Weak topology

We already saw (in an eduApp question) in the Hilbert space setting that convergence of $\lambda(x_k)$ (as $k\to\infty$) for all $\lambda \in X^*$ (with $(x_k)_{k\in\mathbb{N}}$ a sequence in X) is weaker than (i.e. does not imply) convergence of x_k (as $k\to\infty$) when $\dim X=\infty$.

$\rightsquigarrow \exists$ other notions of convergence in infinite dimensions.

Definition Let X be a normed vector space, and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X. Let $x\in X$. Then x_k converges weakly to x, or in symbols $x_k \rightharpoonup x$ (sometimes $x_k \xrightarrow{w} x$), if $\lim_{k\to\infty} \lambda(x_k)=\lambda(x)$ $\forall \lambda\in X^* = L(X,\mathbb{K})$.

Examples H separable Hilbert space with complete ONB $\{e_n\}_{n\in\mathbb{N}}$.
(i) $e_n \xrightarrow{w} 0$. (ii) $\sqrt{n}\, e_n$ does not converge weakly. (iii) $\sin(nt)\xrightarrow{w} 0$ in $L^2([0,2\pi])$.

We collect some basic properties of weak convergence:

Lemma X normed vector space, $(x_k)_{k\in\mathbb{N}}$ sequence in X.

(i) If $\lim_{k\to\infty} x_k = x$, i.e. $x_k \to x$, then also $x_k \rightharpoonup x$.

(Strong convergence implies weak convergence.)

(ii) If (x_k) converges weakly, its weak limit is unique.
(I.e. $x_k \rightharpoonup x$, $x_k \rightharpoonup x' \Rightarrow x=x'$.)

(iii) If $x_k \rightharpoonup x$, then $\sup_{k\in\mathbb{N}} \|x_k\| < \infty$, and $\|x\| \le \liminf_{k\to\infty} \|x_k\|$.

Remark Weak convergence usually does not imply strong convergence when $\dim X = \infty$! E.g. when $X = \ell^2$, $x_k = e_k$. There do exist Banach spaces (e.g. ℓ^1) for which weak conv. $\Rightarrow$ strong conv. (Exercise.)

Proof of the Lemma (i) For $\lambda \in X^*$, we estimate

$$|\lambda(x_k) - \lambda(x)| = |\lambda(x_k - x)| \leq \|\lambda\|_{X^*} \|x_k - x\|_X \xrightarrow{k\to\infty} 0.$$

(ii) If $x_k \rightharpoonup x$ and $x_k \rightharpoonup x'$, then for all $\lambda \in X^*$,

$$\lambda(x - x') = \lambda(x) - \lambda(x') = \lim_{k\to\infty} \lambda(x_k) - \lim_{k\to\infty} \lambda(x_k) = 0.$$

But by Hahn–Banach, this implies $x - x' = 0 \Rightarrow x = x'$.

(iii) · We regard $\{x_k\} \subset X$ via the canonical inclusion map $\iota: X \hookrightarrow X^{**}$ as a family of linear operators

$$A_k := \iota(x_k) \in L(X^*, \mathbb{C}).$$

For $\lambda \in X^*$, $\sup_k |A_k(\lambda)| = \sup_k |\lambda(x_k)| < \infty$ since $\{\lambda(x_k)\}$ is a convergent sequence. Banach–Steinhaus $\Rightarrow \sup_k \|A_k\|_{L(X^*, \mathbb{C})} < \infty$.

That is, $\|\iota(x_k)\|_{X^{**}} = \|x_k\|_X$ is uniformly bounded.

· To get the estimate on the norm of the weak limit, take $\lambda \in X^*$ with $\|\lambda\|_{X^*} = 1$, $\lambda(x) = \|x\|_X$. Then

$$\|x\|_X = \lambda(x) = |\lambda(x)| = \lim_{k\to\infty} |\lambda(x_k)| \leq \liminf_{k\to\infty} \left(\|\lambda\|_{X^*} \|x_k\|_X\right)$$
$$= \liminf_{k\to\infty} \|x_k\|_X. \qquad \square$$

We can "upgrade" this notion of sequential convergence to a topology on X.

