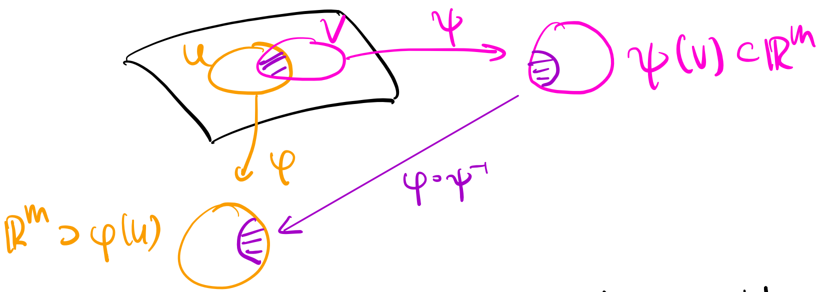
1.1. Recap of basic notions

Definition 1.1. (i) A topological manifold M of dimension m is a second countable Hausdorff topological space s.t. $\forall p \in M \ \exists U \subset M$ open neighborhood of p, $\phi: U \to \phi(U) \subset \mathbb{R}^m$ homeomorphism.
(We call (φ, U) a chart.)

(ii) Two charts (φ, U), (ψ, V) are compatible if the coordinate change

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \to \varphi(U \cap V) \text{ is a (smooth) diffeomorphism.}$$



(iii) A smooth manifold is a topological manifold M equipped with a smooth structure = maximal atlas = maximal system of charts $(U_\alpha, \varphi_\alpha)$ with $\bigcup U_\alpha = M$ s.t. $\varphi_\alpha, \varphi_\beta$ are compatible $\forall \alpha, \beta$.

Remark 1.2. We will not concern ourselves here with limited differentiability: all manifolds (and most objects we will deal with) will be infinitely differentiable (= smooth, C^∞).

From now on, M will denote a C^∞ manifold.

Definition 1.3. (i) For $p \in M$, the tangent space T_pM is defined as

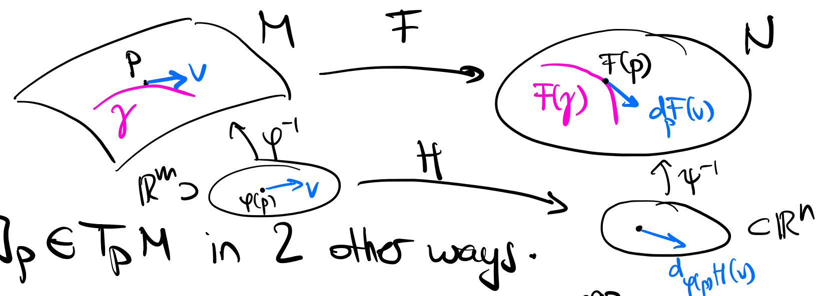
$$T_pM = \{ [\varphi, v] : \varphi \text{ chart}, v \in \mathbb{R}^m \} / \sim ,$$

where $[\varphi,v]\sim[\psi,w] :\Longleftrightarrow d_{\varphi(p)}(\psi\circ\varphi^{-1})(v)=w$.

(Since $d_{\varphi(p)}(\psi\circ\varphi^{-1}):\mathbb{R}^m\to\mathbb{R}^m$ is an invertible linear map, T_pM is an m-dimensional vector space.) Write $[\varphi,v]_p$ for the equivalence class of $[\varphi,v]$.

(ii) For a smooth map $F: M\to N$ between 2 smooth manifolds (i.e. $H=\psi F\varphi^{-1}$ is a smooth map $\forall$ smooth charts φ on M, ψ on N), define the differential $d_pF: T_pM\to T_{F(p)}N$ by

$$d_pF([\varphi,v]_p) = [\psi, d_{\varphi(p)}H(v)]_{F(p)}$$



· We can interpret $[\varphi,v]_p\in T_pM$ in 2 other ways.

(i) Tangent vector of curves through p. If $\gamma:(-1,1)\xrightarrow{C^\infty}M$, $\gamma(0)=p$, then $\gamma'(0)\in T_pM$ is defined as $\gamma'(0):=[\varphi, \underbrace{\tfrac{d}{dt}\varphi(\gamma(t))\big|_{t=0}}_{\text{local coordinate representation of the curve } \gamma}]_p$.

