Having developed a sizeable amount of foundational material, we now turn to the study of linear equations

$$Ax = y : \quad \circledast$$

when do solutions exist? Are they unique? Etc.

- Our first approach to solving $\circledast$ is motivated by the following basic fact in linear algebra: if $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then
$$\operatorname{ran} A = (\ker A^*)^\perp,$$
$$\ker A = (\operatorname{ran} A^*)^\perp.$$
Here $A^* : \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint, $\langle Ax, y\rangle = \langle x, A^* y\rangle \ \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m$.
So $Ax = y$ can be solved for any given $y \in Y \Leftrightarrow \ker A^* = \{0\}$.
In general normed spaces, we do not have an inner product. So the adjoint needs to be defined "better".

Definition Let X, Y be normed $\mathbb{K}$-vector spaces, $A \in L(X, Y)$.
The *adjoint* of A is the map $A^* : Y^* \to X^*$,
$$\lambda \mapsto \lambda \circ A.$$

Thus, if we write $\langle y, \lambda\rangle_{Y, Y^*} := \lambda(y)$ for $\lambda \in Y^*, y \in Y$, then
$$\langle Ax, \lambda\rangle_{Y, Y^*} = \langle x, A^*\lambda\rangle_{X, X^*}.$$

If X, Y are Hilbert spaces, then $A^* : Y \to X$ via Riesz; and $(Ax, y) = (x, A^* y) \quad \forall x \in X, y \in Y.$

Example Let $\Omega \subset \mathbb{R}^n$ be measurable. Let $k \in L^2(\Omega \times \Omega)$, and set $A : L^2(\Omega) \to L^2(\Omega)$,

$$(Au)(x) = \int_\Omega k(x,y)\, u(y)\, dy.$$

(Note that $\|Au\|^2_{L^2(\Omega)} = \int_\Omega \left| \int_\Omega k(x,y) u(y)\, dy \right|^2 dx$

$$\leq \int_\Omega \left(\int_\Omega |k(x,y)|^2 dy \right) \left(\int_\Omega |u(y)|^2 dy \right) dx$$

$$= \|k\|^2_{L^2(\Omega \times \Omega)} \|u\|^2_{L^2(\Omega)},$$

so $A \in L(L^2(\Omega))$ indeed, with $\|A\|_{L(L^2(\Omega))} \leq \|k\|_{L^2(\Omega \times \Omega)}$.)

Claim: $A^* : L^2(\Omega) \to L^2(\Omega)$ is the integral operator with integral kernel $(x,y) \mapsto \overline{k(y,x)}$. That is,

$$(A^* v)(x) = \int_\Omega \overline{k(y,x)}\, v(y)\, dy.$$

Indeed, for $u, v \in L^2(\Omega)$,

$$(Au, v)_{L^2(\Omega)} = \int_\Omega Au(x)\, \overline{v(x)}\, dx$$

$$= \int_\Omega \left(\int_\Omega k(x,y)\, u(y)\, dy \right) \overline{v(x)}\, dx$$

$$= \int_{\Omega \times \Omega} k(x,y)\, \overline{v(x)}\, u(y)\, dx\, dy$$

$$= \int_\Omega \overline{\left(\int_\Omega \overline{k(x,y)}\, v(x)\, dx \right)}\, u(y)\, dy$$

$$\overset{x \leftrightarrow y}{=} \int_\Omega u(x)\, \overline{\left(\int_\Omega \overline{k(y,x)}\, v(y)\, dy \right)}\, dx$$

$$= (u, A^* v)_{L^2(\Omega)}. \qquad \square$$

Lemma (i) For $A \in L(X,Y)$, we have $\|A^*\|_{L(Y^*,X^*)} = \|A\|_{L^2(X,Y)}$.

(ii) $A \in L(X,Y)$, $B \in L(Y,Z)$ (with X, Y, Z normed spaces)

$\Longrightarrow$ $(B \circ A)^* = A^* \circ B^*$.

