7. Space forms

Definition 7.1. A Riemannian manifold (M,g) is called a *space form* if it is *complete* and has *constant sectional curvature* $\kappa \in \mathbb{R}$.

Our goal is to classify all space forms — thus, how (constant) curvature determines the metric: they all "come from" one of the three model spaces

$$(\mathbb{M}^m_\kappa, g^m_\kappa) = \begin{cases} \kappa = 0: & (\mathbb{R}^m, g_{eucl.}) \\ \kappa > 0: & (S^m, \frac{1}{\kappa} g_{std.}) \\ \kappa < 0: & (\mathbb{H}^m, \frac{1}{|\kappa|} g_{std}) \end{cases}$$

($\cong$ m-sphere in $\mathbb{R}^{m+1}$ with radius $\kappa^{-\frac{1}{2}}$)

($\mathbb{H}^m = \{(x,y) \in \mathbb{R}^{m-1} \times (0,\infty)\}$, $g_{std} = \frac{1}{y^2}\left(\sum_{i=1}^{m-1} (dx^i)^2 + dy^2\right)$)

Theorem 7.2. (Killing 1891, Hopf 1926.) Let (M,g) be an m-dimensional space form of curvature $\kappa \in \mathbb{R}$. Then there exists a group

$$\Gamma < \mathrm{Iso}(\mathbb{M}^m_\kappa, g^m_\kappa)$$

that acts freely and properly discontinuously on $\mathbb{M}^m_\kappa$ such that M is isometric to $\mathbb{M}^m_\kappa / \Gamma$. If M is simply connected, then $M = \mathbb{M}^m_\kappa$.

To understand the statement, we need to digress (§7.1 an onwards). For now, we only recall the following:

- Let M be a topological manifold, $\Gamma < \mathrm{Homeo}(M)$ a subgroup acting on M via $\gamma \in \Gamma$, $p \in M \rightsquigarrow \gamma . p = \gamma(p)$.

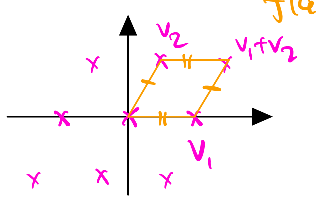
- Γ acts **freely** on M if $\forall \gamma \in \Gamma \setminus \{\mathrm{Id}\}, p \in M$: $\gamma p \neq p$.
- Γ acts **properly discontinuously** on M if $\forall$ compact $K \subset M$ the set $\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\}$ is finite.

Example 7.3. (Euclidean space forms.) (i) Let $\Gamma \subset \mathrm{Iso}(\mathbb{R}^m, g_{eucl.})$ be a group of translations acting freely and discontinuously on $\mathbb{R}^m$. Then $\exists$ linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ $(k \leq m)$ s.t.

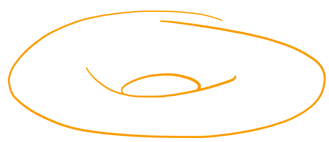
$$\Gamma = \left\{ \mathbb{R}^m \ni x \longmapsto x + \sum_{i=1}^{k} n_i v_i \; : \; n_1, \dots, n_k \in \mathbb{Z} \right\}$$

(exercise).

(i.1) $k = m$: $\mathbb{R}^m/\Gamma$ (with its induced metric, see below) is a flat m-torus, diffeomorphic to $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.



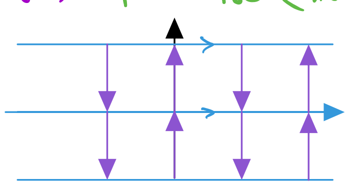
$k = m = 2$:



(i.2) $k < m$: $\mathbb{R}^m/\Gamma$ is isometric to $\mathbb{T}^k \times \mathbb{R}^{m-k}$ ("cylinder") for some flat k-torus $\mathbb{T}^k$.

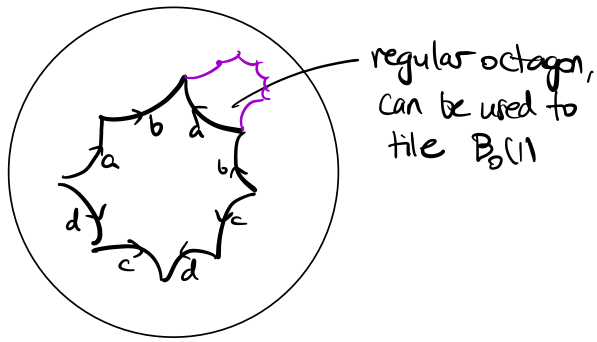
$k = 1 < m = 2$:

(ii) $\Gamma \subset \mathrm{Iso}(\mathbb{R}^2)$ generated by $x \mapsto x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$:



$\Rightarrow$ $\mathbb{R}^2/\Gamma$ is the (flat) Klein bottle.

