4. Geodesics.

We shall now study geodesics ($\gamma : [a,b] \to M$ with $\nabla_{\gamma'}\gamma' = 0$) on a Riemannian manifold (M,g) in depth.

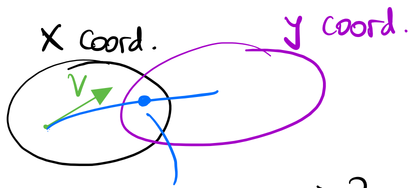
Proposition 4.1 (i) $\forall\, v \in TM$ $\exists !$ maximal geodesic

$$\gamma_v : (a_v, b_v) \to M, \quad \dot\gamma_v(0) = v \qquad (\text{and thus } \gamma_v(0) = \pi(v)).$$

(ii) The set $\mathcal{D} = \{(v,t) \in TM \times \mathbb{R} : t \in (a_v, b_v)\}$ is open, and the map $\mathcal{D} \ni (v,t) \mapsto \gamma_v(t) \in M$ is C^∞.

Proof: This follows from standard ODE theory; the only slightly delicate part is that geodesics need not be contained in a single chart.

Picture:



$\dot\gamma_v(t_0) = w^i \frac{\partial}{\partial x^i} = \tilde w^j \frac{\partial}{\partial y^j}$;

solve geodesic equation with initial condition $\tilde w^j \frac{\partial}{\partial y^j}$ in y-chart.

- An even better proof goes as follows. We define the geodesic vector field $V \in \Gamma(T(TM))$ as follows: in local coordinates $x^1, \dots, x^m$ on M, we get coordinates $v^1, \dots, v^m$ on the fibers of TM by trivializing TM (so $(x^1, \dots, x^m, v^1, \dots, v^m)$ is the vector $v^i \frac{\partial}{\partial x^i} \in T_x M$).

Then $V_{(x,v)} := v^i \frac{\partial}{\partial x^i} - \Gamma_{ij}^k(x)\, v^i v^j \frac{\partial}{\partial v^k} \in T_{(x,v)}(TM)$.

- This is well-defined (i.e. these local coordinate definitions agree on overlaps); indeed, given $(x,v) \in TM$, let γ_v = geodesic with initial direction v at x, and set

$$\tilde{\gamma}_v : (a_v, b_v) \to TM,$$
$$t \mapsto (\gamma_v(t), \tfrac{d}{dt}\gamma_v(t)) \in T_{\gamma_v(t)}M;$$

then $\frac{d}{dt}\tilde{\gamma}_v(t)\big|_{t=0} = V_{(x,v)}$.

– $\mathcal{D}$ is now the maximal domain for the flow of V. $\square$

Definition 4.2. Let $\mathcal{D}_1 = \{ v \in TM : (v,1) \in \mathcal{D}\}$. Then the map

$$\exp : \mathcal{D}_1 \ni v \mapsto \gamma_v(1) \in M$$

is called the exponential map. The map $\exp_p := \exp|_{\mathcal{D}_1 \cap T_pM}$ is the exponential map at p.

Remark 4.3 (i) For $0 \neq v \in T_pM$, $\gamma_v(1) = \gamma_{v/|v|}(|v|)$.

unit vector — time one travels along the geodesic

(ii) For $M = SO(n) \subset \mathbb{R}^{n\times n}$ with the induced metric, consider $p = I \in M$; then $T_pM = \{B \in \mathbb{R}^{n\times n} : B + B^T = 0\}$,

and $\exp_p(B) = \sum_{i=0}^{\infty} \frac{1}{i!} B^i$ (matrix exponential), (Exercise.)

(This generalizes to Lie groups with bi-invariant metrics.)

In local coordinates centered around $p \in M$, we have, for $v \in T_pM$,

$$\gamma_v(t) = tv + O(t^2) \text{ for small } t$$

(simply because γ_v is C^∞ and $\dot{\gamma}_v(0) = v$). This suggests relating T_pM near 0 and M near p:

Proposition 4.4. Let $p \in M$. Then there exists $\varepsilon > 0$ s.t.

$$\exp_p|_{B_\varepsilon(0)} : B_\varepsilon(0) \subset T_pM \to M$$

is a diffeomorphism onto its image.

