2.1. Riemannian manifolds as metric spaces

Definition 2.1. Let (M,g) be a Riemannian manifold. Let $\gamma:[a,b]\to M$ be a smooth curve. Then the length of γ is

$$L_g(\gamma)=L(\gamma):=\int_a^b |\gamma'(s)|_{g_{\gamma(s)}}\,ds=\int_a^b g_{\gamma(s)}(\gamma'(s),\gamma'(s))^{\frac{1}{2}}\,ds.$$

More generally, we define the length of a piecewise smooth curve $\gamma:[a,b]\to M$ — i.e. $\exists\, a=a_0<a_1<\cdots<a_k=b$ s.t. $\gamma|_{[a_i,a_{i+1}]}$ is C^∞, and γ is continuous at all a_i — to be $L(\gamma)=\sum_{i=0}^{k-1} L(\gamma|_{[a_i,a_{i+1}]})$.

Since the continuous function $[a,b]\ni s\mapsto g_{\gamma(s)}(\gamma'(s),\gamma'(s))$ is bounded, we have $L(\gamma)<\infty$.

Remark 2.2. If $\alpha:[\tilde a,\tilde b]\to[a,b]$ is a diffeomorphism, and $\tilde\gamma(\tilde s)=\gamma(\alpha(\tilde s))$ is the reparametrization of γ, then $L(\tilde\gamma)=L(\gamma)$.

Theorem 2.3. Let (M,g) be a connected Riemannian manifold. For $p,q\in M$, set

$$d(p,q):=\inf\{L(\gamma):\gamma:[0,1]\to M \text{ piecewise } C^\infty,\ \gamma(0)=p,\ \gamma(1)=q\}.$$

Then:

(i) (M,d) is a metric space.

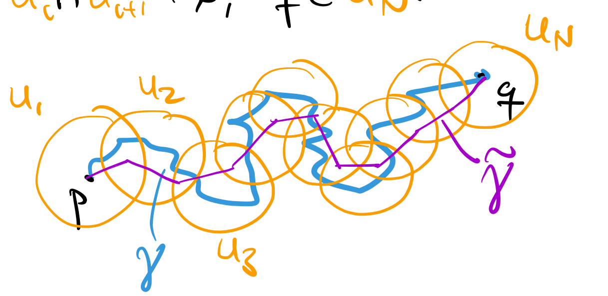
(ii) The topology τ_d induced by d is equal to the topology τ of (the topological manifold) M.

Proof (i) Claim 1: $d(p,q)<\infty$. Pf. Since M is connected and locally path-connected, it is path-connected. So $\exists\,\gamma:[0,1]\to M$ continuous with $\gamma(0)=p$, $\gamma(1)=q$. Cover the compact set $\gamma([0,1])$ with a finite number of charts (φ_i, U_i), $1 \le i \le N$, s.t. $\varphi_i(U_i) = B_1(0)$ (open unit ball in $\mathbb{R}^m$), $p \in U_1$, $U_i \cap U_{i+1} \neq \emptyset$, $q \in U_N$.

Set $p_0 := p$ and pick

$p_i \in U_i \cap U_{i+1}$ $(1 \le i \le N-1)$,

$p_N := q$.

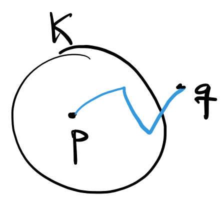


Define $\tilde{\gamma}$ as the piecewise C^∞ curve given by concatenating $\gamma_1, \dots, \gamma_N$ where $\gamma_i(s) = \varphi_i^{-1}\big(\varphi_i(p_{i-1}) + s(\varphi_i(p_i) - \varphi_i(p_{i-1}))\big)$, $0 \le s \le 1$, is the straight line from p_{i-1} to p_i in the chart φ_i.

Then $d(p,q) \le L(\tilde{\gamma}) = \sum_{i=1}^{N} L(\gamma_i(s)) < \infty$.

Claim 2. $d(p,q) \ge 0$, and "=" iff $p = q$. Pf. Suppose $p \neq q$.

