Operators of the form $I-K$, K compact, have nice mapping properties: finite-dimensional kernel, closed range with finite codimension. They are important special cases of:

Definition Let X, Y be Banach spaces, $A \in L(X,Y)$. Then A is a Fredholm operator (or simply "A is Fredholm") if

- $\dim \ker A < \infty$, and
- $\dim \operatorname{coker} A < \infty$, and $\operatorname{ran} A$ is closed.

We then call $\operatorname{ind}(A) := \dim \ker A - \dim \operatorname{coker} A$ the index of A.

Lemma If $A \in L(X,Y)$ has $\dim \operatorname{coker} A < \infty$, then $\operatorname{ran} A$ is closed.

Proof Replacing X by $X/\ker A$, we may assume that A is injective. Indeed, let $Z \subset Y$ be a finite-dimensional subspace with $(\operatorname{ran} A) \cap Z = \{0\}$ and $(\operatorname{ran} A) + Z = Y$. Then

$$\begin{aligned} \tilde{A} : \ & X \oplus Z \to Y, \\ & (x, z) \mapsto Ax + z, \end{aligned}$$

is a bijective map of Banach spaces, hence an isomorphism. Therefore, $\operatorname{ran} A = \tilde{A}(X \oplus \{0\}) \subset Y$ is closed (since $X \oplus \{0\}$ is). $\square$

Thus, the closedness of $\operatorname{ran} A$ in the definition of Fredholm operators is automatic, given the finite codimension of ran A.

Remark Every linear operator A between finite-dimensional vector spaces X, Y is Fredholm. Moreover, since A induces an isomorphism $X/_{\ker A} \longrightarrow \operatorname{ran} A$, we have

$$\begin{aligned} \dim \operatorname{coker} A &= \dim \left(Y/_{\operatorname{ran} A}\right) \\ &= \dim Y - \dim \operatorname{ran} A \\ &= \dim Y - \dim \left(X/_{\ker A}\right) \\ &= \dim Y - \dim X + \dim \ker A \end{aligned}$$

$\Longrightarrow \operatorname{ind}(A) = \dim X - \dim Y$ is independent of A. In infinite dimensions, the situation is of course much more interesting!

Example $R: \ell^2 \longrightarrow \ell^2$ right shift, $R(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then R is Fredholm with index -1; $\operatorname{ind}(R^j) = -j$ $(j \in \mathbb{N})$. Its adjoint $L = R^*$ is the left shift; $\operatorname{ind}(L^j) = j$ $(j \in \mathbb{N})$.

Proposition $A \in L(X, Y)$ Fredholm $\Longrightarrow A^* \in L(Y^*, X^*)$ Fredholm, with $\operatorname{ind}(A^*) = -\operatorname{ind}(A)$.

Proof ran A closed $\Rightarrow$ ran A^* is closed. Thus, $\dim \ker A^* = \dim (\operatorname{ran} A)^\perp = \dim \operatorname{coker} A$, and $\dim \operatorname{coker} A^* = \dim {}^\perp(\operatorname{ran} A^*) = \dim \ker A$. $\square$

The relationship with operators of the form $I - K$, K compact, is:

Proposition X, Y Banach spaces, $A \in L(X, Y)$. Then A is Fredholm $\Longleftrightarrow \exists\, B_1, B_2 \in L(Y, X)$, $K_1 \in L(Y)$ compact, $K_2 \in L(X)$ compact,

s.t. $$AB_1 = I - K_1 \qquad (\circledast)$$
$$B_2 A = I - K_2.$$

Proof Exercise. $\square$

Thus, Fredholm operators are those bounded linear operators which are "invertible modulo compact errors".

Corollary $A \in L(X,Y)$ Fredholm, $K \in L(X,Y)$ compact $\Rightarrow A+K$ Fredholm.

