Application 2: nowhere differentiable continuous functions

Baire category thm

real analysis

We shall prove the following result, due to Banach (1931):

Theorem Let $A = \{u \in C^0([0,1]) : u \text{ is differentiable at least at one point}\}$.

Then A is meager in $(C^0([0,1]), \|\cdot\|_{C^0})$. In other words, a Baire-generic continuous function is nowhere differentiable.

Proof $u \in C^0([0,1])$ being differentiable at $x \in [0,1]$ implies that

$$\exists\, n \;\forall h>0 \;\forall y \in [0,1],\ |x-y|<h : \ |u(x)-u(y)| \leq n|x-y|.$$

Since may as well require $h = \frac{1}{m}$ for some $m \in \mathbb{N}$, we have

$$A \subseteq \bigcup_{n,m\in\mathbb{N}} A_{n,m},$$

where $A_{n,m} = \{u \in C^0([0,1]) : \exists\, x \in [0,1] \text{ s.t. } \forall y \in [0,1] \text{ with } |x-y|<\frac{1}{m}, \text{ we have } |u(x)-u(y)| \leq n|x-y|\}$.

Claim 1: $A_{n,m}$ is closed.

Indeed, suppose $u_k \xrightarrow{k\to\infty} u \in C^0([0,1])$, with $u_k \in A_{n,m}$ $\forall k$.

We get $x_k \in [0,1]$ s.t. $|u_k(x_k) - u_k(y)| \leq n|x_k - y|$ $(\ast)$
$\forall y \in [0,1],\ |x_k - y| < \frac{1}{m}$.

Passing to a subsequence, we may assume $x_k \xrightarrow{k\to\infty} x \in [0,1]$.
Fix $y \in [0,1]$, $|x-y| < \frac{1}{m}$. Then for all large k so that $|x_k - y| < \frac{1}{m}$, we have $(\ast)$; taking the limit $k\to\infty$ (using that $u_k(x_k) \to u(x)$), we obtain $|u(x) - u(y)| \leq n|x-y|$. Therefore, $u \in A_{n,m}$.

Claim 2: $A_{n,m}$ has empty interior ($\Rightarrow$ is nowhere dense).

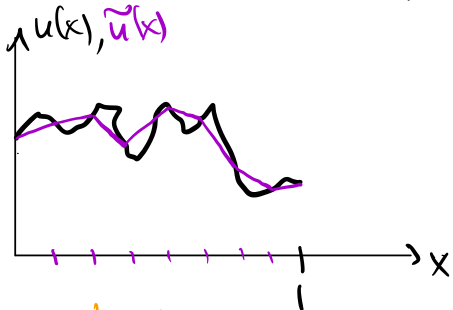
To prove this, we proceed in two steps. Let $u \in A_{n,m}$ and $\varepsilon > 0$.

Step 1. $\exists\, \tilde{u} \in C^0$, piecewise linear, s.t. $\|u - \tilde{u}\|_{C^0} < \frac{\varepsilon}{2}$.

Indeed, for sufficiently large N, we can take

$$\tilde{u}\left(\tfrac{i}{N}\right) = u\left(\tfrac{i}{N}\right), \quad 0 \le i \le N, \text{ and } \tilde{u}\big|_{[\frac{i}{N}, \frac{i+1}{N}]} \text{ linear.}$$

(Exercise.)



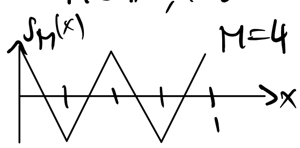
Thus, $\exists\, L > 0$ s.t. $|\tilde{u}(x) - \tilde{u}(y)| \le L|x-y| \ \forall x, y \in [0,1]$. $\circledast$

(Take $L = \max(|\text{slope of } \tilde{u}|) < \infty$.)

Step 2. $\exists\, v \in C^0$, $\|\tilde{u} - v\|_{C^0} \le \frac{\varepsilon}{2}$, s.t. $v \notin A_{n,m}$.

Indeed, we make $\tilde{u}$ "spiky" as follows: for $M \in \mathbb{N}$, let

$$s_M(x) = \begin{cases} (-1)^i, & x = \frac{i}{M},\ 0 \le i \le M, \\ \text{linear interpolation in between} \end{cases}$$



Consider $v = \tilde{u} + \frac{\varepsilon}{2} s_M$. Then $\|v - u\|_{C^0} \le \|v - \tilde{u}\|_{C^0} + \|\tilde{u} - u\|_{C^0}$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

and we claim that for large enough M, $v \notin A_{n,m}$.

Let $x \in [0,1)$. Then $x \in \left[\frac{i}{M}, \frac{i+1}{M}\right)$ for some $i \in \{0, \dots, M-1\}$.

For $0 < \delta < \frac{i+1}{M} - x$, we have

$$|v(x+\delta) - v(x)| \ge \frac{\varepsilon}{2}|s_M(x+\delta) - s_M(x)| - |\tilde{u}(x+\delta) - \tilde{u}(x)|$$

$$\ge \frac{\varepsilon}{2} \cdot \underbrace{2M}_{\left(\text{slope of } s_M \text{ on } [\frac{i}{M}, \frac{i+1}{M})\right)} \cdot \delta - \underbrace{L}_{(\text{from } \circledast)} \cdot \delta = (M\varepsilon - L)\delta.$$

(Similar argument for $x=1$.)

Taking M so large that $M\varepsilon - L > n$, this shows that $v \notin A_{n,m}$.

Conclusion $\bigcup_{n,m\in\mathbb{N}} A_{n,m}$ is meager, and thus so is its subset A. $\square$