Proof. We compute $d_0(\exp_p): T_0(T_pM) \longrightarrow T_pM$.

- Since T_pM is a vector space, $T_pM \cong T_0(T_pM)$ via
$$V \longmapsto \frac{d}{ds}(sV)\Big|_{s=0}.$$ ⊕

- So $d_0\exp_p: T_pM \cong T_0(T_pM) \to T_pM$ is given by
$$d_0\exp_p(V) = \frac{d}{ds}\exp_p(sV)\Big|_{s=0} = \frac{d}{ds}\gamma_{sV}(1)\Big|_{s=0} = \frac{d}{ds}\gamma_V(s)\Big|_{s=0} = V,$$
i.e. the identity map. The claim now follows from the inverse function theorem. $\square$

Definition 4.5. (i) If $V \subset T_pM$ is a neighborhood of $0 \in T_pM$ s.t. $\exp_p|_V: V \to \exp_p(V)$ is a diffeomorphism, then $\exp_p(V)$ is called a normal neighborhood of p, and $\exp_p(B_\varepsilon(0))$ is the geodesic ball around p with radius ε. (Here $\varepsilon > 0$ is s.t. $B_\varepsilon(0) \subset V$.)

(ii) Fix a linear isometry $H: (T_pM, g_p) \xrightarrow{\cong} (\mathbb{R}^m, \text{Euclidean})$. Then
$$\phi: H \circ (\exp_p)^{-1}: \exp_p(B_\varepsilon(0)) \xrightarrow{\cong} B_\varepsilon(0) \subset \mathbb{R}^m$$
defines normal coordinates on M around p.

Remark 4.6. $H \longleftrightarrow$ choice of orthonormal basis of T_pM.

Normal coordinates are very useful for computations involving the metric, partly due to the following result:

Lemma 4.7. In normal coordinates $x^1, \dots, x^m$ around $p \in M$, we have

(i) $g_{ij}(0) = \delta_{ij}$

(ii) $\partial_k g_{ij}(0) = 0$, $\Gamma^k_{ij}(0) = 0$.

Proof (i) $g_{ij}(0) = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}$ since $\left\{H\left(\frac{\partial}{\partial x^i}\right) : 1 \le i \le m\right\}$ is an ONB of $\mathbb{R}^m$, so $\{\frac{\partial}{\partial x^i}|_p\}$ is an ONB of T_pM.

(ii). By definition, geodesics through p take the form

$$\gamma_v(t) = tv \qquad (v \in \mathbb{R}^m \cong T_pM) \qquad (\text{so } \gamma_v^k(t) = tv^k)$$

in normal coordinates. They solve the geodesic equation: $\forall k$,

$$0 = \ddot{\gamma}_v^k(t) + \Gamma_{ij}^k(\gamma_v(t))\,\dot{\gamma}_v^i(t)\,\dot{\gamma}_v^j(t)$$

$$= 0 + \Gamma_{ij}^k(\gamma_v(t))\,v^i v^j.$$

At $t=0$, this gives $\Gamma_{ij}^k(0)\,v^i v^j = 0 \quad \forall v \in \mathbb{R}^m$. Since $\Gamma_{ij}^k = \Gamma_{ji}^k$, this implies (by elementary linear algebra) $\Gamma_{ij}^k(0) = 0 \quad \forall i,j,k$.

- We then compute

$$\partial_k g_{ij}(0) = \partial_k g(\partial_i, \partial_j) = g(\underbrace{\nabla_k \partial_i}_{=\Gamma_{ki}^\ell(0)\partial_\ell = 0}, \partial_j) + g(\partial_i, \underbrace{\nabla_k \partial_j}_{=0}) = 0.$$

$$\left(\text{i.e. } \partial_k g_{ij} = \Gamma_{ik}^\ell g_{\ell j} + \Gamma_{jk}^\ell g_{i\ell} = 0\right)$$

