After developing a fairly large amount of abstract material, we discuss a few applications/examples (while introducing a moderate amount of additional material).

Application 1: Volterra integral equations

operator norm
Neumann series

Let $k \in C^0([0,1]\times[0,1])$, and consider the integral operator

$$A: C^0([0,1]) \longrightarrow C^0([0,1]),$$
$$(Au)(t) = \int_0^t k(t,s)\,u(s)\,ds.$$

Theorem For all $f \in C^0([0,1])$, $\exists!\, u \in C^0([0,1])$ s.t. $f = u - Au$,

i.e. $f(t) = u(t) - \int_0^t k(t,s)\,u(s)\,ds \quad \forall t \in [0,1]$.

We shall give 2 proofs of this Theorem. First, we note:

Claim $A \in L(C^0([0,1]))$, where we use the sup norm $\|\cdot\|_{C^0}$.

Proof · First, we check that $Au \in C^0([0,1])$ for $u \in C^0([0,1])$: for $t \geq t'$,

$$|Au(t) - Au(t')| = \left| \int_0^{t'} (k(t,s) - k(t',s))\,u(s)\,ds + \int_{t'}^{t} k(t,s)\,u(s)\,ds \right|$$
$$\leq \left(\int_0^{t'} |k(t,s) - k(t',s)|\,ds + |t-t'|\,\|k\|_{C^0} \right) \|u\|_{C^0}.$$

- But given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.
$$|t-t'| < \delta \Rightarrow |k(t,s) - k(t',s)| < \varepsilon \quad \forall s \in [0,1].$$
$$\Rightarrow |Au(t) - Au(t')| \leq (\varepsilon t' + \delta \|k\|_{C^0})\,\|u\|_{C^0}, \quad 0 \leq t - t' < \delta.$$

- For any given $\tilde{\varepsilon} > 0$, take $\varepsilon < \frac{\tilde{\varepsilon}}{2}$ and δ as above s.t. also $\delta \|k\|_{C^0} < \frac{\tilde{\varepsilon}}{2}$; then
$$|Au(t) - Au(t')| < \tilde{\varepsilon}\,\|u\|_{C^0} \quad \text{for } t' \leq t \leq t' + \delta.$$

- Similar estimate for $t' - \delta \leq t \leq t'$.

$\Rightarrow Au \in C^0([0,1])$.

- Next, we check that A is bounded. But

$$\|Au\|_{C^0} \le \sup_{t\in[0,1]} \int_0^t |k(t,s)|\,|u(s)|\,ds$$

$$\le \underbrace{\left(\sup_{t\in[0,1]} \int_0^t |k(t,s)|\,ds\right)}_{=:C<\infty}\|u\|_{C^0} = C\|u\|_{C^0}. \qquad \square$$

<u>Proof 1</u> of the Theorem.

- <u>Claim</u> Let $\lambda>0$. Then $\|u\|_\lambda := \sup_{t\in[0,1]} |e^{-\lambda t}u(t)|$ defines a norm on $C^0([0,1])$ which is equivalent to $\|\cdot\|_{C^0}$.

<u>Indeed</u>, $\|u\|_\lambda \le \|u\|_{C^0} \le \sup_{t\in[0,1]} |\underbrace{e^{\lambda} e^{-\lambda t}}_{\ge 1} u(t)| = e^{\lambda}\|u\|_\lambda$.

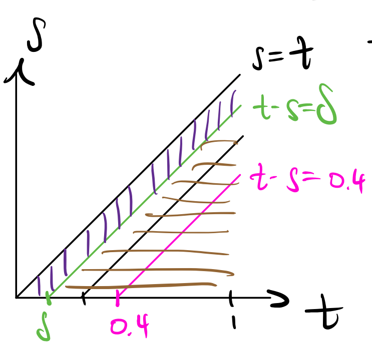
- In particular, $(C^0([0,1]), \|\cdot\|_\lambda)$ is complete.

- <u>Claim</u> $\exists \lambda>0$ s.t. $\|A\|_{L(C^0([0,1]),\|\cdot\|_\lambda))} \le \frac{1}{2}$.

