Analytic Banach space-/operator-valued functions

Definition Let X be a complex Banach space, $\Omega \subset \mathbb{C}$ open, and $f : \Omega \to X$. Then f is *holomorphic* (or *analytic*) if f is complex differentiable at all $z \in \Omega$:

$$\exists \lim_{\substack{h \in \mathbb{C} \\ |h| \to 0}} \frac{f(z+h) - f(z)}{h} =: f'(z).$$

Remarks (i) f is holomorphic $\Longleftrightarrow$ $\forall \lambda \in X^*$, $\lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic.

(Exercise.)

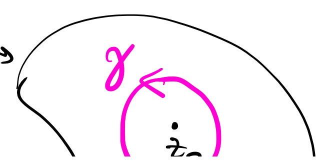
(ii) If $X = L(V,W)$ with V, W Banach spaces:

$f : \Omega \to L(V,W)$ is holomorphic

$\overset{\text{(Exercise)}}{\Longleftrightarrow}$ $\forall v \in V$, the function $\Omega \ni z \mapsto f(z)(v) \in W$ is holomorphic

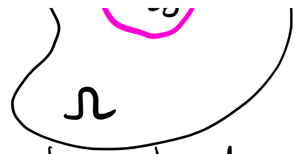
$\overset{\text{(i)}}{\Longleftrightarrow}$ $\forall v \in V, \mu \in W^*$, the function $\Omega \ni z \mapsto \mu(f(v)) \in \mathbb{C}$ is holomorphic.

(iii) The standard results in complex analysis related to Cauchy's Theorem apply in the Banach space-valued setting. E.g. if $f : \Omega \to X$ is holomorphic, and $z_0 \in \Omega$, then $f(z_0) = \oint_\gamma \frac{f(z)}{z - z_0}\, dz$ (limit in X of Riemann sums), where ...



Similarly, if $z_0 \in \Omega$ and

$d = \mathrm{dist}(z_0, \partial\Omega) \in \mathbb{R} \cup \{+\infty\}$,



then $f(z) = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{j!} f^{(j)}(z_0)$, $|z-z_0| < d$,

with uniform convergence in X for $z \in \Omega$, $|z-z_0| < C < d$.

Examples (of analytic families of operators):

(i) $n \times n$ matrices with analytic entries:

$$A(z) = (a_{ij}(z))_{i,j=1,\dots,n}, \qquad a_{ij} : \Omega \to \mathbb{C} \text{ holomorphic.}$$

(ii) Polynomials $\sum_{j=0}^{k} A_j z^j$, $\quad A_j \in L(X,Y)$.

More generally, $\sum_{j=0}^{k} A_j p_j(z)$, $\quad p_j : \Omega \to \mathbb{C}$ holomorphic.

(iii) Lemma If $A : \Omega \to L(X,Y)$ is holomorphic, and $z_0 \in \Omega$ is such that $A(z_0)$ is invertible, then $A(z)^{-1}$ is holomorphic for $z \in B_\varepsilon(z_0)$ when $\varepsilon > 0$ is sufficiently small.

Proof Since the set of invertible maps (between Banach spaces) is open, $\exists\, A(z)^{-1}$ for $z \in B_\varepsilon(z_0)$ when $\varepsilon > 0$ is sufficiently small. Indeed, recall the argument:

$$A(z) = A(z_0) + (A(z) - A(z_0))$$
$$= A(z_0)\,(I + B(z)), \qquad B(z) := A(z_0)^{-1}(A(z) - A(z_0));$$

here $B(z) \in L(X)$ is holomorphic in $z \in \Omega$, and $B(z_0) = 0$.

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $z \in \mathbb{C}$, $|z - z_0| < \varepsilon$ implies $z \in \Omega$, $\|B(z)\|_{L(X)} < 1$.

Since $A(z)^{-1} = (I + B(z))^{-1} A(z_0)^{-1}$, we only need to show that $(I+B(z))^{-1}$ is holomorphic for $|z-z_0|<\varepsilon$. But

$$(I+B(z))^{-1} = \sum_{j=0}^{\infty} (-1)^j B(z)^j$$

converges locally uniformly in $\{z \in \mathbb{C} : |z-z_0| < \varepsilon\}$, and therefore defines a holomorphic family of operators. $\square$

(iv) (Application of (iii).)

