We will deduce the spectral theorem for unbounded self-adjoint operators from results on bounded operators. Our discussion here follows the last part of FA I. Throughout, H denotes a Hilbert space.

Definition (D.7) $A \in L(H)$ is **normal** if $AA^* = A^*A$.

(Equivalently, $A = B + iC$ where $B = B^*$, $C = C^*$, and $BC = CB$)

Lemma (L.4) If $A \in L(H)$ is normal, then $\|A^n\| = \|A\|^n \ \forall n \in \mathbb{N}$, and $\underbrace{r_A := \sup_{z \in \sigma(A)} |z|}_{\text{spectral radius}} = \|A\|$

Proof • We claim that $\|Ax\| = \|A^*x\| \ \forall x \in H$. $(*)$

Indeed, $\|Ax\|^2 = (Ax, Ax) = (A^*Ax, x)$

$= (AA^*x, x) = (A^*x, A^*x) = \|A^*x\|^2$.

• If we have shown $\|A^n\| = \|A\|^n$, then for $x \in H$, $\|x\| = 1$,

$$\|A^n x\|^2 = (A^n x, A^n x) = (A^* A^n x, A^{n-1}x) \leq \|A^* A^n x\| \, \|A^{n-1}x\|$$
$$\overset{(*)}{=} \|A^{1+n}x\| \, \|A^{n-1}x\|$$
$$\leq \|A^{n+1}\| \, \|A^{n-1}\|$$
$$\leq \|A^{n+1}\| \, \|A\|^{n-1}.$$

Taking sup over all x, $\|A\|^{2n} = \|A^n\|^2 \leq \|A^{n+1}\| \, \|A\|^{n-1}$

$\Rightarrow \|A\|^{n+1} \leq \|A^{n+1}\|$.

The converse inequality $\|A^{n+1}\| \leq \|A\|^{n+1}$ is clear.

• Finally, $r_A = \lim_{n \to \infty} \|A^n\|^{1/n} = \|A\|$. $\square$

Theorem (T.6) ("(Continuous) functional calculus")

Let $A \in L(H)$ be self-adjoint. Then there exists a unique continuous *-algebra homomorphism

$$\phi : C^0(\sigma(A)) \longrightarrow L(H)$$

$\left(\begin{array}{l} \phi(\lambda p + \mu q) = \lambda \phi(p) + \mu \phi(q) \\ \phi(1) = I \\ \phi(pq) = \phi(p) \circ \phi(q) \\ \phi(p)^* = \phi(\bar{p}) \end{array} \right)$

so that $\phi(z) = A$ (where $z : \sigma(A) \ni z \mapsto z$).

Moreover, ϕ satisfies:

(i) $\|\phi(f)\|_{L(H)} = \|f\|_{C^0} = \sup_{z \in \sigma(A)} |f(z)|$

(ii) $\sigma(\phi(f)) = f(\sigma(A))$; $\quad Au = zu \Rightarrow \phi(f)Au = f(z)u$.

(iii) $f \geq 0 \Rightarrow \phi(f) \geq 0$, i.e. $(\phi(f)x, x) \geq 0 \ \forall x \in H$.

Proof • If $f = \sum_{j=0}^{n} f_j x^j \in \mathbb{C}[x]$, then $\phi(f) = \sum_{j=0}^{n} f_j A^j$. We show (ii) in this case:

* Let $z_0 \in \sigma(A)$. Write $f(x) - f(z_0) = (x - z_0) g(x)$, $g \in \mathbb{C}[x]$, then

$$\phi(f) - f(z_0) = (A - z_0)\phi(g) = \phi(g)(A - z_0)$$

is not injective or not surjective (or both); so $f(z_0) \in \sigma(\phi(f))$.

* Let $w_0 \in \sigma(\phi(f))$. Factor $f(x) - w_0 = c \prod_{j=1}^{n} (x - x_j)$, $c, x_j \in \mathbb{C}$; thus $f(x_j) = w_0$, and

$$\phi(f) - w_0 = c \prod_{j=1}^{n} (A - x_j).$$

Since $\phi(f) - w_0$ is not invertible, $\exists j$ s.t. $A - x_j$ is not invertible, so $x_j \in \sigma(A)$, and $w_0 = f(x_j)$.

