, $g = \frac{dx^2 + dy^2}{x^2}$,

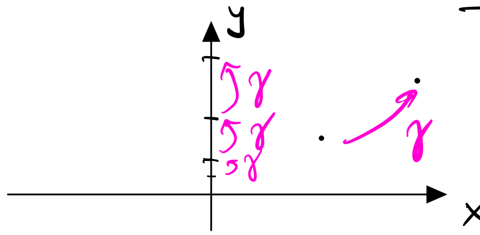
$$\gamma(x,y) = (x+1, y).$$

$(\Rightarrow |\gamma| = 0$, but $d_\gamma(x,y) > 0 \ \forall (x,y) \in \mathbb{H}^2)$.

In Poincaré disc model: "rotation around a boundary point"



· Hyperbolic isometry: $\mathbb{H}^2 = (0,\infty)_x \times \mathbb{R}_y$, $\gamma(x,y) = (2x,2y)$.



Translation along $\underbrace{y\text{-axis}}_{\text{a geodesic!}}$.

This is thus an axial isometry.

Fact: $\mathrm{Min}(\gamma) = y$-axis.

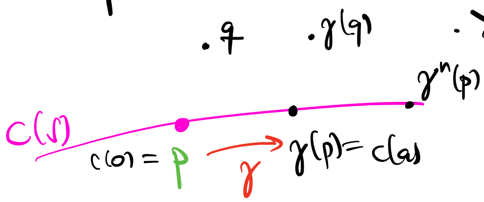
The hyperbolic examples illustrate the following result:

Proposition 8.8. Let (M,g) be a Hadamard manifold, and let $\gamma \in \mathrm{Iso}(M,g)$.

(i) If γ has an axis c, with shift $a > 0$ (i.e. $\gamma(c(s)) = c(s+a)$), then $a = |\gamma|$ (thus γ is hyperbolic) and $c(\mathbb{R}) \subset \mathrm{Min}(\gamma)$.

(ii) If γ is hyperbolic, then $\forall\, p \in \mathrm{Min}(\gamma)\ \exists$ axis c through p.

Proof. (i) Let $q \in M$ be an arbitrary point. Let $p = c(0)$.



(Idea: $d_\gamma(q)$ cannot be $< a$, since otherwise we could get from p to $\gamma^n(p)$ via $q, \gamma(q), \dots$ faster than along the minimizing geodesic c.)

Then for $n \in \mathbb{N}$

$$na = d(p,\gamma^n(p)) \leq d(p,q) + d(q,\gamma(q)) + \dots + d(\gamma^{n-1}(q),\gamma^n(q)) + d(\gamma^n(q),\gamma^n(p))$$

$$\leq 2\,d(p,q) + n\,d_\gamma(q)$$

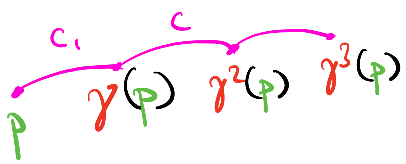
$$\Rightarrow d_\gamma(q) \geq a - \frac{2}{n} d(p,q) \xrightarrow{n\nearrow\infty} a$$

$$\Rightarrow |\gamma| \geq a = d_\gamma(p) \geq |\gamma|$$

$$\Rightarrow |\gamma| = a \Rightarrow \gamma \text{ is hyperbolic, } c(\mathbb{R}) \subset \mathrm{Min}(\gamma).$$

(ii) Let $a := |\gamma|$ and pick $p \in \mathrm{Min}(\gamma)$. Let $c_0 : [0,a] \to M$ be the unit speed geodesic from p to $\gamma(p)$. Define

$$c(na+t) := \gamma^n(c_0(t)) \qquad (n \in \mathbb{Z},\ t \in [0,a)).$$



(Thus $\gamma(c(t)) = c(t+a)$ $\forall t \in \mathbb{R}$.)

$\Rightarrow$ For all $t \in \mathbb{R}$,

$$a = L(c|_{[t,t+a]}) \geq d(c(t), c(t+a)) = d(c(t), \gamma(c(t))) \geq |\gamma| = a.$$

c is parameterized by arc length

$\Rightarrow$ $L(c|_{[t,t+a]}) = d(c(t), c(t+a))$ $\forall t \in \mathbb{R}$.

