$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Vector spaces

$V = \mathbb{K}$-vector space.

Definition V is *infinite-dimensional* if it is not finite-dimensional, i.e. $\forall\ N \in \mathbb{N}\ \exists$ linearly independent $v_1, \dots, v_N \in V$.

Lemma Every vector space V has a basis (in the case $\dim V = \infty$ more precisely called *algebraic basis*, or *Hamel basis*).

Proof (omitted; uses Zorn's Lemma). $\square$

Often much more useful: other notions of basis, where we only want every $v \in V$ to be the limit of linear combinations of basis elements. (E.g. complete orthonormal basis of a Hilbert space.)

Normed vector spaces

Definition A *norm* on a vector space V is a function

$$\|\cdot\| : V \to [0, \infty)$$

with the following properties:

(i) (definiteness) $v \in V,\ \|v\| = 0 \Rightarrow v = 0$

(ii) (absolute homogeneity) $\lambda \in \mathbb{K},\ v \in V \Rightarrow \|\lambda v\| = |\lambda|\ \|v\|$ $\quad (\lambda = x + iy \Rightarrow |\lambda| = (x^2+y^2)^{\frac{1}{2}})$

(iii) (triangle inequality) $v, w \in V \Rightarrow \|v+w\| \leq \|v\| + \|w\|$.

Lemma On a normed vector space $(V, \|\cdot\|)$, the map

$$d : V \times V \to [0, \infty), \qquad d(v,w) = \|v - w\|,$$

gives V the structure of a *metric space* (and thus of a topological space).

Proof Exercise. $\square$

Examples (1) $V=\mathbb{C}^n$, mit $\|(z_1,\dots,z_n)\| := \left(\sum_{j=1}^n |z_j|^2\right)^{\frac{1}{2}}$

(2) $V=\mathbb{C}^n$, mit $\|(z_1,\dots,z_n)\|_\infty := \max_{j=1,\dots,n} |z_j|$.

(3) $V = C^0([0,1]) = \{u : [0,1] \to \mathbb{K} \text{ continuous}\}$.

(i) sup norm: $\|u\|_\infty = \max_{x\in[0,1]} |u(x)| = \|u\|_{C^0}$

(ii) L^p norm: $\|u\|_p = \left(\int_0^1 |u(x)|^p \, dx\right)^{\frac{1}{p}} = \|u\|_{L^p}$ $(1 \le p < \infty)$.

(4) $\ell^p = \{ a = (a_j)_{j\in\mathbb{N}},\ a_j \in \mathbb{C} : \|a\|_p = (\sum_{j=1}^\infty |a_j|^p)^{\frac{1}{p}} < \infty \}$, $1 \le p < \infty$.

$\ell^\infty = \{ a \quad : \|a\|_\infty = \sup_{j\in\mathbb{N}} |a_j| < \infty \}$

(5) $c_0 = \{ a = (a_j)_{j\in\mathbb{N}},\ a_j \in \mathbb{C} : \lim_{j\to\infty} |a_j| = 0 \}$, with norm $\|\cdot\|_\infty$.

Lemma On a normed vector space, the norm and the vector space operations are continuous.

Proof Exercise. $\square$

Definition A Banach space is a normed vector space which is complete as a metric space (i.e. all Cauchy sequences converge).

Examples (1), (2), (3)(i), (4), (5), but not (3)(ii). $(\ell^p, \|\cdot\|_q)$ is complete iff $q \le p$ (Exercise).

In the case $\dim V < \infty$, all norms induce the same topology, as we shall now demonstrate:

Definition Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are equivalent if $\exists\, C > 0$ s.t. $\forall\, v \in V$,

$$C^{-1} \|v\|_1 \le \|v\|_2 \le C \|v\|_1 .$$

The topologies induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ are then the same (exercise).

Proposition Any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a finite-dimensional vector space V are equivalent.

Proof By fixing any basis e_j of V, we may assume $V = \mathbb{K}^n$.

