9. Differential forms

Differential forms are special types of tensors on smooth manifolds. We first discuss the relevant linear algebra before studying their use and properties on manifolds. Initially, we do not need Riemannian metrics.

9.1. Linear algebra

Let V be an m-dimensional $\mathbb{R}$-vector space.

Definition 9.1. Let $k \in \mathbb{N}_0$. Then $\Lambda^k V^*$ is the vector space of all alternating k-multilinear maps $\omega : \underbrace{V \times \cdots \times V}_{k \text{ factors}} \to \mathbb{R}$. That is,

$$\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = -\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k).$$

- For $\sigma \in S_k = \text{Bijections}\{1, \dots, k\}$, we thus have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\, \omega(v_1, \dots, v_k).$$

sign of the permutation $=: \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{pmatrix}$.

- Let $e_1, \dots, e_m \in V$ be a basis, and let $\varepsilon^1, \dots, \varepsilon^m \in V^*$ be the dual basis. Given $\omega \in \Lambda^k V^*$, set $\omega_I = \omega(e_{i_1}, \dots, e_{i_k})$ for $I = (i_1, \dots, i_k) \in \{1, \dots, m\}^k$ which are ordered, i.e. $i_1 < i_2 < \cdots < i_k$.

Then $\omega = \sum_I \omega_I \varepsilon^I$, where $\varepsilon^I := \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \in \Lambda^k V$ is defined via $\varepsilon^I(e_{j_1}, \dots, e_{j_k}) = \operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ (declared to be 0 if $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$).

$\Longrightarrow \{\varepsilon^I : I \subset \{1, \dots, m\}^k \text{ ordered}\}$ is a basis of $\Lambda^k V^*$

$\Rightarrow \dim(\Lambda^k V^*) = \binom{m}{k}$.

- In particular, $\Lambda^k V^* = \{0\}$ for $k > m = \dim V$,
  $\Lambda^1 V^* = V^*$,

Convention: $\Lambda^0 V^* = \mathbb{R}$.

Definition 9.2. Given $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$, define $\omega \wedge \eta \in \Lambda^{k+\ell} V^*$ by

$$(\omega\wedge\eta)(v_1,\dots,v_{k+\ell}) = \frac{1}{k!\,\ell!}\sum_{\sigma\in S_{k+\ell}} (-1)^{\operatorname{sgn}\sigma}\,\omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\,\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}).$$

(Check that this indeed defines an element of $\Lambda^{k+\ell}V^*$.)

Remark 9.3. (i) $(\omega\wedge\eta)\wedge\xi = \omega\wedge(\eta\wedge\xi)$. (Exercise.)

(ii) $\omega\wedge\eta = (-1)^{k\ell}\eta\wedge\omega$. (Exercise.)

(iii) If $\omega_1,\dots,\omega_k \in V^* = \Lambda^1 V^*$, then

$$(\omega_1\wedge\omega_2)(v_1,v_2) = \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1),$$

and more generally

$$(\omega_1\wedge\cdots\wedge\omega_k)(v_1,\dots,v_k) = \sum_{\sigma\in S_k}(-1)^{\operatorname{sgn}\sigma}\,\omega_1(v_{\sigma(1)})\cdots\omega_n(v_{\sigma(n)})$$

$$= \det\big(\omega_i(v_j)\big)_{i,j=1,\dots,k}.$$

In particular, the notation $\varepsilon^I = \varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}$ is consistent with Definition 9.2.

Definition 9.4. (Functoriality of $V\mapsto\Lambda^k V^*$) For a linear map $A: V\to W$, set $A^*: \Lambda^k W^* \to \Lambda^k V^*$,

$$(A^*\omega)(v_1,\dots,v_k) = \omega(Av_1,\dots,Av_k).$$

Example 9.5. If $A: V \to V$, then $A^*: \Lambda^m V^* \longrightarrow \Lambda^m V^*$ is a linear map on a 1-dimensional vector space, and thus multiplication by a real number — $\det(A)$. Indeed,

$$\big(A^*(\varepsilon^1 \wedge \cdots \wedge \varepsilon^m)\big)(e_1, \dots, e_m) = (\varepsilon^1 \wedge \cdots \wedge \varepsilon^m)(Ae_1, \dots, Ae_m)$$

$$\overset{9.3(iii)}{=} \det\big(\varepsilon^i(Ae_j)\big) = \det(A_{ij}).$$

9.2. Differential forms on manifolds; integration

From now on, M denotes an m-dimensional smooth manifold.

Definition 9.6. If $E \xrightarrow{\pi} M$ is a smooth vector bundle of rank r, then $\Lambda^k E^* \xrightarrow{\tilde{\pi}} M$ is the bundle of k-multilinear alternating maps $E_p \times \cdots \times E_p \to \mathbb{R}$. That is: $(\Lambda^k E^*)_p = \Lambda^k (E_p)^*$.

Remark 9.7. (i) If $\{U_\alpha\}$ is a cover of M and $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ are local trivializations of E with transition functions $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^r)$ (i.e. $(\psi_\beta \circ \psi_\alpha^{-1})(q, e) = (q, \tau_{\beta\alpha}(q)v))$, then we get trivializations $\tilde{\psi}_\alpha : \tilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{\binom{r}{k}}$ of $\Lambda^k E^*$ with transition functions

$$\tilde{\tau}_{\beta\alpha} = (\tau^*_{\beta\alpha})^{-1} : U_\alpha \cap U_\beta \to GL(\Lambda^k \mathbb{R}^r).$$

(ii) $\Lambda^k E^*$ is a subbundle of $E^* \otimes \cdots \otimes E^*$.

(iii) A $C^\infty(M)$-multilinear map $\Gamma(E)^k \to C^\infty(M)$ defines a section of $\Lambda^k E^*$ iff it is alternating.

Definition 9.8. • $\Lambda^k T^* M$ = k^{th} exterior power of T^*M

• $\Omega^k(M) = \Gamma(\Lambda^k T^* M)$: space ($C^\infty(M)$-module) of differential k-forms (= alternating $(k,0)$-tensors).

• In local coordinates $x^1, \dots, x^m$ on a chart $U \subset M$, a basis of $\Lambda^k T_p^* M$ for $p \in U$ is given by $d_p x^I = d_p x^{i_1} \wedge \cdots \wedge d_p x^{i_k}$ $(I \subset \{1, \dots, m\}^k$ ordered$)$.

• $\omega \in \Omega^k(M)$ is in such a chart given by

$$\omega_p = \sum_I \omega_I(p)\, d_p x^I, \quad \text{where} \quad \omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right) \in C^\infty(U).$$

(That is, for $V_1, \dots, V_k \in T_pM$, written as $V_j = \sum V_j^i \frac{\partial}{\partial x^i}$,

$$\omega_p(V_1, \dots, V_k) = \sum_I \omega_I(p) \det\left(V_j^{i_\ell}\right)_{j,\ell=1,\dots,k}.)$$

In short, $\omega = \sum_I \omega_I\, dx^I$.

- The linear algebra operations on $\Lambda^k T_q^* M$ can be applied pointwise:

$$\omega \in \Omega^k(M),\ \eta \in \Omega^\ell(M) \implies \omega \wedge \eta \in \Omega^{k+\ell}(M),$$

with the same properties as in Remark 9.3.

Example 9.9. (i) $f \in C^\infty(M) \implies df \in \Gamma(T^*M) = \Omega^1(M)$.

(ii) g Riemannian metric on M; assume that M is orientable.

In local coordinates $x^1, \dots, x^m$ s.t. $\partial_{x^1}, \dots, \partial_{x^m}$ is positively oriented,

define $\omega := \sqrt{\det(g(\partial_{x^i}, \partial_{x^j}))}\, dx^1 \wedge \dots \wedge dx^m$.

If $y = \psi(x)$ with $\det\left(\frac{\partial \psi}{\partial x}\right) > 0$ (so also $\partial_{y^1}, \dots, \partial_{y^m}$ is positively oriented), then $\partial_{x^i} = \frac{\partial \psi^j}{\partial x^i} \partial_{y^j}$

$$\implies \sqrt{\det(g(\partial_{x^i}, \partial_{x^j}))} = \det\left(\frac{\partial \psi}{\partial x}\right) \sqrt{\det(g(\partial_{y^i}, \partial_{y^j}))};$$

$$dy^1 \wedge \dots \wedge dy^m = \left(\frac{\partial \psi^1}{\partial x^{i_1}} dx^{i_1}\right) \wedge \dots \wedge \left(\frac{\partial \psi^m}{\partial x^{i_m}} dx^{i_m}\right)$$

$$= \det\left(\frac{\partial \psi}{\partial x}\right) dx^1 \wedge \dots \wedge dx^n.$$

So ω is well-defined, and it is called the Riemannian volume form of (M, g).