- Let (φ, U) be a chart with $p \in U$, $\varphi(p) = 0$. Let $\varepsilon > 0$ be s.t. $\overline{B_\varepsilon(0)} \subset \varphi(U)$ and $\varphi(q) \notin B_\varepsilon(0)$; set $K = \varphi^{-1}(\overline{B_\varepsilon(0)})$ ($\subset U$, compact).



- Let $\gamma : [0,1] \to M$ be piecewise C^∞, $\gamma(0) = p$, $\gamma(1) = q$,

and set $T = \sup\{t \in [0,1] : \gamma([0,t]) \subset K\}$. Then $\gamma([0,T]) \subset K$ since $\underbrace{K \text{ is closed}}_{\text{(using: } M \text{ is Hausdorff!)}}$, and $\gamma(T) \notin \varphi^{-1}(B_\varepsilon(0))$; so $\gamma(T) \in \varphi^{-1}(\partial B_\varepsilon(0))$.

$\Rightarrow$ $\bar{\gamma}(t) := \varphi(\gamma(t))$, $0 \le t \le T$, is a piecewise C^∞ curve with

$$\bar{\gamma}(0) = 0, \quad |\bar{\gamma}(t)|_{eucl} \le \varepsilon, \quad |\bar{\gamma}(T)|_{eucl} = \varepsilon.$$

$$\Rightarrow L_{eucl}(\bar{\gamma}) = \int_0^T |\bar{\gamma}'(t)|\, dt \ge \int_0^T \bar{\gamma}'(t) \cdot \frac{\bar{\gamma}(T) - \bar{\gamma}(0)}{|\bar{\gamma}(T) - \bar{\gamma}(0)|_{eucl}}\, dt$$

$$= |\bar{\gamma}(T) - \bar{\gamma}(0)|_{eucl} = \varepsilon$$

Set $e := \varphi^*(\text{eucl. metric})$, which is a Riemannian metric on U.

Since K is compact, $\exists\ \lambda > 0$ s.t. $\forall v \in T_pM$, $p \in K$,

$$g_p(v,v) \geq \lambda^2\, e_p(v,v).$$

$$\Rightarrow\ L_g(\gamma) \geq L_g(\gamma|_{[0,T]}) \geq \lambda\, L_e(\gamma|_{[0,T]}) = \lambda\, L_{eucl}(\bar{\gamma}) = \lambda\varepsilon > 0.$$

<u>Claim 3.</u> $d(p,q) \leq d(p,r) + d(r,q)$. <u>Pf</u> Obvious

(ii) <u>$\tau \subseteq \tau_d$:</u> Given $p \in M$, consider a chart (φ, U) with $\varphi(p) = 0$, and $\varepsilon > 0$ s.t. $\overline{B_\varepsilon(0)} \subset \varphi(U)$. The above argument shows that if $\delta < \lambda\varepsilon$, then $q \in M$, $d(p,q) < \delta$ implies $q \in \varphi^{-1}(B_\varepsilon(0))$.

$\Rightarrow$ $B(p,\delta) = \{q \in M : d(p,q) < \delta\} \subset \varphi^{-1}(B_\varepsilon(0))$.

<u>$\tau_d \subseteq \tau$:</u> Conversely, every line segment from 0 to $\varphi(q) \in B_\varepsilon(0)$ has g-length $< \Lambda\varepsilon$ for some $\Lambda < \infty$ (using now $g \leq \Lambda^2 e$ on K).

$\Rightarrow$ $\varphi^{-1}(B_\varepsilon(0)) \subset B(p, \Lambda\varepsilon)$. $\square$