Example 7.4. (Hyperbolic space forms.) Every compact oriented surface of genus $g \geq 2$ (g-holed torus) is diffeomorphic to a quotient $(\mathbb{H}^2, g_{std})/\Gamma$ for some Γ, and thus carries a metric of constant curvature -1. Roughly: in the Poincaré disc model $\left(B_0(1), \frac{dx^2}{4(1-|x|^2)^2}\right)$,



Example 7.5. (Spherical space forms.)

(i) Consider $\Gamma = \{id, p \mapsto -p\} \subset \mathrm{Iso}(S^m)$. Then $S^m/\Gamma = \mathbb{RP}^m$, and the induced metric is called the elliptic metric.

(ii) $m = 2n-1$ odd: lens spaces. Let $p, q_1, \dots, q_n \geq 1$, with p, q_i coprime $\forall i$. View $S^m \subset \mathbb{R}^{2n} = \mathbb{C}^n$ (i.e. $S^m = \{z \in \mathbb{C}^n : |z_1|^2 + \dots + |z_n|^2 = 1\}$). Let

$$\Gamma = \left\{ (z_1, \dots, z_n) \mapsto \left(e^{\frac{2\pi i k q_1}{p}} z_1, \dots, e^{\frac{2\pi i k q_n}{p}} z_n\right) : k = 0, \dots, p-1 \right\}.$$

Then $L(p; q_1, \dots, q_n) := S^m/\Gamma$ is called a lens space.

Regarding Example 7.5(i), we have the following result:

Theorem 7.6. Let (M, g) be a space form with curvature 1 and even dimension. Then (M, g) is isometric to (S^m, g_{std}) or $(\mathbb{RP}^m, g_{ell})$.

Proof. • By Theorem 7.2, we only need to show that if

$\Gamma \subset \mathrm{Iso}(S^m) = O(m+1)$ acts freely and properly discontinuously, then $\Gamma = \{I\}$ or $\Gamma = \{I, -I\}$.

- Let $\gamma \in \Gamma$. Since $m+1$ is odd, γ has an eigenvalue $+1$ or -1.
  - If $+1$ (so $\gamma v = v$ for some $v \in S^m$), then $\gamma = I$ since Γ acts freely.
  - If -1 (so $\gamma v = -v$ for some $v \in S^m$), then $\gamma^2 v = v$, so $\gamma^2 = I$.

    We claim that $\gamma = -I$. If this were false, $\exists\, w \in S^m$, $\gamma w \neq -w$

    $\Rightarrow \tilde{v} := w + \gamma w \neq 0$ satisfies $\gamma \tilde{v} = \gamma w + \gamma^2 w = \gamma w + w = \tilde{v}$

    $\Rightarrow \gamma = I$, contradiction. $\square$

7.1. Covering maps

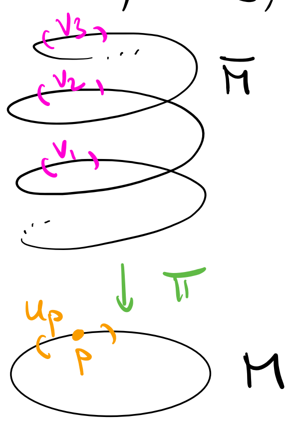
Definition 7.7. Let $\overline{M}, M$ be connected topological spaces. Then a continuous map $\pi : \overline{M} \to M$ is called a covering map if it is surjective, and every $p \in M$ has an open neighborhood $U \subset M$ so that

$$\pi^{-1}(U) = \bigsqcup_{i \in I} V_i \qquad (\text{disjoint union, } I = \text{index set})$$

(A priori, I may depend on p, but one can show that it does not.)

where each $V_i \subset \overline{M}$ is open, and $\pi|_{V_i} : V_i \to U$ is a homeomorphism.

Example 7.8. $\overline{M} = \mathbb{R}$, $M = S^1$, $\pi(x) = e^{ix}$.



Proposition 7.9. Let $\overline{M}$ be a connected topological manifold. Suppose $\Gamma \subset \mathrm{Homeo}(\overline{M})$ acts freely and properly discontinuously on $\overline{M}$. Then $\overline{M}/\Gamma$ is a topological manifold, and the projection $\pi : \overline{M} \to \overline{M}/\Gamma$ is a covering map. (Here $M/\Gamma = M/\sim$, $p \sim q$ iff $\exists \gamma \in \Gamma, \gamma \cdot p = q$.)