(ii) Directional derivative. Given $f: M\to\mathbb{R}$, $V=[\varphi,v]_p\in T_pM$, define

$$Vf := \frac{d}{dt}\underbrace{(f\circ\varphi^{-1})}_{\text{local coordinate representation of the function } f}(\varphi(p)+tv)\Big|_{t=0}.$$

· One writes $\frac{\partial}{\partial\varphi^i}\Big|_p\in T_pM$ for the tangent vector $[\varphi,(0,\dots,\overset{i^{\text{th}}\text{ slot}}{\overset{\downarrow}{1}},\dots,0)]_p$; so every $v\in T_pM$ can be written $v=v^i\frac{\partial}{\partial\varphi^i}\Big|_p$ (implicit summation over repeated index) for $v^i\in\mathbb{R}$, $i=1,\dots,m$ (the coefficients of v in the chart φ).

Remark 1.4. Physics books (on General Relativity) typically define a tangent vector at $p \in M$ to be a collection $(v^i)_{i=1,\dots,n}$ of numbers which "transform contravariantly" when changing coordinates. The math definition above packages all "local coordinate representations" into a single object — the equivalence class $[\varphi, v]_p$.

Definition 1.5. A smooth vector field is a map $V : M \to TM = \bigsqcup_{p \in M} T_pM$, $V_p \in T_pM$, so that in all charts (φ, U) on M, we have

$$V_p = [\varphi, v(p)]_p \quad , p \in U, \text{ with } v : U \to \mathbb{R}^m \text{ smooth}.$$

Write $\Gamma(TM)$ for the space of all vector fields.

- $V \in \Gamma(TM)$ defines a derivation on $C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}$ by $f \mapsto Vf$ with $(Vf)(p) = V_p f$.
- Conversely, given $D : C^\infty(M) \to C^\infty(M)$ linear, $D(fg) = fDg + gDf$, $\exists!\, V \in \Gamma(TM)$ s.t. $Df = Vf \;\forall f \in C^\infty(M)$
- Given $V, W \in \Gamma(TM)$, also $[V, W] \in \Gamma(TM)$ (commutator, Lie bracket) where $[V, W]$ is the derivation $f \mapsto [V, W]f = VWf - WVf$. (Exercise.)

Definition 1.6. The cotangent space of M at p is the dual vector space $T_p^*M = \{\xi : T_pM \to \mathbb{R} \text{ linear}\}$. Its elements are called covectors.

- Given a chart φ around p, a basis of T_p^*M is given by the coordinate differentials

$$d_p\varphi^i : \left.\frac{\partial}{\partial \varphi^j}\right|_p \mapsto \delta^i_j .$$

- More generally, given $f : M \to \mathbb{R}$, define its differential at p by

  $$d_p f(V) = Vf \quad \text{for } V \in T_pM.$$

1.2. Riemannian metrics, first pass.

We can now give a practical definition of a Riemannian metric:

Definition 1.7. (i) A Riemannian metric g on M assigns to every p a positive definite symmetric bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ which depends smoothly on $p \in M$, i.e.

$\forall V, W \in \Gamma(TM)$, the map $M \ni p \mapsto g_p(V_p, W_p) \in \mathbb{R}$ is smooth.

(ii) A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g a Riemannian metric.

- Consider a local coordinate chart $x = (x^1, \dots, x^m)$ (this is more common notation than $\varphi^1, \dots, \varphi^m$). We can then define the metric coefficients

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\right).$$

They satisfy: (i) $g_{ij} = g_{ji}$ (symmetry);

(ii) $v^i g_{ij} v^j = g(V, V) > 0$, $V = v^i \frac{\partial}{\partial x^i}\big|_p$, $\forall\ v \in \mathbb{R}^m, v \neq 0$ (positive definiteness);

(iii) $g_{ij}(p)$ is C^∞ in p.

- We can moreover write

$$g_p = \sum_{i,j=1}^{m} g_{ij}(p)\, d_p x^i \otimes d_p x^j$$

where, for $\xi, \eta \in T_p^*M$, we write $\xi \otimes \eta$ for the bilinear form on T_pM

$$(\xi \otimes \eta)(V, W) := \xi(V)\, \eta(W) \qquad (V, W \in T_pM).$$

Remark 1.8. By polarization, a Riemannian metric is uniquely determined by the associated quadratic form $T_pM \ni V \mapsto |V|^2_{g_p} := g_p(V, V)$, with the interpretation of squared length.