2.2. Integration and volume

In local coordinates $x=(x^1,\dots,x^m)$ on (M,g), we have

$$g = g_{ij}(x)\, dx^i \otimes dx^j .$$

In coordinates $x=\alpha(y)=(\alpha^1(y),\dots,\alpha^m(y))$,

$$\begin{aligned}(\alpha^* g)_y\left(\tfrac{\partial}{\partial y^i},\tfrac{\partial}{\partial y^j}\right) &= g_{\alpha(y)}\left(d_y\alpha\left(\tfrac{\partial}{\partial y^i}\right), d_y\alpha\left(\tfrac{\partial}{\partial y^j}\right)\right) \\ &= \frac{\partial \alpha^k(y)}{\partial y^i}\, g_{kl}(\alpha(y))\, \frac{\partial \alpha^l(y)}{\partial y^j},\end{aligned}$$

so $g = h_{ij}(y)\, dy^i \otimes dy^j$ where $(h_{ij}(y))_{i,j} = \left(\frac{\partial \alpha}{\partial y}\right)^T (g_{ij})_{ij} \left(\frac{\partial \alpha}{\partial y}\right)$.

Lemma 2.4. Let (φ, U), (ψ, V) be two charts and let $f \in C_c^\infty(U\cap V)$.

Let $g_{ij}(x) = g_{\varphi^{-1}(x)}\left(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right)$, $h_{ij}(y) = g_{\psi^{-1}(y)}\left(\frac{\partial}{\partial \psi^i}, \frac{\partial}{\partial \psi^j}\right)$.

Then
$$\int_{\varphi(U\cap V)} (f\circ\varphi^{-1})(x)\sqrt{\det(g_{ij}(x))}\, dx = \int_{\psi(U\cap V)} (f\circ\psi^{-1})(y)\sqrt{\det(h_{ij}(y))}\, dy$$

Proof. Writing $x=\alpha(y)$, $\alpha = \varphi\circ\psi^{-1}$, the left hand side is

$$\int_{\psi(U\cap V)} (f\circ\psi^{-1})(y)\, \underbrace{\sqrt{\det(g_{ij}(\alpha(y)))}\, \left|\det \frac{\partial\alpha}{\partial y}(y)\right|}_{=\left(\det(h_{ij}(y))\right)^{\frac{1}{2}}}\, dy .$$

$\square$

Definition 2.5. For $f \in C_c^\infty(M)$, define

$$\int_M f\, d\mathrm{vol}_g := \sum_i \int_{\varphi_i(U_i)} (f\chi_i)(\varphi_i^{-1}(x)) \sqrt{\det g^{(i)}(x)}\, dx,$$

where:
- (φ_i, U_i) is a finite collection of charts with $\operatorname{supp} f \subset \bigcup U_i$;
- χ_i is a partition of unity subordinate to $\{U_i\}$, i.e. $\operatorname{supp}\chi_i$ is a compact subset of U_i, and $\sum \chi_i = 1$ on $\bigcup U_i$;

- $g^{(i)}(x) = \left(g^{(i)}_{jk}(x)\right)_{jk} = \left(g_{\varphi_i^{-1}(x)}\left(\frac{\partial}{\partial \varphi_i^j}\Big|_{\varphi_i^{-1}(x)}, \frac{\partial}{\partial \varphi_i^k}\Big|_{\varphi_i^{-1}(x)}\right)\right)_{jk}$.

Lemma 2.6. $\int_M f \, d\mathrm{vol}_g$ is independent of the choice of U_i, φ_i, χ_i.

Proof. Exercise. The most interesting aspect of this was already done in Lemma 2.4. $\square$

We can in particular measure the volume of compact (M,g):

$$\mathrm{vol}(M) := \int_M 1 \, d\mathrm{vol}_g.$$

Moreover, if N is a smooth manifold of dimension $n \leq m$,

$F: N \to M$ is an immersion,

$K \subset N$ is compact,

then we can also compute the n-dimensional volume $\mathrm{vol}_{F^*g}(K)$.

When F is an embedding, we write $\mathrm{vol}(F(K)) = \mathrm{vol}_{F^*g}(K)$.

Example 2.7. (i) $\mathrm{vol}(\mathbb{S}^1) = 2\pi$, $\mathrm{vol}(\mathbb{S}^2) = 4\pi$.

(ii) $\gamma: (-\varepsilon, 1+\varepsilon) \to (M,g)$ embedded curve

$\Rightarrow L(\gamma|_{[0,1]}) = \mathrm{vol}(\gamma([0,1]))$.

(iii) areas of embedded surfaces, etc.