Proof. Let $p \in \overline{M}$, and let $\varphi : U \to B_1(0) \subset \mathbb{R}^m$ be a chart around p. It suffices to show: $\exists\, r \in (0,1)$ s.t. the sets $\gamma \cdot \varphi^{-1}(B_r(0))$ are pairwise disjoint. $(*)$

(Then for $\mathcal{U} := \pi(\varphi^{-1}(B_r(0)))$, we have $\pi^{-1}(\mathcal{U}) = \bigsqcup_{\gamma\in\Gamma} \gamma\cdot\mathcal{U}$; and a chart on $\overline{M}/\Gamma$ around $\pi(p)$ is given by the composition

$$\pi(\varphi^{-1}(B_r(0))) \xrightarrow{(\pi|_{\varphi^{-1}(B_r(0))})^{-1}} \varphi^{-1}(B_r(0)) \xrightarrow{\varphi} B_r(0).)$$

- Suppose (*) is wrong. Then for $j\in\mathbb{N}$ $\exists\, p_j, q_j \in B_{\frac{r}{j}}(0)$, $\gamma_j \in \Gamma\setminus\{1\}$, s.t.

$$\gamma_j\, \varphi^{-1}(p_j) = \varphi^{-1}(q_j).$$

Since $\overline{B_{r/2}(0)}$ is compact, so is $K = \varphi^{-1}(\overline{B_{r/2}(0)})$, and since $\gamma_j K \cap K \neq \emptyset$ for all $j \geq 2$, there exists a finite set $G \subset \Gamma\setminus\{1\}$ s.t. $\gamma_j \in G\ \forall j\geq 2$. By passing to a subsequence, $\exists\, \gamma_0 \in G$ s.t. $\gamma_j = \gamma_0\ \forall j$. Taking limits in the equation $\gamma_0\, \varphi^{-1}(p_j) = \varphi^{-1}(q_j)$ gives

$$\gamma_0\, \varphi^{-1}(0) = \varphi^{-1}(0).$$

But the Γ-action is free, so $\gamma_0 = 1$; contradiction. $\square$

Conversely, we have:

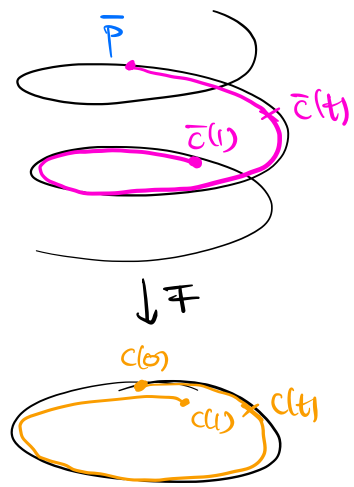
Proposition 7.10. Let $F: \overline{M} \to M$ be a covering map between topological manifolds (Hausdorff). Define the group of deck transformations

$$\Gamma := \{\gamma \in \mathrm{Homeo}(\overline{M}) : F\circ\gamma = F\}.$$

Then Γ acts freely and properly discontinuously on $\overline{M}$. Furthermore, $\overline{M}/\Gamma \cong M$, with the homeomorphism given by $\Gamma\cdot p \mapsto F(p)$ $(p\in\overline{M})$.

Proof. We use the lifting property of coverings: let $c: [0,1] \to M$ be a continuous curve and $\overline{p} \in \overline{M}$, $F(\overline{p}) = c(0)$; then $\exists!\ \overline{c}: [0,1] \to \overline{M}$ s.t.

$$\overline{c}(0) = \overline{p}, \quad F(\overline{c}(t)) = c(t)\ \forall t\in[0,1].$$



(The proof is left as an easy exercise.)

- Γ acts freely. Suppose $\gamma \in \Gamma$, $p \in \overline{M}$, $\gamma(p) = p$. Consider any $q \in \overline{M}$.

  Let $\alpha : [0,1] \to \overline{M}$ be a continuous path from p to q.

  Then $\bar{c} := \gamma \circ \alpha$ is a continuous path from p to $\gamma(q)$

  $\Rightarrow$ $c := F \circ \gamma \circ \alpha$ is a continuous path from $F(p)$ to $F(q)$.

  We can now write down two lifts of c along F with the same starting point p: $\bar{c} = \gamma \circ \alpha$ and α itself. They must agree,

  so $q = \alpha(1) = \bar{c}(1) = \gamma(\alpha(1)) = \gamma(q)$.

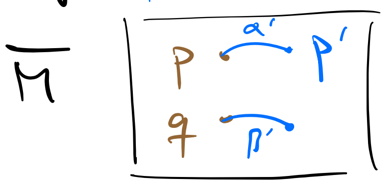
  Since q was arbitrary, we conclude that $\gamma = \mathrm{Id}$.

