We already saw at the beginning of §5 how the curvature of (M,g) influences the behavior of nearby curves / geodesics. We now study this and related relationships in detail.

6.1. Jacobi fields

We already encountered Jacobi fields as variation vector fields of (certain) families of geodesics at the beginning of §5.

Lemma 6.1. Let $\gamma : [0,1] \to M$ be a geodesic, and let $V(t) \in T_{\gamma(t)}M$ be a vector field along γ. Then TFAE:

(i) V is the variation vector field of a variation $\tilde{\gamma} : (-\varepsilon,\varepsilon) \times [0,1] \to M$ of γ by geodesics (i.e. $\tilde{\gamma}(0,t) = \gamma(t)$, and $\forall s \in (-\varepsilon,\varepsilon)$, $\gamma_s := \tilde{\gamma}(s,\cdot)$ is a geodesic).

(ii) V solves the Jacobi equation $V'' + R(V,\gamma')\gamma' = 0$.
(Here $V'' = \frac{D}{dt}\frac{D}{dt}V$, or $\nabla_{\partial_t}\nabla_{\partial_t}V$ if you prefer.)

Proof. (i) $\Rightarrow$ (ii). For all $s \in (-\varepsilon,\varepsilon)$, the geodesic equation for γ_s reads

$$\frac{D}{dt}\frac{\partial}{\partial t}\tilde{\gamma}(s,t) = 0 \qquad (0 \le t \le 1). \qquad \circledast$$

(Note: $\frac{\partial}{\partial t}\tilde{\gamma}(s,t)$ is a vector field along $\tilde{\gamma}$, and so is $\frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}(s,t)$.)

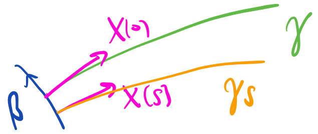
We differentiate $\circledast$ in s:

$$0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial}{\partial t}\tilde{\gamma}(s,t) = \frac{D}{dt}\frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}(s,t) + \left[\frac{D}{ds},\frac{D}{dt}\right]\frac{\partial}{\partial t}\tilde{\gamma}(s,t)$$

$$\overset{\text{exercise}}{=} \frac{D}{dt}\frac{D}{dt}\frac{\partial}{\partial s}\tilde{\gamma} + R\left(\frac{\partial}{\partial s}\tilde{\gamma},\frac{\partial}{\partial t}\tilde{\gamma}\right)\frac{\partial}{\partial t}\tilde{\gamma}.$$

At $s=0$, this gives $0 = \frac{D}{dt}\frac{D}{dt}V + R(V,\gamma')\gamma'$, i.e. the Jacobi equation.

(ii) $\Rightarrow$ (i). Choose any curve $\beta : (-\varepsilon,\varepsilon) \longrightarrow M$ with $\beta(0)=\gamma(0)$ and $\beta'(0)=V(0)$.



Let $X(s) \in T_{\beta(s)}M$ be a vector field along β with $X(0)=\gamma'(0)$, and with $X'(0)=\frac{D}{ds}X(0)$ to be determined later.

Define $\tilde{\gamma}(s,t) = \exp_{\beta(s)}(tX(s))$; this is a variation of γ by geodesics.
We compute its variation vector field $\tilde{V}=\tilde{V}(t)$:

$$\tilde{V}(0) = \frac{\partial}{\partial s}\tilde{\gamma}(0,0) = \frac{\partial}{\partial s}\beta(s)\Big|_{s=0} = V(0),$$

$$\frac{D}{dt}\tilde{V}(0) = \frac{D}{dt}\frac{\partial}{\partial s}\tilde{\gamma}\Big|_{s=t=0} = \frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}\Big|_{s=t=0} = \frac{D}{ds}X(0).$$

If we take $\frac{D}{ds}X(0)$ to satisfy $\frac{D}{ds}X(0) = \frac{D}{dt}V(0)$, then

$$\tilde{V}(0)=V(0), \quad \tilde{V}'(0)=V'(0),$$

and $\tilde{V}$ and V both satisfy the Jacobi equation (for $\tilde{V}$, this follows from part (i)) — which is a (linear) 2nd order ODE. Uniqueness of solutions gives $\tilde{V}(t)=V(t)$ $\forall t \in [0,1]$. $\square$

Remark 6.2. (i) $\tilde{\gamma}(s,t) = \gamma((1+as)t+bs)$ has variation vector field

$$V_{a,b}(t) = \frac{\partial}{\partial s}\tilde{\gamma}(s,t)\Big|_0 = (at+b)\gamma'(t).$$ Tangent to γ. Not interesting.