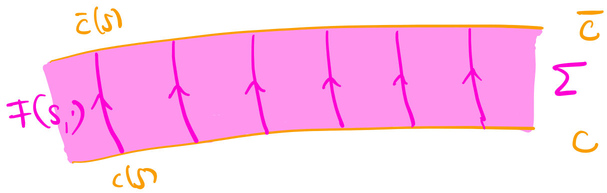
By Theorem 4.11, c is therefore a geodesic ($\Rightarrow$ no kinks in the figure above). $\Rightarrow$ c is an axis through $p = c(0)$. $\square$

In the case of hyperbolic space, there was only one axis (unlike in the Euclidean setting). This is a consequence of the following result.

Proposition 8.9. (Flat strip) Let (M,g) be a Hadamard manifold, and let $c, \bar{c} : \mathbb{R} \to M$ be two unit speed geodesics with $c(\mathbb{R}) \neq \bar{c}(\mathbb{R})$ and $\sup_{s \in \mathbb{R}} d(c(s), \bar{c}(s)) < \infty$. For $s \in \mathbb{R}$, let $v(s) \in T_{c(s)}M$ be such that $\bar{c}(s) = \exp_{c(s)}(v(s))$, and set

$$F(s,t) = \exp_{c(s)}(t\, v(s)), \qquad (s,t) \in \mathbb{R} \times [0,1].$$

Set $\Sigma := F(\mathbb{R} \times (0,1))$. Then $(\Sigma, g|_\Sigma)$ is flat (i.e. its curvature tensor vanishes) and Σ is totally geodesic (i.e. its 2^{nd} fundamental form vanishes), and $K(\Pi)=0$ $\forall$ planes $\Pi \subset T\Sigma$.



<u>Lemma 8.10.</u> Let (M,g) be a Hadamard manifold. Then $\forall$ pairs of geodesics $c, \bar{c} : \mathbb{R} \to M$, the function $h(s) := d(c(s), \bar{c}(s))$ is convex.

<u>Proof.</u> For every $s \in \mathbb{R}$, there exists a unique geodesic $\gamma_s(t) = \exp_{c(s)}(t\, v(s)),\ 0 \le t \le 1$, from $c(s)$ to $\bar{c}(s)$, and $L(\gamma_s) = h(s)$.

· Now, the map

$$\exp : (c,v) \in TM \to (c, \exp_c(v)) \in M \times M \qquad (*)$$

is C^∞, bijective (using Theorem 8.1), and has injective differential at every point and thus is a local diffeomorphism.

$\Rightarrow$ $(*)$ is a diffeomorphism.

· $\Rightarrow$ $\tilde{\gamma}(s,t) := \gamma_s(t)$ is, for s near any $s_0 \in \mathbb{R}$, a C^∞ variation of γ_{s_0}, with variation vector field $\tilde{V}(s,t)$ satisfying

$$\tilde{V}(s,0) = c'(s),$$
$$\tilde{V}(s,1) = \bar{c}'(s),$$

and thus $\frac{D}{ds}\tilde{V}(s,t) = 0$ for $t = 0,1$.

· 2^{nd} variation of length, with $V := \tilde{V}(s_0,\cdot)$ (when $h(s_0) = |\gamma_{s_0}'| \neq 0$):

$$h''(s_0) = \underbrace{\frac{1}{|\gamma_s'|}}_{\text{constant prefactor — note that in the formula in Thm. 6.10, we assumed } |\gamma'|=1} \int_0^1 |V'(t)^\perp|^2 - \underbrace{\langle R(V,\gamma_s')\gamma_s', V\rangle}_{\leq 0} \, dt \geq 0. \qquad \square$$

Proof of Proposition 8.9 Since $d(c(s), \tilde{c}(s))$ is a convex bounded function on $\mathbb{R}$, it is constant.

Step 1: $\langle R(F_* \partial_s, F_* \partial_t) F_* \partial_t, F_* \partial_s\rangle = 0.$

This follows from the 2nd variation formula:

$$0 = \frac{d^2}{ds^2} L(F(s,\cdot))\Big|_{s=s_0} = \int_0^1 \underbrace{|(\tfrac{D}{dt}\partial_s F)^\perp|^2}_{\geq 0} - \underbrace{\langle R(\partial_s F, \partial_t F)\partial_t F, \partial_s F\rangle}_{\geq 0} dt$$

$$\Rightarrow \langle R(\partial_s F, \partial_t F)\partial_t F, \partial_s F\rangle = 0.$$

Step 2: Computation of the metric $F^* g$.