- Construction of deck transformations. Let $p, q \in \overline{M}$, $F(p) = F(q)$. We claim that $\exists\, \gamma \in \Gamma$, $\gamma(p) = q$. (This also proves the injectivity of $\overline{M}/\Gamma \to M$.) Given $p' \in \overline{M}$ and a continuous curve α' from p to p',

  lift $F \circ \alpha'$ (from $F(p)$ to $F(p')$) along F with starting point q

  to get $\beta' : [0,1] \to \overline{M}$ with $F \circ \beta' = F \circ \alpha'$

  $\beta'(0) = q$.



We then set $\gamma(p') := \beta'(1)$.

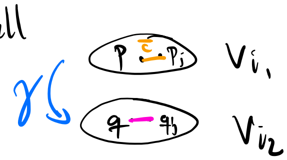
Check: $\gamma \in \Gamma$. (Exercise.) By what we have already shown, this γ is the unique deck transformation with $\gamma(p)=q$.

- Γ acts properly discontinuously. Let $\overline{K} \subset \overline{M}$ be compact. Let $\gamma_j \in \Gamma$, $p_j, q_j \in K$, $\gamma_j(p_j)=q_j$. We need to show that $\{\gamma_j : j \in \mathbb{N}\}$ is a finite set. Suppose not; WLOG, $p_j \to p$, $q_j \to q$, and all γ_j are pairwise distinct. Taking limits in $F(q_j) = F(\gamma_j(p_j)) = F(p_j)$ gives

$$F(q)=F(p).$$

Let $\gamma \in \Gamma$ be the unique deck transformation with $\gamma(p)=q$.

We will show: $\gamma(p_j)=q_j$ for all sufficiently large j.



Let $U \subset M$ be a neighborhood of $F(q)$ s.t. $F^{-1}(U) = \bigsqcup_{i \in I} V_i$, with $F|_{V_i} : V_i \to U$ a homeomorphism. Let $p \in V_{i_1}$, $q \in V_{i_2}$. For all sufficiently large j, there exists a path

$$\tilde{c} : [0,1] \to \overline{M} \text{ from } p \text{ to } p_j \text{ entirely contained in } V_{i_1}.$$

$(F|_{V_{i_2}})^{-1} \circ (F \circ \tilde{c}) : [0,1] \to V_{i_2}$ is a path from q to q_j; thus $\gamma(p_j) = q_j = \gamma_j(p_j)$. This forces $\gamma_j = \gamma$ $\forall j$, contradiction.

- The homeomorphism property of $\overline{M}/\Gamma \to M$ is an easy consequence. $\square$

We now add more structure.

Proposition 7.11. Let $\overline{M}$ be a C^∞ manifold, and suppose $\Gamma \subset \mathrm{Diffeo}(\overline{M})$ acts freely and properly discontinuously. Then $\exists!$ C^∞ structure on $\overline{M}/\Gamma$ so that $\overline{M} \to \overline{M}/\Gamma$ is a local diffeomorphism. If, moreover, $\overline{g}$ is a Riemannian metric on $\overline{M}$ and $\Gamma \subset \mathrm{Iso}(\overline{M})$, then $\exists!$ Riemannian metric g on $\overline{M}/\Gamma$ such that $(\overline{M}, \overline{g}) \to (\overline{M}/\Gamma, g)$ is a local isometry.

Proof. This follows immediately from the description of the charts on $\overline{M}/\Gamma$ in the proof of Proposition 7.9. $\square$

A smooth covering map is a C^∞ map satisfying Definition 7.7, with each $\pi|_{V_i} : V_i \to U$ a diffeomorphism. (Thus, C^∞ covering maps are local diffeomorphisms.)

Definition 7.12. Let $(\overline{M}, \overline{g})$, (M, g) be Riemannian manifolds. A smooth covering map $F : \overline{M} \to M$ is called a Riemannian covering map if $F^* g = \overline{g}$. (This requires $\dim \overline{M} = \dim M$.)

Proposition 7.13. Let $(\overline{M}, \overline{g})$ be complete, and (M, g) connected. Suppose $F : \overline{M} \to M$ is a local isometry. Then F is a Riemannian covering map.

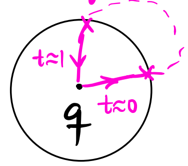
Proof. · Step 1: F is surjective. (i) Since F is a local isometry — in particular $d_pF : T_p\overline{M} \to T_{F(p)}M$ is invertible $\forall p \in \overline{M}$ — F is an open map $\Rightarrow F(\overline{M}) \subset M$ is open.