Note that one can integrate 1-forms along curves, and m-forms over open subsets of an m-dimensional oriented manifold; this follows from the same computation involving the Jacobi determinant as above. Indeed, expressing

$$\omega = \omega_x \, dx^1 \wedge \cdots \wedge dx^m = \omega_y \, dy^1 \wedge \cdots \wedge dy^m, \qquad y = \psi(x)$$

we have $\omega_x = \omega_y \det\left(\frac{\partial \psi}{\partial x}\right)$. For $\chi \in C_c^\infty(M)$ with support in the intersection of both charts ϕ (x-coordinates), $\tilde{\phi}$ (y-coordinates)

$$\int \chi \omega := \int_{\mathbb{R}^m} (\chi \circ \phi^{-1}) \, \omega_x \, dx^1 \cdots dx^m$$

$$= \int_{\mathbb{R}^m} (\chi \circ \phi^{-1}) \, \omega_y \det\left(\frac{\partial \psi}{\partial x}\right) dx^1 \cdots dx^m$$

$$= \int_{\mathbb{R}^m} (\chi \circ \tilde{\phi}^{-1}) \, \omega_y \, dy^1 \cdots dy^m.$$

Using a partition of unity, we can thus define

$$\int_M \omega \qquad \text{for } \omega \in \Omega_c^m(M) \qquad \text{(i.e. } \omega \in \Omega^m(M), \text{ supp } \omega \subset M \text{ compact).}$$

We gain some flexibility as follows:

Definition 9.10. (Pullback of forms) Let $F: M \to N$ be a C^∞ map between C^∞ manifolds. Then $F^*: \Omega^k(N) \to \Omega^k(M)$ is defined by

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_{F(p)}(d_pF(v_1), \ldots, d_pF(v_k))$$

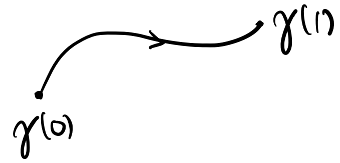
for $\omega \in \Omega^k(N)$, $p \in N$, $v_1, \ldots, v_k \in T_pM$.

If N is an n-dimensional submanifold of M and $i: N \hookrightarrow M$ is the inclusion map, then for $\omega \in \Omega_c^n(M)$, we may define

$$\int_N \omega := \int_N i^* \omega \qquad \text{(integral of } \omega \text{ over } N).$$

Example 9.10. $M = \mathbb{R}^2 \setminus \{0\}$.

(i) $\omega = df$, $\quad N = \gamma([0,1])$ (γ embedding). N is oriented, with $d_t\gamma \in T_{\gamma(t)}N$ being a positively oriented basis.



$$\Rightarrow \int_\gamma \omega := \int_N i^* \omega = \int_N i^* df.$$

In local coordinates, write $\tilde{f}(t) = f(\gamma(t))$, then

$$\begin{aligned}
(i^* df)(d_t\gamma(\partial_t)) &= df(d_t\gamma(\partial_t)) \\
&= \frac{d}{ds} f(\gamma(t+s))\Big|_{s=0} \\
&= \frac{d}{ds} \tilde{f}(t+s)\Big|_{s=0} \\
&= \tilde{f}'(t)
\end{aligned}$$

$$\Rightarrow \quad i^* df = \tilde{f}' \, dt$$

$$\Rightarrow \int_\gamma df = \int_0^1 \tilde{f}'(t)\, dt = \tilde{f}(1) - \tilde{f}(0) = f(\gamma(1)) - f(\gamma(0)).$$

(ii) $\displaystyle\int_{\circlearrowleft} \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi \quad \Rightarrow \quad \frac{x\,dy - y\,dx}{x^2+y^2} \neq df \quad (f \in C^\infty(\mathbb{R}^2 \setminus \{0\}))$.

9.2 Exterior derivative

- Consider a 1-form $\omega \in \Omega^1(M)$ in local coordinates,

$$\omega = \sum_{i=1}^{m} \omega_i \, dx^i , \qquad \omega_i = \omega_i(x) \in C^\infty(\mathbb{R}^m).$$

Then a necessary condition for $\omega = df$ ($\Leftrightarrow \omega_i = \partial_{x^i} f$) is that

$$\circledast \quad \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0 \qquad \forall i,j = 1,\dots,m \qquad \text{(equality of mixed partials of } f\text{).}$$

- $\circledast$ is also a sufficient condition — set $f(x) = \int_\gamma \omega$ $\quad (\gamma(t) = tx, \ 0 \le t \le 1)$

$$= \int_0^1 \sum_{i=1}^{m} \omega_i(tx) \, x^i \, dt.$$

(Indeed, $\partial_{x^j} \int_1^0 \sum_{i=1}^m \omega_i(tx) x^i \, dt = \int_1^0 \omega_j(tx) + (\partial_{x^j}\omega_i)(tx) \, t x^i \, dt$

$$\overset{\circledast}{=} \int_1^0 \omega_j(tx) + (\partial_{x^i}\omega_j)(tx) \, t x^i \, dt$$

$$= \int_1^0 \partial_t \left(t \, \omega_j(tx) \right) dt = \omega_j(x).)$$

- The condition $\circledast$ is coordinate-independent. This follows either by direct calculation, or from the coordinate-independence of the equivalent (local) equation $\omega = df$ $(\exists f)$. This suggest defining

$$d\omega := \sum_{i,j=1}^{m} \frac{\partial \omega_i}{\partial x^j} \, dx^j \wedge dx^i = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j .$$

So $d \circ d = 0$ $\quad (\omega = df$ on $\mathbb{R}^n \Rightarrow d\omega = 0)$.

This admits a vast generalization.

Theorem 9.11. (Exterior derivatives) There exists a unique sequence of differential operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, $k = 0, 1, \dots$, with the following properties:

(i) d on $\Omega^0(M) = C^\infty(M)$ is the usual operator

$$f \mapsto df , \qquad df(V) = V(f) \qquad (V \in \Gamma(TM))$$

(ii) $d^2 = d \circ d = 0$ (as a map $\Omega^k(M) \to \Omega^{k+2}(M)$, for all $k \in \mathbb{N}_0$).

(iii) Leibniz rule: for $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Proof. (a) Uniqueness. In local coordinates, $\omega = \sum_I \omega_I \, dx^I$ for smooth ω_I ($I \subset \{1,\dots,m\}^k$ ordered) has exterior derivative

$$d\omega \underset{\text{d is linear}}{=} \sum_I d(\omega_I \, dx^I) \overset{(iii)}{=} \sum_I d\omega_I \wedge dx^I + \omega_I \underbrace{d(dx^{i_1} \wedge \dots \wedge dx^{i_k})}_{=\sum_{j=1}^{k} (\pm) dx^{i_1} \wedge \dots \wedge \underbrace{d(dx^{i_j})}_{\overset{(ii)}{=}0} \wedge \dots \wedge dx^{i_k}}$$

$$= \sum_I \underbrace{d\omega_I}_{\substack{\text{determined}\\ \text{by (i)}}} \wedge dx^I. \quad (\#)$$

(b) Existence. Straightforward but tedious way: check that (#) transforms correctly under changes of coordinates.

Clever way: find an evidently coordinate-independent expression for $d\omega(V_1,\dots,V_{k+1})$ (tensorial in $V_1,\dots,V_{k+1}$) that gives (#) for $V_1,\dots,V_{k+1}$ = coordinate vector fields. Since for the local coordinate formula (#), properties (i)–(iii) are straightforward to check (exercise), this would finish the proof.

Claim: (✝)

$$d\omega(V_1,\dots,V_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} V_i\big(\omega(V_1,\dots,\underset{\text{omitted}}{\hat{V_i}},\dots,V_{k+1})\big)$$
$$+ \sum_{1\le i<j\le k+1} (-1)^{i+j} \omega([V_i,V_j], V_1,\dots,\hat{V_i},\dots,\hat{V_j},\dots,V_{k+1}). \quad (✝)$$

Check: (a) The right hand side of (✝) is alternating in $V_1,\dots,V_{k+1}$. Easy.

(b) The right hand side of (✝) is tensorial in V_1 (and all other V_j).

Indeed, replacing V_1 by fV_1, we get $f \times$ R.H.S.(✝) plus the terms

$$\sum_{i=2}^{k+1} (-1)^{i+1}(v_i f)\, \omega(v_1,\dots,\hat{v}_i,\dots,v_{k+1}) + \sum_{1<j\le k+1} (-1)^{1+j} \big(-(v_j f)\,\omega(v_1,\dots,\hat{v}_j,\dots,v_{k+1})\big)$$

$[fv_1, v_j] = f[v_1,v_j] - (v_j f)v_1$

$$= 0.$$

(c) In coordinates, with $v_i = \frac{\partial}{\partial x^{\ell_i}}$, the formula $\oplus$ produces, for $\omega = u\, dx^1 \wedge \dots \wedge dx^k$ (general ω by linearity)

$$\sum_{i=1}^{k+1} (-1)^{i+1} (\partial_{\ell_i} u)\, \operatorname{sgn}\begin{pmatrix} 1 & \cdots & i-1 & i & \cdots & k \\ \ell_1 & \cdots & \ell_{i-1} & \ell_{i+1} & \cdots & \ell_{k+1} \end{pmatrix}$$

$$= \sum_{j=1}^{m} (\partial_j u)\, \operatorname{sgn}\begin{pmatrix} j & 1 & 2 & \cdots & k \\ \ell_1 & \ell_2 & \ell_3 & \cdots & \ell_{k+1} \end{pmatrix}$$

$$= (du \wedge dx^1 \wedge \dots \wedge dx^k)(\partial_{\ell_1}, \dots, \partial_{\ell_{k+1}}).$$

only those j survive with $j = \ell_i$ for some $i \in \{1,\dots,k+1\}$; and

$$\operatorname{sgn}\begin{pmatrix} \ell_i & 1 & 2 & \cdots & i-1 & i & \cdots & k \\ \ell_1 & \ell_2 & \ell_3 & \cdots & \ell_i & \ell_{i+1} & \cdots & \ell_{k+1} \end{pmatrix} = (-1)^{i+1} \operatorname{sgn}\begin{pmatrix} 1 & \cdots & i-1 & i & \cdots & k \\ \ell_1 & \cdots & \ell_{i-1} & \ell_{i+1} & \cdots & \ell_{k+1} \end{pmatrix}$$

since the permutation $\begin{pmatrix} 1 & \cdots & i-1 & \ell_i \\ \ell_1 & \cdots & \ell_{i-1} & \ell_i \end{pmatrix}$ arises from $\begin{pmatrix} \ell_i & 1 & \cdots & i-1 \\ \ell_1 & \ell_2 & \cdots & \ell_i \end{pmatrix}$ via $i+1$ transpositions (move ℓ_i in the first row to the very right). $\square$