(ii) If V is a Jacobi field along γ, then $f(t) = \langle V(t), \gamma'(t)\rangle$ satisfies

$$f' = \langle V', \gamma'\rangle + \langle V, \underbrace{\gamma''}_{=0}\rangle = \langle V', \gamma'\rangle \Rightarrow f'' = \langle V'', \gamma'\rangle = \langle R(\gamma', V)\gamma', \gamma'\rangle = 0.$$

$\Rightarrow f$ is an affine function, $f(t) = at+b$. $\Rightarrow V^{\perp} := V - \underbrace{V_{a,b}}_{\text{Jacobi field from (i)}}$ is a Jacobi field with $\langle V^{\perp}, \gamma'\rangle = 0$ $\forall t$.

($V^{\perp}$ = projection of V to $(\gamma')^{\perp}$.)

Thus, one is usually only interested in such normal Jacobi fields.

Example 6.3. If (M,g) has constant sectional curvature κ, then for $V \perp \gamma'$ one has $R(V,\gamma')\gamma' = \kappa V$. (This follows from Proposition 5.9.) The Jacobi equation then reads

$$V'' + \kappa V = 0.$$

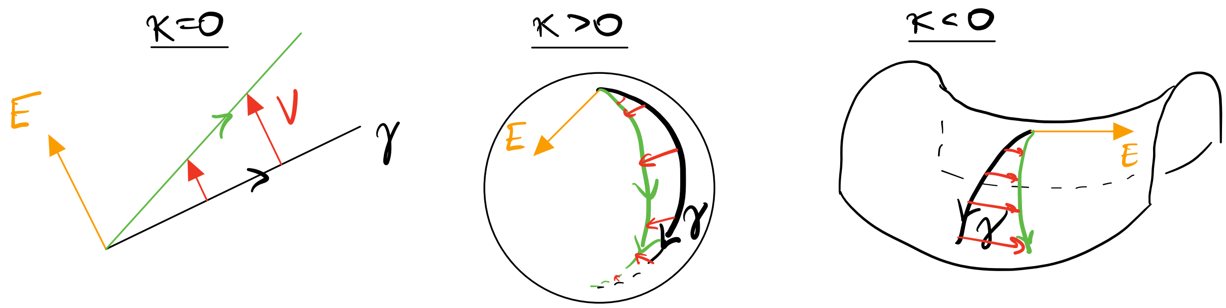
Its solutions are linear combinations of fE where E is a parallel unit vector field along γ normal to γ and $f'' + \kappa f = 0$, so

$$f(t) = a\, sn_\kappa(t) + b\, cn_\kappa(t) \quad \text{where}$$

$$sn_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t) & , \ \kappa > 0 \\ t & , \ \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t) & , \ \kappa < 0. \end{cases} \qquad (\text{so } sn_\kappa(0) = 0,\ sn_\kappa'(0) = 1)$$

$$cn_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa}\, x) & , \ \kappa > 0 \\ 1 & , \ \kappa = 0 \\ \cosh(\sqrt{-\kappa}\, x) & , \ \kappa < 0. \end{cases} \qquad (\text{so } cn_\kappa(0) = 1,\ cn_\kappa'(0) = 0)$$

If $V(0) = 0$, $V'(0) = aE$ (E unit, $\perp \gamma'(0)$), then $V(t) = a \cdot sn_\kappa(t) E$.



Jacobi fields are good tools to probe the curvature of (M,g). One can use them to prove $g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj} x^k x^l + O(|x|^3)$ in normal coordinates. (Exercise.) We can also use them to illustrate the Ricci curvature:

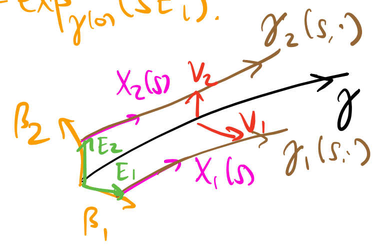
Example 6.4. $\cdot$ Let $\gamma : [0,1] \to M$ be a geodesic,

$E_1, \dots, E_{m-1} \in T_{\gamma(0)}M$ an orthonormal basis of $\gamma'(0)^\perp$,

$X_i(s) =$ parallel transport of $\gamma'(0)$ along $\beta_i(s) = \exp_{\gamma(0)}(sE_i)$.