Proof The "approximate inverses" B_1, B_2 for A in $\circledast$ are also approximate inverses for $A+K$. $\square$

Theorem Let $A \in L(X,Y)$ be a Fredholm operator. If $B \in L(X,Y)$ has sufficiently small operator norm $\|B\|_{L(X,Y)}$, then also $A+B$ is Fredholm, with
$$\operatorname{ind}(A+B) = \operatorname{ind}(A), \qquad \begin{aligned} \dim\ker(A+B) &\le \dim\ker A. \\ \dim\operatorname{coker}(A+B) &\le \dim\operatorname{coker} A. \end{aligned}$$

Corollary The set $\operatorname{Fred}(X,Y)$ of Fredholm operators is open in $L(X,Y)$, and $\operatorname{ind}: \operatorname{Fred}(X,Y) \to \mathbb{Z}$ is constant on each connected component.

Remark $X = Y = H =$ separable Hilbert space
$$\Rightarrow \operatorname{ind}: \pi_0(\operatorname{Fred}(H,H)) \cong \mathbb{Z}.$$

Proof of the Theorem Let $X_0 \subset X$ denote a topological complement of $X_1 := \ker A$, and $Y_1 \subset Y$ a topological complement of $Y_0 := \operatorname{ran} A$. Thus, $\dim X_1 = \dim\ker A$, $\dim Y_1 = \dim\operatorname{coker} A$. The map

$$T_A : X_0 \oplus Y_1 \ni (x,y) \mapsto Ax + y \in Y$$

is an isomorphism; if $(\|B\|_{L(X_0,Y)} \leq) \|B\|_{L(X,Y)}$ is sufficiently small, then so is T_{A+B}. Therefore:

(i) $T_{A+B}(X_0) = (A+B)X_0 \subset Y$ is closed and has codimension $\dim Y_1 = \dim \operatorname{coker} A < \infty \Rightarrow \operatorname{ran}(A+B) \supseteq T_{A+B}(X_0)$ is closed

(ii) $A+B$ is injective on $X_0 \Rightarrow X = X_0 \oplus \underbrace{\ker(A+B) \oplus X_2}_{*}$ for some finite-dimensional space $X_2 \subset X$. But $A+B$ is injective on $X_0 \oplus X_2$, with

$$(A+B)(X) = (A+B)(X_0 \oplus X_2) = T_{A+B}(X_0) \oplus (A+B)(X_2) \subset Y$$

closed (since $T_{A+B}(X_0)$ is closed and of finite codimension).

$$\Rightarrow \dim \operatorname{coker}(A+B) = \operatorname{codim} T_{A+B}(X_0) - \dim X_2 .$$
$$= \dim \operatorname{coker} A - \dim X_2 .$$

(iii) $\dim \ker(A+B) = \overbrace{\dim \ker A}^{*} - \dim X_2$

Taken together, the last two inequalities give the result. $\square$

We can now give the proof of an outstanding claim from the previous part:

Corollary (Fredholm alternative.) Let $K \in L(X)$ be compact. Then either $I-K$ is invertible, or $\exists\, x \in X : (I-K)x = 0$.

More generally, $\dim \ker(I-K) = \dim \operatorname{coker}(I-K)$. #

Proof $A(s) := I - sK$ is a continuous path in $L(X)$ consisting entirely of Fredholm operators

$\Rightarrow \operatorname{ind}(I-K) = \operatorname{ind} A(1) = \operatorname{ind} A(0) = \operatorname{ind}(I) = 0$. ($\Rightarrow$ #)

So: $I-K$ is injective ($\dim\ker(I-K) = 0$)

iff it is surjective ($\dim\operatorname{coker}(I-K) = 0$). $\square$

The same proof implies, more generally:

Corollary $A \in L(X,Y)$ Fredholm, $K \in L(X,Y)$ compact

$\Rightarrow \operatorname{ind}(A+K) = \operatorname{ind}(A)$.

The index is multiplicative:

Proposition $A \in L(X,Y)$, $B \in L(Y,Z)$ Fredholm $\Rightarrow B \circ A$ Fredholm, and $\operatorname{ind}(B\circ A) = \operatorname{ind}(B)\cdot\operatorname{ind}(A)$.

Proof The proof that $B\circ A$ is Fredholm is left as an exercise.