(iii) $A \in L(X,Y) \Longrightarrow A^{**} \in L(X^{**}, Y^{**})$ satisfies

$A^{**}(\imath_X(x)) = \imath_Y(Ax)$, where $\imath_X : X \to X^{**}$, $\imath_Y : Y \to Y^{**}$

are the canonical inclusions.

Proof (i)

$$\begin{aligned}
\|A^*\|_{L(Y^*,X^*)} &= \sup_{\substack{\lambda \in Y^* \\ \|\lambda\|_{Y^*}=1}} \|A^*\lambda\|_{X^*} \\
&= \sup_{\substack{\lambda \in Y^* \\ \|\lambda\|_{Y^*}=1}} \sup_{\substack{x \in X \\ \|x\|=1}} |(A^*\lambda)(x)| \\
&= \sup_{\substack{x \in X \\ \|x\|=1}} \sup_{\substack{\lambda \in Y^* \\ \|\lambda\|_{Y^*}=1}} |\lambda(Ax)| \\
&\overset{\text{Hahn-Banach}}{=} \sup_{\substack{x \in X \\ \|x\|=1}} \|Ax\|_Y = \|A\|_{L(X,Y)}.
\end{aligned}$$

(ii) $(B \circ A)^*\lambda = \lambda \circ B \circ A = A^*(\lambda \circ B) = A^* B^* \lambda$.

(iii) Exercise. $\square$

To study the relationship of (ker/ran)(A/A^*), we first define:

Definition Let X be a normed vector space.

(i) If $A \subset X$ is a subset, then

$$A^\perp := \{\lambda \in X^* : \lambda(a) = 0 \ \forall a \in A\} \subset X^*.$$

(When A is a linear subspace, one also writes $A^\perp = \mathrm{ann}(A)$, the annihilator of A.)

(ii) If $\mathcal{F} \subset X^*$ is a subset, then

$$^\perp\mathcal{F} := \{ x \in X : \lambda(x) = 0 \ \forall \lambda \in X \} \subset X.$$

<u>Remark</u> (i) $A^\perp = (\mathrm{span}\, A)^\perp = (\overline{\mathrm{span}\, A})^\perp$, similarly for $^\perp\mathcal{F}$.

(ii) $A^\perp$ and $^\perp\mathcal{F}$ are always closed.

(iii) $^\perp\mathcal{F} \subset X$, whereas $\mathcal{F}^\perp$ (defined in (i)) $\subset X^{**}$.

<u>Lemma</u> Let $Y \subset X$ be a subspace of the normed space X.
Then $^\perp(Y^\perp) = \overline{Y}$.

<u>Proof</u> "$\supseteq$" follows directly from the definition.

"$\subseteq$" Let $x_0 \in X \setminus \overline{Y}$. Let $\lambda \in X^*$ be such that
$\lambda(x_0) = 1$, $\lambda|_{\overline{Y}} = 0$. Then $\lambda \in Y^\perp$, and
$\lambda(x_0) \neq 0$ implies $x_0 \notin {}^\perp(Y^\perp)$. $\square$

We use these notions to give another more abstract example of adjoints:

<u>Lemma</u> (Inclusions and projections are dual to each other).
Let X = normed vector space, and let $Y \subset X$ be a closed subspace.

(i) Let $i : Y \to X$ denote the inclusion map ($i(y) = y$).
Then $i^* : X^* \to Y^*$ is surjective and induces an isometric isomorphism $X^*/Y^\perp \xrightarrow{\cong} Y^*$.

(ii) Let $\pi : X \to X/Y$ denote the projection $\pi(x) = x + Y = [x]$.

Then $\pi^* : (X/Y)^* \to X^*$ satisfies $\pi^*(\lambda) \in Y^\perp \ \forall \lambda \in (X/Y)^*$,

and π^* induces an isometric isomorphism $(X/Y)^* \xrightarrow{\cong} Y^\perp$.

Remark In (i), $\underbrace{\ker(i^*)}_{=Y^\perp} = (\underbrace{\operatorname{ran} i}_{=Y})^\perp$, $\underbrace{\operatorname{ran}(i^*)}_{=Y^*} = (\underbrace{\ker i}_{=\{0\} \subset Y})^\perp$

Similarly in (ii).