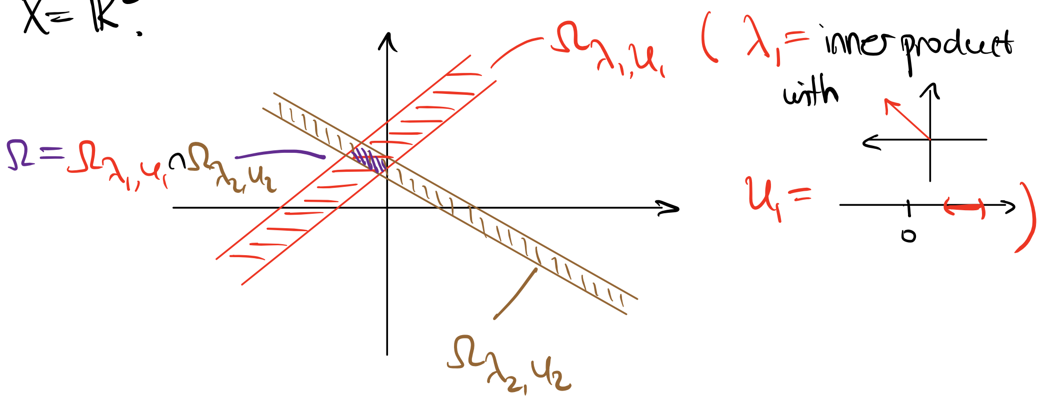
Definition Let X be a normed $\mathbb{K}$-vector space. Then the weak topology τ_w on X is the smallest (i.e. coarsest) topology s.t.

$$\Omega_{\lambda,U} := \lambda^{-1}(U) \text{ is open (i.e. } \in \tau_w) \ \forall \lambda \in X^*, \ U \subset \mathbb{K} \text{ open.}$$

Remark The sets $\Omega_{\lambda,U}$ form a subbasis of the weak topology, and finite intersections $\bigcap_{k=1}^{K} \Omega_{\lambda_k, U_k}$ form a basis.

Facts (i) When $\dim X < \infty$, the weak topology and the norm topology (i.e. the topology with basis {open balls}) are equal.

Picture: $X = \mathbb{R}^2$.



Thus, the sets $\Omega_{\lambda,U}$ for $U = (a,b) \subset \mathbb{R}$ are unbounded slabs between two hyperplanes.

(ii) When $\dim X = \infty$, every non-empty weakly open subset of X is unbounded.

(iii) The weak topology is Hausdorff ($\Leftarrow$ Hahn–Banach, as above). If $\dim X = \infty$, it is strictly weaker than the norm topology.

(iv) Warning: if $\dim X = \infty$, the weak topology is not first countable ($\Rightarrow$ not metrizable). Therefore, one must carefully distinguish e.g. the (topological) closure and the sequential closure w.r.t. weak topology. ($\rightarrow$ Exercises)

Intermezzo: Why should we be interested in the weak topology?

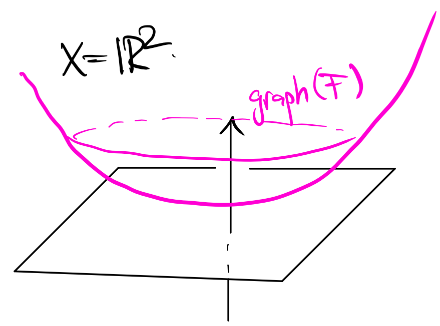
Motivation: the direct method of the calculus of variations.

Setup: $(X, \|\cdot\|)$ Banach space,

$F : X \to \mathbb{R}$ continuous (but not linear),

$$\liminf_{\|x\|\to\infty} |F(x)| = +\infty. \quad (\text{"}F \text{ is coercive."})$$

Set $\lambda := \inf_{x\in X} |F(x)| \in [-\infty, \infty)$.