(2.1) $|\partial_t F(s,t)|^2 = |v(s)|^2 = d(c(s), \tilde{c}(s)) = \text{const.} =: P.$

(2.2) By the 1st variation formula,

$$0 = \frac{d}{ds} L(F(s,\cdot)) = \langle \partial_t F(s,t), \underbrace{\partial_s F(s,t)}_{\text{variation vector field}}\rangle \Big|_0^1 . \quad \oplus$$

But since $\partial_s F$ is a Jacobi field along $F(s,\cdot)$, the function $t \mapsto \langle \partial_t F(s,t), \partial_s F(s,t)\rangle$ is affine. By $\oplus$, it is constant in t, so

$$\langle \partial_t F(s,t), \partial_s F(s,t)\rangle = Q(s).$$

(2.3) We again use that $J_s(t) := \partial_s F(s,t)$ is a Jacobi field, with $f_s(t) := |J_s(t)|^2$ being convex and satisfying

$$f_s(0) = |c'(s)|^2 = 1, \qquad (1)$$

$$f_s(1) = |\tilde{c}'(s)|^2 = 1, \qquad (2)$$

as well as

$$f_s'(t) = \partial_t \langle \partial_s F, \partial_s F\rangle \quad = 2\langle \tfrac{D}{dt}\partial_s F, \partial_s F\rangle$$

$$= 2\langle \tfrac{D}{ds}\partial_t F, \partial_s F\rangle$$

$$= 2\,\partial_s \langle \partial_t F, \partial_s F\rangle - 2\langle \partial_t F, \tfrac{D}{ds}\partial_s F\rangle,$$

so (since $\frac{D}{ds}\partial_s F(s,t) = 0$ for $t=0,1$).

$$f_s'(0) = 2Q'(s) = f_s'(1). \qquad (3)$$

Together, (1)–(3) and convexity imply that $f_s'(t) = 0 \ \forall s,t$

$\Rightarrow f_s(t) = 1 \quad \forall s,t,$

$\overset{(3)}{\Rightarrow} Q'(s) = 0 \quad \forall s.$

(2.4) In summary, $F^*g = \begin{pmatrix} 1 & Q \\ Q & P \end{pmatrix} = \begin{pmatrix} \langle \partial_s F, \partial_s F\rangle & \langle \partial_t F, \partial_s F\rangle \\ \langle \partial_t F, \partial_s F\rangle & \langle \partial_s F, \partial_s F\rangle \end{pmatrix}$

is a constant matrix $\Rightarrow$ flat.

<u>Step 3:</u> Σ is totally geodesic. Let ${}^{\Sigma}\nabla$ = Levi-Civita connection on $(\Sigma, g|_\Sigma)$.

(3.1) $\frac{D}{dt}\partial_t F = 0$ (since $F(s,\cdot)$ = geodesic)

$$= \underbrace{\frac{{}^{\Sigma}D}{dt}\partial_t F}_{=0} + k(\partial_t F, \partial_t F)$$

$\Rightarrow k(\partial_t F, \partial_t F) = 0.$

(3.2) $0 = \langle R(\partial_t F, \partial_s F)\partial_s F, \partial_t F\rangle$

Step 1

$$= \langle {}^{\Sigma}R(\partial_t, \partial_s)\partial_s, \partial_t\rangle + k(\partial_t F, \partial_s F)^2 - \underbrace{k(\partial_t F, \partial_t F)}_{=0} k(\partial_s F, \partial_s F)$$

Gauss equation

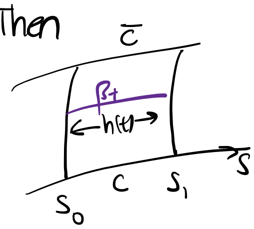
$$= k(\partial_t F, \partial_s F)$$

Σ is flat

$\Rightarrow k(\partial_t F, \partial_s F) = 0.$ ⊕

(3.3) We show $k(\partial_s F, \partial_s F) = 0$ by demonstrating that the curves $\beta_t : s \mapsto F(s,t)$ are geodesics in (M,g) for all $t \in [0,1]$.

Let $s_0 < s_1$ and $h(t) := d(F(s_0,t), F(s_1,t))$. Then



- h is convex (Lemma 8.10),
- $h(0) = s_1 - s_0 = h(1)$,
- $h'(0) = \langle \partial_s F(\cdot, 0), \partial_t F(\cdot, 0) \rangle \Big|_{s_0}^{s_1}$ by the 1st variation formula for the family of geodesics $t \mapsto$ (geodesic from $F(s_0,t)$ to $F(s_1,t)$) which is a variation of the geodesic $s \mapsto F(s,0)$ with variation vector field at $s = s_0, s_1$ given by $\partial_t F(s_{0/1}, 0)$.

  $\Rightarrow h'(0) = 0$ by Step 2.