• For $f \in \mathbb{C}[x]$, we now conclude (since $\phi(f)$ is normal) using Lemma (L.4) that

$$\|\phi(f)\|_{L(H)} = r_{\phi(f)} = \sup_{z\in\sigma(\phi(f))} |z| = \|f\|_{C^0}.$$

- We can extend ϕ from $\mathbb{C}[x]$ to $C^0(\sigma(A))$ by the density of $\mathbb{C}[x] \subset C^0(\sigma(A))$ (Stone-Weierstrass); and (i) holds by continuity.
- For (ii) for general $f \in C^0(\sigma(A))$:
  - * If $z \notin f(\sigma(A))$, then $g(x) := \frac{1}{z - f(x)}$ defines $g \in C^0(\sigma(A))$, and
    $(z - \phi(f)) \circ \phi(g) = \phi(\mathbb{1}) = I = \phi(g) \circ (z - \phi(f))$;
    so $z \notin \sigma(\phi(f))$.
  - * Suppose $z = f(x_0)$, $x_0 \in \sigma(A)$. We claim that $\exists\, u_k \in H$, $\|u_k\| = 1$, so that $\|(z - \phi(f))u_k\| \longrightarrow 0$ as $k \to \infty$; this implies $z \in \sigma(\phi(f))$. To see this:

    Step 1: $\exists\, u_k \in H$, $\|u_k\| = 1$, $\|(x_0 - A)u_k\| \to 0$.

    This is clear when $\ker(x_0 - A) \neq \{0\}$. If $\ker(x_0 - A) = \{0\}$, then $\operatorname{ran}(x_0 - A)$ is dense (since $\operatorname{ran}(x_0 - A)^\perp = \ker(x_0 - A)$), but not closed since $x_0 \in \sigma(A)$. Thus, $\nexists\, C < \infty$ s.t. $\|u\| \le C\|(x_0 - A)u\|$ $\forall u$; this implies the claim.

    Step 2: If $f \in \mathbb{C}[x]$, then $\|(f(x_0) - \phi(f))u_k\| \to 0$.

    Writing $f(x) = \sum_{j=0}^{n} f_j x^j$, we have

    $$(f(x_0) - \phi(f))u_k = \sum_{j=0}^{n} f_j (x_0^j - A^j)u_k$$
    $$= \underbrace{\left[\sum_{j=0}^{n} f_j \left(\sum_{l=0}^{j-1} x_0^l A^{j-1-l}\right)\right]}_{\in L(H)} \underbrace{(x_0 - A)u_k}_{\to 0} \xrightarrow{k\to\infty} 0.$$

    Step 3: Let $f \in C^0(\sigma(A))$ and pick $f_j \in \mathbb{C}[x]$, $f_j \xrightarrow{j\to\infty} f$ in $C^0(\sigma(A))$.

Let $\varepsilon>0$, and let j be s.t. $\|f_j - f\|_{C^0(\sigma(A))} < \varepsilon$.

Let then k_0 be s.t. $\|(f_j(x_0) - \phi(f_j))u_k\| < \varepsilon \quad \forall k \geq k_0$.

Then $\|(f(x_0) - \phi(f))u_k\|$

$$\leq \|(f_j(x_0) - \phi(f_j))u_k\| + \|(f_j(x_0) - f(x_0))u_k\| + \|(\phi(f_j) - \phi(f))u_k\|$$

$$< \varepsilon + 2\|f_j - f\|_{C^0(\sigma(A))}$$

$$< 3\varepsilon.$$

This finishes the proof of the claim.

- For (iii), write $f \geq 0$ as $f = g^2$, $g \geq 0$, then

$$(\phi(f)x, x) = (\phi(g^2)x, x) = (\phi(g)\phi(g)x, x) \overset{\phi(g) = \phi(g)^*}{=\!=} (\phi(g)x, \phi(g)x)$$
$$= \|\phi(g)x\|^2 \geq 0. \qquad \square$$

For $A = A^* \in L(H)$ and $u \in H$, consider the map

$$L_u : C^0(\sigma(A)) \ni f \mapsto (f(A)u, u) \in \mathbb{C}. \qquad \# \qquad \text{Here, } f(A) := \phi(f).$$

Note: (i) $|L_u f| \leq \|f(A)u\| \, \|u\| \leq \|f(A)\|_{L(H)} \|u\|^2$

$$= \|u\|^2 \|f\|_{C^0(\sigma(A))},$$

so $L_u \in (C^0(\sigma(A)))^*$.

(ii) L_u is a positive functional on $C^0(\sigma(A))$, that is,

$$f \geq 0 \Rightarrow L_u(f) \geq 0.$$

(This is Theorem (T.6)(iii).)

Theorem (T.7) (Riesz–Markov.) Let X be a compact Hausdorff space, and let $\lambda \in (C^0(X))^*$ be a positive linear functional. Then there exists a unique Radon measure μ on X s.t.

$$\lambda(f) = \int_X f\, d\mu, \qquad f \in C^0(X). \quad ⊛$$

Conversely, for every Radon measure μ, ⊛ defines a positive linear functional on $C^0(X)$.