Let $\|x\| = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$. The unit sphere

$$S = \{x \in \mathbb{K}^n : \|x\| = 1\}$$

is compact. We claim that $\|\cdot\|_1$ is continuous on S: for $x, y \in S$,

$$\big| \|x\|_1 - \|y\|_1 \big| \le \|x - y\|_1 = \Big\| \sum_{j=1}^n (x_j - y_j) e_j \Big\|_1$$

$$\overset{\text{triangle ineq.}}{\le} \sum_{j=1}^n |x_j - y_j| \underbrace{\|e_j\|_1}_{\le C \ \forall j}$$

$$\le C \sum_{j=1}^n |x_j - y_j|$$

$$\overset{\text{Cauchy-Schwarz}}{\le} C n \|x - y\|.$$

$$\Rightarrow \quad 0 < C_2^{-1} := \min_{x \in S} \|x\|_1 \le \max_{x \in S} \|x\|_1 =: C_1 < \infty,$$

also $C^{-1} \|x\|_1 \le \|x\| \le C \|x\|_1$ for $C = \max\{C_1, C_2\}$.

$\Rightarrow$ $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Same for $\|\cdot\|_2$.

$\Rightarrow$ $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent. (Use: norm equivalence is an equivalence relation on the set of norms.) $\square$

Corollary Finite-dimensional subspaces $(W, \|\cdot\||_W)$ of a normed vector space $(V, \|\cdot\|)$ are complete (and thus closed).

Proof
- Completeness of a normed vector space is independent of the choice of an equivalent norm. Fixing a basis of W, we thus only need to recall that $(\mathbb{K}^n, \|\cdot\|)$ is complete.
- Completeness of $(W, \|\cdot\||_W)$ implies that $W \subset V$ is closed. $\square$

In infinite dimensions, the situation is different:

Example On $C^0([0,1])$, consider $f_n(t) = t^n$, $n \in \mathbb{N}_0$.

$\Rightarrow \|f_n\|_\infty = 1 \quad , \quad \|f_n\|_1 = \int_0^1 t^n \, dt = \frac{1}{n+1}.$

$\Rightarrow \|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent.

The reason why the proof of the Proposition breaks down completely in this case is the following:

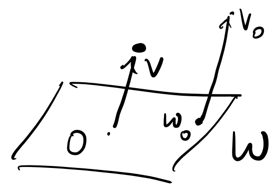
Proposition Let $(V, \|\cdot\|)$ be a normed vector space. Then:

$$\dim(V) < \infty \iff S := \{x \in V : \|x\| = 1\} \text{ is compact.}$$

For the proof, we need:

Lemma Let $(V, \|\cdot\|)$ be a normed vector space, and $W \subsetneq V$ a closed proper linear subspace. Let $\varepsilon > 0$. Then $\exists\, v \in V$ with $\|v\| = 1$ and

$$d(v, W) := \inf_{w \in W} \|v - w\| > 1 - \varepsilon.$$



Proof Choose any $v_0 \in V \setminus W$. Since W is closed, $d_0 := d(v_0, W) > 0$. Choose $w_0 \in W$ s.t. $d(v_0, w_0) = \|v_0 - w_0\| < \frac{d_0}{1-\varepsilon}$, and set $v = \frac{v_0 - w_0}{\|v_0 - w_0\|}$.

Then $d(v, W) = \inf_{w \in W} \left\| \frac{v_0 - w_0}{\|v_0 - w_0\|} - w \right\| = \frac{d(v_0, W)}{\|v_0 - w_0\|} > \frac{d_0}{d_0/1-\varepsilon} = 1 - \varepsilon.$ $\square$

Proof of the Proposition "$\Rightarrow$" Identifying $V \cong \mathbb{K}^n$, $S \subset \mathbb{K}^n$ is closed and bounded $\Rightarrow$ compact.

