Application 4: L^p spaces without measure theory

We study here simple properties of L^p spaces without the use of measure theory. We restrict to L^p spaces on $[0,1]$. We recall

$$\|u\|_{L^p([0,1])} = \left(\int_0^1 |u(x)|^p \, dx \right)^{\frac{1}{p}} \qquad (1 \le p < \infty, \ u \in C^0([0,1]))$$

Riemann integral

Lemma $\|\cdot\|_{L^p} := \|\cdot\|_{L^p([0,1])}$ is a norm on $C^0([0,1])$.

Proof Easy for $p=1$. Consider $1 < p < \infty$

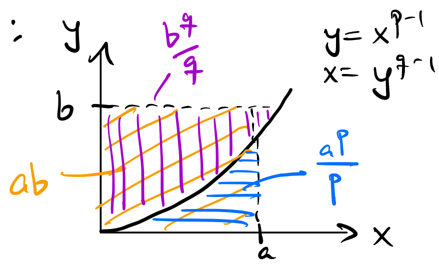
- Step 1. Hölder inequality: $\|uv\|_{L^1} \le \|u\|_{L^p} \|v\|_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

  Proof. · First, Young's inequality: for $a, b > 0$, $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

  To see this, note: $\frac{a^p}{p} = \int_0^a x^{p-1} \, dx$, $\frac{b^q}{q} = \int_0^b y^{q-1} \, dy$,

  and $x \mapsto y = x^{p-1}$ has inverse $y \mapsto x = y^{\frac{1}{p-1}} = y^{q-1}$.

  Thus, proof by picture:



  · Then, for $u, v \ne 0$, replace u, v by $\frac{u}{\|u\|_{L^p}}$, $\frac{v}{\|v\|_{L^q}}$. Then

$$\int_0^1 |u(x)| \, |v(x)| \, dx \le \int_0^1 \frac{|u(x)|^p}{p} \, dx + \int_0^1 \frac{|v(x)|^q}{q} \, dx = 1.$$

- Step 2: $\|u\|_{L^p} = \sup_{\substack{v \in C^0 \\ \|v\|_{L^q} = 1}} \left| \int_0^1 u(x) \, v(x) \, dx \right|$.

  Indeed, the right hand side is $\le \sup_{\|v\|_{L^q}=1} \left(\|u\|_{L^p} \|v\|_{L^q} \right) = \|u\|_{L^p}$ by Hölder.

  For the other inequality, set $\tilde{v}(x) := \begin{cases} \frac{u(x)}{|u(x)|} |u(x)|^{p-1} & , u(x) \ne 0 \\ 0 & , u(x) = 0 \end{cases}$

  with $|\tilde{v}(x)|^q = |u(x)|^{(p-1)q} = |u(x)|^p$. Then $\|\tilde{v}\|_{L^q} = \|u\|_{L^p}^{p/q}$. $\Rightarrow$ For $v = \frac{\tilde{v}}{\|\tilde{v}\|_{L^q}}$:

$$\left| \int_0^1 u(x) \, v(x) \, dx \right| = \frac{1}{\|\tilde{v}\|_{L^q}} \int_0^1 |u(x)|^p \, dx = \frac{1}{\|u\|_{L^p}^{p/q}} \|u\|_{L^p}^p = \|u\|_{L^p}^{p - p/q} = \|u\|_{L^p};$$

so we get "$\geq$" in $\circledast$.

· Step 3 Only check triangle inequality: for $u, v \in C^0([0,1])$,

$$\|u+v\|_{L^p} = \sup_{\|w\|_{L^q}=1} \int_0^1 |u(x)+v(x)|\,|w(x)|\,dx$$
$$\leq \sup_{\|w\|_{L^q}=1} \left(\int_0^1 |u(x)|\,|w(x)|\,dx + \int_0^1 |v(x)|\,|w(x)|\,dx\right)$$
$$= \|u\|_{L^p} + \|v\|_{L^p}. \qquad \square$$

Definition For $1 \leq p < \infty$, we define $L^p([0,1])$ as the completion of $(C^0([0,1]), \|\cdot\|_{L^p})$.

Lemma $L^2([0,1])$ is a Hilbert space with inner product given by continuous extension of $(u,v) = \int_0^1 u(x)\overline{v(x)}\,dx$, $u, v \in C^0([0,1])$.

