Application 5: Fourier transform of 2π-periodic functions

Definition · For a function $u:[0,2\pi]\to\mathbb{C}$, we define

$$(\mathcal{F}u)(n)=\hat{u}(n):=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}e^{-int}u(t)\,dt,\quad n\in\mathbb{Z}.$$

· For a sequence $a=(a_n)_{n\in\mathbb{Z}}$, we define

$$(\mathcal{F}^{-1}a)(x)=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}e^{int}a_n,\quad x\in[0,2\pi]\ \ (\text{or } x\in\mathbb{R}).$$

Thus, $\mathcal{F}u$ is well-defined for $u\in L^1([0,2\pi])$, and
$\mathcal{F}^{-1}a$ is well-defined for $a\in\ell^1(\mathbb{Z})$.

The Fourier transform is particularly well-behaved on $L^2([0,2\pi])$.

Lemma The functions $e_n(t):=\frac{1}{\sqrt{2\pi}}e^{int}\in L^2([0,2\pi])$ are orthonormal.

Proof We compute $(e_n,e_m)=\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-m)t}\,dt=\begin{cases}1, & n=m\\ 0, & n\neq m.\end{cases}$ $\square$

Corollary (i) For $u\in L^2([0,2\pi])$, we have $\|\mathcal{F}u\|_{\ell^2(\mathbb{Z})}\leq\|u\|_{L^2([0,2\pi])}$.

(ii) Let $c_c(\mathbb{Z})=\{a=(a_n)_{n\in\mathbb{Z}}:\ a_n\neq 0 \text{ only for finitely many } n\in\mathbb{Z}\}$.

Then $\|\mathcal{F}^{-1}a\|_{L^2([0,2\pi])}=\|a\|_{\ell^2(\mathbb{Z})}$ for $a\in c_c(\mathbb{Z})$.

Thus, $\mathcal{F}^{-1}$ extends by continuity to $\mathcal{F}^{-1}:\ell^2(\mathbb{Z})\to L^2([0,1])$.

Proof (i) $\sum_{n=-N}^{N}|\mathcal{F}u(n)|^2=\sum_{n=-N}^{N}|(u,e_n)|^2\leq\|u\|^2_{L^2([0,2\pi])}$ for all $N\in\mathbb{N}$.

(Note here that $\sum_{n=-N}^{N}(u,e_n)e_n=$ orthogonal projection of u onto $\operatorname{span}\{e_{-N},\dots,e_N\}$.)

(ii) $\|\mathcal{F}^{-1}a\|^2_{L^2([0,2\pi])}=\|\sum a_ne_n\|^2=\sum|a_n|^2=\|a\|^2_{\ell^2}$. $\square$

Note that $\mathcal{F}(\mathcal{F}^{-1}(a))=a$ for $a\in c_c(\mathbb{Z})$, and therefore for $a\in\ell^2(\mathbb{Z})$. (Check this!) However, we do not yet know if $\mathcal{F}^{-1}(\mathcal{F}u)=u$ for all

$u \in L^2([0,2\pi])$; at this point we only know this for $u \in \mathcal{F}^{-1}(\ell^2(\mathbb{Z}))$, which is a closed subspace of $L^2([0,2\pi])$. That $\mathcal{F}^{-1}(\ell^2(\mathbb{Z})) = L^2$ (and more) follows from:

Theorem The Fourier transform is an isometric isomorphism

$$\mathcal{F} : L^2([0,2\pi]) \longrightarrow \ell^2(\mathbb{Z}),$$ and its inverse is $\mathcal{F}^{-1}$.

In particular, $\{e_n\}_{n\in\mathbb{Z}} \subset L^2([0,2\pi])$ is a complete orthonormal basis.

Proof. We only need to show that $\mathcal{F}^{-1}(\mathcal{F}u) = u \quad \forall u \in L^2([0,2\pi])$. $(\circledast)$

Indeed, together with the Corollary, this gives

$$\|u\|_{L^2} = \|\mathcal{F}^{-1}(\mathcal{F}u)\|_{L^2} = \|\mathcal{F}u\|_{\ell^2}.$$

- Moreover, we only need to show $(\circledast)$ for u lying in some dense subspace of $L^2([0,2\pi])$. To be continued...

Definition We define:

$$C^\infty_{per}([0,2\pi]) = \{u \in C^\infty([0,2\pi]) \text{ (infinitely often differentiable)} : \frac{d^k}{dt^k}u(0) = \frac{d^k}{dt^k}u(2\pi) \quad \forall k \in \mathbb{N}_0\},$$

$$s(\mathbb{Z}) = \{a = (a_n)_{n\in\mathbb{Z}} : \forall k \in \mathbb{N} \ \exists C_k \text{ s.t. } |a_n| \le C_k |n|^{-k} \ \forall n \in \mathbb{Z}\setminus\{0\}\}$$

(rapidly decreasing sequences).

