Application 3: strongly diverging Fourier series

Banach–Steinhaus
Baire categories

Let $X = \{ u : \mathbb{R} \to \mathbb{C}$ continuous 2π-periodic$\}$.
$u(x+2\pi) = u(x)\ \forall x \in \mathbb{R}$.

For $t \in [0,2\pi]$ and $m \in \mathbb{N}_0$, let

$$s_m(u,t) := \frac{1}{2\pi} \sum_{k=-m}^{m} e^{ikt} \underbrace{\left(\int_0^{2\pi} e^{-iks} u(s)\, ds \right)}_{= \hat{u}(k)\ (k^{th} \text{ Fourier coefficient of } u)}$$

Exercise 3.4: the operator norm of $s_m(\cdot,0) \in L(X,\mathbb{C})$ (i.e. of $X \ni u \mapsto s_m(u,0) \in \mathbb{C}$) satisfies $\sup_m \|s_m(\cdot,0)\| = \infty$.

Note: $s_m(u,t) = s_m(\tilde{u},0)$ for $\tilde{u}(s) = u(s+t)$;
thus also $\sup_m \| s_m(\cdot,t) \| = \infty\ \forall t \in [0,2\pi]$. $(*)$

Theorem $\exists\, u \in X$ s.t. $s_m(u,t)$ does not converge for uncountably many $t \in [0,2\pi]$.

Proof • Step 1: Let $\{t_n\}_{n\in\mathbb{N}} \subset [0,2\pi]$ be dense. Then $\exists\, u_0 \in X$ s.t. $\sup_{m\in\mathbb{N}} |s_m(u_0,t_n)| = \infty\ \forall n$.

Indeed, let $\mathcal{L}_n = \{ s_m(\cdot,t_n) : m \in \mathbb{N}\} \subset L(X,\mathbb{C})$. By $(*)$, $\mathcal{L}_n$ is unbounded. By the Condensation of Singularities theorem (exercise 4.2), $\exists\, u_0 \in X$ s.t. $\sup_{A\in\mathcal{L}_n} |Au_0| = \infty\ \forall\, n \in \mathbb{N}$, which precisely the desired statement.

• Step 2: For this $u_0 \in X$, and for $N \in \mathbb{N}$, the set
$A_N := \{ t \in [0,2\pi] : |s_m(u_0,t)| \le N\ \forall m\}$ is closed with empty interior.

Indeed, $A_N = \bigcap_{m \in \mathbb{N}} \{ t \in [0,2\pi] : |s_m(u_0,t)| \leq N \}$. Since $t \mapsto s_m(u_0,t)$ is a continuous function, this is an intersection of closed sets. Moreover, $t_n \notin A_N \ \forall n$, so

$$\underbrace{[0,2\pi] \setminus A_N}_{\text{open}} \supset \underbrace{\{t_n\}_{n \in \mathbb{N}}}_{\text{dense}} \Rightarrow A_N^{\circ} = \emptyset.$$

- Step 3: Conclusion. The set

$$\bigcup_{N \in \mathbb{N}} A_N = \{ t \in [0,2\pi] : \sup_m |s_m(u_0,t)| < \infty \}$$

is meager by Step 2, so its complement — which contains the set of $t \in [0,2\pi]$ where $s_m(u_0,t)$ diverges — cannot be meager (since $[0,2\pi]$ is not meager), and therefore must be uncountable.

$\square$

Remark Carleson's Theorem (1966) $\Rightarrow$ for every $u_0 \in X$, $s_m(u_0,t) \to u_0(t)$ for Lebesgue-almost every $t \in [0,2\pi]$! (This is even true for $u_0 \in L^p([0,2\pi])$, $1 < p < \infty$.)