As a basic guideline, all objects in differential geometry should be sections of vector bundles. The relevant bundles for a Riemannian metric and all further objects associated with it are introduced in the next section.

Example 1.9. (i) $M = \mathbb{R}^m$, chart $(x^1, \ldots, x^m)$ (standard coordinates),

$$g := \sum_{i=1}^m dx^i \otimes dx^i \equiv \sum_{i=1}^m (dx^i)^2 : \text{ Euclidean metric.}$$

(Indeed, $g(\partial_{x^j}, \partial_{x^k}) = \delta_{jk} = e_j \cdot e_k$.)

standard basis vectors, identified with $\partial_{x^j}, \partial_{x^k}$

(ii) $M = (0,\infty) \times (0, 2\pi)$, chart (r, φ), $g := dr^2 + r^2 d\varphi^2$.

Again the Euclidean metric but expressed in polar coordinates.

(iii) On any M, there exists a Riemannian metric. (Exercise.)

Motivated by (ii), we introduce the pullback of a metric g on M along a smooth map $F: N \to M$: for $V, W \in T_pN$,

$$(F^*g)_p(V,W) = g_{F(p)}(d_pF(V), d_pF(W))$$

When F is an immersion, i.e. $d_pF: T_pN \to T_{F(p)}M$ is injective $\forall p$, then F^*g is a Riemannian metric on N.

Definition 1.10. (i) A map $F: (N,h) \to (M,g)$ between Riemannian manifolds is a local isometry if F is a local diffeomorphism (i.e. $\forall p \in N$ $\exists\, U \subset N$ neighborhood of p s.t. $F|_U : U \to F(U)$ is a diffeomorphism) and $F^*g = h$.

(ii) (M,g) and (N,h) are isometric if there exists an isometry $F:(N,h)\to(M,g)$, i.e. F is a diffeomorphism and $F^*g=h$, $(F^{-1})^*h=g$.

Example 1.11. $\big((0,\infty)\times(0,2\pi),\ dr^2+r^2d\varphi^2\big)\ni(r,\varphi)$

$$\mapsto (r\cos\varphi, r\sin\varphi)\in\big(\mathbb{R}^2,(dx^1)^2+(dx^2)^2\big)$$

is a local isometry. (Exercise.)

Example 1.12. (Spheres.) Let $S^m=\{x\in\mathbb{R}^{m+1}:|x|=1\}$, with the induced metric g.

Then $\mathrm{Isom}(S^m):=\{F:S^m\to S^m \text{ isometry}\}\cong O(m+1)$

(Here $A\in O(m+1)$ is identified with the isometry $S^m\ni x\mapsto Ax\in S^m$.)

(Proof: later) In particular, (S^m,g) is ...

(i) homogeneous: $\forall\, p,q\in S^m\ \exists\, F\in\mathrm{Isom}(S^m)$ with $F(p)=q$.

("All points are the same.")

(ii) isotropic: $\forall\, p\in S^m,\ V,W\in T_pS^m,\ |V|_g=|W|_g=1,\ \exists\, F\in\mathrm{Isom}(S^m)$ with $F(p)=p$, $d_pF(V)=W$. ("All directions are the same.")

Example 1.13. (Hyperbolic spaces.) Let $\mathbb{H}^m=\{x\in\mathbb{R}^m:|x|<1\}$, with metric

$$g_{\mathbb{H}^m}:=\frac{4}{(1-|x|^2)^2}dx^2=\frac{4}{(1-\sum(x^i)^2)^2}\sum_{i=1}^m(dx^i)^2.$$

Then $(\mathbb{H}^m,g_{\mathbb{H}^m})$ is homogeneous and isotropic.

- This is the Poincaré disc model of hyperbolic space. It is isometric to the upper half plane model $\{(x',x^m)\in\mathbb{R}^{m-1}\times\mathbb{R}:x^m>0\}$,

$$g=\frac{1}{(x^m)^2}\big(dx'^2+(dx^m)^2\big).$$

(Will be discussed in exercises.)

