· We have already seen some instances of the usefulness of the ability to define functions of a bounded operator, i.e. $f(A)$ for $A \in L(X)$, where $f(x) = e^{itx} = \sum_{j=0}^{\infty} \frac{(itx)^j}{j!}$ or $f(x) = (1-x)^{-1} = \sum_{j=0}^{\infty} x^j$ $(|x|<1)$. We shall now study this procedure $f \longmapsto f(A)$ in detail when

A is a self-adjoint bounded linear operator on a complex separable Hilbert space H.

Definition For $p \in \mathbb{C}[x]$, i.e. $p(x) = \sum_{j=0}^{n} p_j x^j$, we set

$$p(A) := \sum_{j=0}^{n} p_j A^j.$$

Proposition (i) $\varphi: \mathbb{C}[x] \longrightarrow L(H)$,
$p \longmapsto p(A)$,

is an algebra-homomorphism
$\phi(\alpha p) = \alpha \phi(p)$
$\phi(p+q) = \phi(p) + \phi(q)$
$\phi(pq) = \phi(p) \circ \phi(q)$.

(ii) $\phi(p)^* = \phi(\underset{\substack{\uparrow \\ \text{complex conjugation}}}{\bar{p}})$ (i.e. $p(A)^* = \bar{p}(A)$).

(iii) $\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$.

Proof (i) We only check multiplicativity:

$$\phi(pq) = \phi\Big(\sum_{j,\ell} p_j q_\ell x^{j+\ell}\Big) = \sum_{j,\ell} p_j q_\ell A^{j+\ell} = \sum_{j,\ell} p_j A^j q_\ell A^\ell$$
$$= \Big(\sum_j p_j A^j\Big)\Big(\sum_\ell q_\ell A^\ell\Big) = \phi(p)\phi(q).$$

(ii) $(\sum p_j A^j)^* \overset{A=A^*}{=} \sum \overline{p_j} A^j$.

(iii) "$\supseteq$". Let $z_0 \in \sigma(A)$; we claim $p(z_0) \in \sigma(p(A))$, i.e. $p(z_0) - p(A)$ is not invertible. Note that

$p(z_0) - p(z) = (z_0 - z) q(z)$, $\quad q \in \mathbb{C}[x]$, so

$$\underbrace{p(z_0) I - p(A)}_{\text{not invertible}} \Leftarrow \underbrace{(z_0 - A)}_{\text{not invertible}} q(A) = q(A)(z_0 - A).$$

"$\subseteq$". If $w_0 \in \sigma(p(A))$, write $w_0 - p(z) = c \prod_{j=1}^{n} (z_j - z)$, with $w_0 - p(A) = c \prod_{j=1}^{n} (z_j - A)$ not invertible

$\Rightarrow \exists j$ s.t. $z_j - A$ is not invertible.

$\Rightarrow z_j \in \sigma(A)$; but $w_0 - p(z_j) = 0$, so we have proved $w_0 \in p(\sigma(A))$. $\square$

Therefore, the spectrum of $p(A)$ only depends on $p|_{\sigma(A)} : \sigma(A) \to \mathbb{C}$.

Proposition $\| p(A) \|_{L(H)} = \| p \|_{C^0(\sigma(A))} = \sup_{z \in \sigma(A)} |p(z)|$.

Proof For every $B \in L(H)$, we have $\|B\|^2 = \|B^* B\|$; indeed,

$$\|B\|^2 = \sup_{\|x\|=1} |(Bx, Bx)| = \sup_{\|x\|=1} |(B^*Bx, x)| \le \|B^*B\|$$

$$\le \|B^*\| \, \|B\| = \|B\| \cdot \|B\| = \|B\|^2,$$

so we have equality throughout.

$$\Rightarrow \|p(A)\|^2 = \|p(A)^* p(A)\| = \|\bar{p}(A) p(A)\| = \| \overbrace{|p|^2}^{\text{self-adjoint}}(A) \|$$

$$= r_{|p|^2(A)} \quad \text{(spectral radius)}$$

$$= \sup_{z \in \sigma(|p|^2 A)} |z| = \sup_{w \in \sigma(A)} |p(w)|^2 = \|p\|^2_{C^0(\sigma(A))}. \qquad \square$$

Theorem (Functional Calculus.) There exists a (unique) map

$$\phi : C^0(\sigma(A)) \longrightarrow L(\mathcal{H}), \quad f \mapsto f(A) := \phi(f),$$

with the following properties:

(i) ϕ is an algebra-homomorphism, and $\phi(p)^* = \phi(\bar{p})$.

