Definition X, Y normed vector spaces. An operator $A \in L(X,Y)$ is called **compact** if $A(B_1(0)) \subset Y$ is relatively compact (i.e. $\overline{A(B_1(0))}$ is compact).

Example 1 $A: X \to Y$ finite rank, i.e. $\dim \operatorname{ran} A < \infty$.
$\Rightarrow A$ is compact. (Exercise.)

Example 2 $I: X \to X$ is compact $\Leftrightarrow \dim X < \infty$.

Proof "$\Leftarrow$" $\dim X < \infty \Rightarrow I$ is a finite rank operator, hence compact.

"$\Rightarrow$" If $\dim X = \infty$, let $\{x_k\}_{k\in\mathbb{N}} \subset X$, $\|x_k\| = 1$, $\|x_k - x_\ell\| > \frac{1}{2}$ $\forall k \neq \ell$. This has no convergent subsequence $\Rightarrow I$ is not compact. $\square$

Example 3 X normed, Y Banach. Let $A \in L(X,Y)$ be the limit $A = \lim_{k\to\infty} A_k$ of compact operators A_k. Then A is compact.

Proof Let $\{x_j\}_{j\in\mathbb{N}} \subset X$, $\|x_j\| \leq 1$ $\forall j$. Successively pick subsequences $\mathbb{N} \supset J_1 \supset J_2 \supset \cdots$ s.t. $\exists \lim_{\substack{j\to\infty \\ j\in J_k}} A_k x_j$.

Let $J \subset \mathbb{N}$ be the diagonal subsequence, then $\forall k$ $\exists \lim_{\substack{j\to\infty \\ j\in J}} A_k x_j$. We claim that $\{Ax_j\}_{j\in J}$ converges.

Indeed, $\|Ax_j - Ax_\ell\| \leq \|Ax_j - A_k x_j\| + \|A_k(x_j - x_\ell)\| + \|A_k x_\ell - Ax_\ell\|$.

Thus, if $\varepsilon>0$, choose k large s.t. $\|A-A_k\|_{L(X,Y)}<\frac{\varepsilon}{3}$, and then $j_0\in J$ s.t. $\forall j,\ell\in J$, $j,\ell\geq j_0$, $\|A_kx_j-A_kx_\ell\|<\frac{\varepsilon}{3}$. Then $\|Ax_j-Ax_\ell\|<\varepsilon$ for such j,ℓ. Since Y is complete, $\{Ax_j\}_{j\in J}$ converges. $\square$

In particular, the limit of a convergent sequence of finite rank operators is compact.

Example 4 $A:\ell^2\to\ell^2$, $Ae_k=q_ke_k$, where $q=(q_k)\in c_0$.
A is compact since it is the limit of the sequence of finite rank operators $A_n:\ell^2\to\ell^2$, $A_ne_k=\begin{cases} q_ke_k, & k\leq n \\ 0, & k>n. \end{cases}$

Lemma X normed vector space, Y Hilbert space, $A\in L(X,Y)$ compact. Then one can write $A=\lim_{k\to\infty}A_k$, where A_k, $k\in\mathbb{N}$, is a finite rank operator.

Remark Not true for general Banach spaces (very hard! $\rightsquigarrow$ Per Enflo).

Proof Exercise. $\square$

Example 5 Let $A:H\to H$ be a Hilbert–Schmidt operator. (I.e. $\sum\|Ae_i\|^2<\infty$, where $\{e_i\}$ = complete ONB of H.) Then A is compact. (Exercise.)

Example 6 For $s \in \mathbb{R}$, define

$$h^s(\mathbb{Z}) := \left\{ a = (a_n)_{n\in\mathbb{Z}} : \|a\|_{h^s} := \left(\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |a_n|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Thus, $h^0(\mathbb{Z}) = \ell^2(\mathbb{Z})$; and $h^s(\mathbb{Z}) \subset h^t(\mathbb{Z})$ for $s > t$.

Claim Let $s > t$. Then the inclusion map $i : h^s(\mathbb{Z}) \longrightarrow h^t(\mathbb{Z})$ is compact.