Proposition (i) $\mathcal{F} : C^\infty_{per}([0,2\pi]) \longrightarrow s(\mathbb{Z})$

(ii) $\mathcal{F}^{-1} : s(\mathbb{Z}) \longrightarrow C^\infty_{per}([0,2\pi])$.

Proof (i) Let $u \in C^\infty_{per}([0,2\pi])$. For $n \in \mathbb{Z}\setminus\{0\}$,

$$\sqrt{2\pi}\,|\mathcal{F}u(n)| = \left|\int_0^{2\pi} e^{-int}\,u(t)\,dt\right| = \left|\int_0^{2\pi}\frac{1}{(-in)^k}\frac{d^k}{dt^k}(e^{-int})\,u(t)\,dt\right|$$

integration by parts (no boundary terms since $u \in C^\infty_{per}$)

$$= \left|\frac{1}{(-in)^k}\int_0^{2\pi} e^{-int}\,(-1)^k\frac{d^k}{dt^k}u(t)\,dt\right|$$

$$\leq \frac{1}{|n|^k}\underbrace{\int_0^{2\pi}\left|\frac{d^k}{dt^k}u(t)\right|dt}_{=:\,C_k<\infty}$$

$$= \frac{C_k}{|n|^k}.$$

(Philosophy: regularity of $u \rightsquigarrow$ decay of $\mathcal{F}u$ as $|n|\to\infty$.)

(ii) Let $a \in s(\mathbb{Z})$. Put $u_N(t) = \sqrt{2\pi}\sum_{n=-N}^{N} e^{int}a_n \in C^\infty_{per}([0,2\pi])$.

$$\Rightarrow \frac{d^k}{dt^k}u_N(t) = \sqrt{2\pi}\sum_{n=-N}^{N}(in)^k e^{int}a_n.$$

Since $\left\|\frac{d^k}{dt^k}e^{int}\right\|_{C^0} = |n|^k$

and $\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|^k|a_n| \leq \sum_{n\neq 0} C_{k+2}|n|^k|n|^{-k-2} < \infty$, we have uniform

convergence $\frac{d^k}{dt^k}u_N \xrightarrow{N\to\infty} u^{(k)}$ in $C^0([0,2\pi])$,

$u^{(k)}(0) = u^{(k)}(2\pi)$.

One can show (as in the proof that $T_{\frac{d}{dx}}$ is closed) that $u := u^{(0)}$ is differentiable with $\frac{d}{dt}u = u^{(1)}$, and indeed $\frac{d^k}{dt^k}u = u^{(k)}$ $\forall k\in\mathbb{N}$. $\Rightarrow u \in C^\infty_{per}([0,2\pi])$. $\square$

Using various operations on C^∞_{per} and $s(\mathbb{Z})$ — which are not available on L^2 and ℓ^2 — we shall succeed in showing that $\mathcal{F}: C^\infty_{per} \to s(\mathbb{Z})$ and $\mathcal{F}^{-1}: s(\mathbb{Z}) \to C^\infty_{per}$ are inverses of each other:

Lemma Let $u \in C^\infty_{per}([0,2\pi])$, $a \in s(\mathbb{Z})$.

(i) $\mathcal{F}(\frac{d}{dt}u)(n) = in\,(\mathcal{F}u)(n)$,

$\mathcal{F}(e^{it}u)(n) = (\mathcal{F}u)(n-1)$. (Here $e^{it}u \in C^\infty_{per}$ is the function $t \mapsto e^{it}u(t)$.)

(ii) $\mathcal{F}^{-1}((in\,a_n)_{n\in\mathbb{Z}})(t) = \frac{d}{dt}\mathcal{F}^{-1}(a)(t)$,

$\mathcal{F}^{-1}((a_{n-1})_{n\in\mathbb{Z}})(t) = e^{it}(\mathcal{F}^{-1}a)(t)$.

(That is, $\mathcal{F}/\mathcal{F}^{-1}$ intertwine $\frac{d}{dt}$ (differentiation) and $(a_n) \mapsto (in\,a_n)$ (multiplication), and $u \mapsto e^{it}u$ and shifts $(a_n) \mapsto (a_{n+1})$.)