(ii) $p(x) = x \Rightarrow \phi(p) = p(A) = A$.

(iii) $\|f(A)\|_{L(\mathcal{H})} = \|f\|_{C^0(\sigma(A))}$.

(iv) $\sigma(f(A)) = f(\sigma(A))$. Moreover, if $\lambda \in \sigma_p(A)$, $Au = \lambda u$, then $f(\lambda) \in \sigma_p(f(A))$, $f(A)u = f(\lambda)u$.

(v) If A is compact and $f(0) = 0$, then $f(A)$ is compact.

Proof. By (i) & (ii), $\phi(p) = p(A)$ for $p \in \mathbb{C}[x]$. $\oplus$

- $\|\phi(p)\|_{L(\mathcal{H})} = \|p\|_{C^0(\sigma(A))}$ for $p \in \mathbb{C}[x]$. $\otimes$
- Stone–Weierstrass $\Rightarrow \{ p|_{\sigma(A)} : p \in \mathbb{C}[x] \} \subset C^0(\sigma(A))$ is dense. (We use here that $\sigma(A) \subset \mathbb{R}$ is compact.)

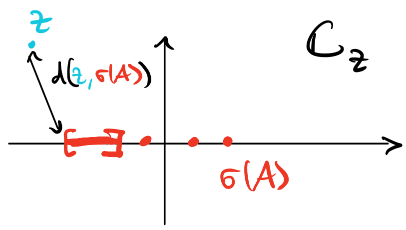
$\Rightarrow \exists!$ extension of $\oplus$ to all $p \in C^0(\sigma(A))$, and (iii) holds by continuity from $\otimes$.

(iv) holds for $f \in \mathbb{C}[x]$. It follows by continuity for all $f \in C^0(\sigma(A))$ (see below).

(v) $f(x) = \sum_{j=1}^{n} f_j x^j \Rightarrow f(A)$ is compact. Since norm limits of compact operators are compact, this holds for $f \in C^0(\sigma(A))$. $\square$

Corollary $A \in L(H)$ self-adjoint, $z \in \rho(A) \Rightarrow \|(z-A)^{-1}\|_{L(H)} = \frac{1}{d(z,\sigma(A))}$.

Proof $(z-A)^{-1} = f(A)$, $f(x) := (z-x)^{-1} \in C^0(\sigma(A))$. Use (iii). $\square$



Corollary $A \in L(H)$ self-adjoint, A positive ($(Ax,x) \geq 0 \ \forall x \in H$) $\Rightarrow \exists B \in L(H)$, $B = B^*$, $B^2 = A$, B positive.

Proof $B = f(A)$ where $f(x) = \sqrt{x} \in C^0(\sigma(A))$. (Note: $\sigma(A) \subset [0,\infty)$.) $\square$

Theorem Let $A \in L(H)$ be compact (but not necessarily self-adjoint). Then $\exists$ orthonormal sets $\{x_n\}$, $\{y_n\}$ (not necessarily complete) and $\{\lambda_n\} \subset (0,\infty)$ s.t.

$$Ax = \sum_n \lambda_n (x, x_n) y_n.$$

The λ_n are the singular values of A; they are the positive eigenvalues of $|A| := (A^*A)^{\frac{1}{2}}$.

Proof · Apply the Hilbert-Schmidt Theorem to A^*A

$\rightsquigarrow$ eigenvalues $\lambda_1^2 \geq \lambda_2^2 \geq \cdots > 0$ and orthonormal eigenvectors $x_1, x_2, \ldots$, and $(\operatorname{span}\{x_n\})^\perp = \ker A^*A = \ker A$.

(Note: $A^*Ax = 0 \Rightarrow 0 = (x, A^*Ax) = (Ax, Ax) \Rightarrow Ax = 0 \Rightarrow A^*Ax = 0$.)