(ii) We claim that $F(\overline{M}) \subset M$ is complete. Let $p \in F(\overline{M})$, so $p = F(\overline{p})$. Let $v \in T_pM$, and set $\overline{v} := (d_{\overline{p}}F)^{-1}(v)$. Since F is a local isometry, the curve $\gamma(t) := F(\exp_{\overline{p}}(t\overline{v}))$ is a geodesic (being the image of a geodesic); since $\gamma(0) = p$, $\gamma'(0) = v$, this shows that $\exp_p$ is defined on all of T_pM, as claimed.

(iii) As a consequence of (ii), $F(\overline{M}) \subset M$ is closed (and hence, by (i) and the connectedness of M, we get $F(\overline{M}) = M$).

- Step 2: F is a covering map. Let $q \in M$, and let $r > 0$ be such that $\exp_q|_{B_r(0)} : B_r(0) \subset T_qM \to M$ is a diffeomorphism. We claim that
$$F^{-1}(B_r(q)) = \bigsqcup_{p \in F^{-1}(q)} B_r(p),$$
$$F|_{B_r(p)} : B_r(p) \to B_r(q) \text{ diffeomorphism.}$$

(i) Let $p, p' \in F^{-1}(q)$, $p \neq p'$. Then $B_r(p) \cap B_r(p') = \emptyset$. Indeed, if $\overline{\gamma}$ is the shortest geodesic in $(\overline{M}, \overline{g})$ from p to p', then $\gamma := F \circ \overline{\gamma}$ is a geodesic loop in (M, g) at q
$$\Rightarrow d(p,p') = L(\overline{\gamma}) \underset{\substack{\uparrow \\ F \text{ is local} \\ \text{isometry}}}{=} L(\gamma) \geq 2r.$$
$\uparrow$ γ must cross $\partial \exp_q(B_r(0))$ twice.



(ii) Let $p \in F^{-1}(q)$. Since F maps geodesics to geodesics,
$$(F \circ \exp_p|_{B_r(0)})(tv) = \text{geodesic in } M \text{ with initial conditions } F(p) = q,\ \tfrac{d}{dt}(\cdot)|_{t=0} = d_pF(v)$$
$$= \exp_q(t\, d_pF(v)).$$

$$\Rightarrow F \circ \exp_p|_{B_r(0)} = \exp_q|_{B_r(0)} \circ d_pF. \qquad \circledast$$

But d_pF is an isometry $B_r(0) \subset T_pM \to B_r(0) \subset T_qM$, and $\exp_q|_{B_r(0)} : B_r(0) \xrightarrow{\cong} B_r(q)$. Therefore, $F|_{B_r(p)} : B_r(p) \to B_r(q)$ is a diffeomorphism (and an isometry).

(iii) It remains to show that $F^{-1}(B_r(q)) \subset \bigcup_{p \in F^{-1}(q)} B_r(p)$.

If $p' \in F^{-1}(B_r(q))$, then $d(F(p'), q) < r$, so

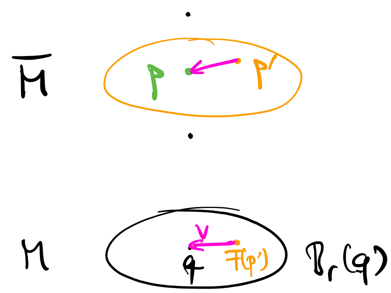
$q = \exp_{F(p')}(v)$ for a unique $v \in T_{F(p')}M$, $|v| < r$.

$\Rightarrow F(\exp_{p'}((d_{p'}F)^{-1}v)) \overset{\circledast}{=} \exp_{F(p')}(v) = q$

so $p := \exp_{p'}((d_{p'}F)^{-1}v) \in F^{-1}(q)$.

$\Rightarrow d(p, p') = |v| < r$

$\Rightarrow p' \in B_r(p)$.



$\square$

7.2. Killing–Hopf I: $\kappa \le 0$.

The strategy is to use the exponential maps of (M,g) and $(\mathbb{M}^m_\kappa, g^m_\kappa)$ and an idenfication $T_pM \cong T_q\mathbb{M}^m_\kappa$ to define a map $\mathbb{M}^m_\kappa \to M$. Since all we have on M is information about the sectional curvature, we first show:

Proposition 7.14. Let $(\bar{M},\bar{g})$, (M,g) have constant sectional curvature κ.
Let $\bar{p}\in\bar{M}$, $p\in M$, and fix an isometry

$$H : (T_{\bar{p}}\bar{M}, \bar{g}_{\bar{p}}) \longrightarrow (T_pM, g_p).$$

Let $r>0$ be s.t. $\exp_{\bar{p}}|_{B_r(0)} : B_r(0) \to B_r(\bar{p})$ is a diffeomorphism, and so that $\exp_p$ is defined on $B_r(0)\subset T_pM$.
Then $F := \exp_p \circ H \circ \exp_{\bar{p}}^{-1} : B_r(\bar{p}) \longrightarrow M$ is a local isometry.