Problem: · $\lambda \in \mathbb{R}$ (i.e. $\lambda > -\infty$)?

· $\exists$ minimizer? $x \in X$ s.t. $F(x) = \lambda$.

Solution attempt ("direct method"):

(i) Let $\{x_k\} \subset X$ be a minimizing sequence, i.e. $F(x_k) \searrow \lambda$ as $k \to \infty$.

Since F is coercive, we have $R := \sup_k \|x_k\| < \infty$.

(ii) We want to prove that a subsequence of $\{x_k\}$ converges. Problem: $\overline{B_R(0)} = \{x \in X : \|x\| \leq R\}$ is not compact ($\Leftrightarrow$ sequentially compact, since X is a metric space) when $\dim X = \infty$. Strategy: use a weaker topology on X (so that "more" sets are compact).

$\rightsquigarrow$ for suitable spaces (e.g. $X =$ reflexive), x_k converges weakly to some $x \in X$ (i.e. $x_k \rightharpoonup x$, and so $\|x\| \leq R$ by the weak lower semicontinuity of $\|\cdot\|$).

(iii) If $\mathcal{F}$ satisfies a lower semicontinuity property w.r.t. weak topology, then $\mathcal{F}(x) \leq \lim\limits_{k\to\infty} \mathcal{F}(x_k) = \lambda \Rightarrow x$ is a minimizer, and $\lambda > -\infty$.

Now back to the general theory.

Lemma X normed vector space, $A \subset X$ subset.

(i) A closed $\Leftrightarrow$ A sequentially closed.

(ii) A weakly closed $\Rightarrow$ A weakly sequentially closed, A closed.

(iii) A weakly sequentially closed $\Rightarrow$ A (sequentially) closed.

Rephrasing in terms of closure operations. For $A \subset X$, write $\overline{A}$ for the closure in the norm topology, further

$$\overline{A}_w := \bigcap_{\substack{C \supset A \\ C \text{ weakly closed}}} C \quad \text{for the weak closure,}$$

and $\overline{A}_{seq} = \{$limits of convergent sequences in X with elements in $A\}$,

$\overline{A}_{w\text{-}seq} = \{$limits of weakly convergent sequences in X with elements in $A\}$.

Above **Lemma** states that for all $A \subset X$,

$$\begin{array}{ccc} \overline{A} & \overset{(ii)}{\subset} & \overline{A}_w \\ {\scriptstyle (i)}\ \| & & \cup\ {\scriptstyle (ii)} \\ \overline{A}_{seq} & \underset{(iii)}{\subset} & \overline{A}_{w\text{-}seq} \end{array}$$

Proof of the Lemma (i) (A closed $\Rightarrow$ A sequentially closed) holds in every topological space. The converse holds in every metric space (more generally, in every first countable space).

(ii) (A weakly closed $\Rightarrow$ A weakly seq. closed) is again true simply because (X, τ_w) is a topological space. Moreover, A is weakly closed $\Rightarrow$ $X \setminus A$ is weakly open $\Rightarrow$ $X \setminus A$ is open $\Rightarrow$ A is closed.

(iii) Suppose A is w.s.c., and let $\{x_k\} \subset A$ with $x = \lim x_k \in X$. Then also $x_k \rightharpoonup x$, and therefore $x \in A$. $\square$

Remark (i) Typically $\overline{A} \subsetneqq \overline{A}_w$. Example: $A = \{x \in X : \|x\| = 1\}$, $\dim X = \infty$. Then $A = \overline{A}$, but $\overline{A}_w$ is the whole closed unit ball $\overline{B_1(0)} = \{x : \|x\| \leq 1\}$.