"$\Leftarrow$" Suppose $\dim V = \infty$. Let $\{y_j\}_{j \in \mathbb{N}}$ be linearly independent. Set $x_1 = \frac{y_1}{\|y_1\|}$. Let $Y_k = \operatorname{span}\{y_1, \dots, y_k\}$, which is a closed proper subspace of V. For $k > 1$, choose $x_k \in Y_k \setminus Y_{k-1}$ s.t. $d(x_k, Y_{k-1}) > \frac{1}{2}$.

Then for $k > \ell$, we have $\|x_k - x_\ell\| \geq d(x_k, Y_\ell) \geq d(x_k, Y_{k-1}) > \frac{1}{2}$.

$\Rightarrow \{x_j\}_{j \in \mathbb{N}} \subset S$ has no convergent subsequence. $\square$

Continuous linear maps Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ denote normed vector spaces.

Proposition For a linear map $A: X \to Y$, the following are equivalent:

(i) A is continuous.

(ii) A is bounded, that is, $\exists\, C>0$ s.t. $\|Ax\|_Y \le C\|x\|_X \;\; \forall x \in X$.

Proof (i)$\Rightarrow$(ii). Since A is continuous at $0 \in X$, and $A(0) = 0 \in Y$, there exists $\varepsilon > 0$ s.t. $B_X(0,\varepsilon) = \{x \in X : \|x\| < \varepsilon\} \subset A^{-1}(B_Y(0,1))$.

Thus, for $x \neq 0$, $\|Ax\|_Y = \Big\| A \underbrace{\frac{\varepsilon x}{2\|x\|_X}}_{\|\cdot\|_X = \frac{\varepsilon}{2} < \varepsilon} \Big\|_Y \frac{2\|x\|_X}{\varepsilon} \le \frac{2}{\varepsilon}\|x\|_X$.

(For $x=0$, $Ax=0$.)

(ii)$\Rightarrow$(i) For $x_0, x_1 \in X$,

$$\|Ax_0 - Ax_1\|_Y = \|A(x_0 - x_1)\|_Y \le C\|x_0 - x_1\|_X. \qquad \square$$

Definition For $A: X \to Y$ continuous and linear, we define its *operator norm* as

$$\|A\|_{L(X,Y)} := \sup_{\|x\|_X = 1} \|Ax\|_Y \;\left(= \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}\right). \;(= \text{the best [smallest] } C \text{ in (ii).})$$

Corollary If $\dim X < \infty$, then every linear $A: X \to Y$ is continuous.

Proof Define on X the *graph norm* $\|x\|_* := \|x\|_X + \|Ax\|_Y$. This is equivalent to $\|\cdot\|_X$, so $\exists\, C>0$ s.t.

$$\|Ax\| = \|x\|_* \le C\|x\|_X. \qquad \square$$

Example • $X = Y = C^0([0,1])$, $\|\cdot\|_X = \|\cdot\|_{C^0}$, $\|\cdot\|_Y = \|\cdot\|_{L^1}$, $A = \mathrm{Id}: X \to Y$. Then

$$\|Au\|_Y = \|u\|_{L^1} = \int_0^1 |u(x)|\,dx \le \|u\|_{C^0},$$

so A is continuous.

• But $B := \mathrm{Id}: Y \to X$ is not continuous: for $u_n(x) = x^n$,

$$\|Bu_n\|_X = \|x^n\|_{C^0} = 1, \text{ while } \|u_n\|_Y = \|x^n\|_{L^1} = \frac{1}{n+1}.$$

Definition $L(X,Y) := \{\text{continuous linear maps } X \to Y\}$.
If $X = Y$ (with same norm), write $L(X) = L(X,X)$.

Proposition $(L(X,Y), \|\cdot\|_{L(X,Y)})$ is a normed space. If Y is a Banach space, then so is $L(X,Y)$.

Proof. Definiteness: if $\|A\|_{L(X,Y)} = 0$, then $\|Ax\|_Y = 0 \ \forall x$, so $A = 0$.

