Functional analysis is most pleasant on Hilbert spaces, a subclass of Banach spaces.

Definition Let X be a $\mathbb{K}$-vector space, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A map

$$(\cdot,\cdot) : X \times X \longrightarrow \mathbb{K}$$

is a scalar product on X if:

(i) $(x,y)=\overline{(y,x)} \quad \forall x,y \in X$;

(ii) $(\alpha x+\beta y, z) = \alpha(x,z)+\beta(y,z) \quad \forall x,y,z\in X, \alpha,\beta \in \mathbb{K}$

(linearity in the first argument),

(iii) $(x,x)\geq 0 \quad \forall x\in X$, and $(x,x)=0 \Leftrightarrow x=0$.

As a consequence, $(x,\alpha y+\beta z) = \overline{(\alpha y+\beta z, x)} = \bar{\alpha}\overline{(y,x)} + \bar{\beta}\overline{(z,x)}$
$$= \bar{\alpha}(x,y)+\bar{\beta}(x,z)$$
("anti-linearity" in the second argument when $\mathbb{K}=\mathbb{C}$).

Warning Physicists typically define scalar products to be linear in the 2nd argument.

Rmk. If $\|\cdot\|$ is a norm on a vector space X, then $\|\cdot\|$ comes from a scalar product iff the parallelogram identity holds:

$$\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2) \quad \forall x,y\in X.$$

"Proof" $\mathbb{R}$-vector space: set $(x,y) := \frac{1}{4}(\|x+y\|^2-\|x-y\|^2)$.

$\mathbb{C}$-vector space: set $(x,y) := \frac{1}{4}(\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2)$. $\square$

From now on, we only consider complex Hilbert spaces (unless explicitly noted otherwise. The case of real Hilbert spaces is the same!

Lemma Let X be a vector space with scalar product $(\cdot,\cdot)$. Set

$$\|x\| := (x,x)^{\frac{1}{2}}, \quad x\in X.$$

(i) (Cauchy-Schwarz inequality.) $\forall x,y \in X$,

$|(x,y)| \leq \|x\| \, \|y\|$. (Equality if and only if x,y are lin. dependent.)

(ii) $\|\cdot\|$ is a norm.

Proof (i) Assume $x \neq 0, y \neq 0$. For all $\alpha \in \mathbb{C}$,

$$0 \leq (x-\alpha y, x-\alpha y) = \|x\|^2 + |\alpha|^2 \|y\|^2 - \alpha (y,x) - \bar{\alpha}(x,y). \quad \circledast$$

Take $\alpha = te^{i\theta}$, $t, \theta \in \mathbb{R}$, where $e^{i\theta}(y,x) = |(x,y)|$; $\circledast$ becomes

$$t^2 \|y\|^2 - 2t|(x,y)| + \|x\|^2 \geq 0.$$

This quadratic expression in t attains its minimum at $t_0 = \dfrac{|(x,y)|}{\|y\|^2}$;

setting $t = t_0 \Rightarrow -\dfrac{|(x,y)|^2}{\|y\|^2} + \|x\|^2 \geq 0$, as desired.

Equality forces $x - t_0 e^{i\theta} y = 0$, i.e. x,y are linearly independent.

(ii) Only check triangle inequality:

$$\begin{aligned}\|x+y\|^2 = (x+y, x+y) &= (x,x) + (y,y) + 2\,\mathrm{Re}\,(x,y) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

□

Definition $(X, (\cdot,\cdot))$ is a Hilbert space if it is complete w.r.t. the canonical norm $\|x\| = (x,x)^{\frac{1}{2}}$.

Examples (i) $\mathbb{K}^n$, $(z,w) = \sum_{j=1}^{n} z_j \overline{w_j}$.

(ii) ℓ^2, with $((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} a_n \overline{b_n}$. $\circledast$

Note that $\left|\sum_{n=1}^{N} a_n \overline{b_n}\right| \leq \left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |b_n|^2\right)^{\frac{1}{2}} \leq \|a\|_{\ell^2} \|b\|_{\ell^2}$

for all N, so $\circledast$ is well-defined.

(iii) Let $\Omega \subset \mathbb{R}^n$. Then $L^2(\Omega)$ is a Hilbert space, with

$$(f,g) = \int_\Omega f(x)\overline{g(x)}\,dx, \qquad f,g \in L^2(\Omega).$$

(iv) Will show: all separable Hilbert spaces are isometric to ℓ^2.