Recall: Let X be a compact Hausdorff space. Let $\mathcal{B} \subset 2^X$ denote the Borel σ-algebra, and let $\mu : \mathcal{B} \to [0,\infty]$ be a measure (i.e. μ is a Borel measure); that is, $\mu(\emptyset)=0$ and $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$ for all pairwise disjoint $E_1, E_2, \ldots \in \mathcal{B}$. Then μ is a Radon measure if μ is (i) finite (i.e. $\mu(X) < \infty$)

(ii) inner regular: $\forall E \in \mathcal{B}$,

$$\mu(E) = \sup_{\substack{K \subseteq E \\ K \text{ compact}}} \mu(K),$$

(iii) outer regular: $\forall E \in \mathcal{B}$,

$$\mu(E) = \inf_{\substack{O \supseteq E \\ O \text{ open}}} \mu(O).$$

We will prove the Riesz–Markov theorem later.

Returning to $L_u(f) = (f(A)u, u)$, $u \in H$, from ⊞, we now get a Radon measure μ_u on $C^0(\sigma(A))$ so that

$$(f(A)u,u) = \int_{\sigma(A)} f\, d\mu_u, \quad f \in C^0(\sigma(A)). \quad ⊛$$

In particular, $\mu_u(\sigma(A)) = (\mathbb{1}(A)u,u) = (u,u) = \|u\|^2$.

Definition (D.8) For $A = A^* \in L(H)$, $u \in H$, the (unique Radon) measure μ_u on $\sigma(A)$ s.t. ⊛ holds is called the *spectral measure associated with u*.

Example (E.11) Let $H=\mathbb{C}^2$, $A=\begin{pmatrix}1&0\\0&2\end{pmatrix} \Rightarrow \sigma(A)=\{1,2\}$.

For $f\in C^0(\sigma(A))$ (i.e. $f=(f(1),f(2))\in\mathbb{C}\times\mathbb{C}$), $f(A)=\begin{pmatrix}f(1)&0\\0&f(2)\end{pmatrix}$ (Exercise). Let $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$; then

$$(f(A)u,u)=\left\langle\begin{pmatrix}f(1)u_1\\f(2)u_2\end{pmatrix},\begin{pmatrix}u_1\\u_2\end{pmatrix}\right\rangle_{\mathbb{C}^2}=f(1)|u_1|^2+f(2)|u_2|^2$$

$$\overset{!}{=}\int_{\sigma(A)}f\,\overset{?}{d\mu_u}=f(1)\mu_u(\{1\})+f(2)\mu_u(\{2\}).$$

$\Rightarrow \mu_u=|u_1|^2\delta_1+|u_2|^2\delta_2$. $\quad$ ($\delta_x(E)=\begin{cases}1, & x\in E\\ 0, & x\notin E\end{cases}$ Dirac measure).

Note: if $\pi_1:\begin{pmatrix}u_1\\u_2\end{pmatrix}\mapsto\begin{pmatrix}u_1\\0\end{pmatrix}$ is the projection map onto the eigenspace of A with eigenvalue 1, then $|u_1|^2=\|\pi_1u\|^2$. $|u_2|^2$ has an analogous interpretation.

So: μ_u records "how much" of u "lives" in the various eigenspaces of A.

Definition (D.9) Let $A\in L(H)$. Then $u\in H$ is a cyclic vector for A if $\operatorname{span}\{u,Au,A^2u,\dots\}$ is dense in H.

Example (E.11, continued) (i) Every $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$ with $u_1\neq0$ and $u_2\neq0$ is cyclic for $A=\begin{pmatrix}1&0\\0&2\end{pmatrix}$.

(ii) $A=\begin{pmatrix}3&0\\0&3\end{pmatrix}$ has no cyclic vector! (Issue: the multiplicity of the spectrum is >1.)

Lemma (L.5) Let $A=A^*\in L(H)$, and suppose $v\in H$ is a cyclic vector for A. Then $\exists$ unitary operator $\mathcal{U}:H\to L^2(\sigma(A),d\mu_v)$ so that $(\mathcal{U}A\mathcal{U}^{-1}f)(\lambda)=\lambda f(\lambda)$, $f\in L^2(\sigma(A),d\mu_v)$.

That is, A is unitarily equivalent to multiplication by λ on $\sigma(A)$.

Recall: H_1, H_2 Hilbert spaces. $U \in L(H_1, H_2)$ is unitary if U is an isometric isomorphism ($\iff UU^* = U^*U = I$).

Proof of Lemma (L.5) Define $U(f(A)v) := f$ for $f \in C^0(\sigma(A))$.

