Banach spaces and continuous maps between them have much better properties than general normed vector spaces.

Recall that a Banach space $(X, \|\cdot\|)$ is a complete metric space with metric $d(x,y) = \|x-y\|$.

**Theorem** (Baire Category Theorem) Let $(X, d)$ be a complete metric space. Let $A_j \subset X$, $j \in \mathbb{N}$, be a nowhere dense set, i.e. $(\overline{A_j})^\circ = \emptyset$.

Then $\bigcup_{j\in\mathbb{N}} A_j \neq X$. ("A complete metric space is not a countable union of nowhere dense sets.")

**Proof** · Since $A_1$ is nowhere dense, $\exists\, x_1 \in X \setminus \overline{A_1}$. Pick $r_1 < \frac{1}{2}$ s.t. $B_1 = B(x_1, r_1) = \{x \in X : d(x, x_1) < r_1\}$ satisfies $B_1 \cap \overline{A_1} = \emptyset$.


· We proceed inductively. Let $j \in \mathbb{N}$, $j \geq 2$.
Since $A_j$ is nowhere dense, $\exists\, x_j \in B_{j-1} \setminus \overline{A_j}$. Pick $r_j < 2^{-j}$ s.t. $B_j := B_{r_j}(x_j)$ satisfies $B_j \cap \overline{A_j} = \emptyset$ and $\overline{B_j} \subset B_{j-1}$.

· $\{x_j\}$ is a Cauchy sequence: for $j \geq \ell$, $x_j, x_\ell \in B_\ell$, so $d(x_j, x_\ell) \leq 2^{-\ell}$.

Let $x = \lim_{j\to\infty} x_j \in X$ (using the completeness of $X$!)

· Fix $\ell \geq 2$. Since $x_j \in B_\ell$ for $j \geq \ell$, we have $x \in \overline{B_\ell} \subset B_{\ell-1}$. Therefore, $x \notin A_{\ell-1}$. Since this is true $\forall \ell \geq 2$, we have $x \in X \setminus \bigcup_{\ell\in\mathbb{N}} A_\ell$. $\square$

**Remark** · $\mathbb{Q}$ (with $d(x,y) = |x-y|$) is not complete, and $\mathbb{Q} = \bigcup_{x\in\mathbb{Q}} \{x\}$ (countable union).

· $\mathbb{R}$ is complete; since $\mathbb{R} = \bigcup_{x\in\mathbb{R}} \{x\}$, the countability of the family $\{A_j\}$ is essential.

**Corollary** Let $X$ be a complete metric space.

(i) Suppose $A_j \subset X$ $(j \in \mathbb{N})$ is closed, and $\bigcup_{j \in \mathbb{N}} A_j = X$. Then $\exists\, j_0$ s.t. $A_{j_0}^\circ \neq \emptyset$.

(ii) Suppose $U_j \subset X$ $(j \in \mathbb{N})$ is open and dense. Then

$$\bigcap_{j \in \mathbb{N}} U_j \subset X \text{ is dense.}$$

**Proof** (i) If $A_j^\circ = \emptyset$ $\forall j$, Baire would imply $\bigcup_{j \in \mathbb{N}} A_j \neq X$, contradiction.

(ii). Let $A_j = X \setminus U_j$; this is closed, and nowhere dense since

$$A_j^\circ = (X \setminus U_j)^\circ = X \setminus \overline{U_j} = X \setminus X = \emptyset. \text{ Therefore,}$$

$$\bigcup_{j \in \mathbb{N}} A_j \neq X; \text{ thus } \bigcap_{j \in \mathbb{N}} U_j = X \setminus \bigcup_{j \in \mathbb{N}} A_j \neq \emptyset.$$

- Replacing $X$ by the complete metric space $X \cap \overline{B_r}(x)$ for arbitrary $x \in X$ and $r > 0$, and $U_j$ by $U_j \cap \overline{B_r}(x)$ (which is open and dense in $X \cap \overline{B_r}(x)$), we conclude that $\overline{B_r}(x) \cap \bigcap_{j \in \mathbb{N}} U_j \neq \emptyset$.

This precisely means that $\bigcap_{j \in \mathbb{N}} U_j$ is dense. $\square$

Indeed, let $x \in \overline{B_r}(x)$, $\varepsilon > 0$. Pick $x' \in B_r(x)$ s.t. $d(x, x') < \frac{\varepsilon}{2}$.
Since $U_j \subset X$ is dense, we can find $y \in U_j$ so that
$d(x', y) < \min\left(\frac{\varepsilon}{2}, r - d(x, x')\right)$.
But then $d(x, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and $d(x, y) < d(x, x') + d(x', y) < r$
$\Rightarrow y \in U_j \cap B_r(x) \subset U_j \cap \overline{B_r}(x)$.