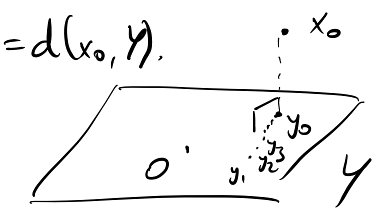
Minimizing distances to closed subspaces of a Hilbert space is better behaved:

Lemma Let $Y \subsetneq X$ be a closed subspace of a Hilbert space X.

Let $x_0 \in X$. Then $\exists!\ y_0 \in Y$ s.t. $\|x_0 - y_0\| = d(x_0, Y) = \inf\limits_{y\in Y} \|x_0 - y\|$.

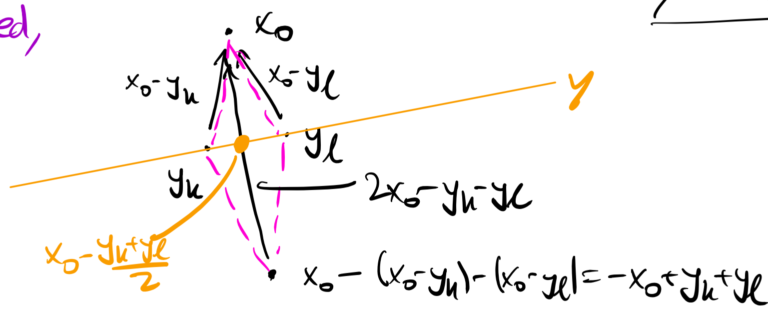
This y_0 satisfies $(x_0 - y_0, y) = 0\ \forall y \in Y$.

Proof Let $\{y_k\} \subset Y$ be s.t. $\|x_0 - y_k\| \xrightarrow{k\to\infty} d := d(x_0, Y)$.



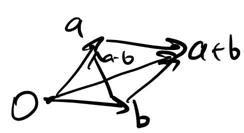
Step 1: $\{y_k\}$ is a Cauchy sequence.

Indeed,



using the parallelogram identity

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$$



with $a = x_0 - y_k$
$b = x_0 - y_\ell$,

we get $\quad \|2(x_0 - \frac{y_k + y_\ell}{2})\|^2 + \|y_k - y_\ell\|^2 = 2(\|x_0 - y_k\|^2 + \|x_0 - y_\ell\|^2)$;

so $\quad 4d^2 + \|y_k - y_\ell\|^2 \le 2(\|x_0 - y_k\|^2 + \|x_0 - y_\ell\|^2)$.

As $k,\ell \to \infty$, the right hand side converges to $2(d^2 + d^2) = 4d^2$, and hence $\|y_k - y_\ell\|^2 \to 0$.

Step 2 Let $y_0 := \lim_{k\to\infty} y_k$; then $\|x_0 - y_0\| = \lim_{k\to\infty} \|x_0 - y_k\| = d$.

The estimates in Step 1 also imply that $y_0 \in Y$ with this property is unique.

Step 3: $(x_0 - y_0, y) = 0 \ \forall y \in Y$.

Indeed, consider for $\alpha \in \mathbb{C}$

$$d^2 = \|x_0 - y_0\|^2 \le \|x_0 - y_0 - \alpha y\|^2$$
$$= \|x_0 - y_0\|^2 + |\alpha|^2 \|y\|^2 - 2\,\mathrm{Re}\big(\alpha (y, x_0 - y_0)\big).$$

Take $\alpha = te^{i\theta}$, $t, \theta \in \mathbb{R}$, s.t. $e^{i\theta}(y, x_0 - y_0) = |(y, x_0 - y_0)|$; get

$$d^2 \le d^2 + t^2 \|y\|^2 - 2t\,|(y, x_0 - y_0)|.$$

The quadratic expression (in t) on the right attains its minimum at $t=0$; therefore, its derivative in t at $t=0$ vanishes

$$\Rightarrow (y, x_0 - y_0) = 0. \qquad \square$$

Definition (i) We write $x \perp y$ iff $(x,y) = 0$.

(ii) Given a subspace $Y \subset X$, we write $Y^\perp = \{x \in X : x \perp y \ \forall y \in Y\}$.

Lemma (i) $Y^\perp$ is closed, and $Y^\perp = \overline{Y}^\perp$. (ii) $(Y^\perp)^\perp = \overline{Y}$.