Absolute homogeneity: $\|\lambda A(x)\|_Y = |\lambda| \, \|Ax\|_Y$, so $\|\lambda A\|_{L(X,Y)} = |\lambda| \, \|A\|_{L(X,Y)}$.

Triangle inequality: $\|(A+B)x\|_Y \leq \|Ax\|_Y + \|Bx\|_Y$
$$\leq \left(\|A\|_{L(X,Y)} + \|B\|_{L(X,Y)}\right) \|x\|_X.$$

- If Y is a Banach space and $\{A_j\} \subset L(X,Y)$ is a Cauchy sequence, then $\forall\, x \in X$, $\{A_j x\} \subset Y$ is a Cauchy sequence. We denote its limit by Ax. Easy check: $A : X \ni x \mapsto Ax \in Y$ is linear.

Moreover: $\|Ax\|_Y = \lim_{j\to\infty} \|A_j x\|_Y \leq \left(\limsup_{j\to\infty} \|A_j\|_{L(X,Y)}\right) \|x\|_X$ implies that A is bounded; so $A \in L(X,Y)$.

Remains to show: $A_j \to A$ in $L(X,Y)$. For $x \in X$,
$$\|(A - A_j)x\|_Y = \left\| \lim_{\ell\to\infty} (A_\ell - A_j)x \right\|_Y$$
$$= \lim_{\ell\to\infty} \|(A_\ell - A_j)x\|_Y$$
$$\leq \left(\limsup_{\ell\to\infty} \|A_\ell - A_j\|_{L(X,Y)}\right) \|x\|_X.$$

But as $j \to \infty$, we have $\limsup_{\ell\to\infty} \|A_\ell - A_j\|_{L(X,Y)} \to 0$ since $\{A_j\}$ is Cauchy. Therefore, $\|A - A_j\|_{L(X,Y)} \to 0$, as desired. $\square$

Corollary Let $(X, \|\cdot\|_X)$ be a normed vector space. Then the dual space $X^* := L(X, \mathbb{K})$ is a Banach space.

We write $\|\lambda\|_{X^*} = \|\lambda\|_{L(X,\mathbb{K})}$. Elements of X^* are called continuous linear functionals on X.

Fun (difficult) game: given X, identify X^* (is it some known space?).

For example: $(c_0)^* \cong \ell^1$, $(\ell^1)^* \cong \ell^\infty$, $(L^p(\mathbb{R}))^* \cong L^{\frac{p}{p-1}}(\mathbb{R})$.

(Here "$\cong$" means "$\exists$ continuous linear isomorphism with continuous inverse.)

Remark At this point, it is not even clear whether, for $\dim X = \infty$, there exist any elements in X^* other than 0!

The space $L(X)$ has more structure: we can compose linear maps. More generally:

Lemma The map $L(X,Y) \times L(Y,Z) \ni (B,A) \mapsto A \circ B \in L(X,Z)$ is continuous, and $\|AB\|_{L(X,Z)} \le \|A\|_{L(X,Y)} \|B\|_{L(Y,Z)}$.

Proof
- $\|ABx\|_Z \le \|A\|_{L(X,Y)} \|Bx\|_Y \le \|A\|_{L(X,Y)} \|B\|_{L(Y,Z)} \|x\|_X$.
- $\|AB - A'B'\|_{L(X,Z)}$

$$\le \|A(B-B')\|_{L(X,Z)} + \|(A-A')B'\|_{L(X,Z)}$$
$$\le \|A\|_{L(X,Y)} \|B-B'\|_{L(Y,Z)} + \left(\|B\|_{L(Y,Z)} + \|B-B'\|_{L(Y,Z)}\right) \|A-A'\|_{L(X,Y)}$$

implies continuity at (B,A). $\square$

With the space $L(X,Y)$ having a norm itself, we can talk about convergence of sequences or series of linear maps.