**Definition** Let $X$ be a metric space, $A \subset X$.

(i) $A$ is called *meager* (or *a set of first category*) if

$$A = \bigcup_{j \in \mathbb{N}} A_j \text{ with } A_j \subset X \text{ nowhere dense.}$$

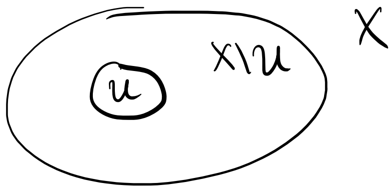
(ii) $A$ is called *nonmeager* (or *a set of second category*) if it is not meager.

(iii) $A$ is called *residual* if $X \setminus A$ is meager ($\Longleftrightarrow A = \bigcap_{j \in \mathbb{N}} U_j$ with $U_j \subset X$ open and dense).

**Example** (i) $\mathbb{Q} \subset \mathbb{R}$ is meager; $\{\text{irrational numbers}\} \subset \mathbb{R}$ are nonmeager and also residual.

(ii) $X$ complete, $A$ residual $\Rightarrow$ $A$ nonmeager. (*Exercise*.)

(ii) $X$ complete $\Rightarrow X$ is nonmeager. Every $\emptyset \neq \mathcal{U} \subset X$ open is nonmeager. When $\overline{\mathcal{U}} \neq X$, then $\mathcal{U}$ is not residual, however.




(iii) $\nexists$ relationship between these notions and Lebesgue measure.

- Claim $\exists$ residual set $S \subset \mathbb{R}$ with $\mathcal{L}^1(S) = 0$ (1-dim'l Lebesgue measure).

  Proof Let $\mathbb{Q} = \{q_j\}_{j \in \mathbb{N}}$ and put $\mathcal{U}_\ell = \bigcup_{j \in \mathbb{N}} (q_j - 2^{-j-\ell}, q_j + 2^{-j-\ell})$

  this is open and dense, and $\mathcal{L}^1(\mathcal{U}_\ell) = \sum_{j=1}^{\infty} 2^{-j-\ell} = 2^{-\ell}$.

  $\Rightarrow S := \bigcap_{\ell \in \mathbb{N}} \mathcal{U}_\ell$ is residual, and $\mathcal{L}^1(S) = \lim_{\ell \to \infty} \mathcal{L}^1(\mathcal{U}_\ell) = 0$. $\square$

- In the other direction, $\exists$ meager sets with positive Lebesgue measure, e.g. fat Cantor sets.

(iv) The set $\{ f \in C^0([0,1]) : f$ is differentiable in at least one point of $[0,1]\}$ is meager. (Banach, 1931. Exercise.)

We now apply the Baire Category Theorem in the context of Banach spaces.

## Principle of Uniform Boundedness (Banach–Steinhaus)

Theorem Let $X$ be a Banach space; let $Y$ be a normed vector space.

Suppose $\{T_\lambda\} \subset L(X,Y)$ is pointwise bounded, i.e.

$$\forall x \in X, \quad \sup_\lambda \|T_\lambda x\|_Y < \infty.$$

Then $\{T_\lambda\}$ is uniformly bounded, i.e.

$$\sup_\lambda \|T_\lambda\|_{L(X,Y)} < \infty.$$

**Proof** For $n \in \mathbb{N}$, let $A_n := \{x \in X : \|T_\lambda x\| \leq n \ \forall \lambda\}$.
By assumption, $X = \bigcup_{n\in\mathbb{N}} A_n$. Moreover, $A_n$ is closed since every $T_\lambda$ is continuous. By the Baire Category theorem, $\exists\, n_0 \in \mathbb{N}$ s.t. $\mathring{A}_{n_0} \neq \emptyset$; that is, $\exists\, x_0 \in X$, $r_0 > 0$ s.t.

