Let $V$ be a normed vector space. One often encounters the following situation: one has a linear functional $\ell : W \to \mathbb{K}$ with $\|\ell\|_{L(W,\mathbb{K})}$ and $W \subsetneqq V$, and wishes to extend it to $\tilde{\ell} : V \to \mathbb{K}$ (that is, $\tilde{\ell}(w) = \ell(w)$ for $w \in W$) with $\|\tilde{\ell}\|_{L(V,\mathbb{K})} = \|\ell\|_{L(W,\mathbb{K})}$.

The Hahn–Banach Theorem does just this.

**Definition** Let $X$ be a $\mathbb{K}$-vector space. A function $p : X \to \mathbb{R}$ is called sublinear if
$$\begin{cases} p(\alpha x) = \alpha p(x) & \forall x \in X, \alpha \geq 0; \quad \mathbb{K} = \mathbb{R}, \\ p(\alpha x) = |\alpha| p(x) & \forall x \in X, \alpha \in \mathbb{C}; \quad \mathbb{K} = \mathbb{C}, \end{cases}$$
and $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$.

**Remarks** (i) Any norm on $X$ is sublinear.

(ii) When $\mathbb{K} = \mathbb{C}$, have $0 = p(0) = p(x-x) \leq p(x) + p((-1)x) = 2p(x)$, so necessarily $p \geq 0$.

**Theorem** (Hahn–Banach, real case) Let $X$ be an $\mathbb{R}$-vector space, and let $p : X \to \mathbb{R}$ be sublinear. Suppose $Y \subset X$ is a linear subspace, and $\lambda : Y \to \mathbb{R}$ is linear and satisfies
$$\lambda(y) \leq p(y) \quad \forall y \in Y.$$
Then $\exists\, \tilde{\lambda} : X \to \mathbb{R}$ linear with $\tilde{\lambda}|_Y = \lambda$, and $\tilde{\lambda}(x) \leq p(x) \; \forall x \in X$.

**Proof** Step 1 If $Y \subsetneqq X$ and $z \in X \setminus Y$, we show how to extend $\lambda$ from $Y$ to $Y' :=$ span of $Y$ and $z$. To this end, let
$$\lambda'(y + tz) = \lambda(y) + t\alpha \quad \text{for } y \in Y, t \in \mathbb{R}, \text{ where } \alpha \in \mathbb{R} \text{ will}$$
be chosen later. Then $\lambda'|_Y = \lambda$. The requirement on $\alpha$ is:

$$(\circledast) \quad \lambda(y) + t\alpha \leq p(y + tz) \quad \forall y \in Y, t \in \mathbb{R}.$$

(i) We first arrange $\circledast$ for $t = 1$. To this end, note that
$p(y+z) - \lambda(y) \geq p(y+z) - p(y) \geq p(z)$, so any

$$\alpha \le \inf_{y\in Y} \left(p(y+z) - \lambda(y)\right) \qquad \dagger$$

works (the point being that this inf is finite).

(ii) The inequality $\circledast$ for $t=-1$ requires

$$\alpha \geq \sup_{x\in Y} \left(\lambda(x) - p(x-z)\right). \qquad \ddagger$$

The supremum is finite, since $\lambda(x) - p(x-z) \le p(x) - p(x-z) \le p(-z)$.

(iii) $\dagger$ and $\ddagger$ can be simultaneously satisfied, since $\forall x, y \in X$, we have

$\lambda(x) - p(x-z) \le p(y+z) - \lambda(y)$. (Indeed, this inequality is equivalent to

$$\begin{aligned} \lambda(x) + \lambda(y) &\overset{!}{\le} p(x-z) + p(y+z) \\ \| \qquad & \quad \ \ \lor\!\!\!\lor \leftarrow \text{sublinearity of } p \\ \lambda\left((x-z)+(y+z)\right) &\le p\left((x-z)+(y+z)\right). \end{aligned}$$

(iv) Fixing $\alpha\in\mathbb{R}$ subject to $\dagger$ and $\ddagger$, we now check $\circledast$ in general: for $t>0$,

$$\begin{aligned} \lambda(y) \pm t\alpha &= t\left(\lambda(t^{-1}y) \pm \alpha\right) \le t^{-1}\, p(t^{-1}y \pm z) \\ &= p(y \pm tz). \end{aligned}$$

<u>Step 2.</u> Now comes the abstract nonsense part. Set

$$P = \{ (\tilde{Y}, \tilde{\lambda}) : \quad \begin{aligned} & Y \subset \tilde{Y} \subset X \text{ linear}, \\ & \tilde{\lambda} : \tilde{Y} \to \mathbb{R} \text{ linear}, \\ & \tilde{\lambda}|_Y = \lambda, \\ & \tilde{\lambda} \le p \text{ on } \tilde{Y} \}; \end{aligned}$$

for $(\tilde{Y}_j, \tilde{\lambda}_j) \in P$, $j=1,2$ we say

$$(\tilde{Y}_1, \tilde{\lambda}_1) \le (\tilde{Y}_2, \tilde{\lambda}_2) \iff \tilde{Y}_1 \subset \tilde{Y}_2,\ \tilde{\lambda}_2|_{\tilde{Y}_1} = \tilde{\lambda}_1.$$

- Then $(P, \le)$ is a partially ordered set. We claim that every totally ordered subset $\{(\tilde{Y}_i, \tilde{\lambda}_i)\} \subset P$ has an upper bound;

to this end, let $\tilde{Y} = \bigcup_\iota \tilde{Y}_\iota$, and set $\tilde{\lambda}|_{\tilde{Y}_\iota} = \lambda_\iota$.
Then $(\tilde{Y}, \tilde{\lambda}) \in \mathcal{P}$, and $(\tilde{Y}_\iota, \tilde{\lambda}_\iota) \leq (\tilde{Y}, \tilde{\lambda})$