Proof Let $a_k = (a_{k,n})_{n\in\mathbb{Z}}, k \in \mathbb{N}$, with $\|a_k\|_{h^s} \le 1$. Then $\forall n \in \mathbb{Z}$, $\{a_{k,n}\}_{k\in\mathbb{N}}$ is bounded, and therefore has a convergent subsequence. By a diagonal subsequence argument, we may assume that $\exists \lim_{k\to\infty} a_{k,n} =: b_n \ \forall n \in \mathbb{Z}$.

- Claim: $b = (b_n)_{n\in\mathbb{Z}} \in h^s(\mathbb{Z})$.

Indeed, $\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |b_n|^2 \le \liminf_{k\to\infty} \sum (1+|n|)^{2s} |a_{k,n}|^2 \le 1$.

- Claim: $a_k \longrightarrow b$ in $h^t(\mathbb{Z})$.

Indeed, let $\varepsilon > 0$. Pick $n_0 \in \mathbb{N}$ s.t. $(1+|n|)^{2t} < \varepsilon (1+|n|)^{2s} \ \forall |n| > n_0$.

Pick k_0 s.t. $\sum_{n=-n_0}^{n_0} (1+|n|)^{2t} |a_{k,n} - b_n|^2 < \varepsilon \quad \forall k > k_0$.

$$\Rightarrow \|a_k - b\|_{h^t}^2 = \sum_{|n|>n_0} (1+|n|)^{2t} |a_{k,n} - b_n|^2 + \varepsilon$$
$$\le \varepsilon \sum_{|n|>n_0} (1+|n|)^{2s} |a_{k,n} - b_n|^2 + \varepsilon$$
$$< \varepsilon \left(\|a_k\|_{h^s} + \|b\|_{h^s} \right)^2 + \varepsilon$$
$$\le 5\varepsilon. \qquad \square$$

Remark Slick proof: $x_n^s := (1+|n|)^{-s} e_n$ is a complete ONB of $h^s(\mathbb{Z})$. $i(x_n^s) = x_n^s = q_n x_n^t$, $q_n = (1+|n|)^{t-s}$. Since $|q_n| \xrightarrow{n\to\infty} 0$, i is compact. (Compare with Example 4.)

Definition For $s \geq 0$, $H^s(S^1) := \mathcal{F}^{-1}(h^s(\mathbb{Z}))$ is the *Sobolev space* of order s on S^1 (= unit circle).

Thus, $H^s(S^1) \hookrightarrow H^t(S^1)$ is compact for $s > t \geq 0$.

Example 7 / Lemma $A \in L(X,Y)$ compact, $B \in L(Z,X)$, $C \in L(Y,Z)$ bounded $\Rightarrow$ $A \circ B \in L(Z,Y)$ and $C \circ A \in L(X,Z)$ are compact.

Now, what are compact operators good for?

Proposition Let X be a Banach space, and let $K \in L(X)$ be compact. Then $A := I - K \in L(X)$ has

(i) $\dim \ker A < \infty$

(ii) $\operatorname{ran} A$ is closed,

(iii) $\dim \operatorname{coker} A = \dim\left(Y/\operatorname{ran} A\right) < \infty$.

Later: (iv) $\dim \operatorname{coker} A = \dim \ker A$.

Lemma Let $A \in L(X,Y)$ be compact. If $\{x_k\}_{k \in \mathbb{N}} \subset X$, $x_k \xrightarrow{w} x_0 \in X$, then $Ax_k \longrightarrow Ax_0$.

Proof We have $Ax_k \xrightarrow[k\to\infty]{w} Ax_0$. Now, $\{x_k\}$ is bounded.

Given any subsequence $\{x_{k_i}\}_{i \in \mathbb{N}}$, $\{Ax_{k_i}\}_{i \in \mathbb{N}}$ has a convergent subsequence $Ax_{k_{i_j}} \xrightarrow{j\to\infty} y$; necessarily $y = Ax_0$.

Therefore, $Ax_k \longrightarrow Ax_0$. $\square$

Proof of the Proposition

(i) $Y := \ker A$ is a Banach space, and $I = K$ on Y, so $I: Y \to Y$ is compact $\Rightarrow \dim Y < \infty$.