$\square$

Remark 4.8. Given how normal coordinates around p are essentially (modulo $O(m+1)$, that is) unique, one should expect the Taylor expansion of $g_{ij}(x)$ around $x=0$ to contain all "local geometric" information about g. And indeed we will see that the quadratic terms

$$\left(g_{ij}(x) = \delta_{ij} + x^k x^\ell (?)_{k\ell ij} + O(|x|^3)\right)$$

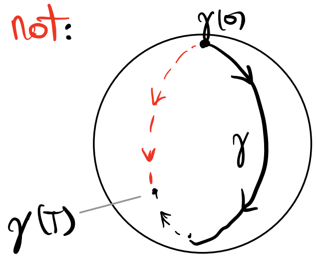
encode the full curvature tensor at p. (Cubic and higher order terms: derivatives of curvature.) $\rightarrow$ Later exercise.

4.1. Minimizing properties of geodesics

On S^2, geodesics (= great circles) $\gamma : \mathbb{R} \to S^m$, parameterized by arc-length ($|\gamma'|=1$) are locally length-minimizing :

$\forall\, T<\pi$, $\gamma|_{[0,T]}$ is the shortest curve joining $\gamma(0)$ and $\gamma(T)$;

but for $T>\pi$, $\gamma|_{[0,T]}$ is not:



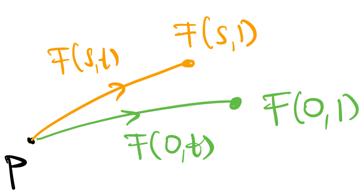
We discuss the locally length-minimizing property on general (M,g) next.
(Global questions are addressed in the next section.)

Lemma 4.8. (Gauss Lemma). Let $v \in T_pM$. Identify $T_v(T_pM) \cong T_pM$ (as in # above). Then $\forall\, w \in T_pM$,

$$\langle \underbrace{d_v \exp_p(v)}_{\substack{\text{tangent vector of the}\\ \text{geodesic } \gamma_v(t) \text{ at } t=1}},\ d_v\exp_p(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p .$$

inner product $g_{\exp_p(v)}$ inner product g_p

Proof. Let $F(s,t) := \exp_p(t(v+sw))$, $0 \le t \le 1$.



Then each $F(s,\cdot)$ is a geodesic, and F is a variation of $\gamma(t) = F(0,t)$ with variation vector field

$$V(t) = \partial_s F(s,t)|_{s=0} = d_{tv}\exp_p(tw).$$

By Lemma 3.17,

$$\frac{d}{ds} L(F(s,\cdot))\Big|_{s=0} = \frac{1}{|v|} \langle \dot\gamma(1), V(1) \rangle$$

$$= \frac{1}{|v|}\langle d_v\exp_p(v),\ d_v\exp_p(w)\rangle. \qquad \circledast$$

On the other hand, $L(F(s,\cdot)) = \int_0^1 \underbrace{|\partial_t F(s,t)|}_{\substack{= \text{const.} = \partial_t F(s,0) = |v+sw| \\ \text{since } F(s,\cdot) \text{ is a geodesic}}} dt = |v+sw|_p$, so

$$\frac{d}{ds} L(F(s,\cdot))\Big|_{s=0} = \frac{d}{ds}\left(g_p(v+sw, v+sw)^{\frac{1}{2}}\right)\Big|_{s=0}$$
$$= \frac{\langle v,w\rangle_p}{|v|_p}.$$

Comparison with $\circledast$ gives the desired result. $\square$

Theorem 4.10. Let $B = \exp_p(B_\varepsilon(0))$ be a geodesic ball contained in a normal neighborhood of p. Let $\gamma : [0,1] \to B$, $\gamma(0) = p$, be a geodesic, and suppose $c : [0,1] \to M$ is a piecewise smooth curve with $c(0) = p$, $c(1) = \gamma(1)$. Then

$$L(c) \geq L(\gamma),$$

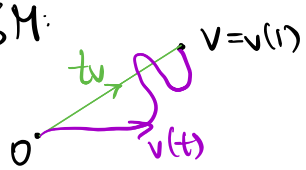
with equality iff c is a monotone reparameterization of γ.

Proof. WLOG $c(t) \neq p$ for $t > 0$. (Otherwise, delete an initial segment of c.)