<u>Indeed</u> $\|A\|_{L((C^0,\|\cdot\|_\lambda))} = \sup_{\|u\|_\lambda=1} \sup_{t\in[0,1]} \left| e^{-\lambda t}\int_0^t k(t,s)\,u(s)\,ds\right|$

$\left(\begin{array}{c}\tilde u(t)=e^{-\lambda t}u(t)\\ \Downarrow \\ \|\tilde u\|_{C^0}=\|u\|_\lambda\end{array}\right)$ $\overset{\Downarrow}{=} \sup_{\|\tilde u\|_{C^0}=1} \sup_{t\in[0,1]} \left| e^{-\lambda t}\int_0^t k(t,s)\,e^{\lambda s}\tilde u(s)\,ds\right|$

$$\le \sup_{t\in[0,1]} \underbrace{\left(\int_0^t e^{-\lambda(t-s)}|k(t,s)|\,ds\right)}_{=:C(\lambda,t)}.$$



Let $\delta > 0$ be such that $|k(t,s)| < \frac{1}{4\delta}$ for $\max(0, t-\delta) \leq s \leq t$.

Then let $\lambda \geq 0$ be s.t. $|e^{-\lambda(t-s)} k(t,s)| < \frac{1}{4}$ for $0 \leq s \leq \max(t-\delta, 0)$

(Note: $e^{-\lambda(t-s)} \leq e^{-\delta\lambda} \searrow 0$ as $\lambda \to \infty$!)

Then
$$C(\lambda, t) = \int_0^{\max(0,t-\delta)} e^{-\lambda(t-s)} |k(t,s)| \, ds$$
$$+ \int_{\max(0,t-\delta)}^{t} e^{-\lambda(t-s)} |k(t,s)| \, ds$$
$$\leq \frac{1}{4} + \delta \cdot \frac{1}{4\delta} = \frac{1}{2} \qquad \forall t \in [0,1].$$

$$\Rightarrow \|A\|_{L((C^0, \|\cdot\|_\lambda))} = \sup_{t \in [0,1]} C(\lambda, t) \leq \frac{1}{2}.$$

Conclusion We can invert $I - A$ using the Neumann series
$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$, with convergence in $L((C^0, \|\cdot\|_\lambda))$. $\square$

Proof 2 of the Theorem. Let us be naive and try to define $(I-A)^{-1}$ as $\sum_{n=0}^{\infty} A^n$ — rather surprisingly, it turns out that this series converges in $L((C^0([0,1]), \|\cdot\|_{C^0}))$! So the yoga in the first proof is actually not necessary (however neat the argument is).

Key idea: $\sum_{n=0}^{\infty} A^n$ converges if $\underbrace{\limsup_{n \to \infty} \|A^n\|_{L(C^0)}^{1/n}}_{\text{"spectral radius of } A\text{"}} < 1$. (Root test.)

Here: $$(A^n u)(t_0) = \int_0^{t_0} k(t_0, t_1)(A^{n-1}u)(t_1) \, dt_1$$
$$= \int_0^{t_0} k(t_0, t_1) \int_0^{t_1} k(t_1, t_2)(A^{n-2}u)(t_2) \, dt_2 \, dt_1$$

$$= \cdots$$

$$= \iint\cdots\int_{0\le t_n\le t_{n-1}\le\cdots\le t_1\le t_0\le 1} k(t_0,t_1)k(t_1,t_2)\cdots k(t_{n-1},t_n)\,u(t_n)\,dt_n\cdots dt_0 .$$

Let $C := \|k\|_{C^0}$, then for $\|u\|_{C^0}=1$, we have

$$\sup_{t\in[0,1]} |A^n u(t)| \le \text{volume}\big(\{0\le t_n\le t_{n-1}\le\cdots\le t_1\le t_0\le 1\}\big)\, C^n .$$

$$\overset{(\text{Exercise})}{=} \frac{1}{(n+1)!}\, C^n ,$$

$$\Longrightarrow \quad \|A^n\|^{1/n}_{L(C^0)} = \frac{C}{\sqrt[n]{(n+1)!}} \xrightarrow{n\to\infty} 0. \qquad \square$$