Proof. F maps geodesics to geodesics; the idea is to bring in sectional curvature via the Jacobi equation. Thus, note that $\forall\, \bar{v},\bar{w}\in T_{\bar{p}}\bar{M}, |\bar{v}|<r,$

$$F(\exp_{\bar{p}}(t(\bar{v}+s\bar{w}))) = \exp_p(t(H(\bar{v})+sH(\bar{w}))).$$

- Compute $\frac{d}{ds}(\dots)|_{s=0}$: for $\bar{J}(t) := d_{t\bar{v}}\exp_{\bar{p}}(t\bar{w})$,

$$J(t) := d_{tH(\bar{v})}\exp_p(tH(\bar{w})),$$

we then have

$$d_{\exp_{\bar{p}}(t\bar{v})}F(\bar{J}(t)) = J(t). \qquad (*)$$

- The Jacobi equation reads

$$\bar{J}'' + \kappa\bar{J} = 0 \qquad (\text{along } \bar{\gamma}(t) = \exp_{\bar{p}}(t\bar{v})),$$

$$J'' + \kappa J = 0 \qquad (\text{along } \gamma(t) = \exp_p(tH(\bar{v}))).$$

Now $\bar{J}(0)=0$, $J(0)=0$;

$\bar{J}'(0)=\bar{w}$ and $J'(0)=H(\bar{w})$ have the same length;

$\langle \bar{J}'(0), \bar{\gamma}'(0)\rangle = \langle \bar{w}, \bar{v}\rangle = \langle H(\bar{w}), H(\bar{v})\rangle = \langle J'(0), \gamma'(0)\rangle =: b|\gamma'(0)|.$

We can solve this explicitly: if $E(t)$ is parallel along $\bar{\gamma}(t)$, with $E(0)=\bar{w}^{\perp}$, and

$$\bar{J}(t) = sn_K(t)\,E(t) + bt\,\bar{\gamma}'(t).$$

(Indeed, the initial conditions of the RHS are $(\cdots)|_{t=0}=0$, $\frac{d}{dt}(\cdots)|_{t=0} = sn_K'(0)\bar{E}(0) + b\bar{\gamma}'(0) = \bar{w}^{\perp} + \langle \bar{w}, \bar{v}\rangle \bar{v} = \bar{w}$.)

Similarly, for $E(t)$ = parallel transport of $H(\bar{w})^{\perp}$ along γ,

$$J(t) = sn_K(t)\,E(t) + bt\,\gamma'(t).$$

$$\Rightarrow \ |\bar{J}(t)|^2 = sn_K^2(t)\,|\bar{w}^{\perp}|^2 + b^2t^2|\bar{v}|^2$$
$$= sn_K^2(t)\,|H(\bar{w})|^2 + b^2t^2|v|^2 = |J(t)|^2.$$

- Returning to $\circledast$, this now implies (for $t=1$) that

$$d_{\exp_{\bar{p}}(\bar{v})}F : S_{\bar{v}} \subset T_{\exp_{\bar{p}}(\bar{v})}\bar{M} \to T_{\exp_p(H(\bar{v}))}M$$

preserves lengths, where

$$S_{\bar{v}} := \{\bar{J}(1) : \bar{J} \text{ is a Jacobi field along } t\mapsto \exp_{\bar{p}}(t\bar{v}) \text{ with } \bar{J}(0)=0\}$$
$$= d_{\bar{v}}\exp_{\bar{p}}(T_{\bar{p}}M).$$

But $d_{\bar{v}}\exp_{\bar{p}}$ is invertible (hence, surjective) for $|v|<r$, so $S_{\bar{v}} = T_{\exp_{\bar{p}}(\bar{v})}\bar{M}$.

$\Rightarrow d_qF$ is an isometry $\forall q \in \exp_{\bar{p}}(B_r(0))$, finishing the proof. $\square$

Proof of Theorem 7.2: $\kappa \leq 0$. Fix any $\bar{p} \in \mathbb{M}^m_\kappa$, $p \in M$, and an isometry $H : (T_{\bar{p}} \mathbb{M}^m_\kappa, g^m_\kappa|_{\bar{p}}) \to (T_p M, g_p)$; set

$$F := \exp_p \circ H \circ \exp_{\bar{p}}^{-1} : \mathbb{M}^m_\kappa \to M.$$

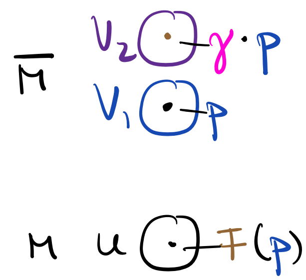
(We use here that $\exp_{\bar{p}} : T_{\bar{p}} \mathbb{M}^m_\kappa \to \mathbb{M}^m_\kappa$ is a diffeomorphism; exercise.) By Proposition 7.14, F is a local isometry; and then Proposition 7.13 says that F is a Riemannian covering map.

