We recall that a normed vector space X is called reflexive if $\iota : X \longrightarrow X^{**}$, $\iota(x) : \lambda \in X^* \longmapsto \lambda(x)$, is an isomorphism.
(ι is always isometric.)

Remark Since dual spaces are always complete, every reflexive space is necessarily a Banach space.

Examples Hilbert spaces (already shown), L^p $(1<p<\infty)$ (later), ℓ^p $(1<p<\infty)$ (exercise).

Theorem Let X be a normed vector space.

(i) If X is reflexive, then so is X^*.

(ii) If X^* is reflexive, then so is $\overline{X}$ (completion of X), and thus X if X is complete.

Proof (i) Exercise.

(ii) • We claim that $X^* = \overline{X}^*$. Indeed, the map $\overline{X}^* \to X^*$, $\lambda \longmapsto \lambda|_X$ is an isometry, and it is surjective since every $\lambda \in X^*$ can be uniquely extended to an element of $\overline{X}^*$ (by continuity and density). So we may replace X by $\overline{X}$ and thus assume that X is complete.

• Suppose now $\iota(X) \subsetneq X^{**}$. Since $\iota(X) \subset X^{**}$ is closed, $\exists$ $\Phi \in (X^{**})^*$ s.t. $\Phi|_{\iota(X)} = 0$ but $\Phi \neq 0$. Since X^* is reflexive, $\exists\, \lambda \in X^*$ s.t. $\Phi = \iota^*(\lambda)$, where $\iota^* : X^* \to X^{***}$ is the canonical inclusion. Therefore, $\forall\, x \in X$,

$$0 = \imath^*(\lambda)(\imath(x)) = \imath(x)(\lambda) = \lambda(x).$$

But this implies $\lambda = 0 \Rightarrow \Phi = 0$, a contradiction. So $\imath(X) = X^{**}$. $\square$

Reflexivity is inherited by closed subspaces:

Theorem Let X be reflexive, and let $Y \subset X$ be a closed subspace. Then Y is reflexive.

Proof • Let $j : Y \to X$ be the inclusion map ($j(y) = y \ \forall y \in Y$), and let $j^* : X^* \to Y^*$ be its dual, defined by

$$j^*(x^*)(y) = x^*(j(y)) = x^*(y) \quad \text{for } x^* \in X^*,\ y \in Y.$$

(Thus $j^*(\lambda) = \lambda|_Y$.) Let $j^{**} : Y^{**} \to X^{**}$ be the dual of j^*,

$$j^{**}(y^{**})(x^*) = y^{**}(j^* x^*) = y^{**}(x^*|_Y) \quad \text{for } y^{**} \in Y^{**}, x^* \in X^*.$$

• Write $\imath : X \to X^{**}$, $\imath^Y : Y \to Y^{**}$ for the canonical inclusions; $\imath$ is surjective by assumption. We need to show that $\imath^Y$ is surjective as well.

• Let $y^{**} \in Y^{**}$. Then $j^{**}(y^{**}) =: x^{**} \in X^{**}$, and therefore $\exists x \in X$ s.t. $x^{**} = \imath(x)$; that is, $\forall x^* \in X^*$,

$$\circledast \quad j^{**}(y^{**})(x^*) = y^{**}(x^*|_Y) = x^{**}(x^*) = \imath(x)(x^*) = x^*(x).$$

We claim that $x \in Y$. If this were false, $\exists x^* \in X^*$ with $x^*(x) = 1$ but $x^*|_Y = 0$. $\circledast$ then gives $0 = 1$, ↯.

• Knowing now that $j^{**}(y^{**}) = \imath(y)$ for $y \in Y$ (formerly called x), we proceed to show that $\iota^Y(y) = y^{**}$. (Thus also $j^{**}\iota^Y = \iota|_Y$.)

To this end, let $y^* \in Y^*$, and let $x^* \in X^*$ denote an extension of y^* (using Hahn–Banach). Then

$$\iota^Y(y)(y^*) = y^*(y) = x^*(y) = \iota(y)(x^*) = j^{**}(y^{**})(x^*)$$
$$= y^{**}(x^*|_Y) = y^{**}(y^*),$$

as desired. $\square$

We shall later prove results giving other useful conditions guaranteeing the reflexivity of a space ($\to$ uniform convexity).

Theorem Let X be a normed vector space.

(i) If X^* is separable, then X is separable.

(ii) If X is separable and reflexive, then X^* is separable.

Proof (i) Exercise.