1.3. Tensor bundles

First, we recall:

Definition 1.14. (i) A vector bundle of rank k is a triple (E, M, π) (often written $\pi: E \to M$) such that

- M is an m-dim. smooth manifold;
- E is an $(m+k)$-dim. smooth manifold (total space)
- π is a smooth map (projection) whose fibers $\pi^{-1}(p) =: E_p$ carry the structure of a k-dimensional real vector space;
- $\forall\, p \in M$ $\exists$ open neighborhood $U \subset M$ of p and a C^∞ diffeomorphism $\psi: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ s.t. $\psi|_{E_q}: E_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism $\forall\, q \in U$. ("Local triviality.")

(ii) A section s of E is a smooth map $s: M \to E$ with $\pi \circ s = \mathrm{id}_M$.

We write $\Gamma(E) = \{ s: M \to E \text{ section} \}$ for the space of all smooth sections.

Example 1.15. (i) $TM \xrightarrow{\pi} M$, $\pi([\varphi, v]_p) = p$, with local trivializations induced by charts φ via

$$\psi: \pi^{-1}(U) = \bigsqcup_{q \in U} T_q U \ni [(\varphi, v)]_q \longmapsto (q, v) \in U \times \mathbb{R}^m.$$

(ii) $T^*M \xrightarrow{\pi} M$ (exercise).

- In a local trivialization $\psi: \pi^{-1}(U) \to U \times \mathbb{R}^k$, a smooth section $s|_U$ is given by $s(q) = \psi^{-1}(q, \tilde{s}(q))$ for a C^∞ function $\tilde{s}: U \to \mathbb{R}^k$.

Using this observation, you can check that $\Gamma(TM)$ given by

Def. 1.14 is the same as $T^*(TM)$ from Def. 1.5.

- Given a cover $\{U_\alpha\}$ of M with local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, we can define the transition functions

$$\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R}) \quad \text{by} \quad \psi_\beta \circ \psi_\alpha^{-1}(q,v) = (q, \tau_{\beta\alpha}(q)v).$$

They satisfy the cocycle condition $\tau_{\gamma\beta} \circ \tau_{\beta\alpha} = \tau_{\gamma\alpha}$, $\tau_{\alpha\alpha} = \mathbb{1}_{k\times k}$.

- Conversely, given such a cover $\{U_\alpha\}$ and collection of functions $g_{\beta\alpha}$, one can construct a rank k vector bundle from these data.

(If the data come from a vector bundle E, this construction produces a vector bundle isomorphic to E.)

Given vector bundles $E, E' \to M$ of rank k, k', one can construct further vector bundles out of them:

(i) Direct sum (or sometimes Whitney sum) $E \oplus E' \to M$, of rank $k+k'$, with total space

$$E \oplus E' = \{ (z,z') \in E \times E' : \pi(z) = \pi'(z') \}$$

and projection $(z,z') \mapsto \pi(z) = \pi'(z')$.

Thus, $(E\oplus E')_p = E_p \oplus E'_p$, which we equip with the linear structure of $E_p \oplus E'_p$. (Exercise: show that $E \oplus E'$ is a smooth manifold.) The transition maps for $E\oplus E'$ are $g_{\beta\alpha}(p) \oplus g'_{\beta\alpha}(p)$ when $g_{\beta\alpha}, g'_{\beta\alpha}$ are transition maps for E, E', for a cover of M by open sets U_α on which both bundles are trivialized: $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^k$, $(\pi')^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^{k'}$.

(ii) Tensor product $E \otimes E' \to M$. This is constructed from local trivializations over a cover $\{U_\alpha\}$ and the transition functions $\tau_{\beta\alpha}, \tau'_{\beta\alpha}$ for E, E' using $U_\alpha \times \mathbb{R}^{kk'}$ and transition functions

$$\tau_{\beta\alpha}(p) \otimes \tau'_{\beta\alpha}(p) \in GL(\mathbb{R}^k \otimes \mathbb{R}^{k'}) \cong GL(kk', \mathbb{R}).$$

Thus, $(E \otimes E')_p = E_p \otimes E'_p$, and $E \otimes E'$ is a rank kk' vector bundle.