We shoot off geodesics

$$\gamma_i(s,t) = \exp_{\beta_i(s)}(t\, X_i(s)).$$



Jacobi vector fields $V_i(t) = \frac{d}{ds}\gamma_i(s,t)\big|_{s=0}$.

- $V_i(0) = \frac{d}{ds}\exp_{\beta_i(s)}(0)\big|_{s=0} = \beta_i'(0) = E_i$

  $V_i'(0) = \frac{D}{ds}\frac{\partial}{\partial t}\gamma_i(s,t)\big|_{s=0} = \frac{D}{ds}X_i(s)\big|_{s=0} = 0$

- Let $E_i(t) =$ parallel transport of E_i along $\gamma(t)$

  $\Rightarrow \{E_1(t), \dots, E_{m-1}(t)\} =$ orthonormal basis of $\gamma'(t)$,

Write $V_i(t) = \sum_j V_{ij}(t)\, E_j(t)$. $\quad (\Rightarrow V_i'' = \sum_j V_{ij}'' E_j.)$

Jacobi equation: $\quad V_i'' + R(V_i, \gamma')\gamma' = 0$

$$\overset{\langle\cdot, E_j\rangle}{\Longrightarrow} \quad V_{ij}'' + \sum_k V_{ik}\langle R(E_k,\gamma')\gamma', E_j\rangle = 0. \quad \circledast$$

At $t=0$: $\quad V_{ij}(0) = \delta_{ij}, \quad V_{ij}'(0) = 0$

$$\overset{\circledast}{\Longrightarrow} \quad V_{ii}''(0) = -\langle R(E_i,\gamma')\gamma', E_i\rangle.$$

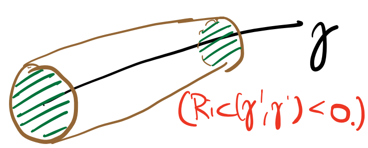
$$\Rightarrow (V_{ij}(t)) = I_{(m-1)\times(m-1)} - \frac{t^2}{2}\begin{pmatrix} \langle R(E_1,\gamma')\gamma', E_1\rangle & & * \\ & \ddots & \\ * & & \langle R(E_{m-1},\gamma')\gamma', E_{m-1}\rangle \end{pmatrix}.$$

- $\mathrm{vol}(B^{m-1}(s))\det(V_{ij}(t)) \approx (m-1)$-dim. area of the cross section at time t of the geodesics with velocity parallel to $\gamma'(0)$ shot off within distance s from $\gamma(0)$

Euclidean ball of radius s

$$\| \quad C_{m-1}s^{m-1}\cdot\left(1 - \frac{t^2}{2}\sum_{i=1}^{m-1}\langle R(E_i,\gamma')\gamma', E_i\rangle + O(t^4)\right)$$

$$= C_{m-1}s^{m-1}\left(1 - \frac{t^2}{2}\mathrm{Ric}(\gamma'(0),\gamma'(0)) + O(t^4)\right).$$



So $\mathrm{Ric}(\gamma'(0),\gamma'(0)) > 0$ means: cross sectional area decreases.

Definition 6.5. Let $\gamma : [a,b] \to M$ be a geodesic, $p = \gamma(a)$, $q = \gamma(b)$. Then q is *conjugate* to p along γ (or a *conjugate point*) if there exists a Jacobi field $J \not\equiv 0$ along γ with $J(0) = 0$, $J(b) = 0$.

Such a Jacobi field J is thus the variation vector field of a variation of γ by geodesics γ_s s.t. $\gamma_s(b) = \gamma(b) + O(s^2)$. Necessarily, J is normal, since the affine function $t \mapsto \langle J, \gamma' \rangle$ vanishes at $t = 0, b$, and is thus $\equiv 0$.

Example 6.6. On $M = S^2$ with the standard metric,

$\gamma =$ great circle (parameterized by arc length),

$\gamma(\pi)$ is conjugate to $\gamma(0)$ along γ.

Lemma 6.7. Let $\gamma(t) = \exp_p(tv)$. Then $\gamma(t_0)$ $(t_0 > 0)$ is conjugate to $\gamma(0)$ if and only if $d_{t_0 v} \exp_p : T_{t_0 v}(T_p M) \to T_{\gamma(t_0)} M$ is not injective.