· Put $y_n = \lambda_n^{-1} A x_n$; then

$$(y_n, y_m) = \lambda_n^{-1}\lambda_m^{-1}(Ax_n, Ax_m) = \lambda_n^{-1}\lambda_m^{-1}(A^*Ax_n, x_m)$$
$$= \lambda_n^{-1}\lambda_m^{-1}\delta_{nm}\lambda_n^2 = \delta_{nm}$$

$\Rightarrow \{y_n\}$ is an orthonormal set.

- $Ax = A\left(\sum_n (x,x_n)x_n\right) = \sum_n \lambda_n (x,x_n) y_n$.
- $\sigma(|A|) = \{\sqrt{\mu} : \mu \in \sigma(A^*A)\} = \{\lambda_n\}$ by the Theorem, part (iv). $\square$

In the "proof" of part (iv) of the Theorem, we use the following notion:

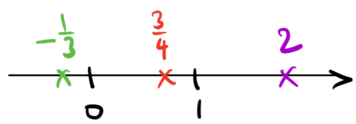
Definition Let (X,d) be a metric space. For $A, B \subset X$ compact, define their **Hausdorff distance** as

best $a \in A$ is that for which the closest point in B is as far away as possible

$$d(A,B) := \max\Big\{ \sup_{a\in A} \underbrace{\inf_{b \in B} d(a,b)}_{\text{closest point to } a \text{ in } B},\ \sup_{b\in B} \underbrace{\inf_{a\in A} d(a,b)}_{\text{closest point to } b \text{ in } A} \Big\}.$$

Examples $X = \mathbb{R}$.

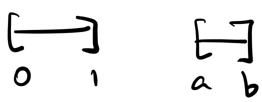
(i) $$d(\{0,1\},\{x\}) = \max\Big\{ \sup\{|x-0|,|x-1|\},\ \inf\{|x-0|,|x-1|\}\Big\}$$



$$= \max\{|x-0|, |x-1|\}$$

$$= \begin{cases} 4/3, & x = -\frac{1}{3} \\ 3/4, & x = \frac{3}{4} \\ 2, & x = 2 \end{cases} \quad \text{(for example).}$$

(ii) $1 < a < b \Rightarrow d([0,1],[a,b]) = \max\{a, b-1\}$.



Proposition The set $\mathcal{C}(X) := \{A \subset X \text{ compact}\}$, equipped with the Hausdorff distance, is a metric space. If X is complete, then so is $\mathcal{C}(X)$.

Proof • Let $A, B \in \mathcal{C}(X)$. Clearly, $d(A,B) \geq 0$. Suppose $\exists\, a_0 \in A \setminus B$;
since B is compact, $\exists\, b_0 \in B$ s.t. $d(a_0, b_0) = d(a_0, B) := \inf_{b \in B} d(a_0, b)$.
$\Rightarrow d(a_0, B) > 0$ (since $b_0 \neq a_0$) $\Rightarrow d(A,B) \geq d(a_0, B) > 0$.
Likewise, $d(A,B) > 0$ if $B \setminus A \neq \emptyset$.

$\Rightarrow d(A,B) = 0$ iff $A = B$.

• Triangle inequality: if $A, B, C \in \mathcal{C}(X)$, then $\forall\, b \in B$:

$$\sup_{a \in A} \inf_{c \in C} d(a,c) \leq \sup_{a \in A} \inf_{c \in C} \big(d(a,b) + d(b,c)\big)$$

$$= \sup_{a \in A} d(a,b) + \inf_{c \in C} d(b,c).$$

Taking $\inf_{b \in B}$ (R.H.S.) gives $\inf_{b \in B} \left(\sup_{a \in A} d(a,b) + \inf_{c \in C} d(b,c) \right)$

$$\leq \inf_{b \in B} \sup_{a \in A} d(a,b) + \sup_{b \in B} \inf_{c \in C} d(b,c),$$

This (and a symmetric argument for $\sup_{c \in C} \inf_{a \in A} d(a,c)$) gives
$d(A,C) \leq d(A,B) + d(B,C)$.

