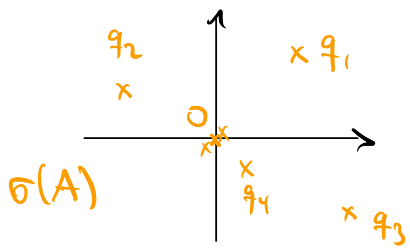
While the spectrum of general bounded linear operators can be very complicated, the situation simplifies dramatically in the case of compact operators.

Example Let $q = (q_n)_{n\in\mathbb{N}} \in c_0$. Define $A: \ell^p \to \ell^p$,
$$(a_n) \mapsto (q_n a_n).$$
Then A is compact, and $\sigma(A) = \{0\} \cup \{q_n : n \in \mathbb{N}\}$.



(Note: $0 \in \overline{\{q_n\}}$.)
Moreover, $\sigma(A)\setminus\{0\} = \sigma_p(A)\setminus\{0\}$;
$0 \in \sigma(A)$ may lie in $\sigma_p(A)$, $\sigma_c(A)$, or $\sigma_r(A)$.
(Exercise.)

It turns out that this example captures the general case:

Theorem Let X be a complex Banach space, $\dim X = \infty$, and let $A \in L(X)$ be compact. Then:

(i) $\sigma(A)\setminus\{0\} = \sigma_p(A)\setminus\{0\}$ is a discrete subset of $\mathbb{C}\setminus\{0\}$ which accumulates at 0.

(ii) For each $\lambda \in \sigma_p(A)\setminus\{0\}$, $\ker(\lambda - A)$ is finite-dimensional (and non-trivial).

Proof (ii) Let $\lambda \in \sigma(A)$, $\lambda \neq 0$. Then $\lambda - A = \lambda\cdot(I - \lambda^{-1}A)$; since $\lambda^{-1}A$ is compact, and $I - \lambda^{-1}A$ fails to be invertible, we conclude that $\ker(I - \lambda^{-1}A) = \ker(\lambda - A)$ is non-trivial and finite-dimensional.

(i) Suppose $\lambda_k \in \sigma(A)$, $k \in \mathbb{N}$, are pairwise distinct, and $\lambda_k \xrightarrow{k\to\infty} \lambda_0 \neq 0$. WLOG, $|\lambda_k| > \frac{1}{2}|\lambda_0|$ $\forall k$.

For each λ_k, pick $u_k \in \ker(\lambda_k - A)$, $\|u_k\|=1$.

- The u_k are linearly independent: if $\sum_{k=1}^{N} c_k u_k = 0$, then

$$0 = \left(\prod_{\ell \neq k_0} (\lambda_\ell - A)\right) \sum_{k=1}^{N} c_k u_k = c_{k_0} \prod_{\ell \neq k_0} (\lambda_\ell - \lambda_{k_0})$$

$$\Rightarrow c_{k_0} = 0.$$

- Let $y_1 = u_1$, and pick $y_k \in Y_k := \operatorname{span}\{u_1, \dots, u_k\}$ with $\|y_k\| = 1$ and $d(y_k, Y_{k-1}) > \frac{1}{2}$. Thus, $y_k = \alpha_k u_k + y_k'$, $0 \neq \alpha_k \in \mathbb{C}$, $y_k' \in Y_{k-1}$.

Moreover, $A(Y_k) = Y_k$ $\forall k$.

- For $k > \ell$, we then have

$$\|Ay_k - Ay_\ell\| = \| A(\alpha_k u_k + y_k') - \overbrace{Ay_\ell}^{\in Y_\ell \subseteq Y_{k-1}} \|$$

$$= \| \lambda_k(\alpha_k u_k + y_k') - \underbrace{(\lambda_k y_k' - Ay_k' + Ay_\ell)}_{=: f_k \in Y_{k-1}} \|$$

$$= \| \lambda_k y_k - f_k \|$$

$$> \tfrac{1}{2}|\lambda_k| > \tfrac{1}{4}|\lambda_0|.$$

$\Rightarrow \{Ay_k\}_{k\in\mathbb{N}}$ has no convergent subsequence, ↯ to the compactness of A. $\square$

The self-adjoint case is even better:

Theorem (Riesz–Schauder.)

H = Hilbert space, $A \in L(X)$ compact and self-adjoint.

Then $\exists$ at most countably many eigenvalues $0 \neq \lambda_k \in \mathbb{R}$ (possibly with multiplicity) and corresponding orthonormal eigenvectors $e_k \in H$, $\|e_k\|=1$, $Ae_k = \lambda_k e_k$, so that

$$Ax = \sum_k \lambda_k (x, e_k) e_k \quad \forall x \in H.$$

Moreover, $H = \overline{\operatorname{span}\{e_k\}} \oplus \ker A$.