Proof. Given $w \in T_p M$, $J(t) := \frac{d}{ds} \exp_p(t(v+sw))\big|_{s=0}$ is the unique Jacobi field with

$$J(0) = 0, \quad J'(0) = \frac{D}{ds}\frac{\partial}{\partial t} \exp_p(t(v+sw))\Big|_{t=s=0} = w.$$

($\Leftarrow$) If $w \in T_p M$, $w \neq 0$, satisfies $d_{t_0 v}\exp_p(w) = 0$, then

$J(t_0) = d_{t_0 v}\exp_p(t_0 w) \overset{(*)}{=} 0$, but $J'(0) = w \neq 0$, so $J \not\equiv 0$.

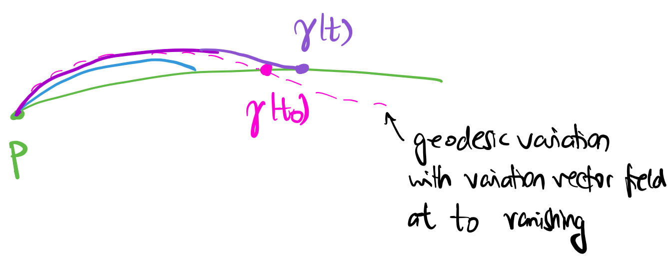
$\Rightarrow \gamma(t_0)$ is conjugate to $\gamma(0)$.

($\Rightarrow$) If $J \not\equiv 0$ is a Jacobi field with $J(0) = 0 = J(t_0)$, it is of the above form with $w = J'(0) \neq 0 \Rightarrow w \in \ker d_{t_0 v}\exp_p$ (see $(*)$). $\square$

We shall prove that the first conjugate point $\gamma(t_0)$ along a unit speed geodesic γ issuing from a point p has the following property:

(i) for $0 \le t \le t_0$, $\gamma|_{[0,t]}$ is the shortest curve from $\gamma(0)$ to $\gamma(t)$ among all nearby curves;

(ii) for $t > t_0$, there exists a curve from $\gamma(0)$ to $\gamma(t)$ near γ with strictly shorter length $< t = L(\gamma|_{[0,t]})$.

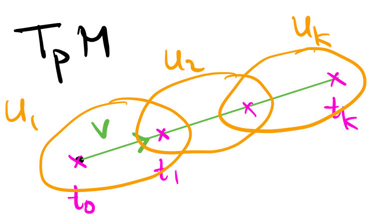


Theorem 6.8. Let $\gamma : [0,\ell] \to M$ be a unit speed geodesic from p to q. Suppose that no $\gamma(t)$ with $t \in (0,\ell]$ is conjugate to p along γ. Then $\exists\ \varepsilon > 0$ s.t. $\forall\ c : [0,\ell] \to M$ piecewise C^∞

$$c(0) = p,\quad c(1) = q,$$
$$d(c(t), \gamma(t)) < \varepsilon \quad \forall\, t,$$

one has $L(c) \ge L(\gamma)$, with equality iff c is a monotone reparameterization of γ.

Proof. • Write $\gamma(t) = \exp_p(tv)$. By assumption, $d_{tv}\exp_p : T_pM \to T_{\gamma(t)}M$ is bijective for all $t \in [0,\ell]$. By the inverse function theorem



and the compactness of $[0,\ell]v$,

$\exists\ 0 = t_0 < t_1 < \cdots < t_k = \ell$,

open sets $U_1, U_2, \ldots, U_k \subset T_pM$, such that $\cdot\ [t_{i-1}, t_i]v \subset U_i \quad \forall\ 1 \le i \le k$,

$\cdot\ \exp_p|_{U_i} : U_i \to V_i := \exp_p(U_i)$ is a diffeomorphism.

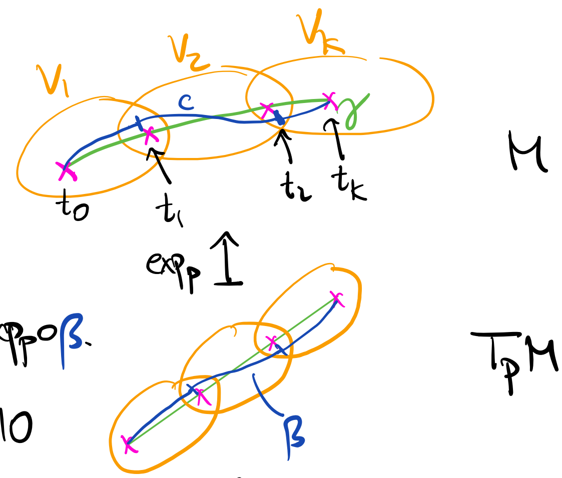
- Choose now $\varepsilon > 0$ so that every V_i contains the closed ε-neighborhood of $\gamma([t_{i-1}, t_i])$.
- Let c be as in the statement of the theorem. Then

$$c([t_{i-1}, t_i]) \subset V_i \quad \forall\ i = 1, \dots, k.$$

Define then $\beta : [0, \ell] \to T_pM$

by $\beta(t) = (\exp_p|_{U_i})^{-1} c(t)$,

$t \in [t_{i-1}, t_i]$.