Lemma Let $\{A_j\} \subset L(X,Y)$, with $\sum_{j=1}^{\infty} \|A_j\|_{L(X,Y)} < \infty$. Suppose Y is complete. Then $\sum_{j=1}^{\infty} A_j = \lim_{n\to\infty} \sum_{j=1}^{n} A_j \in L(X,Y)$ exists.

Proof The sequence of partial sums $\{\sum_{j=1}^{n} A_j\}_{n\in\mathbb{N}}$ is Cauchy. $\square$

Examples Let X = Banach space.

(i) $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} \in L(X)$. $(A^0 := I)$.

(ii) $\mathcal{U}: \mathbb{R} \ni t \mapsto \exp(itA) \in L(X)$ is continuous, and $\mathcal{U}(0) = Id$, $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$.

In fact, U is continuously differentiable (i.e. $\lim_{h\to 0} \frac{U(t+h)-U(t)}{h} \in L(X)$ exists and is continuous in t),

and $\frac{1}{i}\frac{d}{dt}U(t) = A\,U(t)$. (A toy Schrödinger equation!)

(iii) Neumann series.

Lemma Let $A \in L(X)$, $\|A\| < 1$. Then $I - A : X \to X$ is invertible.

Proof Let $B := \sum_{n=0}^{\infty} A^n \in L(X)$. Then

$$(I-A)B = \sum_{n=0}^{\infty} (A^n - A^{n+1}) = A^0 = I,$$

likewise $B(I-A) = I$. $\square$

Quotient spaces

Let V be a normed space, and let $W \subset V$ be a linear subspace. We want to consider elements of V "modulo W".

Define an equivalence relation

$$v_1, v_2 \in V, \qquad v_1 \sim v_2 \iff v_1 - v_2 \in W,$$

with equivalence classes $[v] = v + W$ $(v \in V)$.

Easy: $V/W = \{[v] : v \in V\}$ is a vector space, with

$$\lambda \cdot [v] = [\lambda v] \text{ and } [v_1] + [v_2] = [v_1 + v_2].$$

Proposition Let W be closed, and $W \neq V$. Then:

(i) $\|[v]\|_{V/W} := d(v, W) = \inf_{w \in W} \|v - w\|$ is a norm on V/W.

(ii) The canonical projection $\pi : V \to V/W$, $v \mapsto [v]$, is continuous.

(iii) If V is complete, then so is V/W.

Proof (i) Exercise.

(ii) $\|\pi(v)\|_{V/W} = \|[v]\|_{V/W} \leq \|v-0\| = \|v\|$.

(iii) Let $\{[v_j]\} \subset V/W$ be a Cauchy sequence.

By passing to a subsequence, we may assume that

$$\|[v_{j+1}] - [v_j]\|_{V/W} < 2^{-j} \quad \forall j \in \mathbb{N}.$$

Let $w_1 = 0$, and inductively pick $w_{j+1} \in W$ s.t.

$$\|(v_{j+1} + w_{j+1}) - (v_j + w_j)\|_V < 2 \cdot 2^{-j}.$$

Then $[v_j + w_j] = [v_j]$. But $\{v_j + w_j\} \subset V$ is a Cauchy sequence. Write $v = \lim_{j\to\infty} v_j + w_j$. Since $v_j + w_j \longrightarrow v$, we have $\pi(v_j + w_j) = [v_j] \longrightarrow \pi(v) = [v]$. $\square$

Example (i) $V = (C^0([0,2]), \|\cdot\|_{C^0})$, $W = \{u \in C^0([0,2]) : u(x) = 0 \text{ for } x \in [0,1]\}$.

$\Rightarrow V/W \cong C^0([0,1])$. (Exercise.)

(ii) V normed vector space, $\ell \in V^*$ (i.e. $\ell : V \to \mathbb{K}$ continuous); suppose that $\ell \neq 0$. Then $W = \ker \ell$ is closed, and $V/W \cong \mathbb{K}$.