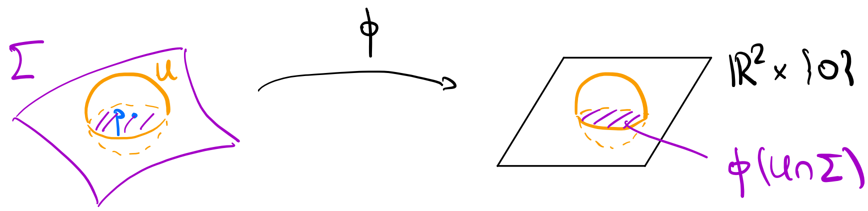
What will this course be about?

- Recall: a surface $\Sigma \subset \mathbb{R}^3$ is a subset s.t.

  $\forall p \in \Sigma$ $\exists$ neighborhood $U \subset \mathbb{R}^3$ of p,
  diffeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^3$,
  s.t. $\phi(U \cap \Sigma) = \phi(U) \cap (\mathbb{R}^2 \times \{0\})$.

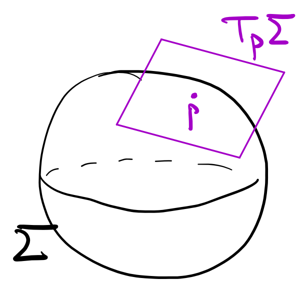


Then: $\phi^{-1}|_{\phi(U\cap\Sigma)} : \phi(U\cap\Sigma) \subset \mathbb{R}^2\times\{0\} \longrightarrow U\cap\Sigma$ gives local coordinates on Σ.

- The tangent space $T_p\Sigma = \{\gamma'(0) : \gamma : (-1,1) \xrightarrow{C^\infty} \Sigma,\ \gamma(0) = p\}$
  $= (d_{\phi(p)}\phi^{-1})(\mathbb{R}^2\times\{0\})$

  inherits from $\mathbb{R}^3$ the Euclidean inner product

  $\rightsquigarrow$ $g_p(X,Y) = X\cdot Y \quad (p\in\Sigma,\ X,Y \in T_p\Sigma \subset \mathbb{R}^3)$.



- Properties of this metric:
  (i) positive definite
  (ii) symmetric
  (iii) smooth (C^∞) in p: if X, Y are smooth vector fields on Σ
  ($X : \Sigma \to \mathbb{R}^3$ smooth, $X(p) \in T_p\Sigma$ $\forall p \in \Sigma$), then
  $p \mapsto g_p(X(p), Y(p))$ is a C^∞ function on Σ.

- What can one do with a metric?
  - (i) measure length of curves, distances between points;
  - (ii) compute curvature of the surface (Theorema egregium).

We will generalize these notions (metrics, curvature) to abstract smooth manifolds, and along the way study

- (iii) tensor bundles;
- (iv) geodesics ("shortest" connections between 2 points);
- (v) covariant differentiation (directional derivatives of vector fields);
- (vi) interplay of curvature and topology (cf. Gauss-Bonnet).

- As a simple examplary result, we shall prove that a simply connected Riemannian manifold with constant sectional curvature is (isometric to) the sphere, Euclidean space, or hyperbolic space (in n dimensions).

- Much of the course will be theory building though.
- Exercises will feature a mix of proofs and examples (where you actually need to compute something!).

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