- The group $\Gamma \subset \mathrm{Homeo}(\mathbb{M}^m_\kappa)$ of deck transformations acts freely and properly discontinuously on $\mathbb{M}^m_\kappa$. (Proposition 7.10.)
- In fact, $\Gamma \subset \mathrm{Diffeo}(\mathbb{M}^m_\kappa)$: if $\gamma \in \Gamma$ and $p \in \mathbb{M}^m_\kappa$, let $U \subset M$ be an open neighborhood of $F(p)$ s.t.

$$F^{-1}(U) \supset V_1 \sqcup V_2, \text{ where } p \in V_1,\ \gamma \cdot p \in V_2,$$
$$F|_{V_i} : V_i \to U \text{ diffeomorphism for } i = 1, 2.$$

$\Rightarrow$ for q in a neighborhood of p, we have

$$\gamma \cdot q = (F|_{V_2})^{-1}(F(q)). \quad (*)$$



Indeed, this is true (by definition) for $q = p$, and by continuity of γ (and its deck transformation property) in a neighborhood of p.

The expression $(*)$ shows that γ is a smooth map near p. Since p is arbitrary, γ is a C^∞ map $\mathbb{M}^m_\kappa \to \mathbb{M}^m_\kappa$. But also $\gamma^{-1} \in \Gamma$ is C^∞; and therefore $\gamma \in \mathrm{Diffeo}(\mathbb{M}^m_\kappa)$.

- By Proposition 7.11., $\mathbb{M}^m_\kappa/\Gamma$ carries a unique C^∞ structure s.t. $\pi : \mathbb{M}^m_\kappa \to \mathbb{M}^m_\kappa/\Gamma$ is a C^∞ covering map. So we have a diagram

$$\begin{tikzcd} \mathbb{M}^m_\kappa \arrow[r, "\pi"] \arrow[d, "F"'] & \mathbb{M}^m_\kappa/\Gamma \\ M & \end{tikzcd}$$

of C^∞ covering maps. By Proposition 7.10, we have a homeomorphism $\tilde{F} : \mathbb{M}^m_\kappa/\Gamma \ni \Gamma\cdot p \mapsto F(p) \in M$; for this $\tilde{F}$, the diagram

$$\begin{tikzcd} \mathbb{M}^m_\kappa \arrow[r, "\pi"] \arrow[d, "F"'] \arrow[dr, phantom, "(\#)"] & \mathbb{M}^m_\kappa/\Gamma \arrow[dl, "\tilde{F}"] \\ M & \end{tikzcd}$$

commutes. (That is, $\tilde{F}\circ\pi \overset{(\#)}{=} F$.) Using the local diffeomorphism property of π, this implies that $\tilde{F}$ is C^∞; using that of F, that $\tilde{F}^{-1}$ is C^∞.

$\Longrightarrow$ $\tilde{F}$ is a diffeomorphism from $\mathbb{M}^m_\kappa/\Gamma$ to M.

- In fact, $\Gamma \subset \mathrm{Iso}(\mathbb{M}^m_\kappa)$. Indeed, for $\gamma\in\Gamma$, we have

$$F\circ\gamma = F, \text{ so } \forall q \in \mathbb{M}^m_\kappa \quad d_{\gamma(q)}F \circ d_q\gamma = d_qF.$$

$\Longrightarrow$ $d_q\gamma = (d_{\gamma(q)}F)^{-1}\circ d_qF$ is an isometry.

- $\Longrightarrow$ $\mathbb{M}^m_\kappa/\Gamma$ carries a Riemannian metric s.t. $\pi : \mathbb{M}^m_\kappa \to \mathbb{M}^m_\kappa/\Gamma$ is a local isometry; but also F is a local isometry, and thus so is $\tilde{F}$ (since $d_{\pi(z)}\tilde{F} = d_zF\circ(d_z\pi)^{-1}$ by $(\#)$).