Proof We already saw that $(\cdot,\cdot)$ is continuous with respect to the topology induced by the associated canonical norm $\|\cdot\|_{L^2}$. $\square$

Proposition (Integral) The Riemann integral $C^0([0,1]) \ni u \mapsto \int_0^1 u(x)\,dx$ extends to a continuous linear functional $\int_0^1 (\cdot)\,dx$ on $L^1([0,1])$.

Proof We only need to show that $\left(u \mapsto \int_0^1 u(x)\,dx\right) \in L(C^0([0,1]), \mathbb{C})$, where we use the $\|\cdot\|_{L^1}$-norm on C^0. But

$$\left|\int_0^1 u(x)\,dx\right| \leq \int_0^1 |u(x)|\,dx = \|u\|_{L^1}. \qquad \square$$

While the definition of L^p via completion does not make it clear why one can regard elements of L^p as functions (or really equivalence classes of functions modulo functions $=0$ a.e.), one can easily see that they behave like functions in many ways. For example:

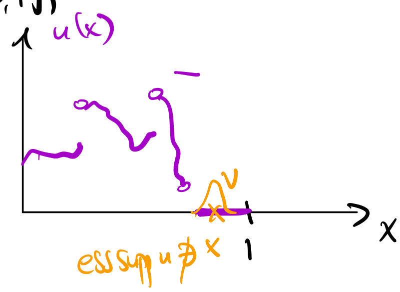
(1) If $u \in L^p([0,1])$ and $v \in C^0([0,1])$, then $uv \in L^p([0,1])$.

More precisely $C^0 \times C^0 \ni (u,v) \mapsto uv \in L^p$ extends by continuity (w.r.t. $\|(u,v)\| = \|u\|_{L^p} + \|v\|_{C^0}$) to a continuous bilinear map $L^p \times C^0 \ni (u,v) \longmapsto uv \in L^p$.

Indeed, this follows from
$$\|uv\|_{L^p} = \left(\int_0^1 |u(x)|^p |v(x)|^p \, dx\right)^{\frac{1}{p}}$$
$$\leq \left(\|v\|_{C^0}^p \int_0^1 |u(x)|^p\right)^{\frac{1}{p}}$$
$$= \|u\|_{L^p} \|v\|_{C^0}.$$

For example, this allows one to define, for $u \in L^p([0,1])$

$$\text{ess supp } u = [0,1] \setminus \{x \in [0,1] : \exists v \in C^0([0,1]), \; v(x) \neq 0, \text{ s.t. } uv = 0 \in L^p([0,1])\}.$$



(2) More generally: if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ $(p,q,r \geq 1)$, then
$L^p \times L^q \ni (u,v) \longmapsto uv \in L^r$. (Exercise.)

Using functional analytic techniques, we can also identify the dual spaces:

Proposition Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then
$$(L^p([0,1]))^* \cong L^q([0,1])$$
Concretely, $j : L^q([0,1]) \ni v \longmapsto j_v \in (L^p([0,1]))^*$,
$$j_v(u) = \int_0^1 u(x) v(x) \, dx,$$
is an isometric isomorphism.

Proof Note that $j_v \in (L^p)^*$ indeed, as proved above.

Step 1: we claim that $\|j_v\|_{(L^p)^*} = \|v\|_{L^q}$.

But $\|j_v\|_{(L^p)^*} = \sup_{\substack{u \in L^p \\ \|u\|_{L^p}=1}} |j_v(u)| = \sup_{\substack{u \in C^0 \\ \|u\|_{L^p}=1}} |j_v(u)|$

$$= \sup_{\substack{u \in C^0 \\ \|u\|_{L^p}=1}} \left| \int_0^1 u(x)v(x)\,dx \right| = \|v\|_{L^q},$$

as we showed in the proof that $\|\cdot\|_{L^p}$ is a norm.

Step 2: j is surjective. To prove this, we need to use a property of L^p which we will only show later: L^p is reflexive, i.e. the canonical inclusion $\iota: L^p \to (L^p)^{**}$ is an isomorphism. (We show this as a consequence of a geometric property of $\|\cdot\|_{L^p}$ called uniform convexity.)