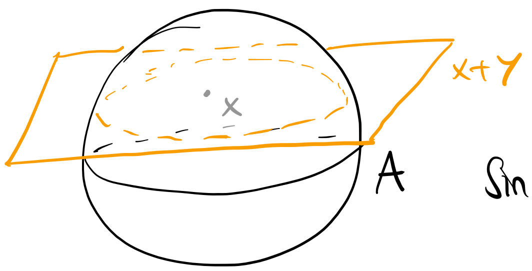
Proof • Since $\|\cdot\|$ is weakly lower semicontinuous, $\overline{A}_w \subseteq \overline{B_1(0)}$.

• Conversely, we shall first show that $0 \in \overline{A}_w$. So suppose we have $N \in \mathbb{N}$, $\lambda_1, \dots, \lambda_N \in X^*$, and $\varepsilon > 0$. Since

$$\mathcal{U} := \bigcap_{n=1}^{N} \{x \in X : |\lambda_n(x)| < \varepsilon\}$$

, as $N, \lambda_n, \varepsilon$ vary, is a neighborhood basis of $0 \in X$ in the weak topology, we need to show that $\mathcal{U} \cap A \neq \emptyset$. But $\mathcal{U} \supset \bigcap_{n=1}^{N} \ker \lambda_n$. Since $\ker \lambda_n \subset X$ has codimension 1, $\bigcap_{n=1}^{N} \ker \lambda_n \subset X$ has codim. $\leq N$, and therefore is a nontrivial linear subspace of X, since $\dim X = \infty$. But every nontrivial subspace of X contains elements of norm 1, as required.

- Let now $x \in \overline{B}_1(0)$ be arbitrary. If $x \in A$, there is nothing to do. Otherwise, every weakly open neighborhood of x contains an affine subspace $x + Y$ where $\{0\} \neq Y \subset X$ is a non-trivial subspace. Claim: $\exists\, y \in Y$ s.t. $x + y \in A$. Indeed,



take $0 \neq y_0 \in Y$ and consider the continuous function $f : \mathbb{R} \ni t \mapsto \|x + t y_0\|$. Since $f(0) = \|x\| < 1$ and $f(t) \geq t\|y_0\| - \|x\| \to \infty$, $t \to \infty$, $\exists\, t_0 \in (0, \infty)$ s.t. $f(t_0) = 1$. Take $y = t_0 y_0$. $\square$

(ii) In general, $\overline{A}_w \neq \overline{A}_{w\text{-seq}}$. So the converse implication in (ii) fails in general. Example: let $X = \ell^2$, and consider the set

$$A = \{\sqrt{n}\, e_n : n \in \mathbb{N}\}.$$

Step 1 A is weakly seq. closed.

Indeed, weakly convergent sequences are bounded. Since $\{\sqrt{n}\, e_n\}$ is unbounded, weakly conv. seq. in A must be eventually constant, so their limit is $= \sqrt{n}\, e_n$ for some $n \in \mathbb{N}$.

Step 2 $0 \in \overline{A}_w$ (but $0 \notin A = \overline{A}_{w\text{-seq}}$!). That is, consider any $\lambda_j \in (\ell^2)^*$, $j = 1, \dots, N \in \mathbb{N}$, so $\lambda_j = (\cdot, y_j)$ for some $y_j \in \ell^2$ by Riesz. Let $\varepsilon > 0$. We need to show that $\exists\, n \in \mathbb{N}$ s.t. $|\lambda_j(\sqrt{n}\, e_n)| < \varepsilon$ $\forall\, j = 1, \dots, N$.