Thus β is piecewise C^∞, and $c = \exp_p \circ \beta$.

Arguing as in the proof of Theorem 4.10 ("Case 1"), we obtain $L(c) \ge L(\gamma)$ from the Gauss Lemma, and the stated characterization of the case of equality. $\square$

Remark 6.9. The conclusion is not true for far-from-γ curves c; example: cylinder.

(or just $c(t) = p\ \forall t$.)

To understand what happens after the first conjugate point (item (ii) above), we need to analyze the variation of lengths to 2^{nd} order. See Theorem 6.12 for the result.

6.2. Second variation of length

Let $\gamma : [a,b] \to M$ be a unit speed geodesic: $\begin{cases} \frac{D}{dt}\gamma'(t) = \nabla_{\partial_t}\gamma'(t) = 0, \\ |\gamma'(t)| = 1 \ \forall t. \end{cases}$

Let $\tilde{\gamma} : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a piecewise C^∞ variation of γ; that is,

- $\tilde{\gamma}(0,t) = \gamma(t)$
- $\tilde{\gamma}$ is continuous
- $\exists\ a = t_0 < t_1 < \cdots < t_k = b$ s.t. $\tilde{\gamma}|_{(-\varepsilon,\varepsilon)\times[t_i,t_{i+1}]}$ is C^∞ $\forall\, 0 \le i < k$.

Write $\gamma_s(t) = \tilde{\gamma}(s,t)$,

$\tilde{V}(s,t) = \frac{\partial}{\partial s}\tilde{\gamma}(s,t)$,

and $V(t) = \tilde{V}(0,t)$ for the variation vector field.

Note that $\frac{d}{ds}L(\gamma_s)|_{s=0} = 0$. Nontrivial information is contained in $\frac{d^2}{ds^2}L(\gamma_s)|_{s=0}$:

Theorem 6.10. (Second variation of length.)

$$\frac{d^2}{ds^2}L(\gamma_s)\Big|_{s=0} = \int_a^b |(V')^\perp|^2 - R(V,\gamma',\gamma',V)\,dt + \left\langle \frac{D}{ds}\tilde{V}\Big|_{s=0}, \gamma' \right\rangle\Big|_a^b.$$

Here, $V'(t) = \frac{D}{dt}V(t)\ (= \nabla_{\gamma'}V)$,

$(V')^\perp = V' - \langle V', \gamma'\rangle\gamma'$ is the component of V' normal to γ'.

Remark 6.11. (i) Normal variation $(V \perp \gamma')$

$$\Rightarrow \langle V', \gamma'\rangle = \frac{d}{dt}\underbrace{\langle V, \gamma'\rangle}_{=0} - \langle V, \underset{0}{\underset{\parallel}{\gamma''}}\rangle = 0, \text{ so } (V')^\perp = V'.$$

(ii) Proper variation $(\tilde{\gamma}(s,a) = \gamma(a),\ \tilde{\gamma}(s,b) = \gamma(b)) \Rightarrow \frac{D}{ds}\tilde{V}|_{s=0} = 0$

$$\Rightarrow \frac{d^2}{ds^2}L(\gamma_s)\Big|_{s=0} = \int_a^b |V'^\perp|^2 - R(V,\gamma',\gamma',V)\,dt.$$

(iii) Normal and proper variation which is moreover C^∞:

$$\frac{d^2}{ds^2} L(\gamma_s)\Big|_{s=0} = -\int_a^b \langle V'' + R(V,\gamma')\gamma', V\rangle \, dt \qquad (*)$$

since $|V'|^2 = \frac{d}{dt}\langle V, V'\rangle - \langle V, V''\rangle$ and $\int_a^b \frac{d}{dt}\langle V, V'\rangle \, dt = 0$.

(Positive sectional curvature of span $\{V, \gamma'\}$ tends to decrease length. Rigorous results later — note that $(*)$ requires $V(a) = V(b) = 0$, so the V'' term typically must be nonzero and competes with the curvature term.)

Proof of Theorem 6.10. Write $\tilde{T}(s,t) = \frac{\partial}{\partial t}\tilde{\gamma}(s,t)$.