Remark H separable $\Rightarrow \exists$ complete ONB of H consisting of eigenvectors of A.

Proof (of the theorem). • Eigenvectors of A corresponding to different eigenvalues are orthogonal. For each eigenvalue $\lambda \neq 0$, pick an ONB of $\ker(\lambda - A)$ (a finite-dim. subspace of H).

$\rightsquigarrow$ Get an orthonormal set $\{e_k\}$ of eigenvectors of A.

• Set $Y = \overline{\operatorname{span}\{e_k\}}$.

Claim: $A|_{Y^\perp} : Y^\perp \to Y^\perp$ is the 0 operator.

Indeed, if $y \perp e_k$ $\forall k$, then $(Ay, e_k) = (y, Ae_k) = (y, \lambda_k e_k) = 0$; so $A(Y^\perp) \subset Y^\perp$. Since $A|_{Y^\perp} \in L(Y^\perp)$ is self-adjoint, we have $\|A|_{Y^\perp}\| = r_{A|_{Y^\perp}}$ (spectral radius). But the compact operator $A|_{Y^\perp}$ cannot have a nonzero eigenvalue by construction of Y. Therefore, $\sigma_p(A|_{Y^\perp}) \subseteq \{0\}$

$\Rightarrow \sigma(A|_{Y^\perp}) \subseteq \{0\}$ (by the preceding Theorem)

$\Rightarrow r_{A|_{Y^\perp}} = 0 \Rightarrow A|_{Y^\perp} = 0$.

· Writing $x \in H$ as $x = \sum_k (x, e_k) e_k + x'$, $x' \in Y^\perp$,
we immediately obtain $Ax = \sum_k (x, e_k) \lambda_k e_k$. $\square$

Example $A = D_\theta^2 + V : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, $V \in L^\infty(\mathbb{S}^1)$ complex-valued. In an earlier example, we had seen that if $B : L^2 \to H^2$ is defined by

$$\mathcal{F}(Bu)(n) = \begin{cases} \frac{1}{n^2} (\mathcal{F}u)(n), & n \neq 0 \\ \mathcal{F}u(0), & n = 0, \end{cases}$$

(so B is an isomorphism), then $AB = I - K$, $K \in L(L^2(\mathbb{S}^1))$ compact. More generally, for $z \in \mathbb{C}$,

$$(A - z)B = I - \underbrace{(zB + K)}_{\text{compact}}.$$

$\Rightarrow A - z : H^2 \to L^2$ is Fredholm of index 0.

Claim 1: $\exists\, r_0 > 0$ s.t. $\operatorname{Re} z < -r_0 \Rightarrow \ker(A - z) = \{0\}$.
(So $A - z$ is invertible.)

Proof $(A - z)u = 0 \Leftrightarrow D_\theta^2 u + (V - z)u = 0$.

Integrate against u in $L^2(\mathbb{S}^1)$; use that

$(D_\theta v, w) = (v, D_\theta w)$ $\left(= \sum_{n \in \mathbb{Z}} n \hat{v}(n) \overline{\hat{w}(n)}\right)$ for $v, w \in H^1(\mathbb{S}^1)$.

$$\Rightarrow 0 = \operatorname{Re}\left[(D_\theta^2 u, u) + ((V - z)u, u) \right]$$
$$= \operatorname{Re}(D_\theta u, D_\theta u) + (\operatorname{Re}(V - z) u, u)$$
$$\geq \left(\min_{x \in \mathbb{S}^1} \operatorname{Re}(V) + r_0 \right) \|u\|^2 \geq \|u\|^2$$

for $r_0 \geq 1 - \min \operatorname{Re} V$. $\Rightarrow u = 0$. $\square$

<u>Claim 2</u> There exists a discrete set $\Lambda \subset \mathbb{C}$, accumulating only at infinity, so that $A - z : H^2 \to L^2$ is invertible for $z \in \mathbb{C} \setminus \Lambda$, while $\ker(A-z)$ is nontrivial (and finite-dimensional) for $z \in \Lambda$.

<u>Proof</u> Fix $z_0 \in \mathbb{C}$, $\operatorname{Re} z_0 < -r_0$. Then

$$R_{z_0} := (z_0 - A)^{-1} : L^2(S^1) \longrightarrow L^2(S^1)$$

is a compact operator (being the composition of $L^2 \xrightarrow{(z_0-A)^{-1}} H^2 \xrightarrow{\text{compact}} L^2$).