(ii) X reflexive $\Rightarrow$ $X^{**} \cong X$, so $X^{**} = (X^*)^*$ is separable since X is $\overset{(i)}{\Rightarrow}$ X^* is separable. $\square$

Application to "calculus of variations"

We return to the "direct method". First, we need a technical result:

Theorem (Eberlein–Šmulian). Let X be reflexive, and let $(x_k)_{k\in\mathbb{N}}$ be a bounded sequence. Then $\exists\, x \in X$ and a subsequence $(x_{k_i})_{i\in\mathbb{N}}$ s.t. $x_{k_i} \xrightarrow[i\to\infty]{w} x$.

Proof. Let $Y = \overline{\operatorname{span}\{x_k : k\in\mathbb{N}\}} \subset X$; this is separable and reflexive. Since the closed unit ball in Y is weakly sequentially compact by Banach–Alaoglu (separable case) (applied to the separable space Y^*, with the unit ball in $(Y^*)^*$ being weak-$*$-sequentially compact) there exist $y\in Y$ and a subsequence (x_{k_i}) s.t.

$$\lambda(x_{k_i}) \xrightarrow{i\to\infty} \lambda(y) \quad \forall\, \lambda\in Y^*.$$

- If $\lambda\in X^*$, then $\lambda|_Y \in Y^*$, and $\lambda(x_{k_i}) = \lambda|_Y(x_{k_i}) \to \lambda|_Y(y) = \lambda(y)$, so we indeed have weak convergence $x_{k_i} \xrightarrow{w} x$ in X. $\square$

Definition Let X be a normed $\mathbb{R}$-vector space, and let $F: M\subset X \to \mathbb{R}$.

(i) F is called wslsc (weakly sequentially lower semicontinuous) if for all $\{x_k\}_{k\in\mathbb{N}} \subset M$ with $x_k \xrightarrow[k\to\infty]{w} x_0 \in M$,

$$F(x_0) \le \liminf_{k\to\infty} F(x_k).$$

(Shorthand notation: $F(x_0) \le \liminf\limits_{x \xrightarrow{w} x_0} F(x)$)

(ii) F is called coercive (w.r.t. $\|\cdot\|$) if

$$F(x) \nearrow \infty \quad \text{for } \|x\| \to \infty,\ x \in M.$$

(i.e. $\forall\, C\ \exists\, C'$ s.t. $x \in M,\ \|x\| \geq C' \Rightarrow F(x) \geq C$.)

Theorem (Variational Principle.) Let X be reflexive, $M \subset X$ non-empty and weakly sequentially closed. Let $F: M \to \mathbb{R}$ be coercive and wslsc. Then $\exists\, x_0 \in X$ s.t. $F(x_0) = \inf_{x \in M} F(x)$.

Proof. Let $\{x_k\}_{k \in \mathbb{N}}$ be a minimizing sequence, i.e.

$$F(x_k) \searrow \alpha_0 := \inf_{x \in M} F(x) \in [-\infty, \infty), \qquad k \to \infty.$$

- Since F is coercive, $\{x_k\}$ is bounded. By Eberlein–Šmulian, we may replace $\{x_k\}$ by a subsequence s.t. we have weak convergence $x_k \xrightarrow[k\to\infty]{w} x_0 \in X$.
- Since F is wslsc, $F(x_0) \leq \liminf_{k \to \infty} F(x_k) = \alpha_0$. This shows that $\alpha_0 > -\infty$, and x_0 is a minimizer. $\square$

Example: variational proof of the Riesz representation theorem.

Let H be a real Hilbert space, $\lambda \in H^*$. Consider

$$F(x) = \tfrac{1}{2}\|x\|^2 - \lambda(x), \qquad x \in H.$$

(i) F is coercive: $F(x) \geq \frac{1}{2}\|x\|^2 - \|\lambda\|_{H^*}\|x\| = \|x\|\left(\frac{1}{2}\|x\| - \|\lambda\|_{H^*}\right)$ when $\|x\| \nearrow \infty$.

(ii) F is wslsc: if $x_k \rightharpoonup x$, then $\|x\| \leq \liminf_{k\to\infty} \|x_k\|$ and $\lambda(x) = \lim_{k \to \infty} \lambda(x_k)$,

so $F(x) \leq \liminf_{k \to \infty} F(x_k)$.

(iii) H is reflexive.

We may thus apply the Theorem and get a minimizer $x_0 \in H$ of F.
For any $x \in H$, $t \in \mathbb{R}$, we then have

$$F(x_0) \leq F(x_0 + tx) = F(x_0) + \frac{t^2}{2}\|x\|^2 + t(x, x_0) - t\lambda(x).$$

The R.H.S is quadratic in t and minimal at $t=0$, so $(x, x_0) = \lambda(x)$.
Since $x \in H$ is arbitrary, we are done.