Example 9.12. On $\mathbb{R}^3$:

$$df = (\partial_1 f)\,dx^1 + (\partial_2 f)\,dx^2 + (\partial_3 f)\,dx^3 \quad \sim \operatorname{grad} f$$

$$d(B_1\,dx^1 + B_2\,dx^2 + B_3\,dx^3) = (\partial_2 B_3 - \partial_3 B_2)\,dx^2\wedge dx^3 + (\partial_3 B_1 - \partial_1 B_3)\,dx^3 \wedge dx^1 + (\partial_1 B_2 - \partial_2 B_1)\,dx^1\wedge dx^2 \quad \sim \operatorname{curl} \vec{B}$$

$$d(E_1\,dx^2\wedge dx^3 + E_2\,dx^3\wedge dx^1 + E_3\,dx^1\wedge dx^2) = (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)\,dx^1\wedge dx^2 \wedge dx^3 \quad \sim \operatorname{div} \vec{E}$$

$d^2 = 0$ reads $\operatorname{curl}\operatorname{grad} = 0$

$d^2 = 0$ reads $\operatorname{div}\operatorname{curl} = 0$.

Proposition 9.13. (Pullback commutes with exterior differentiation)

Let $F: M \to N$ be a C^∞ map between C^∞ manifolds. Then

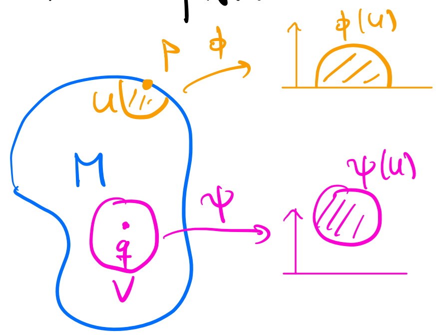
$$d \circ F^* = F^* \circ d \;:\; \Omega^k(N) \to \Omega^{k+1}(M)$$

Proof. Exercise. (This is an application of the chain rule.) $\square$

9.4. Stokes' Theorem

We shall prove a far-reaching generalization of Example 9.4(i).
First, we generalize embedded curve segments like $\gamma([0,1])$.

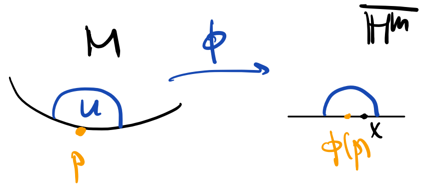
Definition 9.14. (i) A topological manifold with boundary M is a topological space s.t. $\forall p\ \exists$ open set $U \subset M$, $p \in U$, and a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{H}^m := \mathbb{R}^{m-1} \times [0,\infty)$.



(ii) A smooth structure on M is a covering by charts $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{H}^m$ with C^∞ transition functions $\phi_\beta \circ \phi_\alpha^{-1}$.

A C^∞ manifold with boundary is a topological manifold with boundary equipped with a maximal smooth structure.

- We write ∂M for the boundary of M, consisting of all $p \in M$ which get mapped to a point in $\partial \mathbb{H}^m$ by smooth charts. The interior is $M^\circ = M \setminus \partial M$.
- In more detail, by the inverse function theorem, the image of an interior point of $\mathbb{H}^m$ by a coordinate change map must again be an interior point. (Thus, if a point is a boundary point in one chart, this is true in every chart.)
- Given $p \in \partial M$ and a chart $\phi : U \to \phi(U) \subset \mathbb{H}^m$, $\phi(p) \in \partial \mathbb{H}^m$,



then $\phi_\partial : U \cap \partial M \to \phi(U) \cap \partial \mathbb{H}^m$ is a diffeomorphism into an open subset of $\mathbb{R}^{m-1}$.

If $\psi : V \to \psi(V) \subset \mathbb{H}^m$ is another chart, then $\psi_\partial \circ \phi_\partial^{-1}$ is C^∞.

$\Longrightarrow$ ∂M inherits from M the structure of a C^∞ $(m-1)$-dimensional manifold.

Next, we need to understand orientations.

Definition 9.15. Let M be an m-dimensional manifold (possibly with boundary). An orientation on M is an equivalence of nowhere vanishing m-forms $\omega \in \Omega^m(M)$ (i.e. $\omega_p \neq 0 \in \Lambda^m T_p^* M$ $\forall p \in M$), where $\omega \sim \eta$ iff $\exists\, 0 < f \in C^\infty(M)$ s.t. $\omega = f\eta$.

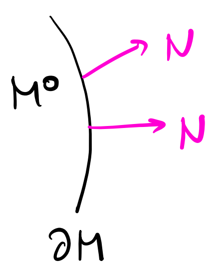
We leave it as an exercise to show that M is orientable if and only if there exists a nowhere vanishing m-form ω (and thus an orientation); and if M is orientable, there exist exactly 2 orientations, $[\omega]$ and $[-\omega]$.

Example 9.16. On $\mathbb{R}^m$, $[dx^1 \wedge \cdots \wedge dx^m]$ is the standard orientation. Likewise on $\mathbb{H}^m$.

The boundary of a manifold with boundary carries an induced orientation:

Lemma 9.17. Let M be a C^∞ manifold with boundary $\partial M \xhookrightarrow{i} M$.

(i) There exists an outward pointing vector field $N \in C^\infty(\partial M; T_{\partial M} M)$, that is, for all $f \in C^\infty(M)$ with $f|_{\partial M} = 0$, $f|_{M^\circ} > 0$ and $d_p f \neq 0$, $p \in \partial M$, we have $Nf(p) < 0$.



(ii) If $[\omega]$ is an orientation on M, then $[i^*(i_N \omega)]$ (where $(i_N\omega)(v_1, \ldots, v_{m-1}) = \omega(N, v_1, \ldots, v_{m-1})$) is an orientation on ∂M which is independent of the choice of outward pointing N.

Proof. (i) In a chart ϕ near a boundary point, let $N_\phi := -\frac{\partial}{\partial x^m}$.

Construct N near ∂M using a partition of unity.

(ii) In positively oriented local coordinates $x^1,\dots,x^{m-1}$ and $x^m \geq 0$,

$\omega = a(x)\, dx^1 \wedge \dots \wedge dx^m$, $\quad 0 < a \in C^\infty$. Every N is of the form

$N = \sum_{i=1}^{m-1} N^i \partial_{x^i} - b\,\partial_{x^m}$ for smooth $N^i = N^i(x^1,\dots,x^{m-1})$, $0 < b = b(x^1,\dots,x^{m-1})$

$$\Rightarrow \quad i^*(i_N \omega)(\partial_{x^1},\dots,\partial_{x^{m-1}}) = \omega(N, \partial_{x^1},\dots,\partial_{x^{m-1}})$$
$$= (-1)^{m-1}\omega(\partial_{x^1},\dots,\partial_{x^{m-1}}, N)$$
$$= (-1)^m\, b.$$

The sign of b does not depend on N. $\square$

Example 9.18. The orientation of $\mathbb{R}^{m-1} = \partial \mathbb{H}^m$ induced by $[dx^1 \wedge \dots \wedge dx^m]$ is $(-1)^m [dx^1 \wedge \dots \wedge dx^{m-1}]$. (So different from the standard orientation for odd m!)

Theorem 9.19. (Stokes' theorem.) Let M be an oriented manifold with (possibly empty) boundary. Let $\omega \in \Omega_c^{m-1}(M)$. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here ∂M carries the induced orientation, and we write $\int_{\partial M} \omega$ for $\int_{\partial M} i^*\omega$, $\quad i : \partial M \hookrightarrow M$ inclusion.

Proof. Step 1: the case $M = \mathbb{R}^m$. To show: $\int_{\mathbb{R}^m} d\omega = 0$. By linearity, it suffices to consider $\omega = f\, dx^2 \wedge \dots \wedge dx^m \Rightarrow d\omega = \frac{\partial f}{\partial x^1} dx^1 \wedge \dots \wedge dx^m$

$\Rightarrow \int d\omega = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \cdots dx^m$. But $f \in C_c^\infty(\mathbb{R}^m)$, so $\int \frac{\partial f}{\partial x^1} dx^1 = 0$

$\Rightarrow \int d\omega = 0.$

Step 2: the case $M = \overline{\mathbb{H}^m}$. • For $\omega = f\, dx^2 \wedge \cdots \wedge dx^m$, get $\int d\omega = 0$ as before, and $i^*\omega = 0$, so $\int_{\mathbb{R}^{m-1}} i^*\omega = 0$. Similarly for all
$\omega = f\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m$, $1 \le i \le m-1$.