- $$\frac{d}{ds} L(\gamma_s) = \int_a^b \partial_s \langle \tilde{T}(s,t), \tilde{T}(s,t)\rangle^{\frac{1}{2}} \, dt$$
$$= \int_a^b \frac{1}{|\tilde{T}(s,t)|} \left\langle \frac{D}{ds}\tilde{T}(s,t), \tilde{T}(s,t)\right\rangle dt$$

- $$\frac{d^2}{ds^2} L(\gamma_s)\Big|_{s=0} = \int_a^b \frac{1}{|\tilde{T}|}\left(\left\langle \frac{D^2}{ds^2}\tilde{T}, \tilde{T}\right\rangle + \left|\frac{D}{ds}\tilde{T}\right|^2\right) - \frac{1}{|\tilde{T}|^3}\left\langle \frac{D}{ds}\tilde{T}, \tilde{T}\right\rangle^2 dt$$

$\tilde{T}(0,t) = \gamma'(t)$, $|\gamma'(t)| = 1$

$$= \int_a^b \left\langle \frac{D}{ds} \underbrace{\frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}}_{= \frac{D}{dt}\frac{\partial}{\partial s}\tilde{\gamma} = \frac{D}{dt}\tilde{V}}, \gamma'\right\rangle + \Big|\underbrace{\frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}}_{= \frac{D}{dt}V}\Big|^2 - \Big\langle \underbrace{\frac{D}{ds}\frac{\partial}{\partial t}\tilde{\gamma}}_{= \frac{D}{dt}V}, \gamma'\Big\rangle^2 dt$$

$$= \int_a^b \left\langle \frac{D}{ds}\frac{D}{\partial t}\tilde{V}, \gamma'\right\rangle + \underbrace{\left(|V'|^2 - \langle V', \gamma'\rangle^2\right)}_{= |(V')^\perp|^2} dt. \qquad (\#)$$

Furthermore, at $s=0$,

$$\left\langle \frac{D}{ds}\frac{D}{dt}\tilde{V}, \gamma'\right\rangle = \left\langle \frac{D}{dt}\frac{D}{ds}\tilde{V}, \gamma'\right\rangle + \left\langle \left[\frac{D}{ds}, \frac{D}{dt}\right]\tilde{V}, \gamma'\right\rangle$$

$$\overset{\text{exercise}}{=} \frac{d}{dt}\left\langle \frac{D}{ds}\tilde{V}, \gamma'\right\rangle - \left\langle \frac{D}{ds}\tilde{V}, \underbrace{\gamma''}_{=0}\right\rangle$$

$$+ \langle R(V,\gamma')V, \gamma'\rangle.$$

Plugging this into $(\#)$ proves the theorem. $\square$

We can now complete our previous analysis of conjugate points.

Theorem 6.12. Let $\gamma : [0,\ell] \to M$ be a unit speed geodesic. Suppose there exists $t_0 \in (0,\ell)$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. Then there exists a piecewise smooth, proper, normal variation $\tilde{\gamma} : (-\varepsilon,\varepsilon) \times [0,\ell] \to M$ of $\gamma = \tilde{\gamma}(0,\cdot)$ s.t.

$$L(\gamma_s) < L(\gamma) \quad \forall \ s \in (-\varepsilon,\varepsilon) \setminus \{0\}.$$

In particular, $\gamma|_{[0,\ell]}$ is not the shortest curve from $\gamma(0)$ to $\gamma(\ell)$.

Proof. • We define the index form

$$I(X,Y) := \int_0^\ell \langle X', Y' \rangle - R(X,\gamma',\gamma',Y) \, dt$$

for any two piecewise C^∞ vector fields X, Y along γ which are normal to γ'. (Thus if X is the variation vector field of a proper, normal variation $\tilde{\gamma}$ of γ, then $\frac{d^2}{ds^2} L(\gamma_s)\big|_{s=0} = I(X,X)$.)

• Let J be a nontrivial Jacobi field along γ with $J(0) = 0 = J(t_0)$.