Proof (i)
$$\mathcal{F}(\tfrac{d}{dt}u)(n) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-int}\frac{d}{dt}u(t)\,dt$$
$$= -\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{d}{dt}(e^{-int})\,u(t)\,dt$$
$$= in\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-int}u(t)\,dt$$
$$= in\,(\mathcal{F}u)(n).$$

$$\mathcal{F}(e^{it}u)(n) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-int}\,e^{it}u(t)\,dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i(n-1)t}u(t)\,dt$$
$$= (\mathcal{F}u)(n-1).$$

(ii) Similar. (Check! One part was essentially part of the proof of the previous Proposition.) $\square$

Therefore, $A := \mathcal{F}^{-1}\circ\mathcal{F} : C^\infty_{per}([0,2\pi]) \longrightarrow C^\infty_{per}([0,2\pi])$ has the properties
$$A\circ\frac{d}{dt} = \frac{d}{dt}\circ A, \quad (*)$$
$$A(e^{it}u)(t) = e^{it}Au(t). \quad (**)$$

Lemma Let $A: C^\infty_{per}([0,2\pi]) \to C^\infty_{per}([0,2\pi])$ be a linear map which satisfies ⊕ and ⊕. Then $\exists c \in \mathbb{C}$ s.t.

$$Au = cu \qquad \forall u \in C^\infty_{per}([0,2\pi]), \text{ i.e. } A = cI.$$

Proof Step 1: $(Au)(t_0)$ only depends on $u(t_0)$ (for any $t_0 \in [0,2\pi]$).

Indeed, suppose $u, v \in C^\infty_{per}$, $u(t_0) = v(t_0)$. Put $w = u - v$; we must show $(Aw)(t_0) = 0$. But we can write $w(t) = (e^{it} - e^{it_0}) f(t)$, $f \in C^\infty_{per}$. (Exercise: use Taylor expansion.) Thus,

$$\begin{aligned}(Aw)(t_0) &= A(e^{it} f)(t_0) - e^{it_0}(Af)(t_0) \\ &\overset{⊕}{=} (e^{it} Af)\big|_{t=t_0} - e^{it_0}(Af)(t_0) \\ &= 0.\end{aligned}$$

Step 2: Put $g(t) := (A\,1)(t)$, where $1 \in C^\infty_{per}$ is the constant function 1. Then $(Au)(t) = g(t) u(t)$ $\forall u \in C^\infty_{per}$. ⊕⊕

Indeed, for $u \in C^\infty_{per}$ and $t_0 \in [0,2\pi]$, $v := u - u(t_0) 1 \in C^\infty_{per}$ satisfies $v(t_0) = 0 \overset{\text{Step 1}}{\Longrightarrow}$

$$\begin{aligned}0 &= (Av)(t_0) \\ &= (Au)(t_0) - u(t_0)(A1)(t_0) \\ &= (Au)(t_0) - g(t_0) u(t_0).\end{aligned}$$

Step 3: $g(t)$ is constant.

Indeed, $g \in C^\infty_{per}$ satisfies

$$\frac{d}{dt} g(t) = \frac{d}{dt}(A1)(t) = A\left(\frac{d}{dt} 1\right)(t) = A(0)(t) = 0.$$

$\Rightarrow \exists c \in \mathbb{C}$ s.t. $g(t) = c$ $\forall t$. Conclusion follows from ⊕⊕. $\square$

For $A = \mathcal{F}^{-1} \circ \mathcal{F} \overset{\text{Lemma}}{=} cI$, we compute $c \in \mathbb{C}$ by evaluating

$$\mathcal{F}^{-1}(\mathcal{F}1) = \mathcal{F}^{-1}\big((\ldots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{\sqrt{2\pi}}, 0, 0, \ldots)\big) = 1 \Rightarrow c = 1.$$

Corollary $\mathcal{F} : C^{\infty}_{per}([0,2\pi]) \to s(\mathbb{Z})$ and $\mathcal{F}^{-1} : s(\mathbb{Z}) \to C^{\infty}_{per}([0,2\pi])$ are inverses of each other.

End of proof of Theorem We have $\mathcal{F}^{-1}(\mathcal{F}u) = u$ for $u \in C^{\infty}_{per}([0,2\pi])$. But $C^{\infty}_{per}([0,2\pi]) \subset L^2([0,2\pi])$ is dense.
(Exercise — it suffices to show that $C^{\infty}_{per}([0,2\pi]) \subset C^{0}_{per}([0,2\pi])$ is dense.) $\Rightarrow$ $\mathcal{F}^{-1}(\mathcal{F}u) = u$ for $u \in L^2([0,2\pi])$. $\square$