• It remains to consider $\omega = f\, dx^1 \wedge \cdots \wedge dx^{m-1}$

$$\Rightarrow d\omega = \frac{\partial f}{\partial x^m} dx^m \wedge dx^1 \wedge \cdots \wedge dx^{m-1} = (-1)^{m-1} \frac{\partial f}{\partial x^m} dx^1 \wedge \cdots \wedge dx^m$$

$$\Rightarrow \int_{\mathbb{H}^m} d\omega = (-1)^{m-1} \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f}{\partial x^m}(x^1, \ldots, x^{m-1}, x^m)\, dx^m \right) dx^1 \cdots dx^{m-1}$$

$\int_0^\infty \frac{\partial f}{\partial x^m}(x^m)\, dx^m = f\Big|_0^\infty = -f(0)$

$$= (-1)^m \int_{\mathbb{R}^{m-1}} f(x^1, \ldots, x^{m-1}, 0)\, dx^1 \cdots dx^{m-1}$$

$$= \int_{\partial \overline{\mathbb{H}^m}} i^*\omega$$

Example 9.18.

Step 3: the general case. We can cover $\operatorname{supp} \omega$ by a finite number of charts $\phi_\alpha : U_\alpha \subset M \to \phi_\alpha(U_\alpha) \subset \overline{\mathbb{H}^m}$. Let χ_α be a subordinate partition of unity. Then

Compute in local coordinates.
Step 1 (for $U_\alpha \subset M^\circ$)
& Step 2 (for $U_\alpha \cap \partial M \neq \emptyset$)

$$\int_M \omega = \sum_\alpha \int_M d(\chi_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\chi_\alpha \omega) = \sum_\alpha \int_{U_\alpha \cap \partial M} i^*(\chi_\alpha \omega)$$

$$= \int_{\partial M} i^*\omega. \qquad \square$$

In the exercises, we will see that this generalizes the standard results from vector calculus such as the divergence theorem ($\int_M \operatorname{div}_g X \, d_{g_M} = \int_{\partial M} X \cdot \nu \, d_{g_{\partial M}}$).

outward pointing unit normal

9.5. de Rham cohomology: definition, Poincaré lemma

- Let M be a C^∞ manifold. Let $\omega \in \Omega^k(M)$. In order for $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$, it is necessary that $d\omega = 0$ (since $d^2=0$).
- For $M = \mathbb{R}^m$, $k=1$, we proved that $d\omega=0$ is also sufficient. Example 9.10(ii) is an example $\left(\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \text{ on } M = \mathbb{R}^2 \setminus \{(0,0)\}\right)$ where $d\omega = 0$ but $\omega \neq df$ for any $f \in C^\infty(M)$.

Definition 9.20. For $k \in \mathbb{N}_0$, the de Rham cohomology group $H^k(M)$ (sometimes $H^k_{dR}(M)$) is defined as

$$H^k(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{ran}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

de Rham cohomology with compact support is defined as

$$H^k_c(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\operatorname{ran}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))}.$$

One calls forms $\omega \in \Omega^k(M)$ with $d\omega = 0$ closed, and those of the form $\omega = d\eta$ exact. Exact forms are always closed; $H^k(M)$ measures to what extent the converse fails.

Example 9.21. (i) $H^0(M) = \ker(d : \Omega^0(M) \to \Omega^1(M)) = \{\text{locally constant functions}\}$

$\cong \mathbb{R}^{\pi_0(M)}$, where $\pi_0(M) = \#$ connected components of M.

(ii) $H^1(\mathbb{R}^m) = \{0\}$

(iii) $H^1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \mathbb{R}$ (later — for now we only know that $\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right] \neq 0 \in H^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

(iv) M compact $\Rightarrow H^k(M) = H^k_c(M)$.

(v) $H^1_c(\mathbb{R}) \cong \mathbb{R}$: indeed, every $\omega = f(x)\,dx$ is closed, but $\omega = d\eta$ (i.e. $f = \eta'$) with $\eta \in C^\infty_c(\mathbb{R})$ iff $\int_{-\infty}^{\infty} f\,dx = 0$.

So $H^1_c(\mathbb{R}) \ni [\omega] \mapsto \int \omega \in \mathbb{R}$ is an isomorphism.

- These examples suggest that $H^k(M)$ captures topological information about M (components, holes, etc.). One can in fact show that $H^k(M)$ is isomorphic to the k-th singular cohomology group. (We will not do this here.) But for our purposes, we shall regard $H^k(M)$ as an invariant of smooth manifolds.

Remark 9.22. (i) If $F: M \to N$ is smooth, then $F^*: \Omega^k(N) \to \Omega^k(M)$ induces a map $F^*: H^k(N) \to H^k(M)$ by Proposition 9.13.

(ii) $H^k(M) \times H^\ell(M) \ni ([\omega],[\eta]) \mapsto [\omega \wedge \eta] \in H^{k+\ell}(M)$ is well-defined (exercise). $\Rightarrow H^*(M) := \bigoplus_{k=0}^{m} H^k(M)$ is the cohomology ring.

(iii) Part (i) only extends to Ω_c, H_c for proper F (i.e. F^{-1}(compact) is compact).

- We can compute H^k for the simplest manifold:

Theorem 9.23. (Poincaré lemma.)

(i) $H^k(\mathbb{R}^m) \cong \begin{cases} \mathbb{R}, & k=0, \\ 0, & k \geq 1. \end{cases}$

(ii) $H^k_c(\mathbb{R}^m) \cong \begin{cases} 0, & k \leq m-1, \\ \mathbb{R}, & k=m. \end{cases}$

Part (i) generalizes the well-known results from vector calculus in $\mathbb{R}^3$ (cf. Example 9.12): for a vector field X on $\mathbb{R}^3$,

- $X = \nabla f \iff \nabla \times X = 0$
- $X = \nabla \times Y \iff \nabla \cdot X = 0$.

Proof of Theorem 9.23. (i) We argue by induction on m, starting with the trivial case $m=0$ (or with the elementary case $m=1$ if you prefer). We relate $\mathbb{R}^m$ and $\mathbb{R}^{m+1} = \mathbb{R}\times\mathbb{R}^m$ via the maps

$$s: \quad \mathbb{R}^m \ni x \quad \mapsto (0,x) \in \mathbb{R}\times\mathbb{R}^m,$$
$$\pi: \mathbb{R}\times\mathbb{R}^m \ni (t,x) \mapsto x \quad \in \mathbb{R}^m.$$

(i.1) Since $\pi\circ s = \mathrm{Id}$, also $(\pi\circ s)^* = s^*\circ\pi^* = \mathrm{Id}$ on $H^k(\mathbb{R}^m)$.

(i.2) But $s\circ\pi \neq \mathrm{Id}$ (since $s(\pi(t,x)) = (0,x)$). Nonetheless, we will show that $(s\circ\pi)^* = \pi^*\circ s^* = \mathrm{Id}$ on $H^k(\mathbb{R}\times\mathbb{R}^m)$ by constructing a chain homotopy $K: \Omega^*(\mathbb{R}\times\mathbb{R}^m) \to \Omega^{*-1}(\mathbb{R}^m)$ s.t.

$$\mathrm{Id} - \pi^*\circ s^* = \pm(d\circ K \pm K\circ d) \quad \text{on } \Omega^k(\mathbb{R}\times\mathbb{R}^m) \qquad \circledast$$

for some choice of signs.

- Granted $\circledast$, note that the right hand side maps closed forms (ω with $d\omega=0$) into exact forms (namely, $\pm d(K\omega)$) and thus to $0\in H^k(\mathbb{R}\times\mathbb{R}^m)$. That is, the right hand side of $\circledast$ induces the 0 map in cohomology, and thus $\pi^*\circ s^* = \mathrm{Id}$ on $H^k(\mathbb{R}\times\mathbb{R}^m)$

- Write $\omega(t,x) = dt\wedge\omega_0(t,x) + \omega_1(t,x)$ where, for $j=0,1$,
$\omega_j(t,x) = \sum_I \omega_{j,I}(t,x)\, dx^I$.

$$\Rightarrow (s^*\omega)(x) = \sum_I \omega_{1,I}(0,x)\, dx^I = \omega_1(0,x)$$

$$\Rightarrow (\omega - \pi^* s^*\omega)(t,x) = dt\wedge\omega_0(t,x) + (\omega_1(t,x) - \omega_1(0,x)). \qquad \#$$

- Set $(K\omega)(t,x) := \int_0^t \omega_0(s,x)\,ds = \int_0^t (i_{\partial_t}\omega)(s,x)\,ds$. Then:
  - $dK\omega_1 = 0$, $(K\,d\omega_1)(t,x) = K(dt\wedge\partial_t\omega_1)(t,x) = \omega_1(t,x) - \omega_1(0,x)$

  $\Rightarrow \# = dK\omega + Kd\omega$ for $\omega=\omega_1$.

$$\cdot\ (dK(dt\wedge\omega_0))(t,x) = dt\wedge\omega_0(t,x) + \int_0^t \underbrace{d_x\omega_0}_{:=\sum_{I,\ell}\partial_\ell\omega_{0,I}\,dx^\ell\wedge dx^I}(s,x)\,ds$$

(i.e. exterior derivative in x-coordinates only)

$$(Kd(dt\wedge\omega_0))(t,x) = K(-dt\wedge d_x\omega_0)(t,x) = -\int_0^t (d_x\omega_0)(s,x)\,ds$$

$\Rightarrow$ (#) $= dK\omega + Kd\omega$ for $\omega = dt\wedge\omega_0$.