$\Rightarrow h = \text{constant} = s_1 - s_0$

$\Rightarrow$ The curve $\beta_t : s \mapsto F(t,s)$, $s \in [s_0, s_1]$, has length $s_1 - s_0$ (since $\langle \partial_s F, \partial_s F \rangle = 1$) and connects $F(t,s_0)$, $F(t,s_1)$ — which have distance $s_1 - s_0$.

$\Rightarrow \beta_t$ is a geodesic by Theorem 4.11. □

Corollary 8.11. A hyperbolic isometry γ of a Hadamard manifold (M,g) with strictly negative curvature has exactly one axis.

Proof. • If γ has distinct axes $c, \bar c$ through $p = c(0)$, $\bar p = \bar c(0)$, then by Proposition 8.8, $\gamma(c(s)) = c(s+a)$ and $\gamma(\bar c(s)) = \bar c(s+a)$ for the same $a = |\gamma| > 0$.

$\Rightarrow d(\gamma^k p, \gamma^k \bar p) = d(p, \bar p)$ (since γ is an isometry)

- For $s \in \mathbb{R}$, write $s = na + t$, $n \in \mathbb{Z}$, $t \in [0,a]$, then

$$d(c(s), \bar{c}(s)) = d(c(t), \bar{c}(t)) \le \max_{t \in [0,a]} d(c(t), \bar{c}(t)),$$

so $\sup_{s \in \mathbb{R}} d(c(s), \bar{c}(s)) < \infty$.

- Proposition 8.9 applies and yields a contradiction to $\sec < 0$. $\square$

Theorem 8.12. (Preissmann, 1942). Let (M,g) be a compact connected Riemannian manifold with $\sec < 0$. Then every nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

As an example, $M = \mathbb{T}^m$ has $\pi_1(M) = \mathbb{Z}^m$, and therefore does not admit a metric with $\sec < 0$.

Proof of Theorem 8.12. · Let $\pi : \widetilde{M} \to M$ be the universal cover, and $\bar{g} := \pi^* g$. Then $(\widetilde{M}, \bar{g})$ is a Hadamard manifold. (Note: $(\widetilde{M}, \bar{g})$ is complete since M is.) $\Rightarrow \pi : (\widetilde{M}, \bar{g}) \to (M,g)$ is a Riemannian covering map.

$\Rightarrow$ There is an isometry $M \cong \widetilde{M}/\Gamma$ where

Γ = group of deck transformations $\cong \pi_1(M)$.

- Fix now $\gamma \in \Gamma \setminus \{Id\}$. By Corollary 8.14 below, γ is hyperbolic.
- Proposition 8.8 produces an axis $L_\gamma \subset M$ which by Corollary 8.10 is unique. Parameterize L_γ via $c : \mathbb{R} \to M$ (so $\gamma(c(s)) = c(s + |\gamma|)$).
- Let $\beta \in \Gamma \setminus \{Id\}$ be an isometry that commutes with γ. Then

$$\gamma(\beta(c(s))) = \beta(\gamma(c(s))) = \beta(c(s+l_\gamma))$$

$\Rightarrow$ $s \mapsto \beta(c(s))$ is the parametrization of another axis $\beta(L_\gamma)$ of γ

$\Rightarrow$ $\beta(L_\gamma) = L_\gamma$.

Similarly, $\gamma(L_\beta) = L_\beta$, and so $L_\beta = L_\gamma$.

- Let $G < \Gamma$ be a nontrivial abelian subgroup, and set $L = L_\gamma$ for any $\gamma \in G \setminus \{Id\}$. Then G acts by isometries on the line L. Since the action is free and properly discontinuous, $G \cong \mathbb{Z}$. (Exercise.) $\square$

To complete the argument, we need:

<u>Lemma 8.13.</u> Let $(\tilde{M}, \tilde{g})$ be a Hadamard manifold, and let $\Gamma < \mathrm{Iso}(\tilde{M}, \tilde{g})$ be a subgroup whose action is properly discontinuous and cocompact, i.e. $\exists$ compact set $\tilde{K} \subset \tilde{M}$ s.t. $\tilde{M} = \bigcup_{\gamma \in \Gamma} \gamma \tilde{K}$.

Then: (i) every $\gamma \in \Gamma$ is elliptic or hyperbolic.