$$B_{r_0}(x_0) \subset A_{n_0}.$$

For $x \in X$, $x \neq 0$, we then find

$$\|T_\lambda x\|_Y = \left\| T_\lambda\left(x_0 + \frac{r_0}{2}\frac{x}{\|x\|_X}\right) - T_\lambda(x_0)\right\|_Y \cdot \frac{2\|x\|_X}{r_0}$$

$$\leq \Bigg( \Big\| T_\lambda \underbrace{\left(x_0 + \frac{r_0}{2}\frac{x}{\|x\|_X}\right)}_{\in B_{r_0}(x_0)} \Big\|_Y + \|T_\lambda \underbrace{(x_0)}_{\in B_{r_0}(x_0)}\|_Y \Bigg) \cdot \frac{2\|x\|_X}{r_0}$$

$$\leq \frac{4n_0}{r_0}\|x\|_X.$$

$$\Rightarrow \sup_\lambda \|T_\lambda\|_{L(X,Y)} \leq \frac{4n_0}{r_0}.$$ ☐

**Corollary** $X$ Banach, $Y$ normed. Suppose $\{A_j\}_{j\in\mathbb{N}} \subset L(X,Y)$ converges pointwise to $A : X \to Y$, i.e. $\lim_{j\to\infty} A_j x = Ax \quad \forall x \in X$.
Then $A \in L(X,Y)$, and $\|A\|_{L(X,Y)} \leq \liminf_{j\to\infty} \|A_j\|_{L(X,Y)} < \infty$.

**Proof** $A$ is certainly linear; we need to show $A$ is bounded.
Banach–Steinhaus implies $\sup_j \|A_j\|_{L(X,Y)} < \infty$. Pick a subsequence $\{A_{j_k}\}_{k\in\mathbb{N}}$ with $\|A_{j_k}\|_{L(X,Y)} \xrightarrow[k\to\infty]{} \liminf_{j\to\infty} \|A_j\|_{L(X,Y)} =: C$.
Then $A_{j_k} \to A$ pointwise, and

$$\|Ax\|_Y = \lim_{k\to\infty} \|A_{j_k} x\|_Y \leq \left(\lim_{k\to\infty} \|A_{j_k}\|_{L(X,Y)}\right)\|x\|_X$$

$$= C\|x\|_X.$$ ☐

**Remark.** The completeness of $X$ is crucial. Take $X = (C^0([0,1]), \|\cdot\|_{L^1})$, $Y = \mathbb{R}$,

and consider $A_j u = j \int_0^{1/j} u(x)\,dx$. Then

$$|A_j u| \le j \|u\|_{L^1},$$

and $A_j u \xrightarrow{j\to\infty} u(0)$. But $A: X \to Y$, $u \mapsto u(0)$, is not continuous.

(E.g. for $u_n(x) = (1-x)^n$, $\|u_n\|_{L^1} = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$, but $Au_n = u_n(0) = 1 \ \forall n$.)

- On the other hand, $|A_j u| \le \|u\|_{C^0}$, and $A: (C^0([0,1]), \|\cdot\|_{C^0}) \to \mathbb{R}$ is continuous.

## Open Mapping Theorem

**Theorem** Let $X, Y$ be Banach spaces, $A \in L(X,Y)$. Suppose $A$ is surjective. Then $A$ is open, i.e. $A(U) \subset Y$ is open for all open $U \subset X$.

**Corollary** $X, Y$ Banach spaces, $A \in L(X,Y)$ bijective $\Rightarrow A^{-1} \in L(Y,X)$.

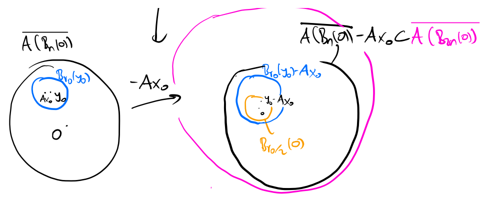
**Proof** $A^{-1}$ is certainly linear; its continuity follows from $(A^{-1})^{-1}(U) = A(U) \subset Y$ being open $\forall\, U \subset X$ open. $\square$

**Proof of Theorem** **Step 1.** Since $A$ is surjective, we have

$$Y = \bigcup_{n=1}^{\infty} A(B_n(0)).$$

Baire (on $Y$) implies: $\exists\, n \in \mathbb{N}$ s.t. $\overline{A(B_n(0))}$ has nonempty interior. Scaling and translating, we infer that $B_\varepsilon(0) \subset \overline{A(B_1(0))}$ for some $\varepsilon > 0$.

(Details: $\exists\, y_0 \in Y$, $r_0 > 0$ s.t. $B_{r_0}(y_0) \subset \overline{A(B_n(0))}$. Let $x_0 \in X$ s.t. $\|x_0\|_X < n$, $\|y_0 - Ax_0\|_Y < \frac{r_0}{2}$; then $B_{r_0}(y_0) - Ax_0 \subset \overline{A(B_n(0))} - Ax_0$. By the triangle inequality, we have $B_{r_0/2}(0) \subset B_{r_0}(y_0) - Ax_0$. Moreover, $\overline{A(B_n(0))} - Ax_0 \subset \overline{A(B_{2n}(0))}$. Thus, $B_{r_0/2}(0) \subset \overline{A(B_{2n}(0))}$, and therefore $B_\varepsilon(0) \subset \overline{A(B_1(0))}$ for $\varepsilon = \frac{r_0}{4n}$.)