- By Zorn's Lemma, $\mathcal{P}$ has a maximal element $(\tilde{Y}, \tilde{\lambda})$. We claim that $\tilde{Y} = X$. If this were false, we could repeat Step 1 and extend $\tilde{\lambda}$ to a functional on a strictly larger space, contradicting the maximality of $(\tilde{Y}, \tilde{\lambda})$. $\square$

**Theorem** (Hahn–Banach, complex case) Let $X$ be a $\mathbb{C}$-vector space, $p: X \to [0, \infty)$ sublinear, $Y \subset X$ linear, and $\lambda: Y \to \mathbb{C}$ linear with $|\lambda(y)| \leq p(y)$ $\forall y \in Y$. Then $\exists \tilde{\lambda}: X \to \mathbb{C}$ with $\tilde{\lambda}|_Y = \lambda$ and $|\tilde{\lambda}(x)| \leq p(x)$ $\forall x \in X$.

**Proof**. Let $\lambda_1 = \operatorname{Re} \lambda$, $\lambda_2 = \operatorname{Im} \lambda$, then

$$\lambda(ix) = \lambda_1(ix) + i\lambda_2(ix) = i\lambda(x) = -\lambda_2(x) + i\lambda_1(x) \quad \forall x \in X,$$

so $\lambda_2(x) = -\lambda_1(ix)$.

- Let $\tilde{\lambda}_1: X \to \mathbb{R}$ be an $\mathbb{R}$-linear extension of $\lambda_1$, with
  $\tilde{\lambda}_1(x) \leq p(x)$ $\forall x \in X$.
- Put $\tilde{\lambda}(x) = \tilde{\lambda}_1(x) - i\tilde{\lambda}_1(ix)$; this is $\mathbb{R}$-linear, and indeed $\mathbb{C}$-linear (since $\tilde{\lambda}(ix) = i\tilde{\lambda}(x)$), with $\tilde{\lambda}|_Y = \lambda$.
- Finally, let $x \in X$ and choose $\alpha \in \mathbb{C}$, $|\alpha| = 1$, with
  $0 \leq |\tilde{\lambda}(x)| = \alpha\tilde{\lambda}(x) = \tilde{\lambda}(\alpha x) = \tilde{\lambda}_1(\alpha x)$; we thus get

$$|\tilde{\lambda}(x)| = \tilde{\lambda}_1(\alpha x) \leq p(\alpha x) = |\alpha| p(x) = p(x). \qquad \square$$

**Corollary** (Extension of continuous linear functionals)

Let $X$ be a normed vector space, $Y \subset X$, and let $\lambda \in Y^*$. Then $\exists\, \tilde{\lambda} \in X^*$ with $\tilde{\lambda}|_Y = \lambda$ and $\|\tilde{\lambda}\|_{X^*} = \|\lambda\|_{Y^*}$.

**Proof** Apply Hahn-Banach with $p(x) := \|\lambda\|_{Y^*} \|x\|_X$. $\square$

**Corollary** (Many continuous linear functionals) Let $X$ be a normed vector space, and let $0 \neq x \in X$. Then $\exists\, \lambda \in X^*$ with $\lambda(x) = \|x\|$ and $\|\lambda\|_{X^*} = 1$.

**Proof** Define $\lambda_0 : Y := \operatorname{span}\{x\} \to \mathbb{K}$ by

$$\lambda_0(\alpha x) = \alpha \|x\|_X.$$

Hahn-Banach (with $p = \|\cdot\|_X$) produces the desired $\lambda$. $\square$

Thus, for $X \neq \{0\}$, the dual space $X^*$ always contains many elements.

**Proposition** Let $X$ be a normed vector space, and define the linear map $\iota : X \to X^{**} = (X^*)^*$ ("double dual of $X$") as follows:

$$\iota(x)(\lambda) = \lambda(x) \quad \text{for } x \in X \text{ and } \lambda \in X^*.$$

Then $\iota$ is injective, and $\|\iota(x)\|_{X^{**}} = \|x\|_X$.

**Proof** We have $\|\iota(x)\|_{X^{**}} = \sup\limits_{\substack{\lambda \in X^* \\ \|\lambda\|_{X^*}=1}} \underbrace{|\iota(x)(\lambda)|}_{} \leq \|x\|_X$.

$\left( = |\lambda(x)| \leq \|\lambda\|_{X^*} \|x\|_X = \|x\|_X \right)$

On the other hand, given $x \in X$, $\exists\, \lambda \in X^*$ such that $\|\lambda\|_{X^*} = 1$ and $\lambda(x) = \|x\|_X$; so $\|\iota(x)\|_{X^{**}} \geq |\iota(x)(\lambda)| = \|x\|_X$.

This gives $\|\iota(x)\|_{X^{**}} = \|x\|_X$, which in turn implies the injectivity of $\iota$. $\square$

**Corollary** Every normed vector space $(X, \|\cdot\|_X)$ has a completion $(\tilde{X}, \|\cdot\|_{\tilde{X}})$, i.e. a Banach space for which there exists a continuous injective isometric linear map $i : X \to \tilde{X}$ with dense range.

Proof We may take $\tilde{X}$ = closure of $\iota(X) \subset X^{**}$, with
$$\|\cdot\|_{\tilde{X}} = \|\cdot\|_{X^{**}}\big|_{\tilde{X}}. \qquad \square$$

Exercise The completion is unique up to isometric isomorphism.