- Well-definedness: $\|f(A)v\|_H^2 = (f(A)v, f(A)v) = (f(A)^* f(A)v, v)$

$$= (|f|^2(A)v, v) = \int_{\sigma(A)} |f|^2 \, d\mu_v$$

$$= \|f\|^2_{L^2(\sigma(A), d\mu_v)}.$$

- Since v is cyclic for A, $\{f(A)v : f \in C^0(\sigma(A))\} \subset H$ is dense, and thus U extends to a continuous isometry $H \to L^2(\sigma(A), d\mu_v)$.
- Since $C^0(\sigma(A)) \subset L^2(\sigma(A), d\mu_v)$ is dense, $\operatorname{ran} U = L^2(\sigma(A), d\mu_v)$, so U is an isometry.
- For $f \in C^0(\sigma(A))$,

$$(UAU^{-1}f)(\lambda) = (UAf(A)v)(\lambda)$$

$$= (U\tilde{f}(A))(\lambda) \qquad (\tilde{f}(\lambda) := \lambda f(\lambda))$$

$$= \tilde{f}(\lambda)$$

$$= \lambda f(\lambda).$$

This extends by continuity to all $f \in L^2(\sigma(A), d\mu_v)$. $\square$

Example (E.11, continued') $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = A^* \in L(\mathbb{C}^2)$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ cyclic (so $v_1 \neq 0, v_2 \neq 0$), $d\mu_v = |v_1|^2 \delta_1 + |v_2|^2 \delta_2$.

$\Rightarrow L^2(\sigma(A), d\mu_v) \cong (\mathbb{C}^2, \|\cdot\|_v)$ where $\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\|_v^2 = |v_1|^2|a|^2 + |v_2|^2|b|^2$.

- The map $U : (\mathbb{C}^2, \|\cdot\|) \to (\mathbb{C}^2, \|\cdot\|_v)$ maps

$$f(A)v = \begin{pmatrix} f(1)v_1 \\ f(2)v_2 \end{pmatrix} \longmapsto f = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix},$$

i.e. $U\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a/v_1\\b/v_2\end{pmatrix} = (1 \mapsto \frac{a}{v_1},\ 2 \mapsto \frac{b}{v_2})$.

- Thus, $UAU^{-1}: L^2(\sigma(A), d\mu_v) \ni (1 \mapsto a, 2 \mapsto b) \longmapsto (1 \mapsto a, 2 \mapsto 2b)$
$\in L^2(\sigma(A), d\mu_v)$.

Lemma (L.6) Let $A = A^* \in L(\mathcal{H})$, $\mathcal{H}$ separable. Then $\exists N \in \mathbb{N} \cup \{+\infty\}$ and closed pairwise orthogonal subspaces $\{\mathcal{H}_n\}_{n=1}^N$ (i.e. $v \perp w\ \forall v \in \mathcal{H}_n, w \in \mathcal{H}_m, n \neq m$) s.t. $\mathcal{H} = \bigoplus_{n=1}^N \mathcal{H}_n$, and

(i) $A(\mathcal{H}_n) \subset \mathcal{H}_n$

(ii) $\forall n\ \exists v_n \in \mathcal{H}_n$ which is cyclic for $A|_{\mathcal{H}_n} \in L(\mathcal{H}_n)$, i.e. $\mathrm{span}\{v_n, Av_n, A^2v_n, \dots\} \subset \mathcal{H}_n$ is dense.

Proof (a) Take any $0 \neq v_1 \in \mathcal{H}$ and let $\mathcal{H}_1 := \overline{\mathrm{span}\{A^j v_1 : j \in \mathbb{N}_0\}}$.
Then (i) and (ii) hold for $\mathcal{H}_1, v_1$.

(b) $\mathcal{H}_1^\perp$ is A-invariant: if $u \in \mathcal{H}_1^\perp$ and $v \in \mathcal{H}_1$, then
$$(Au, v) = (u, \underbrace{Av}_{\in \mathcal{H}_1}) = 0, \text{ so } Au \in \mathcal{H}_1^\perp.$$

- Continue with $\mathcal{H}_1^\perp$ in place of $\mathcal{H}$. More precisely: apply Zorn's Lemma using the ideas (a) & (b). $\square$

Theorem (T.8) (Spectral Theorem, multiplication operator form.)
Let $A = A^* \in L(\mathcal{H})$, $\mathcal{H}$ separable. Then $\exists$ Radon measures $\{\mu_n\}_{n=1}^N$ ($N \in \mathbb{N} \cup \{\infty\}$) and a unitary operator
$$U: \mathcal{H} \longrightarrow \bigoplus_{n=1}^N L^2(\mathbb{R}, d\mu_n)$$
so that $(UAU^{-1}f)_n(\lambda) = \lambda f_n(\lambda)$, $f = (f_n)_{n=1}^N \in \bigoplus_{n=1}^N L^2(\mathbb{R}, d\mu_n)$.

Proof Get $\mathcal{H}_n$ from Lemma (L.6), and apply Lemma (L.5) to each $A|_{H_n}$. $\square$

Remark (R.7) Write $M := \bigsqcup_{n=1}^{N} \mathbb{R}$ (disjoint union) $\equiv \bigcup_{n=1}^{N} \{n\} \times \mathbb{R}$,

$$\mu = \bigsqcup_{n=1}^{N} \mu_n \quad (\text{i.e. } \mu|_{\{n\}\times\mathbb{R}} = \mu_n),$$

$$g((n,\lambda)) = \lambda \quad (\text{so } g : M \to \mathbb{R}),$$

then $U : H \to L^2(M, d\mu)$ has the property that

$$UAU^{-1}f = gf, \quad f \in L^2(M, d\mu).$$

Corollary (C.1) $A = A^* \in L(H)$. Then $\exists$ finite measure space (M,μ), a bounded measurable function f on M, and a unitary map

$$U : H \to L^2(M, d\mu) \text{ s.t. } \quad UAU^{-1} = T_f : g \mapsto fg \quad (g \in L^2(M,\mu)).$$

(Compare this with Proposition (P.2)!)