So suppose $j: L^q \to (L^p)^*$ is not surjective. Since

$\operatorname{ran} j = j(L^q) \subsetneq (L^p)^*$ is a closed subspace, Hahn-Banach

$\Rightarrow \exists\, \Phi \in (L^p)^{**}$, $\Phi \neq 0$, s.t. $\Phi|_{\operatorname{ran} j} = 0$. ⊛

But by the reflexivity of L^p, $\Phi = \iota(u)$ for some $0 \neq u \in L^p$.

So ⊛ gives $\iota(u)(j_v) = 0$ $\forall v \in L^q$, i.e. $j_v(u) = 0$.

To show: if $u \in L^p([0,1])$ is such that $j_v(u)=0$ $\forall v \in C^0$, then $u=0$.

To do this, assume to the contrary that $u \neq 0$; we may assume $\|u\|_{L^p}=1$. Take $\tilde{u} \in C^0([0,1])$ with $\|\tilde{u}-u\|_{L^p} < \frac{1}{2}$, $\|\tilde{u}\|_{L^p}=1$.

$\Rightarrow \forall v \in C^0$: $|j_v(\tilde{u})| = |j_v(\tilde{u}-u)| \le \|j_v\|_{(L^p)^*} \|\tilde{u}-u\|_{L^p} < \frac{1}{2}\|v\|_{L^q}$.

For $v(x) = \frac{\overline{\tilde{u}(x)}}{|\tilde{u}(x)|} |\tilde{u}(x)|^{p-1}$, this gives

$$1 = \int |\tilde{u}(x)|^p dx < \frac{1}{2}\|v\|_{L^q} = \frac{1}{2}\|\tilde{u}\|_{L^p}^{q/p} = \frac{1}{2},$$

a contradiction.

Therefore, $\Phi = \iota(u) = 0$, contradicting $\Phi \neq 0$.

$\Rightarrow$ ran $j = (L^p)^*$. $\square$

We have carefully avoided the case $p=\infty$ so far.

<u>Definition</u> We set $L^\infty([0,1]) := (L^1([0,1]))^*$.

How is this a space of functions? We can regard $u \in C^0([0,1])$ as an element $T_u \in L^\infty([0,1])$ by

$$T_u : L^1([0,1]) \ni v \longmapsto \int_0^1 u(x)v(x)\,dx.$$

Thus, $u \mapsto T_u$ is an injective map $C^0([0,1]) \hookrightarrow L^\infty([0,1])$.

Its range, however, is not dense. The identification of L^∞ with essentially bounded measurable functions requires measure theory.

- Nonetheless, we can do many things with L^∞ that we are used to. E.g. we can define a "pointwise multiplication" map

$$L^p \times L^\infty \ni (u,v) \longmapsto uv \in L^p, \quad 1 < p < \infty, \quad (*)$$

as follows: for $\frac{1}{p}+\frac{1}{q}=1$, we define $uv \in L^p = (L^q)^*$ as

$$L^q \ni w \longmapsto u(vw); \quad \text{note: } vw \in L^1, \text{ so } u(vw) \in \mathbb{C}$$

is well-defined. (Check that for $u, v \in C^0$, this definition of uv agrees with the pointwise product $u \cdot v$.)

- Have $L^\infty \times L^\infty \ni (u,v) \longmapsto uv \in L^\infty$ via $uv : w \in L^1 \mapsto u(vw)$; note that $vw \in L^1$ by ⊛.

Remark It looks like L^∞ is somehow different than L^p, $p<\infty$: we could define L^p $(p<\infty)$ as a completion of C^0, but not L^∞.
(Exercise: $\{T_u : u \in C^0([0,1])\} \subset (L^1)^* = L^\infty$ is closed and not dense.)

- But the "correct" perspective is that $L^p([0,1])$, for all $1 \le p \le \infty$, is a space of distributions ("generalized functions"): $u \in L^p([0,1])$ induces $T_u : C^0([0,1]) \ni v \longmapsto \int_0^1 u(x) v(x) dx$, i.e.

$$T_u \in (C^0([0,1]))^*.$$

(T_u assigns to a test function v a complex number.)

- More general distributions: e.g. Dirac distribution δ_{x_0}, $x_0 \in [0,1]$:

$$\delta_{x_0} : v \in C^0([0,1]) \longmapsto v(x_0).$$

Exercise: $\delta_{x_0} \notin L^p([0,1])$ $\forall\, 1 \le p \le \infty$
(i.e. $\nexists\, u \in L^p$ s.t. $\delta_{x_0} = T_u \in (C^0([0,1]))^*$.)