Well, we have $\|y_j\|^2 = \sum_{n=1}^{\infty} |(y_j, e_n)|^2$, and therefore also

$$\sum_{n=1}^{\infty} \sum_{j=1}^{N} |(y_j, e_n)|^2 < \infty \text{ for all } k \in \mathbb{N}. \text{ Thus, } \exists\, n \in \mathbb{N}$$

s.t. $\displaystyle\sum_{j=1}^{N} |(y_j, e_n)|^2 < \frac{\varepsilon^2}{n}$. (Otherwise, $\displaystyle\sum_{n=1}^{\infty}\sum_{j=1}^{N} |(y_j,e_n)|^2 \geq \sum_{n=1}^{\infty} \frac{\varepsilon^2}{n} = \infty$ ↯)

For this n, $|\lambda_j(\sqrt{n}\, e_n)| = \sqrt{n}\, |(e_n, y_j)| < \sqrt{n} \sqrt{\frac{\varepsilon^2}{n}} = \varepsilon \;\; \forall j = 1, \dots, N.$

$\square$

(iii) Exercise: X = separable Hilbert space, A = unit sphere

$\Rightarrow \overline{A}_{w\text{-seq}} = \overline{B_1(0)}$.

But A is sequentially closed! So $\overline{A}_{seq} (= \overline{A}) \neq \overline{A}_{w\text{-seq}}$, showing that the converse implication in (iii) fails in general.

- Returning to our desire for compactness in infinite-dimensional spaces, the weak topology is fairly good; e.g. the closed unit balls in Hilbert spaces or L^p-spaces $(1<p<\infty)$ are weakly (sequentially) compact (later). On the other hand, the unit ball of c_0 is not weakly (sequentially) compact (exercise). To understand this phenomenon better, we need to take a detour.
- It turns out that on dual spaces of normed vector spaces, one can define yet another topology which has excellent compactness properties.

Weak-$*$-topology

Definition Let X be a normed vector space, and let $\{\lambda_k\}_{k\in\mathbb{N}} \subset X^*$. Then λ_k converges in the weak-$*$-topology to $\lambda \in X^*$ if $\lim_{k\to\infty} \lambda_k(x) = \lambda(x)$ $\forall\ x \in X$; we write $\lambda_k \xrightarrow{w^*} \lambda$.

Remarks (i) The weak-$*$-limit of $\{\lambda_k\}$, if it exists, is unique.

(ii) $\lambda_k \xrightarrow{w^*} \lambda \Rightarrow \sup \|\lambda_k\|_{X^*} < \infty$. (Banach–Steinhaus on X; exercise.)

(iii) Using the canonical inclusion $\imath: X \longrightarrow X^{**}$, weak-$*$-convergence $\lambda_k \xrightarrow{w^*} \lambda$ is equivalent to

$$\Phi(\lambda_k) \to \Phi(\lambda) \qquad (*)$$

$\forall\ \Phi \in \operatorname{ran} \imath \subset (X^*)^*$. Compare this with weak convergence of λ_k, which requires $(*)$ for all $\Phi \in X^{**}$. Thus:

Lemma $\lambda_k \xrightarrow{w} \lambda$ implies $\lambda_k \xrightarrow{w^*} \lambda$. The converse is true when X is reflexive.

We can upgrade the notion of sequential weak-$*$-convergence to a topology on X^*:

Definition The weak-$*$-topology τ_{w^*} on X^* (for X = normed vector space) is the smallest topology containing the sets

$$\{\lambda \in X^* : |(\lambda - \lambda_0)(x_0)| < \varepsilon\} \quad \text{for all } \lambda_0 \in X^*,\ x_0 \in X,\ \varepsilon > 0.$$

(These sets form a subbasis of τ_{w^*}.)

Remark We have seen that $\tau_{w*} \subset \tau_w$ on X^*. Therefore, for $A \subset X^*$,
$$\overline{A} \subset \overline{A}^{w} \subset \overline{A}^{w*}.$$

The key compactness result is:

Theorem (Banach-Alaoglu). Let X be a normed vector space.

(i) The closed unit ball $B^* := \{\lambda \in X^* : \|\lambda\|_{X^*} \leq 1\}$ is compact in the weak-$*$-topology.

(ii) If X is separable, then B^* is weak-$*$-sequentially compact.