Proof Follow Remark (R.7). If we choose the vectors v_n generating the cyclic subspaces of H so that $\|v_n\| = 2^{-n}$, then

$$\mu_n(\mathbb{R}) = \|v_n\|^2 = 2^{-2n}, \quad \text{so } \mu(M) = \sum_{n=1}^{N} \mu_n(\mathbb{R}) < \infty. \qquad \square$$

Example (E.12) Let $H = \ell^2(\mathbb{Z})$ and $(Au)_n = i(u_{n+1} - u_{n-1})$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$.

(This is a discrete version of $i\frac{d}{dt}$.) Check that $A = A^* \in L^2(\ell^2(\mathbb{Z}))$.

Recall that $\mathcal{F}^{-1} : \ell^2(\mathbb{Z}) \to L^2([0,2\pi])$, $\mathcal{F}^{-1}u(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} u_n e^{in\xi}$, is unitary. Let $\tilde{A} = \mathcal{F}^{-1} A \mathcal{F}$, so

$$\begin{aligned}
\tilde{A}u(\xi) &= \frac{1}{\sqrt{2\pi}} \sum_n i(\mathcal{F}u(n+1) - \mathcal{F}u(n-1)) e^{in\xi} \\
&= \frac{1}{\sqrt{2\pi}} \sum_n i e^{-i\xi} \mathcal{F}u(n+1) e^{i(n+1)\xi} - i e^{i\xi} \mathcal{F}u(n-1) e^{i(n-1)\xi} \\
&= i(e^{-i\xi} - e^{i\xi}) u(\xi)
\end{aligned}$$

$= 2\sin(\xi)\, u(\xi).$

So $\mathcal{U} = \mathcal{F}$ is a unitary map as in Corollary (C.1) with

$$\mathcal{U} A \mathcal{U}^{-1} = \tilde{A} = T_g \text{ on } L^2([0,2\pi]), \quad g = 2\sin(\xi).$$

(Finding explicit $\mathcal{U}$ is only possible in very special, even if important, circumstances...)

We shall now extend the continuous functional calculus, Theorem (T.6), to a more general functional calculus:

Theorem (T.9) ("Borel functional calculus")

Let $A = A^* \in L(H)$. Let $B^\infty(\sigma(A)) := \{u : \sigma(A) \to \mathbb{C}$ Borel-measurable, $\|u\|_\infty = \sup_{x \in \sigma(A)} |u(x)| < \infty\}$. Then there exists a unique continuous $*$-algebra-homomorphism $\phi : B^\infty(\sigma(A)) \to L(H)$ so that

(i) $\phi(f) = A$ for $f(x) = x$;

(ii) if $f_n(x) \to f(x)$ $\forall x \in \sigma(A)$ $(f_n, f \in B^\infty(\sigma(A)))$ and $\|f_n\|_\infty$ is bounded, then $\phi(f_n) \to \phi(f)$ strongly in $L(H)$, i.e. $\phi(f_n)u \to \phi(f)u$ $\forall u \in H$.

Moreover, ϕ has the properties

(iii) $Au = \lambda u \Rightarrow \phi(f)u = f(\lambda)u$,

(iv) $f \geq 0 \Rightarrow \phi(f) \geq 0$.

(v) If $B \in L(H)$ commutes with A (so $AB = BA$), then B commutes with $\phi(f)$ $\forall f \in B^\infty(\sigma(A))$.

Remark (R.8) Instead of $B^\infty(\sigma(A))$, one can use $L^\infty(\sigma(A))$, if one defines this as the space of equivalence classes of $B^\infty(\sigma(A))$ where $u \sim v$ if $u(x) = v(x)$ for almost all x w.r.t. all spectral measures μ_{v_n} (v_n = cyclic vectors from before).

Remark (R.9) The point of (ii) is that $B^\infty(\sigma(A))$ is the smallest space of functions $\sigma(A) \to \mathbb{C}$ which (α) contains $C^0(\sigma(A))$ and (β) is closed under pointwise limits of uniformly bounded sequences (as in part (ii)). (Exercise)

Proof of Theorem (T.9)

- Existence of ϕ: * We apply Corollary (C.1), so $U : H \to L^2(M, d\mu)$ is unitary, $g : M \to \mathbb{R}$ is bounded and measurable,

  $UAU^{-1} = T_g$ (multiplication by g) on $L^2(M, d\mu)$.