(ii) If Γ acts freely, every $\gamma \in \Gamma \setminus \{Id\}$ is hyperbolic (thus axial)

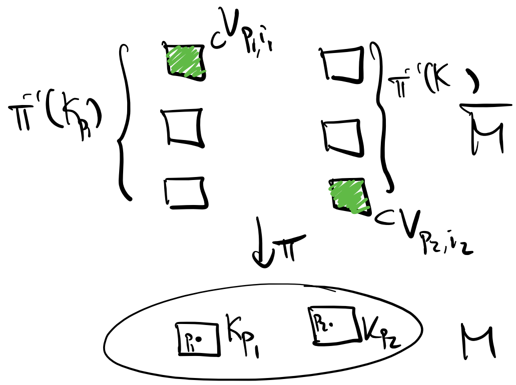
<u>Corollary 8.14.</u> If, in the notation of Lemma 8.13, Γ acts freely and $\tilde{M}/\Gamma$ is compact, then every $\gamma \in \Gamma \setminus \{Id\}$ is hyperbolic.

<u>Proof.</u> We only need to show that Γ is cocompact. Write $\pi: \tilde{M} \to \tilde{M}/\Gamma = M$.

Now, for all $p \in M$ $\exists$ $U_p \subset M$ open neighborhood s.t.

$$\pi^{-1}(U_p) = \bigsqcup_{i \in I_p} V_{p,i}, \qquad \pi|_{V_{p,i}}: V_{p,i} \to U_p \text{ diffeomorphism.}$$

Let $K_p \subset U_p$ be a compact neighborhood of K_p. Since M is



compact, we can select finitely many points $p_1,\dots,p_N \in M$ s.t. $M = \bigcup_{j=1}^{N} K_{p_j}$.

- For each $j=1,\dots,N$, fix $i_j \in I_{p_j}$. Set $\overline{K} := \bigcup_{j=1}^{N} \left(\pi|_{V_{p_j,i_j}}\right)^{-1}(K_{p_j})$.

- Given any $\overline{p} \in \overline{M}$, let j be such that $\pi(\overline{p}) \in K_{p_j}$, and let $\overline{p}' \in \overline{M}$ be such that $\pi(\overline{p}') = \pi(\overline{p})$ and $\overline{p}' \in V_{p_j,i_j}$. For $\gamma \in \Gamma$ with $\gamma\overline{p}' = \overline{p}$, we then have $\gamma^{-1}\overline{p} \in \overline{K} \Rightarrow \overline{p} \in \gamma\overline{K}$. $\square$

Proof of Lemma 8.13. Let $\gamma \in \Gamma$.

(i) • Let $\overline{p}_i \in \overline{M}$, $i \in \mathbb{N}$, be s.t. $d_\gamma(\overline{p}_i) \searrow |\gamma|$ as $i \to \infty$.

Let $\alpha_i \in \Gamma$, $\overline{q}_i := \alpha_i^{-1}\overline{p}_i \in \overline{K}$.

Set $\gamma_i = \alpha_i^{-1}\gamma\alpha_i$. Then $d_{\gamma_i}(\overline{q}_i) = d(\alpha_i^{-1}\overline{p}_i, \alpha_i^{-1}\gamma\overline{p}_i) = d(\overline{p}_i, \gamma\overline{p}_i)$

$= d_\gamma(\overline{p}_i) \searrow |\gamma|$.

Without loss of generality, $d_{\gamma_i}(\overline{q}_i) < |\gamma|+1 \ \forall i$.

- Since $\widetilde{K} := \{\overline{p} \in \overline{M} : d(\overline{p}, \overline{K}) \leq |\gamma|+1\}$ is compact, and Γ acts properly discontinuously, the set $\{\gamma_i\} \subset \Gamma$ is finite. (Note that $\gamma_i\widetilde{K} \cap \widetilde{K} \ni \gamma_i\overline{q}_i \ \forall i$.)

- Passing to a subsequence, we may assume $\gamma_i = \gamma_0 \in \Gamma \ \forall i$, and $\overline{q}_i \to \overline{q} \in \overline{K}$. Then

$$d_{\gamma_0}(\overline{q}) = d(\overline{q}, \gamma_0\overline{q}) = \lim_{i\to\infty} d(\overline{q}_i, \gamma_i\overline{q}_i) = \lim_{i\to\infty} d_{\gamma_i}(\overline{q}_i) = |\gamma|.$$

$\Longrightarrow \operatorname{Min}(\gamma) \ni \bar{q}$ is nonempty.

(ii) If $\gamma \in \Gamma$ has $|\gamma| = 0$, then $\exists\ \bar{q} \in \bar{M}$ s.t. $\gamma \bar{q} = \bar{q}$. If Γ acts freely, this forces $\gamma = \mathrm{Id}$. Thus $\gamma \neq \mathrm{Id}$ implies that γ is hyperbolic. $\square$