(i.3) We can now conclude: $H^k(\mathbb{R}^m) = H^k(\mathbb{R}^{m-1}) = \dots = H^k(\{0\})$

$$\cong \begin{cases} \mathbb{R}, & k=0 \\ 0, & k\geq 1. \end{cases}$$

□

(ii) Exercise.

The same arguments show, more generally, that we have isomorphisms

$$H^k(\mathbb{R}\times M) \underset{\pi^*}{\overset{s^*}{\rightleftarrows}} H^k(M), \qquad H^k_c(\mathbb{R}\times M) = H^{k-1}_c(M).$$

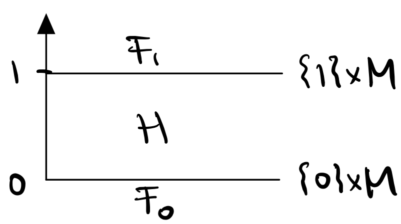
Corollary 9.24. Let $F_0, F_1 : M \to N$ be smoothly homotopic maps, i.e. $\exists\ H : [0,1]\times M \xrightarrow{C^\infty} N$, $H(0,\cdot) = F_0$, $H(1,\cdot) = F_1$. Then

$$F_0^* = F_1^* \text{ as maps } H^k(N) \to H^k(M).$$

(i.e. "smoothly homotopic map induce the same map in cohomology.")

Proof. Write $s_j : M \to [0,1]\times M$,
$p \mapsto (j, p)$

for $j = 0,1$; so $F_0 = H\circ s_0$
$F_1 = H\circ s_1$.



$\Rightarrow F_0^* = s_0^* \circ H^*$. Since s_0^*, s_1^* are both inverses of $\pi^* : [0,1] \times M \to M$ in cohomology, they are equal $\Rightarrow F_0^* = s_1^* \circ H^* = F_1^*$. $\square$

<u>Corollary 9.25.</u> Let M, N be two manifolds (not necessarily of the same dimension). Suppose they have the same homotopy type in the C^∞ sense, i.e. $\exists\, F: M \to N$, $G: N \to M$ s.t.

$F \circ G$ is smoothly homotopic to Id_N, and

$G \circ F$ is smoothly homotopic to Id_M.

Then $H^k(M) \cong H^k(N)$.

<u>Proof.</u> $F^* \circ G^*$ and $G^* \circ F^*$ are the identity maps on $H^k(M)$ and $H^k(N)$, respectively; so the maps

$$H^k(M) \underset{F^*}{\overset{G^*}{\rightleftarrows}} H^k(N)$$

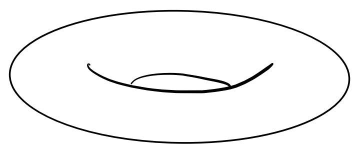
are isomorphisms. $\square$

<u>Example 9.26.</u> (i) M contractible (same homotopy type as $\{*\}$)

$$\Rightarrow H^k(M) \cong \begin{cases} \mathbb{R}, & k=0, \\ 0, & k \geq 1. \end{cases}$$

(ii) $H^k(\mathbb{R}^2 \setminus \{(0,0)\}) \cong H^k(S^1)$ ($\cong \mathbb{R}$).

$\cong H^k(\text{cream cheese bagel})$.



9.6. Computing H^*: Mayer–Vietoris sequence

Manifolds being "glued together" from open subsets (balls, if one likes) of $\mathbb{R}^m$, it would be convenient to be able to keep track of de Rham cohomology as one attaches additional open sets (starting with a single one). This is what the long exact sequence in cohomology does.

We write $\Omega^*(M) := \bigoplus_{k\in\mathbb{N}_0} \Omega^k(M)$.

Lemma 9.27. Suppose $M = U \cup V$ where U, V are open. Write

$i_U : U \hookrightarrow M$, $i_V : V \hookrightarrow M$ and $i_1 : U\cap V \hookrightarrow U$, $i_2 : U \cap V \hookrightarrow V$ for the inclusion maps.

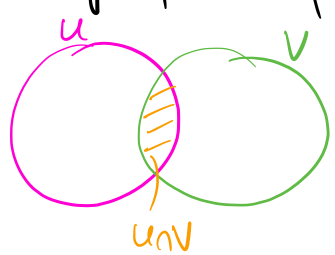
Then we have a short exact sequence

$$0 \to \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U\cap V) \longrightarrow 0. \quad \circledast$$

$$\omega \longmapsto (i_U^*\omega, i_V^*\omega)$$

$$(\eta, \zeta) \longmapsto i_1^*\eta - i_2^*\zeta$$

(That is, the range of one map is equal to the kernel of the next.)



Proof. • Exactness at $\Omega^*(M)$: if $i_U^*\omega = 0$, $i_V^*\omega = 0$ then $\omega_p = 0 \in \Lambda^* T_p^* M$ $\forall\, p \in U \cup V = M \Rightarrow \omega = 0$.

• Exactness at $\Omega^*(U)\oplus\Omega^*(V)$: if $(\eta, \zeta) = (i_U^*\omega, i_V^*\omega)$, then $i_1^*\eta - i_2^*\zeta = i^*\omega - i^*\omega = 0$ where $i : U\cap V \overset{i_1}{\underset{i_2}{\rightrightarrows}} {U \atop V} \overset{i_U}{\underset{i_V}{\rightrightarrows}} M$ is the inclusion map.

· Conversely suppose $\eta \in \Omega^*(U)$, $\zeta \in \Omega^*(V)$ satisfy $i_1^*\eta = i_2^*\zeta$.

Then $\omega_p := \begin{cases} \eta_p, & p \in U \\ \zeta_p, & p \in V \end{cases}$ is well-defined; and $\omega \in \Omega^*(M)$

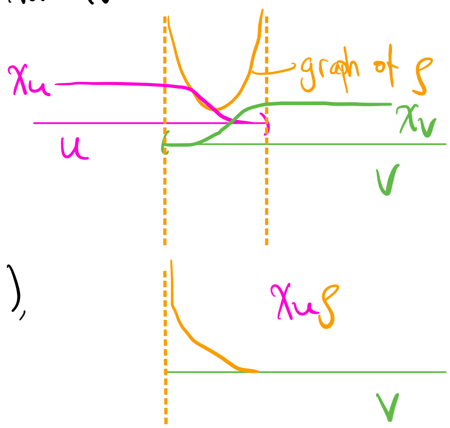
maps to (η, ζ) indeed.

· Exactness at $\Omega^*(U \cap V)$: let $\rho \in \Omega^*(U \cap V)$. ($\rho$ may blow up at $\partial(U \cap V) \subset M$!)

Let $\chi_U, \chi_V \in C^\infty(M)$ be a partition of unity subordinate to $\{U, V\}$. (So $\operatorname{supp} \chi_U \subset U$, $\operatorname{supp} \chi_V \subset V$, $\chi_U + \chi_V = 1$). Then

$\zeta := -\chi_U \rho \in \Omega^*(V)$

$\eta := \chi_V \rho \in \Omega^*(U)$



Since

$$\rho = i_1^*(\overbrace{\chi_V \rho}^{\eta}) - i_2^*(\overbrace{-\chi_U \rho}^{\zeta}),$$

we are done. □

Corollary 9.28. The short exact sequence ⊛ induces a long exact sequence in cohomology (called Mayer–Vietoris sequence)

$$\begin{tikzcd}
 & H^2(M) \arrow[r] & \cdots & \\
 & H^1(M) \arrow[r] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[llu, "d^*"'] \\
0 \arrow[r] & H^0(M) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[llu, "d^*"']
\end{tikzcd}$$

.

We describe d^* explicitly: the short exact sequence ⊛ induces a commutative diagram with exact rows

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{k+1}(M) \arrow[u] \arrow[r, "i_U^*\oplus i_V^*"] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[u] \arrow[r, "(i_1^*,-i_2^*)"'] & \Omega^{k+1}(U\cap V) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{k}(M) \arrow[u, "d"] \arrow[r] & \Omega^{k}(U)\oplus\Omega^{k}(V) \arrow[u, "d\oplus d"] \arrow[r, "(i_1^*,-i_2^*)"'] & \Omega^{k}(U\cap V) \arrow[u, "d"] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

$\kappa \qquad \eta \qquad \zeta \qquad \omega$

- Let $\omega \in \Omega^k(U\cap V)$ be closed ($d\omega = 0$), and write

$$\omega = i_1^*\eta - i_2^*\zeta = (i_1^*, -i_2^*)\begin{pmatrix}\eta\\ \zeta\end{pmatrix}$$

Then $(i_1^*, -i_2^*)\begin{pmatrix}d\eta\\ d\zeta\end{pmatrix} = d\left((i_1^*,-i_2^*)\begin{pmatrix}\eta\\ \zeta\end{pmatrix}\right) = d\omega = 0$,

so by exactness of the row $(k+1)$,

$$(i_1^*, -i_2^*)\begin{pmatrix}d\eta\\ d\zeta\end{pmatrix} = \begin{pmatrix} i_U^*\kappa \\ i_U^*\kappa\end{pmatrix}.$$

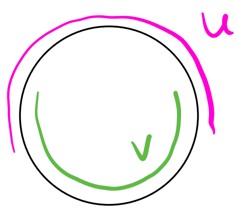
Then $\kappa := [d^*\omega]$.