Case 1: $c([0,1]) \subset B$. Write $c(t) = \exp_p(v(t))$; so $v(0) = 0$, $|v(1)| = L(\gamma)$, $v(t) \neq 0$ $\forall\, t > 0$.

(Indeed, $\gamma(t) = \exp_p(tv)$, and $\gamma(1) = c(1)$ implies, by the injectivity of $\exp_p$ on $B_\varepsilon(0)$, that $v = v(1)$; but $L(\gamma) = |v|$.)

T_pM:



- For $t>0$, write $v'(t) = \lambda(t)v(t) + w(t)$ $\quad$ ($\lambda : (0,1] \xrightarrow{C^\infty} \mathbb{R}$, $T_pM \ni w(t) \perp v(t)$).

Then
$$|c'(t)|^2_{g_{c(t)}} = |d_{v(t)}\exp_p(v'(t))|^2 = |d_v\exp_p(\lambda v + w)|^2$$
$$= \lambda^2|v|^2 + \underbrace{|d_v\exp_p(w)|^2}_{\geq 0} \quad \text{by the Gauss lemma.}$$

- $\Rightarrow$
$$L(c) = \int_0^1 |c'(t)|_{g_{c(t)}}\, dt \overset{(\dagger)}{\geq} \int_0^1 |\lambda(t)v(t)|_{g_p}\, dt.$$

- But for $t>0$, $\frac{d}{dt}|v(t)| = \frac{d}{dt}\langle v(t), v(t)\rangle^{\frac{1}{2}} = \frac{\langle v, v'\rangle}{|v|} = \lambda|v|$

$\Rightarrow$ $L(c) \overset{(\ddagger)}{\geq} \int_0^1 \frac{d}{dt}|v(t)|\, dt = |v(1)| = L(\gamma)$

- Equality requires:

  $(\dagger)$: $d_{v(t)}\exp_p(w(t)) = 0$ $\forall t \in (0,1]$. Since $\exp_p$ is a diffeomorphism on $B_\varepsilon(0)$, this forces $w(t) \equiv 0$.

  $(\ddagger)$: $\lambda = |\lambda|$, i.e. $\lambda(t) \geq 0$ $\forall t$.

$\Rightarrow$ $t \mapsto v(t)$ is a monotone reparameterization of $t \mapsto tv$.

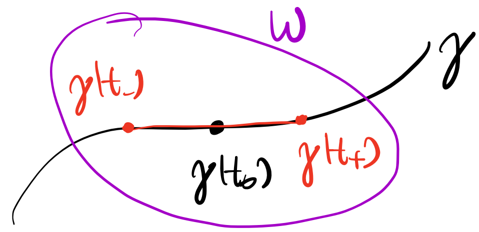
<u>Case 2:</u> $c([0,1]) \not\subset B$. Let $T = \sup\{t \in [0,1] : \gamma(t) \in B\} > 0$; then $c(T) \in \partial B$. For $\delta > 0$, let $\gamma_\delta : [0,1] \to B$ be the geodesic with $\gamma_\delta(0) = p$, $\gamma_\delta(1) = c(T-\delta)$. Then
$$L(c) \geq L(c|_{[0,T-\delta]}) \geq L(\gamma_\delta) \xrightarrow{\delta \to 0} \varepsilon > L(\gamma). \qquad \square$$

Thus for fixed $p \in M$, the unique shortest curve from p to a point $q \in B$ is the radial geodesic from p to q. This can be strengthened:

Theorem 4.11. Let $\gamma : [a,b] \to M$ be a piecewise smooth curve parameterized by arc length. Suppose that $L(\gamma) \le L(\tilde{\gamma})$ $\forall$ piecewise differentiable $\tilde{\gamma}$ joining $\gamma(a)$ to $\gamma(b)$. Then γ is a geodesic.

Proof Let $t_0 \in [a,b]$. There exist $\delta > 0$ and a neighborhood $W \subset M$ of $\gamma(t_0)$ s.t. $\exp_q(B_\delta(0)) \supset W$ $\forall$ $q \in W$ is a geodesic neighborhood. (I.e. W is a normal neighborhood of each of its points.) (Exercise.) For all $a \le t_- \le t_0 \le t_+ \le b$, $t_- \ne t_+$, with $\gamma(t_-), \gamma(t_+) \in W$, Theorem 4.10 then implies that

$\gamma|_{[t_-,t_+]}$ = radial geodesic from $\gamma(t_-)$ to $\gamma(t_+)$.