Proof (i) If $(x, y_k) = 0$, $y_k \xrightarrow{k\to\infty} y$, then $(x,y) = \lim_{k\to\infty} (x, y_k) = 0$. We use here the continuity of $(\cdot,\cdot): X \times X \to \mathbb{K}$, which follows from

$$|(x,y) - (x',y')| = |(x-x', y) + (x', y-y')| \le \|x - x'\|\,\|y\| + \|x'\|\,\|y - y'\|.$$

(ii) Let $y \in Y$, then $(x,y) = 0 \ \forall x \in Y^\perp \Rightarrow y \in (Y^\perp)^\perp$. So $Y \subset (Y^\perp)^\perp$, and hence $\overline{Y} \subset (Y^\perp)^\perp$ by (i). If $x \notin \overline{Y}$, let $y_0 \in \overline{Y}$ be s.t. $x - y_0 =: y_1 \in \overline{Y}^\perp$; then $y_1 \ne 0$, and $(x, y_1) = (x - y_0, y_1) = (y_1, y_1) \ne 0 \Rightarrow x \notin (\overline{Y}^\perp)^\perp = (Y^\perp)^\perp$. $\square$

Theorem. X Hilbert space, $Y \subsetneq X$ closed subspace. Then $X = Y \oplus Y^\perp$. Every $x \in X$ has a unique decomposition

$x = y_0 + y_1$, $\quad y_0 \in Y$, $y_1 \in Y^\perp$, and $\|x\|^2 = \|y_0\|^2 + \|y_1\|^2$. $\circledast$

- Moreover, the orthogonal projection $\begin{cases} \pi_Y : X \to Y, \\ \quad x \mapsto y_0, \end{cases}$ is continuous.

Proof
- $\|x\|^2 = (y_0 + y_1, y_0 + y_1) = \|y_0\|^2 + \|y_1\|^2 + \underbrace{2\operatorname{Re}(y_0, y_1)}_{=0}$.
- $\|\pi_Y(x)\| = \|y_0\| \leq \|x\|$ by $\circledast$. $\square$

Corollary X Hilbert space, $Y \subset X$ closed $\Rightarrow$ $X/Y \cong Y^\perp$ (isometric as Banach spaces)

Proof The map $A : Y^\perp \ni y_1 \mapsto [y_1] \in X/Y$ is bijective, and

$$\|[y_1]\|^2_{X/Y} = \inf_{y_0 \in Y} \|y_1 + y_0\|^2 = \inf_{y_0 \in Y} \left(\|y_1\|^2 + \|y_0\|^2\right) = \|y_1\|^2$$

$\Rightarrow A$ is an isometry. $\square$

Definition Let X be a separable Hilbert space. Then a complete orthonormal basis of X is a set $\{e_1, e_2, e_3, \dots\} \subset X$ s.t.

(i) $(e_i, e_j) = \delta_{ij} \quad \forall i, j$ (orthonormal)

(ii) $X = \overline{\operatorname{span}\{e_j\}}$.

finite if $\dim X < \infty$, countably infinite if $\dim X = \infty$

Proposition Every separable Hilbert space X has a complete ONB.

Proof
- Let $\{a_j\} \subset X$ be dense. By passing to a subsequence, we may assume that $a_{j+1} \notin \operatorname{span}\{a_1, \dots, a_j\}$ $\forall j = 0, 1, 2, \dots$ (For $j = 0$: read this as $a_1 \neq 0$.) Then $\operatorname{span}\{a_1, \dots\} \subset X$ is dense.
- Using Gram-Schmidt, define a sequence $\{e_j\}_{j \in \mathbb{N}}$ of orthonormal vectors s.t. $\operatorname{span}\{e_1, \dots, e_j\} = \operatorname{span}\{a_1, \dots, a_j\}$ $\forall j$.
- Since $\operatorname{span}\{a_j\} = \operatorname{span}\{e_j\}$ is dense, we are done. $\square$

Proposition Let $\{e_j\}$ be a complete ONB of X, and let $x \in X$. Then there exist unique $x_j \in \mathbb{C}$ s.t. $x = \sum_{j=1}^{\infty} x_j e_j$. In fact, $x_j = (x, e_j)$, and
$\|x\|^2 = \sum_j |x_j|^2$. $\circledast$ (Parseval Identity.)