**Step 2** We claim that $B_\varepsilon(0) \subset A(B_2(0))$.

Let $y \in B_\varepsilon(0)$, and pick $x_0 \in B_1(0)$ s.t. $y_1 := y - Ax_0 \in B_{\varepsilon/2}(0)$.

Since $B_{\varepsilon/2}(0) \subset \overline{A(B_{1/2}(0))}$, we can pick $x_1 \in B_{1/2}(0)$ s.t.

$$y_2 := y_1 - Ax_1 \in B_{\varepsilon/4}(0).$$

Continuing in this fashion, we construct $x_j \in B_{2^{-j}}(0)$ s.t.

$$y_{j+1} = y_j - Ax_j \in B_{2^{-j-1}\varepsilon}(0).$$

Let $x = \sum_{j=0}^{\infty} x_j \in B_2(0)$ (note: $\sum_{j=0}^{\infty} \|x_j\|_X < \sum_{j=0}^{\infty} 2^{-j} = 2$, and $X$ is complete),

then $y - Ax = \lim_{J\to\infty} \left(y - \sum_{j=1}^{J} Ax_j\right) = \lim_{J\to\infty} y_{J+1} = 0$ since $y_{J+1} \in B_{\underbrace{2^{-J-1}\varepsilon}_{\to 0}}(0)$,

**Step 3** Let $U \subset X$ be open and $y = Ax \in A(U)$, $x \in U$.

If $r > 0$ is s.t. $B_r(x) \subset U$, then
$$\begin{aligned} A(U) \supset A(B_r(x)) &= Ax + A(B_r(0)) \\ &\supset Ax + B_{\varepsilon r/2}(0) \\ &= B_{\varepsilon r/2}(y). \end{aligned}$$

$\Rightarrow A(U) \subset Y$ is open. $\square$

**Example** The following example demonstrates the uselessness (for analytic purposes) of algebraic bases in infinite dimensions.

Let $X = Y = \ell^2$, and extend the linearly independent family $\{e_i\}_{i\in\mathbb{N}} \subset \ell^2$, $(e_i)_k = \delta_{ik}$, to an algebraic basis $(b_\lambda)_{\lambda\in\Lambda}$ with $\|b_\lambda\|_{\ell^2} = 1$.

Coming from linear algebra, the norm

$$\|a\|_1 := \sum_{\lambda\in\Lambda} |a_\lambda| \,, \qquad a = \sum_{\lambda\in\Lambda} a_\lambda b_\lambda \quad \text{(with only finitely many } a_\lambda \text{ nonzero)}$$

looks sensible. We have

$$\|a\|_{\ell^2} \le \sum |a_\lambda| \, \|b_\lambda\|_{\ell^2} = \|a\|_1, \qquad (\ast)$$

so $A = \mathrm{Id} : (\ell^2, \|\cdot\|_1) \to (\ell^2, \|\cdot\|_{\ell^2})$ is a continuous bijection.

But for $a_k := \frac{1}{\sqrt{k}} \sum_{j=1}^{k} e_j$, we have
$$\|a_k\|_1 = \tfrac{1}{\sqrt{k}} \cdot k = \sqrt{k},$$
$$\|a_k\|_{\ell^2} = \left(\tfrac{1}{\sqrt{k}}\right)^2 \cdot k = 1,$$

so $A^{-1}$ is not continuous. Therefore $A$ is not open. $(\ast)$ thus implies that $(\ell^2, \|\cdot\|_1)$ is not complete:

**Lemma** If $X$ is complete with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$,
and $\exists\, C > 0$ s.t. $\|\cdot\|_2 \le C \|\cdot\|_1$,
then $\exists\, C' > 0$ s.t. $\|\cdot\|_1 \le C' \|\cdot\|_2$. $(\#)$

**Proof** $A = \mathrm{Id} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous and bijective, hence open; so $A^{-1} = \mathrm{Id} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is bounded, which gives $(\#)$. $\square$

**Lemma** Let $X$ be a Banach space. Suppose $A \in L(X)$ is invertible. If $B \in L(X)$ satisfies $\|B - A\|_{L(X)} < \|A^{-1}\|^{-1}$, then also $B \in L(X)$.
(So: the set of invertible bounded linear maps on a Banach space is open.)