  We then define $\phi(f)$, $f \in B^\infty(\sigma(A))$, via

  $$U\phi(f)U^{-1} = T_{f\circ g}.$$

  (Idea/motivation: If A = multiplication by λ on $L^2(\sigma(A), d\mu)$, then $\phi(f)$ = multiplication by $f(\lambda)$.)

* It is easy to check that ϕ is a $*$-algebra-homomorphism, and that (i), (iii), and (iv) hold (Exercise).

* To check (ii), let $u \in L(M, d\mu)$; then $|f_n(g(x))u(x)| \le \|f_n\|_\infty |u(x)|$ is dominated by $v := (\sup_n \|f_n\|_\infty)|u| \in L^2(M, d\mu)$ and

converges pointwise to u; so $T_{f_n}u = f_n u \longrightarrow fu = T_f u$ in $L^2(M, d\mu)$ by the Dominated Convergence Theorem.

Since U is unitary, this implies the strong convergence

$$\phi(f_n) = U^{-1} \circ T_{f_n} \circ U \longrightarrow U^{-1} \circ T_f \circ U = \phi(f).$$

* To check (v), note that $T_g \widetilde{B} = \widetilde{B} T_g$ where $\widetilde{B} = UBU^{-1} \in L(L^2(M, d\mu))$.

This implies $T_{g^2}\widetilde{B} = T_g T_g \widetilde{B} = T_g \widetilde{B} T_g = \widetilde{B} T_g T_g = \widetilde{B} T_{g^2}$, and more generally $T_{f\circ g}\widetilde{B} = \widetilde{B} T_{f \circ g}$ (*) for $f \in \mathbb{C}[x]$.

By continuity of $C^0(\sigma(A)) \ni f \longmapsto T_{f\circ g} \in L(L^2(M, d\mu))$, we get (*) for all $f \in C^0(\sigma(A))$.

Finally, using property (ii) and Remark (R.9), we get (*) for all $f \in B^\infty(\sigma(A))$.

· Uniqueness of ϕ: The $*$-algebra-homomorphism and continuity requirements fix ϕ on $C^0(\sigma(A))$ (cf. Theorem (T.6)). The uniqueness on $B^\infty(\sigma(A))$ follows from Remark (R.9). $\square$

Why would one want to do this? The main point in applications is that we can take $f = 1_\Omega$, $\Omega \subset \sigma(A)$ measurable (typically even just an interval). The operator

$$P(\Omega) := 1_\Omega(A) := \phi(1_\Omega) \in L(H)$$

is then called the spectral projector to Ω. (Note indeed that $P(\Omega)^2 = P(\Omega)$ since $1_\Omega^2 = 1_\Omega$; and $P(\Omega)^* = P(\Omega)$, so $P(\Omega)$ is in fact an orthogonal projection!)

We moreover have the relationship between $P(\Omega)$ and spectral measures:

$v \in H$, $\Omega \subset \sigma(A)$ measurable $\Rightarrow$ $\mu_v(\Omega) = \int_{\sigma(A)} 1_\Omega \, d\mu_v = (1_\Omega(P)v, v) = (P(\Omega)v, v)$
$= \|P(\Omega)v\|^2.$

Example (E.13) (i) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\Omega = \{2\} \Rightarrow P(\Omega) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
(orthogonal projection onto the 2-eigenspace of A).

(ii) Let $A = T_f$ on $L^2([0,1])$, where $f(x) = x$. (Cf. Proposition (P.2).) So $Au(x) = xu(x)$. (In Quantum Mechanics, this is the position operator.) Let $\Omega \subset \mathbb{R}$ be measurable. Then $P(\Omega)u(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$ $\otimes$

NB: A has no eigenvalues (Exercise: $\sigma(A) = \sigma_c(A) = [0,1]$). Physicists would tell you $|x\rangle = \delta_x$ (Delta distribution at x) is an eigenfunction of A with $A|x\rangle = x|x\rangle$. But of course $|x\rangle \notin L^2([0,1])$. But $\otimes$ does look like $P(\Omega)$ projects u onto the "span of all eigenspaces of A with eigenvalues in Ω"; it does so, however, in a mathematically meaningful way!

Part due to intrinsic interest, but also to prepare the unbounded case, we generalize Theorem (T.8) to families of commuting self-adjoint operators — compare with the simultaneous diagonalizability of commuting self-adjoint matrices.