• If X is complete and $\{A_j\} \subset \mathcal{C}(X)$ is Cauchy, then
$A := \{$ limits of convergent sequences $\{a_j\}$ with $a_j \in A_j \; \forall j \}$
is the limit of A_j. (Exercise.) $\square$

Thus, part (iv) of the Theorem follows from 2 separate statements:

Lemma $K \subset \mathbb{R}$ compact, $\{f_j\}_{j \in \mathbb{N}} \subset C^0(K; \mathbb{R})$ convergent with
$f = \lim_{j \to \infty} f_j \in C^0(K; \mathbb{R}) \Rightarrow f_j(K) \longrightarrow f(K)$ in $\mathcal{C}(\mathbb{R})$.

Proof $\sup_{x\in K}\inf_{y\in K}|f_j(x)-f(y)| \le \sup_{x\in K}|f_j(x)-f(x)| = \|f_j-f\|_{C^0(K)}$,

similarly $\sup_{y\in K}\inf_{x\in K}|f_j(x)-f(y)| \le \|f_j-f\|_{C^0(K)}$.

$\Rightarrow d(f_j(K), f(K)) \le \|f_j-f\|_{C^0(K)} \xrightarrow{j\to\infty} 0$. □

Theorem (i) Let X = complex Banach space, $\{A_j\}_{j\in\mathbb{N}} \subset L(X)$ convergent with $A = \lim_{j\to\infty} A_j \in L(X)$. Then $\forall \varepsilon>0\ \exists j_0\in\mathbb{N}$ s.t.

$$\sup_{z\in\sigma(A_j)}\inf_{w\in\sigma(A)}|z-w| < \varepsilon \quad \forall j\ge j_0.$$

("The spectrum depends upper semicontinuously on the operator" — in the limit $A_j\to A$, spectrum may suddenly appear.)

(ii) If X = complex Hilbert space, $A_j = A_j^*$, then also $\forall\varepsilon>0\ \exists j_0$:

$$\sup_{z\in\sigma(A)}\inf_{w\in\sigma(A_j)}|z-w|<\varepsilon \quad \forall j\ge j_0.$$

Thus, $\sigma(A_j)\to\sigma(A)$. ("$\sigma(A)$ depends continuously on $A=A^*$.")

Proof (i) Let $\mathcal{U}_\varepsilon = \{z\in\mathbb{C} : d(z,\sigma(A))<\varepsilon\}$. We need to show:

$\exists j_0\in\mathbb{N}$ s.t. $\sigma(A_j)\subset\mathcal{U}$ $\forall j\ge j_0$.

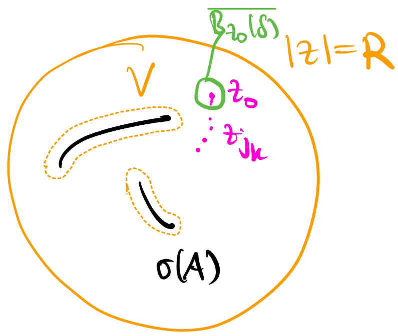
If this were not true, then $\exists z_{j_k}\in\sigma(A_{j_k})$ $(j_k\ge j_0,\ j_k\xrightarrow{k\to\infty}\infty)$ s.t. $d(z_{j_k},\sigma(A))\ge\varepsilon$.

But since $\sup\|A_j\|_{L(X)} =: R<\infty$, z_{j_k} lies in the compact set $V=\{z\in\mathbb{C}: |z|\le R\}\setminus\mathcal{U}$. WLOG $z_{j_k}\to z_0\in V$. ⊕

Then $z_0\notin\sigma(A) \Rightarrow \exists\ \delta>0,\ j_0'\in\mathbb{N}$ s.t.

$z\in\mathbb{C}$, $|z-z_0|<\delta$, $j\ge j_0'$ implies $z\notin\sigma(A_j)$. (This uses that

the subset of $L(X)$ consisting of all invertible operators is open.)
Contradiction to $\oplus$.



(ii). If this were false: $\exists\ \varepsilon>0$ and $j_k \xrightarrow{k\to\infty} \infty$, $z_k \in \sigma(A)$ s.t.