Definition 1.16. Let $r, s \in \mathbb{N}_0$. Then the (r,s)-tensor bundle is

$$T_{r,s}M := \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ factors}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ factors}}.$$

Thus, $(T_{r,s}M)_p = T_pM \otimes \cdots \otimes T_pM \otimes T_p^*M \otimes \cdots \otimes T_p^*M$ is the space of $(r+s)$-multilinear forms

$$T_p^*M \otimes \cdots \otimes T_p^*M \otimes T_pM \otimes \cdots \otimes T_pM \to \mathbb{R}.$$

Lemma 1.17. Every $T \in \Gamma(T_{r,s}M)$ defines a $C^\infty(M)$-multilinear map

$$(\underbrace{\Gamma(T^*M)}_{\substack{\text{space of} \\ \text{1-forms}}})^r \times (\underbrace{\Gamma(TM)}_{\substack{\text{space of} \\ \text{vector fields}}})^s \longrightarrow C^\infty(M) \quad \text{via}$$

$$(\omega_1, \dots, \omega_r, V_1, \dots, V_s) \mapsto T(\omega_1, \dots, \omega_r, V_1, \dots, V_s).$$

Conversely, every such map defines (uniquely) a section of $T_{r,s}M$.

Proof. Exercise. $\square$

Remark 1.18. A $(1,s)$-tensor T can equivalently be regarded as a $C^\infty(M)$-multilinear map $(\Gamma(TM))^s \to \Gamma(TM)$. (Later: the curvature tensor of a Riemannian metric is a $(1,3)$-tensor.)

Directly from Lemma 1.17, we get:

Lemma 1.19. (Riemannian metrics, second pass.) A Riemannian metric g on M in the sense of Definition 1.7 defines an element of

$$\Gamma(T_{0,2}M) = \Gamma(T^*M \otimes T^*M).$$

Conversely, $g \in \Gamma(T_{0,2}M)$ is a Riemannian metric iff g_p is symmetric and positive definite for all $p \in M$.

(Thus, the "smoothness in p" is built into the definition of $\Gamma(T_{0,2}M)$.)

Remark 1.20. Having gone through all this trouble to define Riemannian metrics in an abstract setting, it turns out that it is enough to study submanifolds of Euclidean space (with the induced metric):

Theorem (Nash 1956). $\exists\ f: \mathbb{N} \to \mathbb{N}$ ($f(m) = \frac{m(m+1)}{2} + 2m + 3$ works) s.t. $\forall\ (M,g)$, $\dim M = m$, $\exists$ submanifold $\tilde{M} \subset \mathbb{R}^{f(m)}$ and an isometry $(M,g) \to (\tilde{M}, \text{induced metric})$.

— Nonetheless, it is in general a good idea to avoid the introduction of extraneous structures (e.g. embeddings into Euclidean space), and thus we shall happily continue with a detailed study of (abstract) Riemannian manifolds.

On a Riemannian manifold, one has a preferred identification of TM, T^*M:

Lemma 1.21. Let (M,g) be a Riemannian manifold. Then

$$T_pM \ni V \mapsto V^\flat := g_p(V, -) \in T_p^*M, \quad p \in M,$$

defines an isomorphism $TM \cong T^*M$. (The inverse of this map is denoted $T_p^*M \ni \xi \mapsto \xi^\# \in T_pM$.)

Proof. Follows from the non-degeneracy of the bilinear form g_p. $\square$

The maps $V \mapsto V^\flat$, $\xi \mapsto \xi^\#$ are called musical isomorphisms (for physicists: index lowering, index raising.) In local coordinates,

$$g = g_{ij}\, dx^i \otimes dx^j, \quad V = V^i \frac{\partial}{\partial x^i} \implies V^\flat = g_{ij} V^i\, dx^j \equiv V_j\, dx^j,$$

where $V_j := g_{ij} V^i$.

Write $(g^{ij})_{i,j=1,\dots,m} = (g_{ij})^{-1}$ for the inverse matrix; then

$$\xi = \xi_i\, dx^i \implies \xi^\# = g^{ij} \xi_i \frac{\partial}{\partial x^j} \equiv \xi^j \frac{\partial}{\partial x^j},$$

where $\xi^j = g^{ij} \xi_i$.

Using $\#$, we can define a positive definite symmetric bilinear form on T_p^*M via $g_p^{-1}(\xi, \eta) := g_p(\xi^\#, \eta^\#)$. We then have

$$g_p^{-1}(dx^i, dx^j) = g_p\left(g^{ik} \frac{\partial}{\partial x^k}, g^{jl} \frac{\partial}{\partial x^l}\right) = g^{ik} g_{kl} g^{jl} = (g^{-1} g g^{-1})^{ij} = g^{ij}.$$

Corollary 1.22 $\quad g_p^{-1} = g^{ij}(p) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ (local coordinate expression for g^{-1}).