(All other curves are longer!)

$\Rightarrow$ γ is C^∞ at t_0, and a geodesic near $t = t_0$. $\square$

4.2. Global existence of geodesics; completeness.

An obvious question which our discussion of local properties of geodesics has not addressed is:

When can 2 points on (M,g) be joined by a geodesic?

As the example $M = \mathbb{R}^2 \setminus \{(0,0)\}$, $p = (-1,0)$, $q = (1,0)$ indicates, this is related to completeness properties of (M,g).

Theorem 4.12. (Hopf–Rinow.) Let (M,g) be a connected Riemannian manifold. Then the following are equivalent:

(i) (M,d) is a complete metric space (i.e. all Cauchy sequences converge).

(ii) (M,g) is geodesically complete, i.e. $\exp$ is defined on all of TM. (So $\mathcal{D} = TM$ in the notation of Prop. 4.1(ii).)

(iii) $\exists\, p \in M$ s.t. $\exp_p$ is defined on all of T_pM.

(iv) Closed and bounded subsets of (M,d) are compact.

Furthermore, if (any one of) these conditions hold, then

(v) $\forall p, q \in M$ $\exists$ geodesic γ from p to q with $L(\gamma) = d(p,q)$.

Proof. (iii) $\Rightarrow$ (v). Let $q \neq p$. Let $\varepsilon > 0$ be such that $\exp_p(B_{2\varepsilon}(0))$ is a normal neighborhood of p; and $q \notin B = \exp_p(B_\varepsilon(0))$ (i.e. $d(p,q) \geq \varepsilon$). Let $x \in \partial B$ be a point with

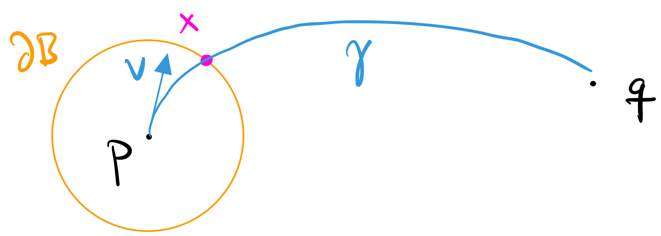
$$d(x,q) = \inf_{\partial B} d(\cdot, q).$$

(Use: $d(\cdot,q)$ is continuous; and ∂B is compact, being the continuous image of $\partial B_\varepsilon(0)$.)

Let $v \in T_pM$, $|v|=1$, be s.t. $x = \exp_p(\varepsilon v)$, and set

$\gamma(t) = \gamma_v(t) = \exp_p(tv)$.

We claim that $\gamma(r) = q$ where $r = d(p,q)$.



Proof: Let $A = \{ s \in [0,r] : d(\gamma(s), q) = r - s \}$. Then:

- $0 \in A$ (by definition of r).
- A is closed (since d, γ are continuous).
- A is open. Let $s_0 \in A$, $s_0 < r$. We claim that $s_0 + \delta \in A$ $\forall$ small $\delta > 0$. Let $B_{2\delta}(\gamma(s_0))$ be a normal neighborhood, and let $B' = B_\delta(\gamma(s_0))$, $\gamma(q) \notin B'$,

$$x' \in \partial B' \text{ s.t. } d(x', q) = \inf_{\partial B'} d(\cdot, q).$$

It suffices to show that $x' = \gamma(s_0 + \delta)$. (*)

(Indeed, this would give $\quad d(\gamma(s_0), q) = \inf_{x' \in \partial B'} \left(\overbrace{d(\gamma(s_0), x')}^{=\delta} + d(x', q) \right)$

$$d(\gamma(s_0+\delta), q) = d(x', q) \overset{\#}{=} \underbrace{d(\gamma(s_0), q)}_{= r - s_0} - \delta$$

$$= r - (s_0 + \delta).)$$

To show (*), note that

$$d(p, x') \geq d(p,q) - d(x', q) \overset{\#}{=} r - (r - s_0 - \delta) = s_0 + \delta.$$