If $A \in L(X)$, $\lambda_0 \in \mathbb{C}$, and $A-\lambda_0 : X \to X$ is invertible, then $\lambda \mapsto (A-\lambda)^{-1}$ is holomorphic for λ in a small neighborhood of λ_0.

Spectrum and resolvents

Definition Let X be a complex Banach space, $A \in L(X)$.

(i) The *resolvent set* of A is

$$\rho(A) := \{ z \in \mathbb{C} : \quad zI_X - A \in L(X) \text{ is invertible} \}.$$

The map $\rho(A) \ni z \mapsto R_z := (z-A)^{-1}$ is the *resolvent* of A.

(ii) The *spectrum* of A is

$$\sigma(A) := \mathbb{C} \setminus \rho(A) = \{ z \in \mathbb{C} : z - A \text{ is not injective, or not surjective, or both} \}.$$

Remark Since the space of invertible elements of $L(X)$ is open, $\rho(A) \subset \mathbb{C}$ is open. $\Rightarrow$ $\sigma(A) \subset \mathbb{C}$ is closed.

Examples (i) $A \in \mathbb{C}^{n\times n}$ (matrix). Then $\sigma(A) = \{\text{eigenvalues of } A\}$,

$$\rho(A) = \mathbb{C} \setminus \sigma(A).$$

(ii) $A =$ right shift $\in L(\ell^2(\mathbb{N}))$: $A(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$.

Exercise: • $z \in \mathbb{C}$, $|z| > 1 \Rightarrow z \in \rho(A)$.

• $|z| = 1 \Rightarrow z - A$ is injective, has dense range, but $z \notin \rho(A)$ $(\operatorname{ran}(z-A) \neq \ell^2)$

• $|z| < 1 \Rightarrow z - A$ is injective, but its range is not dense.

$$\Rightarrow \sigma(A) = \{z \in \mathbb{C} : |z| \leq 1\},$$
$$\rho(A) = \{z \in \mathbb{C} : |z| > 1\}.$$

(But for no $z \in \sigma(A)$ does there exist an eigenvector with eigenvalue z!)

(iii) Let $X = C^0(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{C} : \|u\|_{C^0} = \sup_{x \in \mathbb{R}} |u(x)| < \infty\}$ and $g \in C^0(\mathbb{R})$. Put $A : X \to X$,
$u \mapsto gu$ (pointwise product).

Claim: $\sigma(A) = \overline{\operatorname{range}(g)} = \overline{\{g(x) : x \in \mathbb{R}\}}$.

Indeed, if $z \notin \overline{\operatorname{range}(g)}$, $\exists\, \varepsilon > 0$ s.t. $|z - g(x)| > \varepsilon$ $\forall x \in \mathbb{R}$.

$\Rightarrow z - A : X \to X$ is invertible, with inverse given by
$$((z-A)^{-1} v)(x) = \frac{1}{z - g(x)} v(x) \qquad (v \in X),$$
so $z \in \rho(A) \Rightarrow \sigma(A) \subset \overline{\operatorname{range}(g)}$

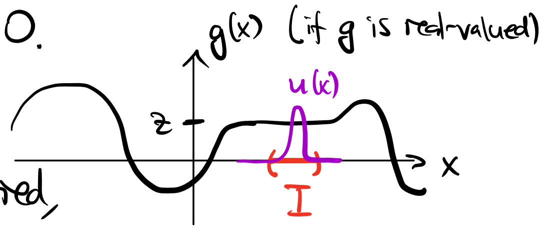
• On the other hand, if $z = g(x_0) \in \operatorname{range}(g)$, we claim that $z - A$ is not surjective; indeed, for all $u \in C^0(\mathbb{R})$, we have
$$((z-A)u)(x_0) = (z - g(x_0)) u(x_0) = 0; \text{ so } 1 \notin \operatorname{ran}(z-A).$$

$\Rightarrow \operatorname{range}(g) \subset \sigma(A)$.

· Since $\sigma(A)$ is closed, we are done. $\square$

Two more comments about this example:

- If $g(x)=z$ for x in a non-empty open interval $I\subset\mathbb{R}$, then $\exists\, u\in X$: $(z-A)u=0$.