Define for $t \in \mathbb{R}$

$$C(t) := \underbrace{\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}}_{\substack{Y \\ \oplus \\ Z} \leftarrow \substack{Y \\ \oplus \\ Y}} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \underbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}_{\substack{Y \\ \oplus \\ Y} \leftarrow \substack{X \\ \oplus \\ Y}} : \begin{matrix} X \\ \oplus \\ Y \end{matrix} \to \begin{matrix} Y \\ \oplus \\ Z \end{matrix}.$$

$$\begin{matrix} Y \\ \oplus \\ Z \end{matrix} \longleftarrow \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \longleftarrow \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

Then $C(t)$ is Fredholm $\forall t$; and

$C(0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ has $\operatorname{ind}(C(0)) = \operatorname{ind}(A) + \operatorname{ind}(B)$,

$$C(\tfrac{\pi}{2}) = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -I \\ B\circ A & 0 \end{pmatrix}$$

has $\operatorname{ind}(C(\pi/2)) = \operatorname{ind}(B\circ A)$.

Since $\operatorname{ind}(C(t))$ is constant, we are done. $\square$

Example Let $a, b \in C^0([0,1])$. We study the boundary value problem

$$\begin{cases} u''(x) + a(x)u'(x) + b(x)u(x) = f(x), & x \in [0,1] \\ u(0) = u(1) = 0. \end{cases} \quad (*)$$

Here, $f \in C^0([0,1])$ is given, and we seek a solution $u \in C^2([0,1])$.

Claim (i) The number of linearly independent conditions on f for solvability is equal to the number of linearly independent solutions of the homogeneous equation (i.e. $f=0$). This number is at most 1.

(ii) If a is real-valued and $b<0$, then $(*)$ has a unique solution $u \in C^2([0,1])$ for all $f \in C^0([0,1])$.

Proof. Let $X = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$, with norm

$$\|u\|_X = \|u\|_{C^2([0,1])} := \|u\|_{C^0} + \|u'\|_{C^0} + \|u''\|_{C^0}.$$

Let $Y = C^0([0,1])$, with $\|f\|_Y = \|f\|_{C^0}$.

(a) The operator $A : X \to Y$,
$u \mapsto u''$,

is an isomorphism. (In particular, it is Fredholm with index 0.)

Indeed: $Au = 0 \Rightarrow u'' = 0$, so u is linear; since $u(0) = u(1) = 0$, we get $u = 0$.

· Given $f \in Y$, set $u_c(x) := cx + \int_0^x \int_0^s f(r)\,dr\,ds$, $c \in \mathbb{C}$;

then $u_c(0)=0$, $u_c''=f$, and $u_c(1)=c+\int_0^1\int_0^s f(r)\,dr\,ds=0$

for the (unique) choice $c=-\int_0^1\int_0^s f(r)\,dr\,ds$.

(b) The operator $K: X \to Y$,

$$u \mapsto au'+bu,$$

is compact. Indeed, K is the composition of the compact map $X \to C^1([0,1])$ (exercise) and the bounded operator

$$C^1([0,1]) \to C^0([0,1]), \quad u \mapsto au'+bu.$$

(i) $A+K: X \to Y$ is Fredholm of index 0.

A solution $u \in C^2([0,1])$ of $u''+au'+b=0$ with $u(0)=0$ is uniquely determined by the value of $u'(0)\in\mathbb{C}$. Depending on whether for some choice of $u'(0)\in\mathbb{C}$ we have $u(1)=0$,

$$\dim\ker(A+K) = 0 \text{ or } 1.$$

(ii) If $u\in\ker(A+K)$, and $x_0\in[0,1]$ is a maximum, then $u'(x_0)=0$, $u''(x_0)\le 0$, and

$$0=\underbrace{u''(x_0)}_{\le 0}+\underbrace{a(x_0)u'(x_0)}_{=0}+\underbrace{b(x_0)}_{<0}u(x_0).$$

Thus, we cannot have $u(x_0)>0 \Rightarrow u(x)\le 0\ \forall x$.

Similarly arguing at a minimum of u gives $u(x)\ge 0\ \forall x$

$\Rightarrow u=0. \Rightarrow \ker(A+K)=\{0\} \Rightarrow A+K: X\to Y$ is surjective. $\square$