(ii) · Let $V \subset X$ be a topological complement to $\ker A$. We claim that $\exists\, C>0$ s.t. $\forall\, x \in V$: $\|x\| \leq C\|Ax\|$ ⊛

If this were false, $\exists \{x_k\}_{k\in\mathbb{N}} \subset V$: $\|x_k\|=1$, $\|Ax_k\| \leq \frac{1}{k}$.

Passing to a subsequence, we may assume $Kx_k \longrightarrow y$;

since $x_k - Kx_k \longrightarrow 0$, this implies $x_k \longrightarrow y$; thus $\|y\|=1$.

Since V is closed, $y \in V$. But $Ay = \lim\limits_{k\to\infty} Ax_k = 0$, so

$y \in \ker A \Rightarrow y \in V \cap \ker A = \{0\}$, ↯ to $\|y\|=1$.

· ⊛ $\Rightarrow$ ran A is closed.

(iii) We use a result from the future, which tells us that also K^* is compact $\Rightarrow (\operatorname{ran} A)^\perp = \ker A^* = \ker(I-K^*)$ is finite-dimensional. $\square$

Example · On the Sobolev space $H^2(\mathbb{S}^1) = \mathcal{F}^{-1} h^2(\mathbb{Z})$, we define $D_\theta^2 u$ via $\mathcal{F}(D_\theta^2 u)(n) = n^2 (\mathcal{F}u)(n)$.

(Recall $\mathcal{F}(\frac{d}{d\theta}u)(n) = in(\mathcal{F}u)(n)$ for $u \in C^\infty_{per}$, so
$D_\theta^2 = -\frac{d^2}{d\theta^2}$ on C^∞_{per}.)

· Moreover, for $V \in L^\infty(\mathbb{S}^1)$ $(= L^\infty([0,2\pi]))$, we define
$Vu \in L^2(\mathbb{S}^1)$ for $u \in H^2(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$
as the pointwise product.

· Consider then
$$A := D_\theta^2 + V : H^2(\mathbb{S}^1) \longrightarrow L^2(\mathbb{S}^1),\quad u \longmapsto D_\theta^2 u(\theta) + V(\theta)u(\theta).$$

A good approximate inverse to A is, perhaps, the isomorphism

$$B : L^2 \xrightarrow{\cong} H^2, \quad \mathcal{F}(Bf)(n) = \begin{cases} \frac{1}{n^2}(\mathcal{F}f)(n), & n \neq 0 \\ (\mathcal{F}f)(0), & n = 0. \end{cases}$$

Indeed, $A \circ B = D_\theta^2 \circ B + V \circ B$, with

$$\mathcal{F}(D_\theta^2 Bu)(n) = \begin{cases} (\mathcal{F}u)(n), & n \neq 0 \\ 0, & n = 0. \end{cases}$$

$\Rightarrow$ $A \circ B = I - K$, where $Ku = (\mathcal{F}u)(0) - V \circ B : L^2 \to L^2$.

But $L^2 \ni u \mapsto (\mathcal{F}u)(0) \in L^2$ (constant function!) has finite rank, and $B : L^2 \to H^2 \hookrightarrow L^2$ is compact, thus so is $V \circ B$.

$\Rightarrow$ K is compact.

· Theorem implies: (i) $\dim \overbrace{\ker A}^{\subset H^2(S^1)} < \infty$,

(ii) $\operatorname{ran} A \subset L^2(S^1)$ is closed,

(iii) $\dim \operatorname{coker} A < \infty$.

(& (iv) A is surjective $\Leftrightarrow$ A has trivial kernel.)

This example is a first demonstration of the usefulness of compact maps which are given by inclusions of function spaces (here $H^2(S^1) \hookrightarrow L^2(S^1)$). Two more important such results are:

Theorem (Arzelà–Ascoli.) Let $K \subset \mathbb{R}^n$ be compact. Let $\mathcal{F} \subset C^0(K)$. The following are equivalent:

(i) $\overline{\mathcal{F}}$ is sequentially compact.

(ii) $\mathcal{F}$ is bounded and equicontinuous: $\sup_{f \in \mathcal{F}} \|f\|_{C^0(K)} < \infty$, and $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\forall x, y \in K$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Proof

(i) $\Rightarrow$ (ii) · $\mathcal{F}$ compact $\Rightarrow$ $\mathcal{F}$ bounded.