Let $X(t) = \begin{cases} J(t), & 0 \le t \le t_0 \\ 0, & t_0 \le t \le \ell. \end{cases}$

Let $Y(t)$ be a normal vector field, $Y(0)=0$, to be chosen momentarily.

$$\Rightarrow I(X,X) = 0,$$

$$I(X,Y) = \int_0^{t_0} \frac{d}{dt}\langle X', Y \rangle - \langle \overbrace{X'' + R(X,\gamma')\gamma'}^{=0}, Y \rangle \, dt$$

$$= \langle X'(t_0), Y(t_0) \rangle.$$

Let us require $Y(t_0) = -X'(t_0)$, then (since $X'(t_0) \neq 0$ in view of

$X \not\equiv 0$)

$$I(X,Y) = -|X'(t_0)|^2 < 0.$$

$$\Rightarrow I(X+\lambda Y, X+\lambda Y) = I(X,X) + 2\lambda I(X,Y) + \lambda^2 I(Y,Y)$$
$$= -2\lambda |X'(t_0)|^2 + \lambda^2 I(Y,Y)$$
$$< 0$$

for all sufficiently small $\lambda > 0$.

$\Rightarrow \frac{d^2}{ds^2} L(\gamma_s)\Big|_{s=0} < 0$ for $\tilde{\gamma}$ = piecewise C^∞ variation of γ with variation vector field $X + \lambda Y$. $\square$

6.3. Further applications of the 2nd variation formula

We present some results relating geometric and topological information.

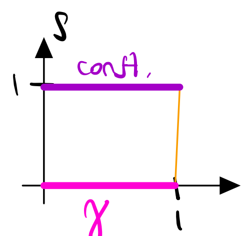
Theorem 6.13. (Synge 1936.) Let M be a compact and orientable Riemannian manifold with $\dim M$ even and $K>0$ (positive sectional curvature). Then $\pi_1(M)=\{0\}$, i.e. M is simply connected.

Recall: (i) Orientable means: $\exists$ cover $M=\bigcup_{i\in I} U_i$ by charts (U_i,ϕ_i) s.t. $\det(d(\phi_i\circ\phi_j^{-1}))>0$ $\forall\, i,j$ with $U_i\cap U_j\neq\emptyset$.

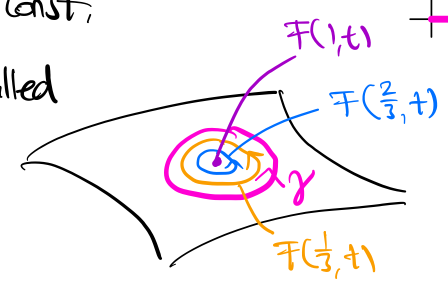
(ii) Simply connected means: $\forall$ continuous curves $\gamma:[0,1]\to M$, $\gamma(0)=\gamma(1)$, $\exists$ continuous map $F:[0,1]_s\times[0,1]_t\to M$ s.t. $F(0,t)=\gamma(t)$,

$F(s,0)=F(s,1)$ $\forall\, s\in[0,1]$,

$F(1,t)=\text{const}$.



(Such an F is called a homotopy.)



Remark 6.14. (i) $\dim M$ even is necessary: $\mathbb{RP}^{2k+1}=S^{2k+1}/_{x\sim -x}$ (equivalently, the space of lines through 0 in $\mathbb{R}^{2k+2}$), equipped with the standard metric, is orientable and has $K=1$ (just like S^{2k+1}), but $\pi_1(\mathbb{RP}^{2k+1})=\mathbb{Z}/_{2\mathbb{Z}}$. A nontrivial loop is

$$\gamma(t)=[(\cos t,\sin t,0,\dots,0)],\quad 0\le t\le\pi.$$

(ii) The theorem implies that even-dimensional compact manifolds M with $\pi_1(M)\neq\{0\},\ \mathbb{Z}/_{2\mathbb{Z}}$ do not admit any metric with positive sectional curvature. (E.g. $M = \mathbb{T}^2$, with $\pi_1 = \mathbb{Z}^2$) (Indeed, if M is orientable, this is immediate. Otherwise, consider the 2-sheeted orientable covering $\tilde{M} \xrightarrow{\pi} M$, which if $\pi_1(M) \neq \{0\}, \mathbb{Z}/2\mathbb{Z}$ has non-trivial fundamental group $\pi_1(\tilde{M}) \neq 0$; a metric g on M with $K > 0$ would pull back to $\tilde{g} = \pi^* g$, also with $\tilde{K}_p = K_{\pi(p)} > 0$, contradicting the theorem.)

Proof of Theorem 6.13. • Suppose, for the sake of contradiction, that M is not simply connected. Then there exists a closed geodesic $\gamma : [0,\ell] \to M$ i.e. $\gamma(0) = \gamma(\ell)$ and $\gamma'(0) = \gamma'(\ell)$, which has minimal length among all variations and is not null-homotopic. (In fact, there exists such a γ in every free homotopy class $[\alpha] \in \pi_1(M)$, $[\alpha] \neq 0$.)