Proof of the Lemma (i). Given $\lambda \in Y^*$, let $\tilde{\lambda} \in X^*$ be an extension of λ (given by Hahn–Banach). Then for $y \in Y$

$$i^*(\tilde{\lambda})(y) = \tilde{\lambda}(i(y)) = \tilde{\lambda}(y) = \lambda(y). \quad \circledast$$

$\Rightarrow i^*(\tilde{\lambda}) = \lambda$, so i^* is surjective.

- $i^*(\tilde{\lambda}) = 0 \iff i^*(\tilde{\lambda})(y) = 0 \quad \forall y \in Y$

$\qquad \overset{\circledast}{\iff} \tilde{\lambda}(y) = 0 \quad \forall y \in Y$

$\qquad \iff \tilde{\lambda} \in Y^\perp$.

$\Rightarrow \ker i^* = Y^\perp$. Thus i^* induces an isomorphism

$$X^*/\ker i^* \to \operatorname{ran} i^*, \text{ i.e. } j^* : X^*/Y^\perp \to Y^*.$$

- Let $[\lambda] \in X^*/Y^\perp$. Then $\|j^*[\lambda]\|_{Y^*} = \sup_{\substack{y \in Y \\ \|y\|=1}} |(j^*[\lambda])(y)|$

$$= \sup_{\substack{y \in Y \\ \|y\|=1}} |(i^*\lambda)(y)| = \sup_{\substack{y \in Y \\ \|y\|=1}} |\lambda(y)| = \|\lambda|_Y\|_{Y^*}$$

We thus need to show $\|\lambda|_Y\|_{Y^*} = \inf_{\mu \in Y^\perp} \|\lambda + \mu\|_{X^*}$ (for $\lambda \in X^*$).

"$\leq$" is clear since $(\lambda+\mu)|_Y = \lambda|_Y \ \forall \mu \in Y^\perp$.

"$\geq$": Given $\lambda \in X^*$ let $\tilde{\lambda} \in X^*$ be an extension of $\lambda|_Y$ with $\|\tilde{\lambda}\|_{X^*} = \|\lambda|_Y\|_{Y^*}$. Then $(\tilde{\lambda}-\lambda)|_Y = 0$, so $\mu_0 := \tilde{\lambda}-\lambda \in Y^\perp$ and

$$\inf_{\mu \in Y^\perp} \|\lambda+\mu\|_{X^*} \leq \|\lambda+\mu_0\|_{X^*} = \|\tilde{\lambda}\|_{X^*} = \|\lambda|_Y\|_{Y^*}.$$

(ii) Exercise. $\square$

Theorem X, Y Banach spaces. Let $A \in L(X,Y)$. If $\operatorname{ran} A \subset Y$ is closed, then (i) $(\operatorname{ran} A)^\perp = \ker A^*$, $\operatorname{ran} A = {}^\perp(\ker A^*)$.

(ii) $\ker A = {}^\perp(\operatorname{ran} A^*)$, $(\ker A)^\perp = \operatorname{ran} A^*$.

(iii) $\operatorname{ran} A^* \subset X^*$ is closed.

Proof Case 1: A is bijective. By the Open Mapping Theorem, $B := A^{-1} \in L(Y,X)$. Since

$$BA = I_X \longleftarrow \text{identity map on } X$$
$$AB = I_Y,$$

we get

$$A^*B^* = (I_X)^* = I_{X^*},$$
$$B^*A^* = (I_Y)^* = I_{Y^*}$$

$\Rightarrow A^* \in L(Y^*, X^*)$ is invertible. The conclusion is now obvious.

Case 2: $A \in L(X,Y)$. We factor $A = i \circ \tilde{A} \circ \pi$,

$\pi : X \longrightarrow X/\ker A$ projection,

$i : \operatorname{ran} A \longrightarrow Y$ inclusion,

$\tilde{A} : X/\ker A \longrightarrow \operatorname{ran} A$ induced by A, bijection.