- Explicitly, $\eta = \chi_V\,\rho$ and $\zeta = -\chi_U\,\rho$

$$\Rightarrow d^*[\omega] = \text{equivalence class of } \begin{cases} d(\chi_V\rho) & \text{on } U \\ d(-\chi_U\rho) & \text{on } V\end{cases} \quad \text{mod } d(\Omega^k(M)). \quad (**)$$

(Sanity check: $d(\chi_V\rho) = d((1-\chi_U)\rho) = d(-\chi_U\rho)$ on $U\cap V$.)

Example 9.29: Cohomology of S^1.



Cover $S^1 = U \cup V$ as follows:

Then $U\cap V \cong \mathbb{R} \sqcup \mathbb{R}$, so Mayer-Vietoris gives

$$
\begin{tikzcd}
 & & 0\oplus 0 & 0 & \\
 & & H^1(U)\oplus H^1(V) \arrow[u, equal, "\text{Poincaré}"'] & H^1(U\cap V) \arrow[u, equal, "P."'] & \\
 & H^1(S^1) \arrow[r] & H^1(U)\oplus H^1(V) \arrow[r] & H^1(U\cap V) \arrow[r, dotted] & \cdots \\
0 \arrow[r] & H^0(S^1) \arrow[r] & H^0(U)\oplus H^0(V) \arrow[r, "\circledast"] & H^0(U\cap V) \arrow[llu, bend right, "d^*"'] & \\
 & \mathbb{R} \arrow[u, equal, "9.21(i)"] & \mathbb{R}\oplus\mathbb{R} \arrow[u, equal, "P."] & \mathbb{R}\oplus\mathbb{R} \arrow[u, equal, "P."] &
\end{tikzcd}
$$

The map ⊛ has 1-dim. kernel $\Rightarrow 1 = (2-1)$-dim. range

$\Rightarrow d^*$ has 1-dim. kernel $\Rightarrow 1 = (2-1)$-dim. range $= H^1(S^1)$

$\Rightarrow H^1(S^1) \cong \mathbb{R}$.

We have an analogue for compactly supported cohomology:

Lemma 9.30. The inclusion maps $U \cap V \underset{j_2}{\overset{j_1}{\rightrightarrows}} \begin{matrix} U \\ V \end{matrix} \underset{j_V}{\overset{j_U}{\rightrightarrows}} M$ induce a short exact sequence

$$0 \to \Omega_c^*(U \cap V) \xrightarrow{(j_1)_* \oplus (j_2)_*} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{(j_U)_* - (j_V)_*} \Omega_c^*(M) \to 0.$$

(Here $(j_1)_* \omega$, for $\omega \in \Omega_c^*(U \cap V)$, is the differential form on U given by extension by 0, i.e. $((j_1)_* \omega)_p = \begin{cases} \omega_p, & p \in U \cap V \\ 0, & p \in U \setminus (U \cap V). \end{cases}$)

Notice that the arrows go "the other way" compared to Lemma 9.27.
The proof of Lemma 9.30 is left as an exercise.

Corollary 9.31. Long exact sequence in cohomology with compact support:

$$\begin{tikzcd}
& H_c^2(M) & \cdots & \\
& H_c^1(M) \arrow[u] & H_c^1(U) \oplus H_c^1(V) \arrow[l] & H_c^1(U \cap V) \arrow[l] \arrow[from=1-3, bend left] \\
0 & H_c^0(M) \arrow[l] \arrow[u] & H_c^0(U) \oplus H_c^0(V) \arrow[l] & H_c^0(U \cap V) \arrow[l] \arrow[from=2-2, bend left, "d^*"]
\end{tikzcd}$$

Here, $d^*[\omega]$ for $\omega \in \Omega_c^k(M)$, $d\omega = 0$, is the equivalence class of $d(\chi_U \omega) = d(-\chi_V \omega)$. (This is closed and supported in $\operatorname{supp} \omega \cap U \cap V$, so lies in $\Omega_c^{k+1}(U \cap V)$.) ⊛⊛$_c$

9.7. Good covers and finite-dimensionality of $H^*(M)$

In general, $H^k(M)$ may be an ∞-dimensional vector space ($M = S^1 \sqcup S^1 \sqcup \cdots$, and connected examples exist too). When M admits a finite good cover, we shall show that $H^k(M)$ and $H^k_c(M)$ are finite-dimensional.

Definition 9.32. Let M be a C^∞ manifold. Then a cover $\{U_\alpha\}$ of M by open sets is a good cover if all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_N}$ $(N = 1, 2, \ldots)$ are diffeomorphic to $\mathbb{R}^m$.

Example 9.33. (i) S^1:

(ii) All (compact) manifolds admit a (finite) good cover. (This is Theorem 9.40)

Theorem 9.34. Suppose M admits a finite good cover $M = \bigcup_{i=1}^{N} U_i$. Then $\dim H^k(M), \dim H^k_c(M) < \infty$ $\forall k$.

Proof by induction on the cardinality N of a finite good cover.

Base case: $N=1$. Then $M \cong \mathbb{R}^m$, so the result follows from the Poincaré lemma.

Inductive step: suppose all C^∞ manifolds admitting a good cover with $N-1$ charts have finite-dimensional $H^*_{(c)}$. Consider now our $M = \bigcup_{i=1}^{N} U_i$. Apply Mayer-Vietoris with $U = \bigcup_{i=1}^{N-1} U_i$, $V = U_N$; note that $U \cap V = \bigcup_{i=1}^{N-1} (U_i \cap V)$ has a good cover with $N-1$ charts.

$$\Rightarrow \quad \cdots \longrightarrow H^{k-1}(U \cap V) \longrightarrow H^k(U \cup V) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow \cdots$$

Exactness at $H^k(U\cup V)=H^k(M)$ implies that

$$\dim H^k(U\cup V) \le \dim H^{k-1}(U\cap V)+\dim H^k(U)+\dim H^k(V) < \infty$$ by the inductive hypothesis. $\square$

The construction of charts with good intersection properties relies on the following result in Riemannian geometry.

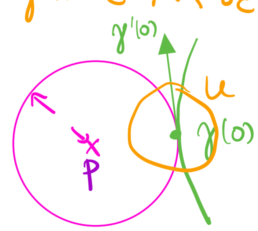
Definition 9.35. Let (M,g) be a Riemannian manifold. Then an open set $W\subset M$ is called strongly convex if for all $q_0,q_1\in \overline{W}$ there exists a unique minimizing geodesic $\gamma:[0,1]\to M$ with $\gamma(0)=q_0$, $\gamma(1)=q_1$, and $\gamma((0,1))\subset W$.

Examples include balls in $(\mathbb{R}^m, g_{eucl})$ and the intersection $S^m\cap\{x^{m}>\varepsilon\}$ in (S^m, g_{std}) for any $\varepsilon>0$.

Proposition 9.36. (Geodesically convex neighborhoods.) Let (M,g) be a Riemannian manifold. Let $p\in M$. Then there exists $r_0>0$ s.t. the geodesic ball $B_{r_0}(p)$ is strongly convex.

We first need:

Lemma 9.37. For all p in (M,g), $\exists\, c>0$ s.t. if $r<c$ and $\gamma: I\subseteq\mathbb{R}\to M$ is a geodesic with $\gamma(0)\in\partial B_r(p)$, $\gamma'(0)\in T(\partial B_r(p))$, then $\exists\, U\subset M$ open neighborhood of $\gamma(0)$ s.t. $\gamma(t)\in M\setminus B_c(p)$ $\forall t\neq 0$ s.t. $\gamma([0,t])\subset U$.



Proof. • Let $W\subset M$ be a totally normal neighborhood of p. Denote by

$$T_1W := \{ (q,v) \in TW :\ q \in W,\ v \in T_qM,\ |v| = 1 \}$$

the unit tangent bundle. For $\varepsilon > 0$ small, the map

$$\gamma : (-\varepsilon, \varepsilon) \times T_1W \ni (t, q, v) \longmapsto \exp_q(tv)$$

is well-defined and C^∞; and for small ε, $\gamma((-\varepsilon,\varepsilon) \times T_1W)$ is contained in a normal neighborhood of every point in W.

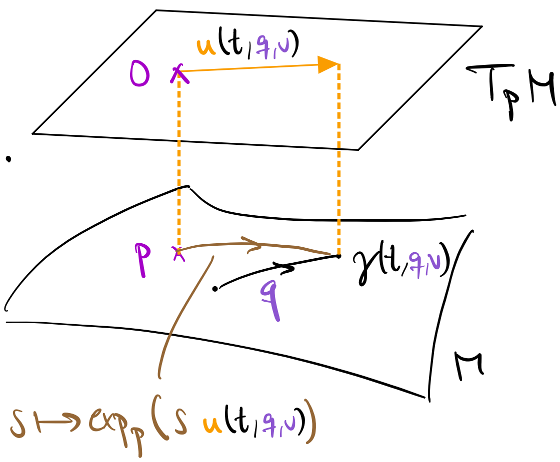
- Define

$$u : (-\varepsilon, \varepsilon) \times T_1W \ni (t, q, v) \longmapsto \exp_p^{-1}(\gamma(t, q, v)),$$

$$F(t,q,v) := |u(t,q,v)|^2_{g_p}$$

(= squared distance from p to $\gamma(t,q,v)$).