- If $z\in\overline{\text{range}(g)}$ but the above condition is not satisfied, then $z-A$ is injective but not surjective; if $z\in\text{range}(g)$, then $\text{ran}(z-A)$ has positive codimension, if $z\in\overline{\text{range}(g)}\setminus\text{range}(g)$, then $\text{ran}(z-A)$ is dense.

(iv) $g\in L^\infty(\Omega)$, $\Omega\subset\mathbb{R}^n$ open and non-empty.

$A: L^p(\Omega)\to L^p(\Omega)$, $u\mapsto gu$.

$\Rightarrow \sigma(A)=\text{ess ran }g=\{z\in\mathbb{C}: \forall\varepsilon>0,\ \mathcal{L}^n(g^{-1}(B_\varepsilon(z)))>0\}$.

(Exercise.)

(v) Let $(g_n)_{n\in\mathbb{N}}\in\ell^\infty(\mathbb{N})$. Define $A:\ell^p(\mathbb{N})\to\ell^p(\mathbb{N})$,

$$(a_n)\mapsto(g_n a_n).$$

Then $\sigma(A)=\overline{\{g_n : n\in\mathbb{N}\}}\subset\mathbb{C}$. (Exercise.)

We see that $z\in\mathbb{C}$ can lie in $\sigma(A)$ for a variety of reasons.

Definition Let $A\in L(X)$.

(i) $\sigma_p(A)=\{z\in\sigma(A): \ker(z-A)\neq\{0\}\}$.

"Pure point spectrum."

(ii) $\sigma_c(A) = \{z \in \sigma(A) : \ker(z-A) = \{0\},\ \operatorname{ran}(z-A) \text{ is dense}\}$.

"Continuous spectrum."

(iii) $\sigma_r(A) = \sigma(A) \setminus (\sigma_p(A) \cup \sigma_c(A))$. "Residual spectrum."

The spectrum and its various pieces are invariant under conjugation by isomorphisms:

Lemma X, Y Banach spaces, $A \in L(X)$. If $U : X \to Y$ is invertible, then $\sigma(A) = \sigma(UAU^{-1})$.

Proof $(z-A)u = f \iff (z - UAU^{-1})Uu = Uf$. $\square$

Example (i) $X = \ell^2(\mathbb{Z})$; $A \in L(X)$ discrete 2^{nd} derivative:

$(Au)_n = u_{n+1} - 2u_n + u_{n-1}$. Claim: $\sigma(A) = \sigma_c(A) = [0,4]$.

Indeed, consider $\mathcal{F}^{-1} A \mathcal{F} : L^2([0,2\pi]) \to L^2([0,2\pi])$;

$$(\mathcal{F}^{-1} A \mathcal{F} u)(x) = \mathcal{F}^{-1}\left((\hat{u}(n+1) - 2\hat{u}(n) + \hat{u}(n-1))_{n\in\mathbb{Z}} \right)(x)$$
$$= e^{-ix} u(x) - 2u(x) + e^{ix} u(x)$$
$$= 2(\cos(x) - 1)\, u(x).$$

$$\Rightarrow \sigma(A) = \sigma(\mathcal{F}^{-1} A \mathcal{F}) = \overline{\text{range } 2(\cos - 1)} = [0,4]. \quad \square$$

(ii) $\partial_\theta : H^1(S^1) \to L^2(S^1)$ has $\sigma(\partial_\theta) = \sigma_p(\partial_\theta) = i\mathbb{Z}$.

Indeed, $\mathcal{F} \partial_\theta \mathcal{F}^{-1} : \ell^2(\mathbb{Z}) \ni (a_n)_{n\in\mathbb{Z}} \mapsto (in\, a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

This gives the desired result by a previous example.

For all $n \in \mathbb{Z}$, $\ker(in - \partial_\theta) = \operatorname{span}\{e^{in\theta}\}$.