• Let $H = \gamma'(0)^\perp \subset T_{\gamma(0)}M$; this is an odd-dimensional hyperplane.

Parallel transport along γ defines an isometry $P : H \to H$.

(Indeed, $P : T_{\gamma(0)}M \to T_{\gamma(0)}M$ is an isometry which maps $\gamma'(0)$ to $\gamma'(\ell) = \gamma'(0)$, and thus H to H.) Since the parallel transport of a positively oriented ONB of $T_{\gamma(0)}M$ remains positively oriented, P is orientation-preserving (i.e. $\det P = +1$).

Therefore P has an eigenvector $0 \neq v \in H$ with $Pv = v$.

• Let $V(t) =$ parallel transport of v along γ

$(\Rightarrow V(0) = V(\ell) = v)$,

$\tilde{\gamma} =$ variation of γ through closed curves with variation vector field V.

$$\Rightarrow \frac{d^2}{ds^2} L(\gamma_s)\Big|_{s=0} = -\int_0^\ell \langle \underbrace{V''}_{=0} + \underbrace{R(V,\gamma')\gamma', V\rangle}_{>0 \text{ since } (M,g) \text{ has positive sectional curvature}} dt < 0,$$

contradicting the length minimization property of γ. □

Theorem 6.15. (Myers 1941.) Let (M,g) be a complete connected Riemannian manifold. Suppose $\exists$ constant $\kappa > 0$ s.t. $\mathrm{Ric} \geq (m-1)\kappa$, i.e.

$$\mathrm{Ric}(X,X) \geq (m-1)\kappa \quad \forall\ X \in TM,\ |X| = 1.$$

Then

$$\mathrm{diam}(M) := \sup_{p,q \in M} d(p,q) \leq \frac{\pi}{\sqrt{\kappa}}. \qquad (*)$$

In particular, M is compact.

Remark 6.16. The estimate $(*)$ is sharp for $M =$ m-sphere with radius $\kappa^{-\frac{1}{2}}$.

Proof of Theorem 6.15. • Pick $p, q \in M$ with $d(p,q) =: \ell$. Let $\gamma : [0,\ell] \to M$ be the unit speed geodesic with $\gamma(0) = p$, $\gamma(\ell) = q$.

• Let $E_1, \dots, E_m$ be a parallel orthonormal frame of TM along γ, with $E_m(t) = \gamma'(t)$. (Construction: pick $E_1(0), \dots, E_m(0) = \gamma'(0) \in T_pM$ to be an orthonormal basis, and define $E_i(t)$, $i = 1, \dots, m$, by parallel transport along γ. Use Lemma 3.17.)

For $i = 1, \dots, m-1$, consider a proper normal variation of γ with variation vector field $f(t)E_i(t)$ where $f(0)=f(\ell)=0$. By Theorem 6.10,

$$0 \leq \int_0^\ell |f'(t)E_i(t)|^2 - R(f(t)E_i(t), \gamma'(t))\gamma'(t), f(t)E_i(t)\rangle \, dt.$$

Summing over $i=1,\dots,m-1$ gives

$$0 \leq \int_0^\ell (m-1) f'(t)^2 - Ric(\gamma'(t),\gamma'(t)) f(t)^2 \, dt$$

$$\leq (m-1)\int_0^\ell (f'(t)^2 - \kappa f(t)^2)\, dt$$

$$\Rightarrow \kappa \leq \frac{\int_0^\ell f'(t)^2 \, dt}{\int_0^\ell f(t)^2 \, dt}. \qquad (*)$$

· Here, $f \in C^\infty([0,\ell])$ with $f(0)=f(\ell)=0$ is arbitrary. Write $f(t)=f_0(t/\ell)$, then $f'(t)=\ell^{-1} f_0'(t/\ell)$, so $(*)$ is equivalent to

$$\kappa \leq \ell^{-2} R(f_0) \quad \text{where} \quad R(f_0) := \frac{\int_0^1 f_0'(t)^2 dt}{\int_0^1 f_0(t)^2 dt}.$$

$$\Leftrightarrow \ell \leq \sqrt{\frac{R(f_0)}{\kappa}}. \qquad (**)$$

· To get the sharpest possible bound on ℓ from $(**)$, we want to minimize $R(f_0)$ (under the condition $f_0(0)=f_0(\ell)=0$).

Fact: $f_0(t)=\sin(\pi t)$ is optimal, and $R(f_0)=\pi^2$. $\square$