$\Rightarrow \sigma(R_{z_0})$ is a bounded discrete subset of $\mathbb{C} \setminus \{0\}$ accumulating at 0. But $f \in \ker(\lambda - R_{z_0})$, $\lambda \neq 0$,

$$\text{iff } \underbrace{R_{z_0} f}_{\in H^2} = \lambda f \iff f = \lambda (z_0 - A) f$$
$$\iff A f = (z_0 - \tfrac{1}{\lambda}) f$$
$$\iff \ker\left((z_0 - \tfrac{1}{\lambda}) - A \right) \neq \{0\}.$$

Thus $\Lambda = \{ z_0 - \frac{1}{\lambda} : \lambda \in \sigma(R_{z_0}) \}$. $\square$

<u>Remark</u> $u \in \ker(A - z) \iff D_\theta^2 u + V u = z u$.

<u>Claim 3</u> If V is real-valued, there exists an orthonormal basis of $L^2(S^1)$ consisting of eigenvalues of $D_\theta^2 + V$.

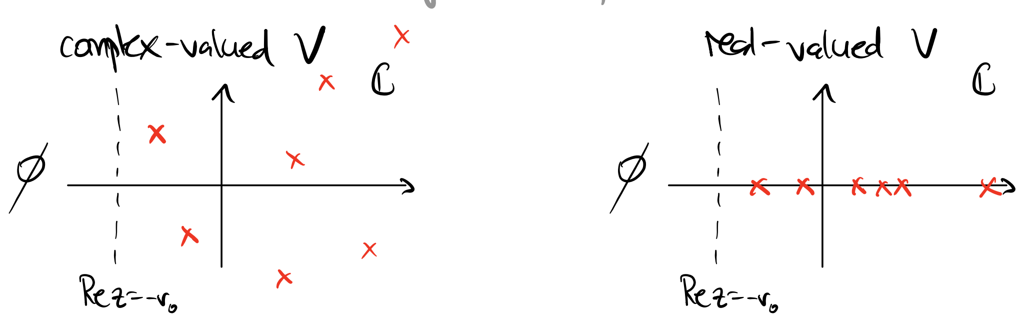
<u>Proof</u> For $z \in \mathbb{R}$, $z < -r_0$, R_{z_0} is self-adjoint (and

still compact) since for $u, v \in L^2(\mathbb{S}^1)$ and writing $u = (z_0 - A)\tilde{u}$, $v = (z_0 - A)\tilde{v}$ with $\tilde{u}, \tilde{v} \in H^2(\mathbb{S}^1)$,

$$
\begin{aligned}
(R_{z_0} u, v) &= (\tilde{u}, (z_0 - A)\tilde{v}) = (\tilde{u}, z_0 \tilde{v}) - (\tilde{u}, D_\theta^2 \tilde{v}) - (\tilde{u}, V\tilde{v}) \\
&\overset{\left(\substack{z_0 \in \mathbb{R}, \\ V \text{ real-valued}}\right)}{=} (z_0 \tilde{u}, \tilde{v}) - (D_\theta^2 \tilde{u}, \tilde{v}) - (V\tilde{u}, \tilde{v}) \\
&= ((z_0 - A)\tilde{u}, \tilde{v}) \\
&= (u, R_{z_0} v).
\end{aligned}
$$

A complete ONB of $L^2(\mathbb{S}^1)$ consisting of eigenfunctions of R_{z_0} thus does the job. $\square$

"Eigenvalues" of A



Special case: $V = 0$, so $A = D_\theta^2$. If $0 \neq u \in H^2(\mathbb{S}^1)$, $\lambda \in \mathbb{R}$ solve $Au = \lambda u$, then $n^2 \hat{u}(n) = \lambda \hat{u}(n)$ $\forall n \in \mathbb{Z} \Leftrightarrow \lambda = n_0^2$ for some $n_0 \in \mathbb{Z}$, and $\hat{u}(n) = 0$ $\forall n \in \mathbb{Z} \setminus \{\pm n_0\}$.

$\Leftrightarrow$ $\lambda = n_0^2$, and $u \in \operatorname{span} \begin{cases} 1, & n_0 = 0 \\ e^{\pm i n_0 \theta}, & n_0 \neq 0. \end{cases}$

Thus, we conclude that $\{e^{ik\theta}\}_{k \in \mathbb{Z}}$ is a complete ONB of $L^2(\mathbb{S}^1)$. (This is a bit circuitous since we used this fact in our definition of $H^2(S^1)$; but one can give more direct definitions of $H^2(S^1)$, for which there is no circuitous logic...)

