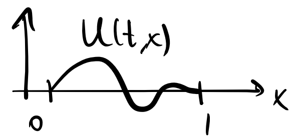
**A little motivation** Vibrating string with restoring force:

$$(\circledast) \quad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) U(t,x) = -V(x)U(t,x),$$

$$U(t,0) = U(t,1) = 0.$$




$V$ continuous, real-valued.

- Modes of vibration: $\omega \in \mathbb{C}$ s.t. $\exists\, u(x)$ (twice continuously differentiable) s.t. $U(t,x) = e^{i\omega t}u(x)$ solves $\circledast$.

$$\iff \circledast \quad \begin{cases} -\frac{d^2u}{dx^2}(x) + V(x)u(x) = \lambda u(x) & (\lambda = \omega^2) \\ u(0) = u(1) = 0. \end{cases}$$

- $\circledast$ is an eigenvalue problem: for what $\lambda$ is there a nontrivial solution $u$?

Note: $-\frac{d^2}{dx^2} + V(x)$ is a linear operator between $\infty$-dim'l function spaces!

Will see (FA 2):
- all such $\lambda$ are real
- $\exists$ infinite sequence $\lambda_j \in \mathbb{R}$, $u_j(x) \neq 0$ of solutions, with $\lambda_j \longrightarrow \infty$
- $\{u_j\}$ is an orthonormal basis of $L^2([0,1])$.

Remark $L^2$-spaces arise naturally e.g. in relation with the energy functional $\frac{1}{2}\int_0^1 \left|\frac{du}{dx}(x)\right|^2 + V(x)|u(x)|^2\,dx$.

Long, windy, beautiful road to get there...