- Let $\tilde{\varepsilon} > 0$ be s.t. $\exp_p(B_{\tilde{\varepsilon}}(0)) = $
$= B_{\tilde{\varepsilon}}(p) \subset W$.



- For a geodesic $\gamma(t) = \gamma(t,q,v)$ with $d(p, \overset{q}{\overset{\parallel}{\gamma(0)}}) = \tilde{\varepsilon}$ and $\gamma'(0) \in T(\partial B_{\tilde{\varepsilon}}(p))$, the Gauss Lemma gives $\left\langle \frac{\partial u}{\partial t}(0,q,v), u(0,q,v) \right\rangle_{g_p} = 0 \Rightarrow \frac{\partial F}{\partial t}(0,q,v) = 0$

tangent vector of the geodesic from p to $\gamma(0) = q$

- Remains to show: for $\tilde{\varepsilon} > 0$ sufficiently small, the critical point $(0,q,v)$ of F is a strict minimum, i.e. $\frac{\partial^2 F}{\partial t^2}(0,q,v) > 0$. (Then $\gamma(t)$ leaves $\overline{B_{\tilde{\varepsilon}}(q)}$ for all small $t \neq 0$.)

To this end, note that $u(t,p,v) = tv \Rightarrow F(t,p,v) = t^2|v|^2 = t^2 \Rightarrow \frac{\partial^2 F}{\partial t^2}(0,p,v) = 2.$

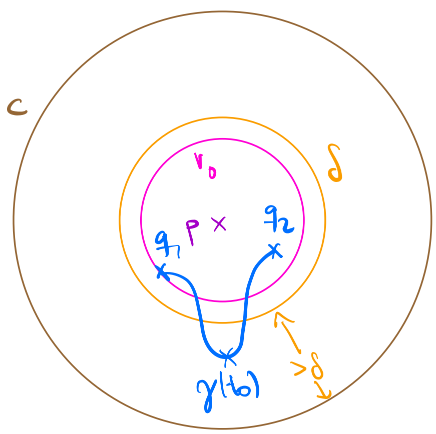
Therefore, $\exists$ neighborhood $V \subset W$ of p s.t.

$$\frac{\partial^2 F}{\partial t^2}(0, q, v) > 0 \quad \forall q \in V,\ v \in T_q M,\ |v| = 1.$$

For $c > 0$ with $B_c(p) \subset V$, we are done. $\square$

<u>Proof of Proposition 9.36.</u> • Let $c > 0$ be as in Lemma 9.37. Let W be a totally normal neighborhood of p, with $\delta > 0$ s.t. $\forall q \in W$, $\exp_q(B_\delta(0)) \supset W$ is a normal neighborhood of q. By shrinking W and δ, we can arrange $\delta < \frac{c}{2}$. Let $r_0 < \delta$. We claim that $B_{r_0}(p)$ is strongly convex.

• Let $q_1, q_2 \in \overline{B_{r_0}(p)}$, and let $\gamma : [0,1] \to M$ ($q_1 \to p \to q_2$) be the unique minimizing geodesic from q_1 to q_2; thus $L(\gamma) \le 2 r_0 < 2\delta$, and γ is contained in $B_c(p)$ (since curves from $\partial B_\delta(p)$ to $\partial B_c(p)$ have length $\ge c - \delta > \delta$, and thus a curve from q_1 to q_2 not contained in $B_c(p)$ has length $> 2\delta$).



Now, if $\gamma((0,1))$ were not contained in $B_{r_0}(p)$, there would exist $t_0 \in (0,1)$ with $r := d(p, \gamma(t_0)) = \max_{t \in [0,1]} d(p, \gamma(t)) \in (r_0, c)$.

The points $\gamma(t)$ for t near t_0 remain in $\overline{B_r(p)}$, in contradiction to Lemma 9.37. $\square$

Lemma 9.38. Strictly convex sets W are diffeomorphic to $\mathbb{R}^m$.

Proof. Fix $p \in W$. For all $q \in W$, the unique minimizing geodesic from p to q remains in W; so W is diffeomorphic to

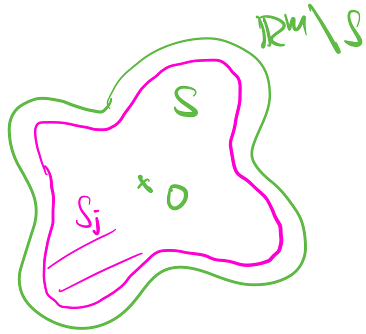
$$S := \exp_p^{-1}(W) \subset T_pM$$

is open and star-shaped around $0 \in T_pM$. Use then Proposition 9.39. $\square$

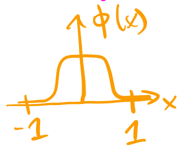
Proposition 9.39. Let $S \subset \mathbb{R}^m$ be star-shaped around $0 \in S$, i.e. $\forall p \in S$, also $tp \in S$. If S is open, then S is diffeomorphic to $\mathbb{R}^m$.

Proof. · **Step 1**: $\exists\, f \in C^\infty(\mathbb{R}^m)$ s.t. $f > 0$ on S,
$f = 0$ on $\mathbb{R}^m \setminus S$.

Indeed, let $S_j := \{ p \in S : \mathrm{dist}(p, \mathbb{R}^m \setminus S) > \frac{1}{j} \}$.



Let $\phi(x) = \begin{cases} \exp(-\frac{1}{1-|x|^2}), & |x| < 1, \\ 0, & |x| \geq 1; \end{cases}$



thus $\phi \in C^\infty(\mathbb{R}^m)$, $\phi|_{B_1(0)} > 0$, $\phi|_{\mathbb{R}^m \setminus B_1(0)} = 0$.

Set $\phi_j(x) = \phi(jx)$ and

$$f_j(x) = 2^{-j} (\phi_j * 1_{S_j})(x) = 2^{-j} \int_{S_j} \phi_j(x-y)\, dy.$$

Then (a) $f := \sum_{j=1}^{\infty} f_j$ converges in $C^k(\mathbb{R}^m)$ $\forall k$ (exercise) $\Rightarrow f \in C^\infty(\mathbb{R}^m)$.

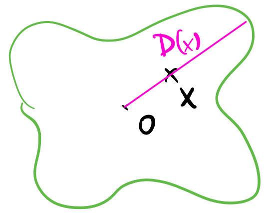
(b) $f = 0$ on $\mathbb{R}^m \setminus S$ since this is true for all f_j.

(c) Let $p \in S$, and let $j \in \mathbb{N}$ be s.t. $j > \frac{1}{\mathrm{dist}(p, \mathbb{R}^m \setminus S)}$. Then $f_j(p) > 0$. Since p was arbitrary, $f|_S > 0$.

· **Step 2.** Define $\lambda(x) := 1 + \left(\int_0^{|x|} \frac{dt}{f(t\frac{x}{|x|})} \right)^2$ for $x \in S$. We will show that $F(x) := \lambda(x)\, x$ is a diffeomorphism $S \to \mathbb{R}^m$.

(a) Since $\lambda(x) \underset{t=|x|s}{=} 1 + \left(\int_0^1 \frac{ds}{f(sx)}\right)^2 |x|^2$, we see that $\lambda \in C^\infty(S)$
$\Longrightarrow F : S \to \mathbb{R}^m$ is C^∞.

(b) Set $D(x) := \sup\{\mu > 0 : \mu \frac{x}{|x|} \in S\}$.



We claim that for all $x \neq 0$, the map $[0, D(x))\frac{x}{|x|} \ni y \mapsto F(y) \in [0,\infty)\frac{x}{|x|}$ is a bijection.

- Injectivity is a consequence of $f|_S > 0$, so $\mu \mapsto \lambda(\mu\frac{x}{|x|})$ is strictly increasing in μ, and thus so is $\mu \longmapsto |F(\mu\frac{x}{|x|})| = \mu\lambda(\mu\frac{x}{|x|})$.
- For surjectivity, suppose first that $D(x) = \infty$; then
$$|F(\mu\tfrac{x}{|x|})| \underset{\lambda \geq 1}{\geq} \mu \longrightarrow \infty \text{ as } \mu \to D(x).$$
If $D(x) < \infty$, then since f is C^1, $\exists\, C < \infty$ s.t.
$$f(\mu\tfrac{x}{|x|}) = f(\mu\tfrac{x}{|x|}) - f(D(x)\tfrac{x}{|x|}) \leq C(D(x) - \mu)$$
$\Longrightarrow \int_0^{D(x)} \frac{dt}{f(t\frac{x}{|x|})}$ diverges, and hence $\lambda(\mu\frac{x}{|x|}) \nearrow \infty$ as $\mu \to D(x)$, again giving $|F(\mu\frac{x}{|x|})| \longrightarrow \infty$.