Since $\operatorname{ran} A$ is a Banach space, $\tilde{A}^*$ is a bijection ($\Leftarrow$ Case 1).

Now $A^* = \pi^* \circ \tilde{A}^* \circ i^*$, with

$\pi^* : \left(X/_{\ker A}\right)^* \to (\ker A)^\perp \subset X^*$ injective

$i^* : Y^* \to Y^*/_{(\operatorname{ran} A)^\perp}$ surjective.

$\Rightarrow$ $\operatorname{ran} A^* = \operatorname{ran} \pi^* = (\ker A)^\perp$ (which is closed),

$\ker A^* = \ker i^* = (\operatorname{ran} A)^\perp$. $\square$

Remark If $A \in L(X,Y)$ and $\operatorname{ran} A^*$ is closed, then $\operatorname{ran} A$ is closed. This is much harder to prove, but it is obvious when X is reflexive.

- An obvious question is now: are there some general classes of operators which have closed range (or whose adjoints have closed range)? We shall study an important such class ("Fredholm operators") later in detail.

- For now, here is an easy result:

Corollary X reflexive, Y Banach, $A \in L(X,Y)$. Suppose we have the following estimate: $\exists\, C>0 \;\; \forall\, y^* \in Y^*$: $\|y^*\|_{Y^*} \leq C \|A^* y^*\|_{X^*}$. (*)

Then A is surjective.

Proof. (*) implies that $\operatorname{ran} A^*$ is closed: if $\{A^* y_k^*\}_{k\in\mathbb{N}}$ is a Cauchy sequence, then so is $\{y_k^*\}$, with limit $y^* \in Y^*$, and thus $A^* y_k^* \xrightarrow{k\to\infty} A^* y^*$.

- A^* is injective: $A^* y^* = 0 \Rightarrow \|y^*\|_{Y^*} = 0 \Rightarrow y^* = 0$.
- The previous Theorem implies that

$$\operatorname{ran} A^{**} = (\ker A^*)^\perp = \{0\}^\perp = Y^{**}.$$

Given $y \in Y$, $\exists\, x \in X$ s.t. $A^{**}(\iota(x)) = \iota(y)$. (This uses the reflexivity of X.) $\iff$ $Ax = y$. $\square$

Thus, solvability of $Ax = y$ follows from an estimate for A^*, a rather concrete task!

Example. $X = Y = \ell^2(\mathbb{Z})$, $V \in \ell^\infty(\mathbb{Z})$,

$$(Au)_n = \underbrace{(u_{n-1} - 2u_n + u_{n+1})}_{\text{"discrete Laplacian"}\ \left(-\frac{d^2}{dx^2}\right)} + V_n u_n.$$

If $\operatorname{Re} V_n \geq c > 0\ \forall n \in \mathbb{Z}$, then we have an estimate

$$\|u\|_{\ell^2} \leq \frac{1}{c} \|A^* u\|_{\ell^2}. \quad (*)$$

$\Rightarrow$ A is surjective. (Could also use Lax–Milgram here.)

To prove $(*)$, set $(Du)_n = u_{n+1} - u_n \Rightarrow (D^* u)_n = u_{n-1} - u_n$,

and $(D^* D u)_n = (Du)_{n-1} - (Du)_n = u_{n-1} - 2u_n + u_{n+1}$

$\Rightarrow A = D^* D + V$.

$$\begin{aligned}
\Rightarrow \quad \operatorname{Re}(Au, u) &= \operatorname{Re}(D^* D u, u) + \operatorname{Re}(Vu, u) \\
&= \operatorname{Re}(Du, Du) + \sum_{n \in \mathbb{Z}} (\operatorname{Re} V_n) |u_n|^2 \\
&\geq \operatorname{Re} \|Du\|^2 + c\|u\|^2 \\
&\geq c\|u\|^2
\end{aligned}$$

$\Rightarrow c\|u\|^2 \leq \|Au\|\,\|u\| \Rightarrow (*)$.