· Let $\varepsilon>0$, and choose $f_1, \dots, f_N \in \mathcal{F}$ s.t. $\mathcal{F} \subset \bigcup_{j=1}^{N} B_\varepsilon(f_j)$.

Since $f_j \in C^0(K)$ $\forall j$, $\exists\ \delta>0$ s.t.

$$x,y \in K,\ |x-y|<\delta \Rightarrow |f_j(x)-f_j(y)|<\varepsilon\ \forall j=1,\dots,N$$

Let now $f \in \mathcal{F}$, and pick j s.t. $\|f-f_j\|_{C^0}<\varepsilon$.

For $x,y \in K$, $|x-y|<\delta$, we then estimate

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_j(x)| + |f_j(x)-f_j(y)| + |f_j(y)-f(y)| \\ &< \quad \varepsilon \quad\quad + \quad \varepsilon \quad\quad + \quad \varepsilon \\ &= 3\varepsilon.\end{aligned}$$

$\Rightarrow$ $\mathcal{F}$ is equicontinuous.

(ii) $\Rightarrow$ (i) Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ be a sequence. Let $\{x_j\}_{j\in\mathbb{N}} \subset K$ be dense. A diagonal argument (using only the boundedness of $\mathcal{F} \subset C^0(K)$) allows us to pass to a subsequence, which we denote $\{f_k\}$ again, so that $\exists\, f(x_j) := \lim_{k\to\infty} f_k(x_j)$.

Claim. f has a (unique) continuous extension to an element of $C^0(K)$.

Indeed, given $\varepsilon>0$, and choosing $\delta>0$ from the equicontinuity condition on $\mathcal{F}$, we have, for $|x_j - x_\ell|<\delta$:

$$|f(x_j)-f(x_\ell)| \le \liminf_{k\to\infty} |f_k(x_j)-f_k(x_\ell)| \le \varepsilon.$$

Claim $f_k \to f$ in $C^0(K)$.

Indeed, given $\varepsilon>0$, choose $\delta>0$ as above. For some $J\in\mathbb{N}$, we have $K\subset\bigcup_{j=1}^{J}B_\delta(x_j)$. Let $k_0\in\mathbb{N}$ be s.t.

$$|f(x_j)-f_k(x_j)|<\varepsilon \quad \forall\, j=1,\dots,J,\quad k\geq k_0.$$

Given any $x\in K$, pick $j\in\{1,\dots,J\}$ s.t. $|x-x_j|<\delta$, we then estimate (for $k\geq k_0$)

$$\begin{aligned}|f(x)-f_k(x)| &\leq |f(x)-f(x_j)|+|f(x_j)-f_k(x_j)|+|f_k(x_j)-f_k(x)|\\ &< \varepsilon + \varepsilon + \varepsilon\\ &= 3\varepsilon,\end{aligned}$$

thus, $\|f-f_k\|_{C^0(K)}<3\varepsilon$. $\square$

Theorem (Fréchet–Kolmogorov) Let $K\subset\mathbb{R}^n$ be compact, $1\leq p<\infty$. Let $\mathcal{F}\subset L^p(K)$. The following are equivalent:

(i) $\overline{\mathcal{F}}$ is sequentially compact.

(ii) $\mathcal{F}$ is bounded and "equicontinuous in L^p", i.e.

$$\sup_{f\in\mathcal{F}}\|f\|_{L^p(K)}<\infty,\qquad \sup_{f\in\mathcal{F}}\|f-\tau_h f\|_{L^p}\xrightarrow[h\in\mathbb{R}^n]{h\to 0}0,$$

where $(\tau_h f)(x)=\begin{cases} f(x+h): & x+h\in K\\ 0 & \text{otherwise.}\end{cases}$

Proof Omitted. (See Struwe's lecture notes.) $\square$

Example 1 Ω = bounded domain with C^1 boundary $\Rightarrow$ $C^1(\bar{\Omega}) \to C^0(\bar{\Omega})$ is compact.