(Thus, if $\{\lambda_k\}_{k\in\mathbb{N}} \subset X^*$ is bounded, then there exist a subsequence $\{\lambda_{k_n}\}_{n\in\mathbb{N}}$ and $\lambda \in X^*$ s.t.
$$\lambda_{k_n} \xrightarrow[n\to\infty]{w*} \lambda, \quad \text{i.e.} \quad \lambda_{k_n}(x) \xrightarrow{n\to\infty} \lambda(x) \quad \forall x \in X.)$$

For the proof, we need Tychonov's Theorem: if I is a set and X_i is a compact topological space for all $i \in I$, then $X = \prod_{i\in I} X_i$ is compact. (Here, a subbasis of the topology of X is given by the sets $\pi_i^{-1}(U)$ for $i \in I$ and $U \subset X_i$ is open, where $\pi_i : X \to X_i$ is the projection.)

Proof of Banach-Alaoglu

(i) Define
$$\Phi : B^* \to \mathcal{F} := \prod_{x\in X} \{z \in \mathbb{K} : |z| \leq \|x\|\}$$
$$\lambda \mapsto (x \mapsto \lambda(x)).$$

By Tychonov, $\mathcal{F}$ is compact. We need to show that $\Phi(B^*)$ is closed. Note: $\Phi(B^*) = \{f : X \to \mathbb{K} : |f(x)| \leq \|x\| \ \forall x,$
f is linear$\}$.

So if $f \in \mathcal{F} \setminus \Phi(B^*)$, then $\exists\, x, y \in X, \alpha \in \mathbb{K}$ s.t.

$$f(x+\alpha y) \neq f(x) + \alpha f(y).$$

There exists an open neighborhood $\mathcal{U}$ of f in $\mathcal{F}$ s.t. this failure of linearity persists for $g \in \mathcal{U}$.

(ii) When X is separable, we can prove the weak sequential compactness more directly.

Let $\{x_i\}_{i \in \mathbb{N}} \subset X$ be dense. Select subsequences $\mathbb{N} \supset I_1 \supset I_2 \supset \cdots$ s.t. for all $i \in \mathbb{N}$ we have

$$\lambda_k(x_i) \longrightarrow a_i =: \lambda(x_i), \quad k \in I_i,\ k \to \infty.$$

This is possible since $\{\lambda_k(x_i)\}_{k \in \mathbb{N}}$ is bounded for all i.

Let $I = (1^{st}$ element of I_1, 2^{nd} element of $I_2, \ldots)$ be the diagonal sequence; thus $\lambda_k(x_i) \xrightarrow[k \in I]{k \to \infty} \lambda(x_i) \quad \forall\, i$.

Claim 1: λ can be extended to an element of X^*.

Indeed, we can extend λ to $\operatorname{span}\{x_i\}_{i \in \mathbb{N}}$ by linearity. Moreover,

$$|\lambda(x)| = \lim_{\substack{k \in I \\ k \to \infty}} |\lambda_k(x)| \leq \Big(\limsup_{\substack{k \in I \\ k \to \infty}} \|\lambda_k\|_{X^*}\Big) \|x\| \quad \forall\, x \in \operatorname{span}\{x_i\}$$

$\Rightarrow \lambda$ has a (unique) extension to $\lambda \in X^*$.

Claim 2: $\lambda_k \xrightarrow[\substack{k \in I \\ k \to \infty}]{w^*} \lambda$.

Indeed, let $x \in X$ and $\varepsilon > 0$. Pick $i \in \mathbb{N}$ s.t. $\|x - x_i\| < \varepsilon$, and pick k_0 s.t. $|\lambda(x_i) - \lambda_k(x_i)| < \varepsilon \ \forall\, k \in I,\ k \geq k_0$.