Proof We only consider the case $\dim X = \infty$. For $J \in \mathbb{N}$, define
$\Pi_J x = \sum_{j=1}^{J} (x, e_j) e_j$. This is the orthogonal projection onto $\operatorname{span}\{e_1, \dots, e_J\}$
since for $\ell = 1, \dots, J$, $(x - \Pi_J x, e_\ell) = (x, e_\ell) - \sum_{j=1}^{J} (x, e_j) \underbrace{(e_j, e_\ell)}_{=\delta_{j\ell}}$
$$= (x, e_\ell) - (x, e_\ell) = 0.$$
Thus, $\|\Pi_J x\|^2 \le \|x\|^2$; but $\|\Pi_J x\|^2 = \sum_{j,\ell=1}^{J} (x, e_j) \overline{(x, e_\ell)} (e_j, e_\ell) = \sum_{j=1}^{J} |(x, e_j)|^2$.
$\Rightarrow \left((x, e_j) \right)_{j \in \mathbb{N}} \in \ell^2$. $\circledast$

Next, we infer that $\{\Pi_J x\}_{J \in \mathbb{N}}$ is Cauchy since for $J > K$
$$\|\Pi_J x - \Pi_K x\|^2 = \Big\| \sum_{j=K+1}^{J} (x, e_j) e_j \Big\|^2 = \sum_{j=K+1}^{J} |(x, e_j)|^2 \xrightarrow{J, K \to \infty} 0 \text{ by } \circledast.$$
Let $\bar{x} = \lim_{J \to \infty} \Pi_J x = \sum_{j=1}^{\infty} (x, e_j) e_j$. Then $\forall \ell \in \mathbb{N}$:
$$(\bar{x}, e_\ell) = \lim_{J \to \infty} \underbrace{(\Pi_J x, e_\ell)}_{= (x, e_\ell) \text{ for } J \ge \ell} = (x, e_\ell) \Rightarrow \bar{x} - x \perp \operatorname{span}\{e_1, e_2, \dots\}.$$
$\Rightarrow \bar{x} - x = 0$ since $\operatorname{span}\{e_1, \dots\}$ is dense
$\Rightarrow x = \bar{x}$.

Finally, $\|x\|^2 = \|\Pi_J x\|^2 + \|x - \Pi_J x\|^2 = \sum_{j=1}^{J} |(x, e_j)|^2 + \underbrace{\|x - \Pi_J x\|^2}_{\xrightarrow{J \to \infty} 0} \xrightarrow{J \to \infty} \sum_{j=1}^{\infty} |(x, e_j)|^2$.

Uniqueness of x_j: take inner product of $x = \sum_{\ell=1}^{\infty} x_\ell e_\ell$ with $e_j \Rightarrow x_j = (x, e_j)$. $\square$

Corollary (i) If $n = \dim X < \infty$, then $X \cong \mathbb{C}^n$ (isomorphism of Hilbert spaces).
(ii) If $\dim X = \infty$, then $X \cong \ell^2$ via $x \mapsto (x_j)$.

Proof of (ii) Note that $A: X \ni x \mapsto \left((x, e_j) \right)_{j \in \mathbb{N}} \in \ell^2$ is surjective: given $(x_j)_{j \in \mathbb{N}} \in \ell^2$, the series $\sum_{j=1}^{\infty} x_j e_j$ is Cauchy and thus converges.
A is an isometry by Parseval, and thus we are done. $\square$

Therefore, all separable Hilbert spaces are "the same" up to isometry.

We now turn to the discussion of dual spaces of Hilbert spaces.

Theorem (Riesz representation theorem.) Let $\mathcal{H}$ be a Hilbert space and $\lambda \in \mathcal{H}^*$. Then $\exists!\, y \in \mathcal{H}$ s.t. $\lambda(x) = (x,y)\ \forall x \in X$.

Corollary The map $j : \mathcal{H} \ni y \longmapsto j_y \in \mathcal{H}^*$,

$$j_y(x) = (x,y),$$

is an antilinear isometry. (That is, $j_{\alpha y + z} = \bar{\alpha} j_y + j_z$ for $y, z \in \mathcal{H}$, $\alpha \in \mathbb{C}$.)

Proof · For $y \in Y$, $\|j_y\|_{\mathcal{H}^*} = \sup\limits_{\|x\|=1} |(x,y)| \le \sup\limits_{\|x\|=1} (\|x\|\,\|y\|) = \|y\|$.

For $y \neq 0$, $\|j_y\|_{\mathcal{H}^*} \geq \left| j_y\left(\frac{y}{\|y\|}\right)\right| = \left(\frac{y}{\|y\|}, y\right) = \|y\|$

$\Rightarrow \|j_y\|_{\mathcal{H}^*} = \|y\|$.

· Surjectivity follows from Riesz; antilinearity follows from the properties of $(\cdot,\cdot)$. □

Proof of the Riesz representation theorem. Let $Y = \ker\lambda$; this is a closed subspace of $\mathcal{H}$.

Case 1. $Y = \mathcal{H}$. This means $\lambda \equiv 0$, so $y = 0$ is the unique choice that works.