Example 2 Volterra operators, again. Let $K \subset \mathbb{R}^n$ be compact, $k \in C^0(K \times K)$. Define $A: C^0(K) \to C^0(K)$ by

$$(Au)(x) = \int_K k(x,y)\, u(y)\, dy.$$

Claim: A is compact.

Indeed, let B = closed unit ball in $C^0(K)$. We claim that $A(B) \subset C^0(K)$ is bounded and equicontinuous. (Then Arzelà–Ascoli implies the claim.)

- Since $A \in L(C^0(K))$, $A(B)$ is certainly bounded.
- Let $\varepsilon > 0$. Since $K \times K$ is compact, $\exists\, \delta > 0$ s.t. $(x,y), (x',y') \in K \times K$, $|x-x'| + |y-y'| < \delta \Rightarrow |k(x,y) - k(x',y')| < \varepsilon$.

For $u \in B$, this implies for $x, x' \in K$, $|x-x'| < \delta$:

$$|Au(x) - Au(x')| \le \int_K |k(x,y) - k(x',y)|\,|u(y)|\, dy$$

$$\le \varepsilon\, \mathrm{vol}(K)\, \|u\|_{C^0(K)}. \qquad \square$$

Therefore, $I - A : C^0(K) \to C^0(K)$ has properties (i)–(iv) above.

Finally, to tie up a loose end:

Theorem (Schauder) Let X, Y be Banach spaces, $A \in L(X,Y)$. Then A is compact $\Leftrightarrow$ $A^* \in L(Y^*, X^*)$ is compact.

Proof ($\Rightarrow$) Let $\{\lambda_k\}_{k\in\mathbb{N}} \subset Y^*$, $\|\lambda_k\|_{Y^*} \le 1$; we need to find a convergent subsequence of $\{A^*\lambda_k\}_{k\in\mathbb{N}}$.

Now $K := A(B_X) \subset Y$ is compact ($B_X \subset X$: unit ball).

Consider $\mathcal{F} = \{f_k : K \ni y \mapsto \lambda_k(y)\} \subset C^0(K)$.

- Since $\|f_k\|_{C^0(K)} = \sup\limits_{x\in B_X} |\lambda_k(Ax)| \le \|A\|_{L(X,Y)} \|\lambda_k\|_{Y^*} = \|A\|_{L(X,Y)}$,

  $\mathcal{F}$ is bounded.

- Since $y, y' \in K$, $\|y - y'\|_Y < \delta$ implies

  $$|f_k(y) - f_k(y')| = |\lambda_k(y) - \lambda_k(y')| \le \|\lambda_k\|_{Y^*} \|y - y'\| \le \delta,$$

  $\mathcal{F}$ is equicontinuous.

Arzelà–Ascoli $\Rightarrow$ $\overline{\mathcal{F}}$ is sequentially compact; so $f_{k_i} \xrightarrow{i\to\infty} f$ in $C^0(K)$.

$$\Rightarrow \sup_{x\in B_X} |(A^*\lambda_{k_i})(x) - f(Ax)|$$
$$= \sup_{y\in K} |f_{k_i}(y) - f(y)|$$
$$= \|f_{k_i} - f\|_{C^0(K)} \xrightarrow{i\to\infty} 0$$

$$\Rightarrow \|A^*\lambda_{k_i} - A^*\lambda_{k_j}\|_{X^*} \le \sup_{x\in B_X} |(A^*\lambda_{k_i})(x) - f(Ax)| + \sup_{x\in B_X} |f(Ax) - (A^*\lambda_{x_j})(x)| \xrightarrow{i,j\to\infty} 0.$$

($\Leftarrow$) A^* compact $\Rightarrow$ A^{**} compact; but $A^{**} \circ \iota_X = \iota_Y \circ A$ on X.

$$\Rightarrow A^{**}(\iota_X(B_X)) \subset Y^{**} \text{ has compact closure.}$$

$$\begin{array}{c} A^{**}(\iota_X(B_X)) \\ \| \\ \iota_Y(A(B_X)) \subset \iota_Y(Y). \end{array}$$

Since $\iota_Y(Y) \subset Y^{**}$ is closed, $\overline{A(B_X)} \subset Y$ is compact. $\square$