Then $|\lambda(x)-\lambda_k(x)|$

$$\leq |\lambda(x)-\lambda(x_i)| + |\lambda(x_i)-\lambda_k(x_i)| + |\lambda_k(x_i)-\lambda_k(x)|$$

$$< \|\lambda\|_{X^*}\|x-x_i\| + \varepsilon + \|\lambda_k\|_{X^*}\|x-x_i\|$$

$$< (\|\lambda\|_{X^*} + 1 + \sup\|\lambda_k\|_{X^*})\,\varepsilon.$$

Since ε was arbitrary, this finishes the proof. $\square$

Examples (i) $L^1([0,1])$ is separable, so Banach–Alaoglu (ii) applies: if $\{f_k\}\subset L^\infty=(L^1)^*$ is bounded, there exists $f\in L^\infty$ s.t. $f_{k_i}\xrightarrow[i\to\infty]{w^*}f$; that is,

$$\int_0^1 f_{k_i}(x)g(x)\,dx \xrightarrow{i\to\infty} \int_0^1 f(x)g(x)\,dx \quad \forall\, g\in L^1.$$

(ii) $L^\infty([0,1])$ on the other hand is not separable, and weak sequential compactness of the unit ball in $(L^\infty)^*$ fails.

Example: Let $\lambda_k\in(L^\infty)^*$, $\lambda_k(f)=k\int_0^{1/k}f(x)\,dx$. Then $\|\lambda_k\|_{(L^\infty)^*}=1$. But $\nexists\ \lambda\in(L^\infty)^*$ s.t. $\lambda_{k_i}\xrightarrow[i\to\infty]{w^*}\lambda$ for some subsequence $\{k_i\}_{i\in\mathbb{N}}$.

Proof. WLOG $\frac{k_{i+1}}{k_i}\nearrow\infty$ as $i\to\infty$. Put

$$f(x):=\sum_{j=1}^{\infty}(-1)^j\,\mathbb{1}_{\left[\frac{1}{k_{j+1}},\frac{1}{k_j}\right]}(x)\in L^\infty,\ \|f\|_{L^\infty}=1.$$

$$\text{Then } \lambda_{k_i}(f) = k_i \sum_{j=i}^{\infty} (-1)^j \left(\frac{1}{k_j} - \frac{1}{k_{j+1}}\right)$$

$$= \underbrace{(-1)^i \left(1 - \frac{k_i}{k_{i+1}}\right)}_{\text{term for } j=i} + \underbrace{k_i \int_0^{\frac{1}{k_{i+1}}} f(x)\,dx}_{\text{all remaining terms } (j>i)}.$$

$$\Rightarrow \left|\lambda_{k_i}(f) - (-1)^i\right| \leq \frac{k_i}{k_{i+1}} + k_i \cdot \frac{1}{k_{i+1}} \xrightarrow[i\to\infty]{} 0.$$

Since $\lambda_{k_i}(f)$ accumulates at both $+1$ and -1, it cannot converge. $\square$

(iii) On the other hand, $C^0([0,1])$ is separable, and the same $\lambda_k \in (C^0)^*$ converge in weak-$*$ to δ_0: $\forall f \in C^0([0,1])$,

$$\lambda_k(f) = k\int_0^{1/k} f(x)\,dx \xrightarrow{k\to\infty} f(0) = \delta_0(f).$$

Consider now a normed vector space X. If X is reflexive, then $\iota: X \xrightarrow{\cong} X^{**}$, and the weak-$*$-topology on X^{**} (or more precisely its preimage under ι) is the same as the weak topology on X.

$$\begin{tikzcd} X^{**} \\ X^* \\ X \arrow[uu, bend left, "\iota", "\cong"'] \end{tikzcd}$$

X^* used to define weak-$*$-topology on X^{**}

$\|$

X^* used to define weak topology on X

Corollary If X is reflexive, then the closed unit ball $\overline{B} \subset X$ is weakly compact. (So this applies to X = Hilbert space, ℓ^p, L^p $(1<p<\infty)$.)

Later: (i) weak sequential compactness of $\overline{B}$. (ii) Converse of Corollary is true.

We shall now study reflexivity and separability in some depth.