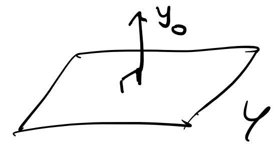
Case 2. $Y \neq \mathcal{H}$. Since λ induces a non-zero injective map $\mathcal{H}/Y \longrightarrow \mathbb{K}$, $[y] \longmapsto \lambda(y)$, we have $\dim(\mathcal{H}/Y) = \dim \mathbb{K} = 1$.

But $\mathcal{H}/Y \cong Y^\perp$, so $Y^\perp = \operatorname{span}\{y_0\}$ for some $y_0 \neq 0$;

by definition of y_0, we have $\lambda(y_0) \neq 0$.

Claim: $\lambda(x) = \dfrac{\lambda(y_0)}{\|y_0\|^2}(x, y_0) \quad \forall x \in H.$ $\circledast$

($\Leftarrow$ $y = \dfrac{\overline{\lambda(y_0)}}{\|y_0\|^2} y_0$ works.)



Indeed, the right hand side of $\circledast$ defines an element of H^*

which – vanishes on $(Y^\perp)^\perp = \overline{Y} = Y$, just like λ;

– evaluates to $\lambda(y_0)$ for $x = y_0$, just like λ.

Since $H = Y \oplus \operatorname{span}\{y_0\}$, this gives $\circledast$.

Uniqueness of y. If $(x, y) = (x, y')$ $\forall x \in H$, then

$$(x, y - y') = 0 \quad \forall x \in H.$$

Plug in $x = y - y'$ $\Rightarrow$ $\|y - y'\|^2 = 0$ $\Rightarrow$ $y = y'$. $\square$

Corollary Every Hilbert space H is reflexive; that is, the canonical inclusion $\iota : H \to H^{**}$, $x \mapsto (\lambda \mapsto \lambda(x))$, is an isomorphism.

Proof We only need to show the surjectivity of ι.

- Since $H \ni y \mapsto j_y = (\cdot, y) \in H^*$ is an antilinear isometry, the dual space H^* is itself a Hilbert space with scalar product $(j_y, j_{y'}) = (y', y)$, $y, y' \in H$. (The switch of y, y' on the right ensures that $(j_y, j_{y'})$ is linear in the first argument: $(\alpha j_y, j_{y'}) = (j_{\bar{\alpha} y}, j_{y'}) = (y', \bar{\alpha} y) = \alpha (y', y) = \alpha (j_y, j_{y'})$.)
- If $\Phi \in H^{**}$, then Riesz gives us an element $j_x \in H^*$ (with $x \in H$) s.t. $\Phi(\lambda) = (\lambda, j_x)$ $\forall \lambda \in H^*$,

i.e. $\forall y \in H$, $\Phi(j_y) = (j_y, j_x) = (x, y) = j_y(x) = \iota(x)(j_y)$.

$\Rightarrow$ $\Phi = \iota(x)$, as desired. $\square$

The Riesz representation theorem has many applications; for applications to PDE, wait until Functional Analysis 2. The interested student is encouraged to look up applications to measure theory ("von Neumann's proof of the Radon–Nikodym Theorem").

- Next, we turn to some of the earlier 'Big Theorems'.
- Observation The Hahn–Banach Theorem, with respect to the sublinear function $\|\cdot\|$, is trivial on Hilbert spaces:

  if $\lambda : Y \subset X \to \mathbb{C}$ is continuous and linear, we can extend it by continuity to $\lambda' : \overline{Y} \subset X \to \mathbb{C}$. Then define

  $$\tilde{\lambda} : X = \overline{Y} \oplus \overline{Y}^{\perp} \ni (y_0, y_1) \longmapsto \lambda'(y_0).$$

  Fact. $\tilde{\lambda}$ is the unique extension of λ with $\|\tilde{\lambda}\|_{X^*} = \|\lambda\|_{Y^*}$.
- An important application of the Closed Graph Theorem:

Theorem (Hellinger–Toeplitz) Let X be a Hilbert space, and suppose $A : X \to X$ is linear and symmetric, i.e. $(Ax, y) = (x, Ay)$ $\forall x, y \in X$. Then A is continuous.

Proof We show that Γ_A is closed. Thus, suppose $x_k \to x$, $Ax_k \to y$. For all $z \in X$, we then have

$$(y, z) = \lim_{k\to\infty} (Ax_k, z) = \lim_{k\to\infty} (x_k, Az) = (x, Az) = (Ax, z).$$

Therefore, $y = Ax$, as desired. $\square$