Theorem (T.10) H = separable Hilbert space; $A_1, \dots, A_n \in L(H)$ commuting self-adjoint operators (i.e. $A_j^* = A_j$, $A_j A_k = A_k A_j$).
Then $\exists$ · finite measure space (M, μ),
· bounded measurable functions $f_1, \dots, f_n : M \to \mathbb{R}$,
· a unitary map $U : H \to L^2(M, d\mu)$,
so that
$$(U A_j U^{-1} g)(x) = f_j(x) g(x), \qquad g \in L^2(M, d\mu),\ x \in M,\ j = 1, \dots, n.$$

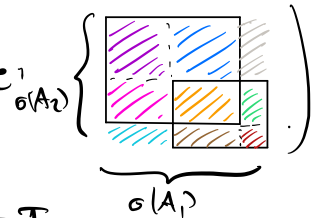
Proof We will construct a functional calculus for $(A_1, \dots, A_n)$ and use Theorem (T.7) (Riesz–Markov) to produce (M, μ).
Let $K := \prod_{j=1}^{n} \sigma(A_j) \in \mathbb{R}^n$.

Step 1: Define $f(A_1, \dots, A_n)$ for simple functions $f \in S$, where
$$S := \left\{ \sum_{j=1}^{m} \alpha_j 1_{R_j}(x) : m \in \mathbb{N},\ \alpha_j \in \mathbb{C},\ R_j \in \mathcal{R} \right\},$$
$$\mathcal{R} := \{ E_1 \times \dots \times E_n : E_j \subset \sigma(A_j) \text{ measurable} \}.$$

To do this, we only need to define $f(A_1, \dots, A_n)$ for $f = 1_R$, $R = E_1 \times \dots \times E_n \in \mathcal{R}$. We set
$$f(A_1, \dots, A_n) = \underbrace{1_{E_1}(A_1)}_{\text{spectral projection}} \cdots 1_{E_n}(A_n) \in L(H).$$

Step 2: For $f \in S$, $\| f(A_1, \dots, A_n) \|_{L(H)} \le \| f \|_\infty := \sup_{x \in K} |f(x)|$.
Indeed, we may write $f = \sum_{j=1}^{m} \alpha_j 1_{R_j}$ where $\alpha_j \in \mathbb{C}$ and the $R_j \subset \mathcal{R}$ are pairwise disjoint, and $\cup R_j = K$.

$$\Longrightarrow \|f\|_\infty = \max_{1\le j\le m} |\alpha_j| .$$

(Exercise; picture: )

· On the other hand,

$$\sum_{j=1}^m 1_{R_j}(A_1,\dots,A_n) = \underbrace{1_{\sigma(A_1)}(A_1)}_{=I}\cdots \underbrace{1_{\sigma(A_n)}(A_n)}_{=I} = I,$$

and writing $R_j = \prod_{p=1}^n E_{j,p}$

$$1_{R_j}(A_1,\dots,A_n)\,1_{R_k}(A_1,\dots,A_n) = \prod_{p=1}^n 1_{E_{j,p}}(A_p)\prod_{p=1}^n 1_{E_{k,p}}(A_p)$$

$\left(\begin{array}{c}\text{all operators}\\ 1_{E_{j,p}}(A_p)\\ \text{commute}\end{array}\right)$ $\Uparrow$ (Theorem (T.9)(v) & Exercise)

$$\overset{!}{=} \prod_{p=1}^n 1_{E_{j,p}\cap E_{k,p}}(A_p) = 0 \text{ if } j\neq k \text{ (since then } \exists\, p \text{ s.t. } E_{j,p}\cap E_{k,p} = \emptyset).$$

· Therefore, for $v\in H$,

$$\|f(A_1,\dots,A_n)v\|^2 = \Big\|\sum_{j=1}^m \alpha_j \overbrace{1_{R_j}(A_1,\dots,A_n)v}^{\text{pairwise orthogonal}}\Big\|^2$$

$$= \sum_{j=1}^m |\alpha_j|^2 \|1_{R_j}(A_1,\dots,A_n)v\|^2$$

$$\le \Big(\max_{1\le j\le m} |\alpha_j|^2\Big)\sum_{j=1}^m \|\overbrace{1_{R_j}(A_1,\dots,A_n)v}^{\text{pairwise orthogonal}}\|^2$$

$$= \|f\|_\infty^2 \Big\|\underbrace{\sum_{j=1}^m 1_{R_j}(A_1,\dots,A_n)}_{=I}v\Big\|^2$$

$$= \|f\|_\infty^2 \|v\|^2 .$$

Step 3: $f(A_1,\dots,A_n)$ for $f\in C^0(K)$.

Well, given $f\in C^0(K)$, pick $f_j\in S$, $j\in\mathbb{N}$, s.t. $\|f-f_j\|_\infty \xrightarrow{j\to\infty} 0$.