(iii) $\partial_\theta^2 : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ has $\sigma(\partial_\theta^2) = \sigma_p(\partial_\theta^2) = \{-k^2 : k \in \mathbb{N}_0\}$.

$$\ker(-k^2 - \partial_\theta^2) = \begin{cases} \operatorname{span}\{1\}, & k=0 \\ \operatorname{span}\{e^{-ik\theta}, e^{ik\theta}\}, & k \geq 1. \end{cases}$$

We next prove some general results about the spectrum of bounded operators.

Theorem Let $A \in L(X)$. Then $\rho(A) \ni z \mapsto R_z = (z-A)^{-1}$ is holomorphic.

Moreover, if $z_0 \in \rho(A)$, then $B_{\|R_{z_0}\|^{-1}_{L(X)}}(z_0) \subset \rho(A)$, and thus $\|R_z\|_{L(X)} \geq \dfrac{1}{d(z, \sigma(A))}$.

Proof The first part was already proved. For the second part, note that $z - A = z_0 - A + z - z_0 = (z_0 - A)(I + R_{z_0}(z - z_0))$, and $I + R_{z_0}(z-z_0)$ is invertible for $|z - z_0| \, \|R_{z_0}\|_{L(X)} < 1$. $\square$

Theorem Let $A \in L(X)$, where X is a complex Banach space, $\dim X \geq 1$.

(i) $\sigma(A) \neq \emptyset$ and $\rho(A) \neq \emptyset$.

(ii) Define the spectral radius of A as

$$r_A := \sup_{z \in \sigma(A)} |z|.$$

Then $r_A = \limsup_{n \to \infty} \|A^n\|^{1/n}$.

Proof (i). If $z \in \mathbb{C}$, $|z| > \|A\|_{L(X)}$, then $z - A = z(I - z^{-1}A)$ is invertible since $\|z^{-1}A\|_{L(X)} < \infty$. Thus, $z \in \rho(A)$.

· This also shows: $\|R_z\|_{L(X)} \leq |z|^{-1} \|(I - z^{-1}A)^{-1}\|_{L(X)} \xrightarrow{|z|\to\infty} 0.$ ⊛

· Suppose $\sigma(A) = \emptyset$. Then for all $x \in X$, $\lambda \in X^*$, the map $f_{x,\lambda}: \mathbb{C} \ni z \mapsto \lambda((z-A)^{-1}x) \in \mathbb{C}$ is holomorphic.

But $|f_{x,\lambda}(z)| \xrightarrow{|z|\to\infty} 0$ by ⊛. Liouville's Theorem implies that $f_{x,\lambda} = 0$. Since λ, x are arbitrary, this implies $(z-A)^{-1} = 0 \in L(X)$, which is absurd since $(z-A)^{-1} \in L(X)$ is a bijection and $X \neq \{0\}$.

(ii) · We first prove "$\leq$" in (#): if $|z| > \limsup_{n\to\infty} \|A^n\|^{1/n}$, then by the root test, the series

$$\sum_{j=0}^{\infty} (z^{-1}A)^j \in L(X)$$

converges; the limit is $(I - z^{-1}A)^{-1} = z(z-A)^{-1}$.

· For the proof of "$\geq$" in (#), let $r > r_A$; we need to show that $\|A^n\|^{1/n} \leq r$ for large n.

To this end, we shall show that

$$A^n = \frac{1}{2\pi i} \int_{\partial B_r(0)} z^n (z-A)^{-1}\, dz. \quad (‡)$$

Indeed, this integral is independent of the choice of $r > r_A$ by Cauchy's Theorem (since $z^n(z-A)^{-1}$ is holomorphic); and if we increase r to a value $r > \|A\|$, then

$$z^n(z-A)^{-1} = z^n \sum_{j=0}^{\infty} z^{-j-1} A^j \quad \text{for } |z| \geq r,$$

so $\frac{1}{2\pi i}\int_{\partial B_r(0)} z^n (z-A)^{-1}\,dz = [\text{coefficient of } z^{-1}] = A^n$ indeed.

(†) now gives $\|A^n\| \leq r^{n+1} \max_{|z|=r} \|R_z\|_{L(X)}$

$$\Rightarrow \limsup_{n\to\infty} \|A^n\|^{1/n} \leq r. \qquad \square$$

Remark In fact, $\exists \lim_{n\to\infty} \|A^n\|^{1/n}$ (which is thus equal to the lim sup, and thus to the spectral radius of A).