On the other hand, the piecewise C^∞ curve

$\tilde{\gamma}: \quad p \xrightarrow{\gamma} \gamma(s_0) \xrightarrow[\text{radial geodesic}]{} x'$ (parameterized by arc length)

has length $s_0 + \delta$. By Theorem 4.11, $\tilde{\gamma}$ must be a geodesic, and in particular does not have a break point at $\gamma(s_0)$

$\Rightarrow x' = \tilde{\gamma}(s_0+\delta) = \exp_p((s_0+\delta)v) = \gamma(s_0+\delta)$.

- A is therefore all of $[0,r]$. But $r \in A$ means $d(\gamma(r), q) = 0$, so $\gamma(r) = q$.

- (ii) $\Rightarrow$ (iii): obvious.
- (iii) $\Rightarrow$ (iv): let $K \subset M$ be closed and bounded. Then $\exists\, r > 0$ s.t. $K \subset \exp_p(\overline{B_r(0)})$ (since K is bounded). But $\exp_p(\overline{B_r(0)})$ is compact. Since K is a closed subset of a compact set, it is itself compact.
- (iv) $\Rightarrow$ (i): if $\{p_i\}_{i\in\mathbb{N}} \subset M$ is a Cauchy sequence in (M,d), then $K := \overline{\{p_i : i \in \mathbb{N}\}}$ is bounded and closed, hence compact. Compact subsets of metric spaces are sequentially compact; therefore, $\{p_i\}$ has a subsequential limit which (since (M,d) is Hausdorff) must be its (unique) limit.
- (i) $\Rightarrow$ (ii): • Suppose (M,g) is not geodesically complete. Then some arc-length-parameterized geodesic γ is defined for $0 \le s < \bar{s}$ but not for $\bar{s}$. Let $\{s_i\}_{i\in\mathbb{N}}$ be a sequence with $0 \le s_1 < s_2 < s_3 < \cdots < s_i \nearrow \bar{s}$. Given $\varepsilon > 0$, $\exists\, N$ s.t. $i,j \ge N \Rightarrow |s_i - s_j| < \varepsilon$, so $d(\gamma(s_i), \gamma(s_j)) \le L(\gamma|_{[s_i,s_j]}) = s_j - s_i < \varepsilon$.

$\Rightarrow \{\gamma(s_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in (M,d).

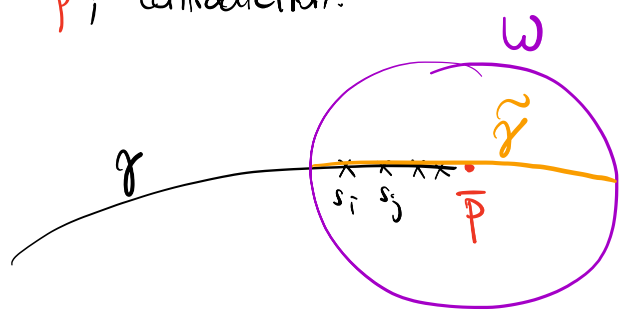
Since (M,d) is complete,

$$\gamma(s_i) \xrightarrow{i\to\infty} \bar{p} \in M.$$

- Let $W \subset M$ be a totally normal neighborhood of $\bar{p}$, with $W \subset \exp_q(B_\delta(0))$ $\forall\, q \in W$, and $\exp_q(B_\delta(0))$ is a normal neighborhood. Let N be s.t. $|s_i - s_j| < \delta$ $\forall\, i,j \geq N$,

  $\gamma(s_i) \in W$ $\forall\, i \geq N$.

  Fix $i,j > N$ with $\gamma(s_i) \neq \gamma(s_j)$, and let $\tilde{\gamma}$ = geodesic in W through $\gamma(s_i), \gamma(s_j)$ (which thus has minimal length); we must have $\gamma = \tilde{\gamma}$ where the former is defined; and thus $\tilde{\gamma}$ extends γ past $\bar{p}$; contradiction.



$\square$