By Step 2, $f_j(A_1,\dots,A_n)\in L(H)$ is a Cauchy sequence, and we can set

$$f(A_1,\dots,A_n) = \lim_{j\to\infty} f_j(A_1,\dots,A_n) \in L(H).$$

The map

$$\phi : C^0(K) \ni f \longmapsto f(A_1,\dots,A_n) \in L(H)$$

* is linear (by construction)
* is continuous, and indeed $\|\phi(f)\|_{L(H)} = \lim_{j\to\infty} \|\phi(f_j)\|_{L(H)}$ $\overset{\text{Step 2}}{=} \lim_{j\to\infty} \|f_j\|_\infty = \|f\|_\infty = \|f\|_{C^0(K)}$.
* $\phi(x_j) = A_j$ if $x_j : (x_1,\dots,x_n) \mapsto x_j$ (Exercise)
* $\phi(fg) = \phi(f)\circ\phi(g)$ (by approximation of f, g by elements of S),
* $\phi(\bar{f}) = \phi(f)^*$.

Step 4: spectral measures.

- Let $v \in H$ and set $H_v = \{ f(A_1,\dots,A_n)v : f \in C^0(K)\}$.

  Let us assume $\overline{H_v} = H$. (If this is not the case, use a Zorn's Lemma argument, cf. Lemma (L.6).) Since

  $$\lambda : C^0(K) \longrightarrow \mathbb{R}, \quad \lambda(f) = (f(A_1,\dots,A_n)v, v),$$

  is a positive bounded linear functional, Riesz-Markov (Theorem (T.7)) produces a unique Radon measure μ_v on K, with $\mu_v(K) \le \|v\|^2$, s.t. $\lambda(f) = \int_K f \, d\mu_v$, $f \in C^0(K)$.

- Define $U : H_v \longrightarrow L^2(K, d\mu_v)$ by

  $$f(A_1,\dots,A_n)v \longmapsto f.$$

Then: * $\|U f(A_1,\dots,A_n)v\|_{L^2}^2 = \int_K |f|^2 \, d\mu_v = \lambda(|f|^2) = \lambda(\bar{f}f)$
$= ((\bar{f}f)(A_1,\dots,A_n)v, v)$

$$= (f(A_1,\dots,A_n)^* f(A_1,\dots,A_n) v, v)$$
$$= \| f(A_1,\dots,A_n) v \|^2_{\mathcal{H}}$$

$*$ $\mathcal{U}$ has dense range.

$\Rightarrow \mathcal{U}$ extends uniquely to a unitary map $\overline{\mathcal{H}_u} = \mathcal{H} \to L^2(K, d\mu_u)$.

· Lastly, for $f \in C^0(K)$: $\mathcal{U} A_j \mathcal{U}^{-1} f = \mathcal{U} A_j \mathcal{U}^{-1} (\mathcal{U}\phi(f) v)$
$$= \mathcal{U}(\phi(x_j)\phi(f) v)$$
$$= \mathcal{U}(\phi(x_j f) v) = x_j f. \qquad \square$$

Remark (R.10) The proof (incl. the omitted Zorn's Lemma argument) also shows that $\exists$ unitary $\mathcal{U} : \mathcal{H} \to \bigoplus L^2(\mathbb{R}^n; d\mu_{v_n})$ s.t.

$\mathcal{U} A_j \mathcal{U}^{-1}$ is multiplication by x_j on each $L^2(\mathbb{R}^n; d\mu_{v_n})$.

So if $(A_1,\dots,A_n)$ has a cyclic vector v, then $\mathcal{U} A_j \mathcal{U}^{-1}$ is multiplication by x_j on $L^2(\mathbb{R}^n, d\mu_v)$.

Corollary (C.2) $\mathcal{H}$ = separable Hilbert space, $A \in \mathcal{L}(\mathcal{H})$ normal ($AA^* = A^*A$). Then $\exists$ finite measure space (M, μ), $g : M \to \mathbb{C}$ bounded measurable, $\mathcal{U} : \mathcal{H} \to L^2(M, d\mu)$ unitary s.t. $(\mathcal{U} A \mathcal{U}^{-1} f)(x) = g(x) f(x)$, $f \in L^2(M, d\mu)$.

Proof Let $A_1 = \frac{1}{2}(A + A^*)$,
$$A_2 = \frac{1}{2i}(A - A^*).$$

Then $A_1^* = A_1$ and $A_2^* = A_2$ and $A_1 A_2 = A_2 A_1$.

$\Rightarrow$ Theorem (T.10) applies, giving (M,μ), $g_1, g_2 : M \to \mathbb{R}$, and U; and $UAU^{-1}f = UA_1U^{-1}f + iUA_2U^{-1}f = g_1 f + ig_2 f = gf$ for $g := g_1 + ig_2$. □

Remark (R.11) Following Remark (R.10), there exists, in the setting of Corollary (C.2), a unitary map $U : H \to \bigoplus_n L^2(\mathbb{C}; d\mu_{u_n})$ s.t. $(UAU^{-1}(g_n))_n(z) = z\, g_n(z)$ $\forall n$.

In case $\exists$ cyclic vector for (A_1, A_2), A is unitarily equivalent to multiplication by z on $L^2(K, d\mu_v)$ where

$$K = \{x + iy : x \in \sigma(A_1),\ y \in \sigma(A_2)\} \Subset \mathbb{C}.$$