This finishes the proof. $\square$

7.3. Killing-Hopf I: $\kappa > 0$.

The proof idea is essentially the same; only now $\exp_{\bar p}$ is not a diffeomorphism $T_{\bar p}\mathbb{S}^m \to \mathbb{S}^m$. Instead, we need to use 2 maps F_j like F above, each defined using

$$\exp_{\bar p_j}^{-1} : \mathbb{S}^m_\kappa \setminus \{-p_j\} \to B_{\frac{\pi}{\sqrt{\kappa}}}(0) \subset T_{p_j}\mathbb{S}^m_\kappa \qquad (j=1,2)$$

where $\bar p_1 \neq \pm \bar p_2$; i.e. we consider

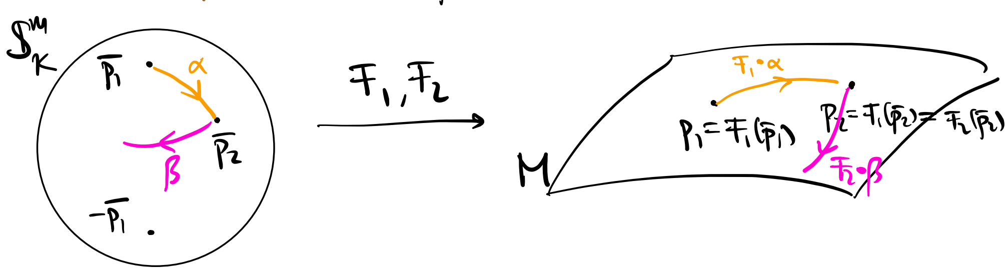
$$F_j := \exp_{p_j} \circ H_j \circ \left(\exp_{\bar p_j}\big|_{B_{\frac{\pi}{\sqrt{\kappa}}}(0)}\right)^{-1} : \mathbb{S}^m_\kappa \setminus \{-\bar p_j\} \to M$$

where $H_j : (T_{\bar p_j}\mathbb{S}^m_\kappa ; g^m_\kappa|_{\bar p_j}) \longrightarrow (T_p M, g_p)$ is an isometry;

by choosing $p_2 = F_1(\bar p_2)$, $H_2 := d_{\bar p_2} F_1$, we ensure that

$$F_2(\bar p_2) = p_2 = F_1(\bar p_2)$$

$$d_{\bar p_2} F_2 = d_0 \exp_p \circ H_2 \circ \mathrm{Id} = H_2 = d_{\bar p_2} F_1 .$$



The situation is thus: $\begin{cases} F_1, F_2 \text{ are local isometries } \mathbb{S}^m_\kappa \setminus \{-\bar p_j\} \to M, \\ F_1(\bar p_2) = F_2(\bar p_2),\ d_{\bar p_2} F_1 = d_{\bar p_2} F_2. \end{cases}$ (*)

We then apply the following result.

Lemma 7.15. Let $(\bar M, \bar g)$, (M, g) be two Riemannian manifolds, with $\bar M$ connected. Let $F_1, F_2 : \bar M \to M$ be two local isometries s.t., for some $\bar p \in \bar M$, $F_1(\bar p) = F_2(\bar p)$ and $d_{\bar p} F_1 = d_{\bar p} F_2$. Then

$F_1 = F_2$ on all of $\overline{M}$.

Proof. Let $A = \{\bar{q} \in \overline{M} : F_1(\bar{q}) = F_2(\bar{q}), d_{\bar{q}}F_1 = d_{\bar{q}}F_2\}$. Then:

(i) A is nonempty since $\bar{p} \in A$.

(ii) A is closed since F_1, F_2 are continuous.

(iii) A is open: let $\bar{q} \in A$. Since F_1, F_2 map geodesics to geodesics,

$$\gamma_j(t) := F_j(\exp_{\bar{q}}(t\bar{v})) = \exp_{F_j(\bar{q})}(t\, d_{\bar{q}}F_j(\bar{v})) \qquad (j=1,2).$$

But $\gamma_1(0) = \gamma_2(0)$, $\gamma_1'(0) = \gamma_2'(0) \Rightarrow \gamma_1(t) = \gamma_2(t)$ on their common domain of definition. Since $\exp_{\bar{q}}$ is a local diffeomorphism near $0 \in T_{\bar{q}}M$, this gives $F_1 = F_2$ on a small neighborhood of $\bar{q}$.

(iv) $A = \overline{M}$ since $\overline{M}$ is connected. $\square$

- Therefore, $F_1 = F_2$ on $S^m_K \setminus \{-\bar{p}_1, -\bar{p}_2\}$, and thus they can be glued together to a single map $F : S^m_K \to M$ via

$$F|_{S^m_K \setminus \{-p_j\}} := F_j.$$

- The same arguments as in §7.2 show that F is a Riemannian covering map, and S^m_K/Γ is isometric to (M,g) where $\Gamma < \mathrm{Iso}(S^m_K, g^m_K)$ is the group of deck transformations of F.
- This completes the proof of Theorem 7.2. $\square$