$\sigma(A_{j_k}) \cap B_\varepsilon(z_k) = \emptyset \quad \forall k$. Since $\sigma(A)$ is compact,

WLOG $z_k \to z_0 \in \sigma(A)$; then $B_{\varepsilon/2}(z_0) \subset B_\varepsilon(z_k)$ for $k \geq k_0$

k_0 large, so $\sigma(A_{j_k}) \cap B_{\varepsilon/2}(z_0) = \emptyset$ then.

Since the A_{j_k} are self-adjoint, this implies

$$\|(z_0 - A_{j_k})^{-1}\|_{L(H)} \leq \frac{1}{\varepsilon/2} \qquad (k \geq k_0). \quad \circledast$$

- We claim that $(z_0 - A_{j_k})^{-1}$ is a Cauchy sequence; indeed,

$$\|(z_0 - A_{j_k})^{-1} - (z_0 - A_{j_\ell})^{-1}\|_{L(H)}$$

$$= \| \underbrace{(z_0 - A_{j_k})^{-1}}_{\text{uniformly bounded}} \underbrace{(A_{j_k} - A_{j_\ell})}_{\xrightarrow{k,\ell\to\infty} 0} \underbrace{(z_0 - A_{j_\ell})^{-1}}_{\text{unif. bdd.}} \|_{L(H)}$$

$$\xrightarrow{k,\ell\to\infty} 0.$$

Let $B := \lim_{k\to\infty} (z_0 - A_{j_k})^{-1}$; then

$$B(z_0 - A) = B \cdot \lim_{k\to\infty} (z_0 - A_{j_k}) = I,$$
$$(z_0 - A)B = I$$

$\Rightarrow$ $z_0 - A$ is invertible; contradiction to $z_0 \in \sigma(A)$. $\square$

Remark Part (ii) is not true for general operators A and sequences $A_j \to A$. E.g. let $H = \ell^2(\mathbb{Z})$,

$$A e_n = \begin{cases} e_{n-1} : & n \in \mathbb{Z},\ n \neq 0 \\ 0 : & n = 0. \end{cases}$$

Then $\forall z \in \mathbb{C}$, $|z|<1$, $(A-z)\underbrace{(\ldots, 0, \underset{\substack{\uparrow \\ 0^{th}}}{1}, \underset{\substack{\uparrow \\ 1^{st}}}{z}, z^2, \ldots)}_{\in \ell^2(\mathbb{Z})} = 0$,

so $B_0(1) \subset \sigma_p(A)$.

- But if $A_j = A + \frac{1}{j} B$, $B e_n = \begin{cases} 0 : & n \in \mathbb{Z}, n \neq 0 \\ e_{-1} : & n = 0, \end{cases}$

then $B_0(1) \subset \rho(A_j)$ $\forall j$. (Exercise.)

Remark We have really only proved Theorem (iv) for real-valued f ($\Rightarrow f(A)^* = f(A)$.) The general case follows from a (not very hard) extension of this entire section to normal operators ($AA^* = A^*A$). (Note: $f(A)^* f(A) = \overline{f}(A) f(A) = |f|^2(A) = f(A) f^*(A)$ $\forall f \in C^0(\sigma(A); \mathbb{C})$.)

- In Functional Analysis 2, we will push the functional calculus further to yield the Spectral Theorem. Here, we only briefly indicate the situation in the finite-dimensional case.

Thus, let $H = \mathbb{C}^n$, $A \in L(H)$ self-adjoint; thus the continuous functional calculus for A is an algebra homomorphism

$$\phi : C^0(\sigma(A)) \longrightarrow L(H)$$

$$C^0(\sigma(A)) = \{ f : \sigma(A) \to \mathbb{C} \} \quad (\text{since } \sigma(A) \subset \mathbb{C} \text{ is a finite set}).$$

Explicitly, if $\sigma_1, \ldots, \sigma_m \in \mathbb{R}$ are the distinct eigenvalues, and $\Pi_j : \mathbb{C}^n \to E_j = \ker(\sigma_j - A)$ is the orthogonal projection to the j-th eigenspace, then $A = \sum_{j=1}^m \sigma_j \Pi_j$, and $f(A) = \sum_{j=1}^m f(\sigma_j) \Pi_j$.

Let us look at this more abstractly.