Remark One important take-away from this example is the usefulness of an operator (such as D_θ^2+V) to have compact "resolvent". Beware though: D_θ^2+V is not a bounded linear map on one Banach space, but rather between two spaces X $(=H^2)$ and Y $(=L^2)$, with (compact!) inclusion $X \hookrightarrow Y$. Hence the quotation marks and my avoiding the word "spectrum" for the set $\Lambda \subset \mathbb{C}$ above.

One can remove the quotation marks if one regards D_θ^2+V as an unbounded self-adjoint operator on $L^2(S^1)$ with domain $H^2(S^1)$; we shall not discuss this for now, however.

In special self-adjoint cases, one can characterize the eigenvalues variationally:

Theorem (Courant–Fischer min-max principle.) Let $A \in L(H)$ be a compact self-adjoint operator on a separable Hilbert space H, $\dim H = \infty$, which is positive: $\langle x, Ax\rangle > 0$ $\forall x \in H, x \neq 0$. Then:

(i) $\ker A = \{0\}$

(ii) The eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$, $\lambda_k \searrow 0$ as $k \to \infty$, and $\lambda_k = \mu_k := \sup_{\substack{Y \subset H \text{ subspace} \\ \dim Y \geq k}} \inf_{\substack{x \in Y \\ \|x\|=1}} (x, Ax)$ $\circledast$

Proof Only $\circledast$ does not follow immediately from the previous Theorem and the positivity of A. Let $\{e_k\}_{k \in \mathbb{N}}$ be a complete ONB of H with $Ae_k = \lambda_k e_k$.

"$\lambda_k \leq \mu_k$". Take $Y_k := \operatorname{span}\{e_1, \dots, e_k\}$. For $x = \sum_{j=1}^{k} x_j e_j \in Y_k$ with $x_j = (x, e_j)$, $\|x\|^2 = \sum_{j=1}^{k} |(x,e_j)|^2 = 1$, we have

$$\mu_k \geq (x, Ax) = \left(\sum_{j=1}^{k} x_j e_j , \sum_{\ell=1}^{k} x_\ell \lambda_\ell e_\ell \right) = \sum_{j=1}^{k} \lambda_j |(x,e_j)|^2$$

$$\geq \lambda_k \sum_{j=1}^{k} |(x,e_j)|^2 = \lambda_k .$$

"$\lambda_k \geq \mu_k$". Let $Y \subset H$ be a subspace with $\dim Y \geq k$.

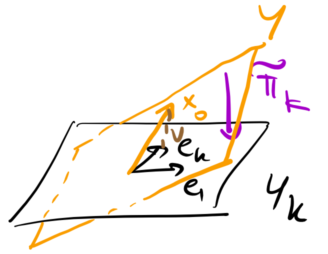
Case 1 $\exists\, x_0 \in Y$, $x_0 \perp Y_k$, $\|x_0\| = 1$. Then

$$\inf_{\substack{x \in H \\ \|x\|=1}} (x, Ax) \leq (x_0, Ax_0) = \left(x_0, \sum_{j=k+1}^{\infty} \lambda_j (x_0, e_j) e_j \right)$$

$$= \sum_{j=k+1}^{\infty} \lambda_j |(x_0,e_j)|^2 \leq \lambda_k \sum_{j=k+1}^{\infty} |(x_0,e_j)|^2 \leq \lambda_k .$$

Case 2 The orthogonal projection $\pi_k : H \to Y_k$ restricts to an injective map $\tilde{\pi}_k : Y \to Y_k$. Since $\dim Y \geq \dim Y_k$,

$\tilde{\pi}_k$ is then a bijection.



Let $x_0 := \tilde{\pi}_k^{-1}(e_k) = e_k + v$,

where $v \in Y_k^\perp$. Then

$$(x_0, x_0) = 1 + \|v\|^2$$

and

$$\begin{aligned}
(x_0, Ax_0) &= (e_k + v, \lambda_k e_k + Av) \\
&= \lambda_k + \lambda_k \underbrace{(v, e_k)}_{=0} + \underbrace{(e_k, Av)}_{\substack{=(Ae_k, v) \\ = \lambda_k (e_k, v) = 0}} + \overbrace{(v, Av)}^{\text{estimate using Case 1}} \\
&\leq \lambda_k + 0 + 0 + \lambda_k \|v\|^2 \\
&= \lambda_k (x_0, x_0),
\end{aligned}$$

$$\Longrightarrow \inf_{\substack{x \in Y \\ \|x\| = 1}} (x, Ax) = \inf_{\substack{x \in Y \\ x \neq 0}} \frac{(x, Ax)}{(x, x)} \leq \frac{(x_0, Ax_0)}{(x_0, x_0)} \leq \lambda_k. \quad \square$$