**Proof** Let $D = B - A$ and write $B = A + D = A(I + A^{-1}D)$.
Since $\|A^{-1}D\|_{L(X)} \le \|A^{-1}\|_{L(X)} \|D\|_{L(X)} < 1$, the map $I + A^{-1}D$ is invertible, and therefore $B^{-1} = (I + A^{-1}D)^{-1} A^{-1} \in L(X)$. $\square$

# Closed Graph Theorem

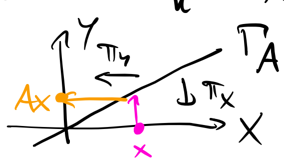
**Definition** • $X, Y$ vector spaces, $A : X \to Y$ linear. The graph of $A$ is

$$\Gamma(A) := \{(x, Ax) : x \in X\} \subset X \times Y.$$

• When $X, Y$ are normed, we say that $A : X \to Y$ is closed if $\Gamma(A) \subset X \times Y$ is closed. (Here $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.)

**Theorem** Let $X, Y$ be Banach spaces, and let $A : X \to Y$ be a linear map. Then $A$ is continuous $\iff$ $A$ is closed.

**Proof** "$\Rightarrow$" $(x_k, Ax_k)$ Cauchy $\Rightarrow$ $x_k$ is Cauchy, so with $x = \lim x_k$ also $\lim Ax_k = Ax$ $\Rightarrow$ $\lim (x_k, Ax_k) = (x, Ax) \in \Gamma(A)$.

"$\Leftarrow$" 


$\Gamma_A \subset X \times Y$ is a closed linear subspace, i.e. a Banach space in its own right.

$\pi_X|_{\Gamma_A} : \Gamma_A \to X$ is continuous and bijective $\Rightarrow$ open,

and $\quad Ax = \pi_Y|_{\Gamma_A}\left((\pi_X|_{\Gamma_A})^{-1}(x)\right)$. Thus, $A$ is continuous as the composition of $(\pi_X|_{\Gamma_A})^{-1} \in L(X, \Gamma_A)$ and $\pi_Y|_{\Gamma_A} \in L(\Gamma_A, Y)$. $\square$

**Remark** For linear $A : X \to Y$, consider three statements:

(i) $x_n \to x$

(ii) $Ax_n \to y$

(iii) $Ax = y$

To prove that $A$ is continuous one usually needs to show (i) $\Rightarrow$ (ii) & (iii). When $X, Y$ are Banach spaces the Closed Graph Theorem says we need only check (i) & (ii) $\Rightarrow$ (iii).

Example Completeness of $X$ is crucial. Let $X = C^1([0,1])$, $\|\cdot\|_X = \|\cdot\|_{C^0}$, and $Y = C^0([0,1])$, $\|\cdot\|_Y = \|\cdot\|_{C^0}$, and consider $A = \frac{d}{dt} : X \to Y$.

(i) $\Gamma_A \subset X \times Y$ is closed: if $\{u_k\} \subset C^1([0,1])$ and $\{\frac{du_k}{dt}\} \subset C^0([0,1])$ are Cauchy sequences, then $u_k(0) \longrightarrow u_0 \in \mathbb{C}$,

$$\frac{du_k}{dt} \longrightarrow v \in C^0([0,1]).$$

The candidate for the limit of $(u_k, \frac{du_k}{dt}) \in X \times Y$ is $(u,v)$, where $u(t) = u_0 + \int_0^t v(x)\,dx$. And indeed,

$$\begin{aligned}
\|u - u_k\|_X &= \sup_{t\in[0,1]} \left| u_0 + \int_0^t v(x)\,dx - u_k(0) - \int_0^t \frac{du_k}{dt}(x)\,dx \right| \\
&\le |u_0 - u_k(0)| + \sup_{t\in[0,1]} \int_0^t \left| v(x) - \frac{du_k}{dt}(x) \right| dx \\
&\le |u_0 - u_k(0)| + \left\| v - \frac{du_k}{dt} \right\|_{C^0} \\
&\xrightarrow{k\to\infty} 0.
\end{aligned}$$

(ii) $A$ is unbounded: for $u_k(x) = x^k$, $\|u_k\|_X = 1$,

but $\|Au_k\|_Y = \|k x^{k-1}\|_{C^0} = k \nearrow \infty$ as $k \to \infty$.

"Unfortunately", $A$ is a very important operator. We may end up saying more about such unbounded closed operators towards the end of the course.