Let $x \in H$; then $C^0(\sigma(A)) \ni f \mapsto (f(A)x, x) \in \mathbb{C}$ is a continuous linear map, and thus

$$(f(A)x,x) = \sum f(\sigma_j)\, \mu_x(\sigma_j).$$

(Explicitly, $\mu_x(\sigma_j) = (\underbrace{1_{\{\sigma_j\}}}_{f = \begin{cases} 1 & \text{at } \sigma_j \\ 0 & \text{at } \sigma_\ell \neq \sigma_j \end{cases}}(A)x, x) = (\Pi_j x, x) = \|\Pi_j x\|^2.$)

Set $d\mu_x = \sum_{j=1}^{m} \mu_x(\sigma_j)\, \delta_{\sigma_j}$, then $(f(A)x,x) = \int_{\sigma(A)} f\, d\mu_x$.

$d\mu_x$ is the spectral measure of x. (Here: a counting measure.)

By polarization, $\forall x,y \in H\ \exists$ measure $d\mu_{x,y}$ on $\sigma(A)$ s.t.

$$(f(A)x,y) = \int_{\sigma(A)} f\, d\mu_{x,y} \quad \forall f \in C^0(\sigma(A)).$$

(Explicitly, $d\mu_{x,y} = \left(\sum_{j=1}^{m} (\Pi_j x, y)\, \delta_{\sigma_j}.\right)$

Dropping x,y: $\quad f(A) = \int_{\sigma(A)} f(z)\, d\mu(z)$

where $d\mu$ is a projection-valued measure.

(Explicitly, $d\mu = \sum_{j=1}^{m} \Pi_j\, \delta_{\sigma_j}$.)

Note how the projections Π_j drop out of our abstract arguments!

Examples (i) $I = 1(A) = \int_{\sigma(A)} d\mu$

(ii) $A = \int_{\sigma(A)} z\, d\mu(z)$

(iii) An example for a bounded operator on $\ell^2(\mathbb{Z})$:

$(Au)(n) = u(n+1) - 2u(n) + u(n-1)$; $\sigma(A) = [-4,0]$. $\psi \in \ell^2(\mathbb{Z})$

$\Rightarrow \ (f(A)\psi, \psi) = \int_{-4}^{0} f(\lambda)\, d\mu_\psi(\lambda),$

$d\mu_\psi(\lambda) = \left(\left| \hat{\varphi}\left(\arccos\left(\frac{\lambda+2}{2} \right)\right)\right|^2 + \left| \hat{\varphi}\left(-\arccos\left(\frac{\lambda+2}{2} \right)\right)\right|^2 \right) \frac{d\lambda}{\sqrt{-\lambda^2 - 4\lambda}}.$

If we are brave and use $f(x) = 1_{[a,b]}(x) = \begin{cases} 1, & a \le x \le b \\ 0 & \text{otherwise}, \end{cases}$

then $f(A)\psi$ projects ψ onto its "components" corresponding to spectrum in $[a,b]$.

Theorem (Spectral Theorem.) Let $A \in L(H)$, $A = A^*$, and suppose $\exists\, \psi \in H$ s.t. $\operatorname{span}\{ p(A)\psi : p \in \mathbb{C}[x]\} \subset H$ is dense. Then $\exists$ unitary operator $U : H \longrightarrow L^2(\sigma(A); d\mu_\psi)$ s.t.

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda).$$

Proof sketch $U(\phi(f)\psi) := f \quad (f \in C^0(\sigma(A)))$ does the job; $U(A\,\phi(f)\psi) = U(\phi(\lambda f)\psi) = \lambda f \quad (f = f(\lambda))$ $\square$

$\left(\|\phi(f)\psi\|^2 = (\phi(f)^*\phi(f)\psi, \psi) = (\phi(\bar{f}f)\psi, \psi) = \int_{\sigma(A)} |f|^2\, d\mu_\psi. \right)$

Remark (i) When $\dim H = \infty$, I still need to explain where $d\mu_\psi$ comes from.

(ii) When $\dim H < \infty$, the existence of ψ is equivalent to the property that all eigenvalues of A are distinct.

(If this fails, we get unitary equivalence of A to a direct sum of $L^2(\sigma(A), d\mu_{\psi_j})$'s for appropriate ψ_j's.)