For the proof, note that $x_n := \log \|A^n\|$ is subadditive:

$$\begin{aligned} x_{n+m} = \log \|A^{n+m}\| &\leq \log(\|A^n\|\,\|A^m\|) \\ &= \log\|A^n\| + \log\|A^m\| \\ &= x_n + x_m. \end{aligned}$$

(We set $x_n = -\infty$ if $A^n = 0$.) But if $\{x_n\}_{n\in\mathbb{N}_0}$ is any subadditive sequence, then $\liminf \frac{x_n}{n} = \limsup \frac{x_n}{n}$ $\left(= \lim \frac{x_n}{n} \in \mathbb{R} \cup \{-\infty\}\right.$, so $= \log\left(\lim_{n\to\infty} \|A^n\|^{1/n}\right)$ in our application).

Indeed, for any fixed $m \in \mathbb{N}$, we have

$$x_{km+\ell} \leq k x_m + x_\ell \qquad (k\in\mathbb{N},\ 0\leq \ell \leq m-1)$$

$$\Rightarrow \frac{x_{km+\ell}}{km+\ell} \leq \frac{1}{m+\ell/k} x_m + \frac{x_\ell}{km+\ell}.$$

$$\Rightarrow \underbrace{\limsup_{\substack{k\to\infty \\ 0\leq \ell\leq m-1}} \frac{x_{km+\ell}}{km+\ell}}_{= \limsup_{n\to\infty} \frac{x_n}{n}} \leq \frac{1}{m} x_m.$$

Taking the $\liminf_{m\to\infty}$ of this estimate proves the claim. —

Much as in the case of matrices, the spectrum of self-adjoint operators has special properties:

Theorem Let X be a complex Hilbert space, $A \in L(X)$ self-adjoint (i.e. $A = A^*$). Then:

(i) $\sigma(A) \subset \mathbb{R}$;

(ii) $r_A = \|A\|$;

(iii) $\|(z-A)^{-1}\|_{L(X)} \leq \frac{1}{|\operatorname{Im} z|}$ $\quad (z \in \mathbb{C} \setminus \mathbb{R})$.

Proof (i). If $z \in \mathbb{C} \setminus \mathbb{R}$, then $\ker(z-A) = \{0\}$; indeed,

$$(z-A)x = 0 \Rightarrow \big((z-A)x, x\big) = z\|x\|^2 - (Ax,x) = 0. \quad \circledast$$

But $\overline{(Ax,x)} = (x, Ax) = (A^*x, x) = (Ax, x)$ is real. So $\circledast$ gives $\operatorname{Im}\big(z\|x\|^2\big) = 0$ and thus $\|x\|^2 = 0 \Rightarrow x = 0$.

- From $\circledast$, we obtain

$$|\operatorname{Im} z|\,\|x\|^2 = \big|\operatorname{Re}\big((z-A)x, x\big)\big|$$
$$\leq \|(z-A)x\|\,\|x\|$$
$$\Rightarrow \|x\| \leq \frac{1}{|\operatorname{Im} z|}\|(z-A)x\| \qquad (\Rightarrow \text{(iii)})$$

$\Rightarrow$ $z - A$ has closed range; and

$(\operatorname{ran}(z-A))^{\perp} = \ker(\bar{z} - A^*) = \{0\}$ by what we have already shown.

$\Rightarrow$ $z - A$ is invertible.

(ii). We have $\|A^2\| = \|A^*A\| = \sup\limits_{\|x\|=1} \|A^*Ax\|$

$$\geq \sup_{\|x\|=1} (A^*Ax, x) = \sup_{\|x\|=1} (Ax, Ax) = \|A\|^2.$$

Conversely, $\|A^2\| \leq \|A\| \, \|A\| = \|A\|^2$. <u>So</u> $\|A^2\| = \|A\|^2$.

· Inductively, $\|A^{2^k}\| = \|A\|^{2^k} \Rightarrow \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \lim\limits_{k\to\infty} \underbrace{\|A^{2^k}\|^{\frac{1}{2^k}}}_{=\|A\| \ \forall k} = \|A\|.$ $\square$

We shall prove much more in this case later on!