(c) We have so far established that $F : S \to \mathbb{R}^m$ is a C^∞ bijection. To finish the proof that it is a diffeomorphism, it suffices to show that $d_x F \neq 0$ $\forall\, x \in S$ (since this implies that it is a local diffeomorphism).
So suppose $d_x F(v) = (d_x\lambda)(v)\, x + \lambda(x) v = 0 \Longrightarrow v = \mu x$ for some $\mu \in \mathbb{R}$, and thus $\mu(\lambda'(x) + \lambda(x)) = 0$. Since $\lambda \geq 1$ and $\lambda' \geq 0$, this forces $\mu = 0$. Therefore, $d_x F$ is injective. $\square$

We can now return to good covers:

Theorem 9.40. Every C^∞ manifold M admits a good cover. If M is compact, it admits a finite good cover.

Proof. Fix any C^∞ Riemannian metric on M. Cover M by geodesically strictly convex neighborhoods. The intersection of any two such sets is again strictly convex. By Lemma 9.38, we are done. $\square$

9.8. Poincaré duality

Let M be an oriented C^∞ m-dimensional manifold (without boundary). Then for closed $\omega \in \Omega^k(M)$, $\eta \in \Omega_c^{m-k}(M)$, the integral

$$\int_M \omega \wedge \eta$$

is unchanged when adding exact forms to ω, η; indeed, if $\rho \in \Omega^{k-1}(M)$, then $\int_M d\rho \wedge \eta = \int_M d(\rho \wedge \eta) + \underbrace{(-1)^k \rho \wedge d\eta}_{=0} = 0$ by Stokes' theorem, similarly $\int_M \omega \wedge d\tilde{\rho} = 0 \quad \forall \tilde{\rho} \in \Omega_c^{m-k-1}(M)$.

$\Rightarrow$ We have a well-defined map

$$H^k(M) \times H_c^{m-k}(M) \ni ([\omega], [\eta]) \longmapsto \int_M \omega \wedge \eta \in \mathbb{R}. \qquad (\circledast)$$

Theorem 9.41. (Poincaré duality.) Suppose M admits a finite good cover. Then $(\circledast)$ is a nondegenerate pairing, and thus induces an isomorphism

$$H^k(M) \cong (H_c^{m-k}(M))^*.$$

Recall here that a nondegenerate pairing $b: V \times W \to \mathbb{R}$ of 2 finite-dimensional vector spaces is a bilinear map s.t.

$$\begin{cases} v \in V, \; b(v,w) = 0 \;\forall w \;\Rightarrow v = 0 \\ w \in W, \; b(v,w) = 0 \;\forall v \;\Rightarrow w = 0. \end{cases}$$

Equivalently, the maps

$$\begin{cases} V \ni v \mapsto b(v,\cdot) \in W^* \\ W \ni w \mapsto b(\cdot,w) \in V^* \end{cases}$$

are injective, and thus $V \cong W^*$ (by dimension counting).

Example 9.42. (i) If M is connected, then $H^m_c(M) \cong (H^0(M))^* \cong \mathbb{R}$.

There is a canonical element $\lambda : H^0(M) = \{\text{constant functions}\} \to \mathbb{R}$ given by $\lambda(1) = 1$. Its Poincaré dual is denoted $[M] \in H^m_c(M)$.

The isomorphism $H^m_c(M) \cong \mathbb{R}$ is given by $[\omega] \mapsto \int_M \omega$.

(ii) If M is compact, $\dim H^k(M) = \dim H^{m-k}(M)$.

(iii) If M is noncompact and connected: $H^0_c(M) = 0 \Rightarrow H^m(M) = 0$.

(iv) Let $N \subset M$ be an orientable submanifold of dimension $n \leq m$, and assume N is a closed subset of M. Then

$$H^n(M) \ni [\omega] \longmapsto \int_N \omega \in \mathbb{R}$$

is well-defined. Its Poincaré dual (or the Poincaré dual of N) is the unique element $[N] \in H^{m-n}_c(M)$ s.t.

$$\int_N \omega = \int_M \omega \wedge [N] \qquad \forall\, \omega \in \Omega^n(M),\ d\omega = 0.$$

The proof of Theorem 9.41 uses:

Lemma 9.43. (5-lemma.) Suppose

$$\begin{array}{ccccccccccc} \cdots \to & A & \to & B & \to & C & \to & D & \to & E & \to \cdots \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} & \\ \cdots \to & A' & \to & B' & \to & C' & \to & D' & \to & E' & \to \cdots \end{array}$$

is a commutative diagram of abelian groups (e.g. vector spaces) with exact rows and s.t. $\alpha, \beta, \delta, \varepsilon$ are isomorphisms. Then γ is an isomorphism. (For this to hold, it suffices to assume that the diagram commutes up to signs.)

Proof. Exercise. $\square$

Proof of Theorem 9.41. We use induction on the number N of elements of a good cover.

Base case: $N=1$. This follows from the Poincaré lemma for $k \neq 0$.

For $k=0$, $H^0(\mathbb{R}^m)$ is spanned by $[1]$,

$H^m_c(\mathbb{R}^m)$ by $[\omega]$ for any $\omega \in \Omega^m_c(\mathbb{R}^m)$ with $\int_{\mathbb{R}^m} \omega \neq 0$.

$$\Rightarrow \int_{\mathbb{R}^m} [1] \wedge [\omega] = \int_{\mathbb{R}^m} \omega \neq 0.$$

Inductive step. Suppose the theorem is proved for all manifolds admitting a good cover with $N-1$ elements. Let $M = \bigcup_{i=1}^{N} U_i$ be a good cover. Mayer–Vietoris for $U = \bigcup_{i=1}^{N-1} U_i$, $V = U_N$ gives exact rows in

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^{k-1}(U) \oplus H^{k-1}(V) \arrow[r] \arrow[d, "\alpha"] & H^{k-1}(U\cap V) \arrow[r, "d^*"] \arrow[d, "\beta"] \arrow[dr, phantom, "\#"] & H^k(U\cup V) \arrow[r, "\text{restriction}"] \arrow[d, "\gamma"] & H^k(U)\oplus H^k(V) \arrow[r, "\text{difference}"] \arrow[d, "\delta"] & H^k(U\cap V) \arrow[r] \arrow[d, "\varepsilon"] & \cdots \\
\cdots \arrow[r] & H^{m-k+1}_c(U)^* \oplus H^{m-k+1}_c(V)^* \arrow[r] & H^{m-k+1}_c(U\cap V)^* \arrow[r, "\delta"] & H^{m-k}_c(U\cup V)^* \arrow[r] & H^{m-k}_c(U)^* \oplus H^{m-k}_c(V)^* \arrow[r] & H^{m-k}_c(U\cap V)^* \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are given by

$$H^k(U) \ni [\omega] \mapsto \left(H^{m-k}_c(U) \ni [\eta] \mapsto \int_U \omega \wedge \eta \right) \in H^{m-k}_c(U)^*$$

etc.

- By the inductive hypothesis, $\alpha, \beta, \delta, \varepsilon$ are isomorphisms. To conclude the proof using the 5-lemma, we only need to show that the above diagram commutes (up to signs). We shall only check this for the second square $\#$.
- Recall from (**):

$$d^*[\omega] = \left[\begin{cases} d(\chi_V \omega) & \text{on } U \\ d(-\chi_U \omega) & \text{on } V \end{cases} \right], \quad [\omega] \in H^{k-1}(U \cap V);$$

and from (**c) (by taking adjoints) for $\lambda : H^{m-k+1}_c(U\cap V) \to \mathbb{R}$

$$(\delta\lambda)([\rho]) = \lambda(d^*[\rho]) = \lambda([d(\chi_U\rho)])$$
$$= \lambda([-d(\chi_V\rho)]), \quad [\rho]\in H_c^{m-k}(U\cup V)$$

- For $[\omega]\in H^{k-1}(U\cap V)$, $[\rho]\in H_c^{m-k}(U\cup V)$, we thus compute

$$\gamma(d^*[\omega])([\rho]) = \int_{U\cup V} d^*[\omega]\wedge[\rho]$$

$d^*[\omega]=[\eta]$ where

$$\eta = \begin{cases} d(\chi_V\omega) & \text{on } U \\ d(-\chi_U\omega) & \text{on } V \end{cases}$$

$$\Rightarrow \eta = \chi_U\eta + \chi_V\eta = \chi_U d(\chi_V\omega) - \chi_V d(\chi_U\omega)$$

$$= \int_{U\cup V} \big(\chi_U\, d(\chi_V\omega) - \chi_V\, d(\chi_U\omega)\big)\wedge\rho$$

$$= \int_{U\cup V} \Big(d(\chi_V\omega\wedge\chi_U\rho) - (-1)^{k-1}\chi_V\omega\wedge d(\chi_U\rho) - d(\chi_U\omega\wedge\chi_V\rho) + (-1)^{k-1}\chi_U\omega\wedge d(\chi_V\rho)\Big)$$

$$= (-1)^k\int_{U\cup V}\big(\chi_V\omega\wedge d(\chi_U\rho) - \chi_U\omega\wedge d((1-\chi_U)\rho)\big).$$

$$= (-1)^k\int_{U\cup V}\omega\wedge d(\chi_U\rho)$$

On the other hand,

$$\delta(\beta[\omega])([\rho]) = \beta[\omega]([d(\chi_U\rho)])$$
$$= \int_{U\cup V}\omega\wedge d(\chi_U\rho).$$

The two expressions are thus equal (up to a sign $(-1)^k$), as desired. □