As seen in exercises, one can use spectral methods (eigen-function decompositions) to solve evolution equations on smooth bounded domains $\Omega \in \mathbb{R}^n$, such as the

heat equation: $\begin{cases} \partial_t u(t,x) = \Delta u(t,x) \\ u(0,\cdot) = u_0 \quad \in L^2(\Omega) \\ u(t,\cdot)|_{\partial\Omega} = 0, \end{cases}$

wave equation: $\begin{cases} \partial_t^2 u(t,x) = \Delta u(t,x) \\ u(0,\cdot) = u_0 \in H_0^1(\Omega) \\ \partial_t u(0,\cdot) = u_1 \in L^2(\Omega) \\ u(t,\cdot)|_{\partial\Omega} = 0. \end{cases}$

We want to study here a bit the case of waves on unbounded domains, and indeed we shall focus on the case of $\mathbb{R}^n$.

Theorem (T.36) Let $V \in C_c^\infty(\mathbb{R}^n)$ be real-valued. Then the operator $-\Delta + V$ is self-adjoint on $H^2(\mathbb{R}^n)$.

Proof We already know that $-\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is self-adjoint. (Indeed, $\mathcal{F}^{-1} \circ (-\Delta) \circ \mathcal{F}$ is a multiplication operator by $|\xi|^2$ on $L^2(\mathbb{R}^n_\xi)$, where $\mathcal{F} : L^2(\mathbb{R}^n) \xrightarrow{\cong} L^2(\mathbb{R}^n)$ is the Fourier transform.)

Thus, $-\Delta + i : H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is invertible. We thus have, as operators $H^2 \rightarrow L^2$,

$$-\Delta + V + i = (-\Delta + V + i)(-\Delta + i)^{-1}(-\Delta + i)$$

$$= \left(I + V(-\Delta+i)^{-1}\right) \circ (-\Delta+i).$$

We claim that $I+V(-\Delta+i)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible.
(This implies that $-\Delta+V+i : H^2 \to L^2$ is invertible. Similarly for $-\Delta+V-i : H^2 \to L^2 \implies -\Delta+V$ is self-adjoint with domain $H^2(\mathbb{R}^n)$.)

To prove the claim, note that $V\circ(-\Delta+i)^{-1} : L^2 \to L^2$ is a compact operator: this follows from the fact that multiplication by V, as a map $H^2(\mathbb{R}^n)$ $(= \operatorname{ran}(-\Delta+i)^{-1}) \to L^2(\mathbb{R}^n)$, factors through

$$\begin{aligned} H^2(\mathbb{R}^n) &\xrightarrow{\text{restriction}} H^2(B_R(0)) \underset{\text{(Rellich)}}{\overset{\text{compact}}{\hookrightarrow}} L^2(B_R(0)) \\ &\xrightarrow{V} L^2(B_R(0)) \xrightarrow[\text{by } 0]{\text{extension}} L^2(\mathbb{R}^n) \end{aligned}$$

where $R>0$ is s.t. $\operatorname{supp} V \subset B_R(0)$.

$\implies I+V(-\Delta+i)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a compact perturbation of I, and therefore Fredholm of index 0. We therefore only need to check its injectivity.

But if $f \in L^2(\mathbb{R}^n)$ satisfies $(I+V(-\Delta+i)^{-1})f = 0$, then $u := (-\Delta+i)^{-1} f \in H^2(\mathbb{R}^n)$ solves $(-\Delta+i+V)u = 0$

$$\Rightarrow 0 = ((-\Delta + i + V)u, u)_{L^2(\mathbb{R}^n)} = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} V|u|^2\,dx + i\|u\|^2_{L^2(\mathbb{R}^n)}.$$

Taking imaginary parts gives $u=0$. $\square$

Lemma (L.22) $\sigma(-\Delta+V) \subset [\min(V), \infty)$.

Proof We claim that for $\lambda \in \mathbb{R}$ with $\lambda < \min(V) \le 0$,

$$-\Delta + V - \lambda : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is invertible. Using the Fourier transform, $-\Delta - \lambda : H^2 \to L^2$ is invertible, so we only need to prove the invertibility of $I + V(-\Delta-\lambda)^{-1} : L^2 \to L^2$. This is a Fredholm operator of index 0, and it is injective since $(I + V(-\Delta-\lambda)^{-1})f = 0$ implies for $u := (-\Delta-\lambda)^{-1} f \in H^2$ the equation

$$(-\Delta + V - \lambda)u = 0$$

$$\Rightarrow 0 = ((-\Delta + V - \lambda)u, u)_{L^2(\mathbb{R}^n)} = \|\nabla u\|^2_{L^2} + \|\sqrt{\underbrace{V-\lambda}_{>0}}\, u\|^2_{L^2}$$

$$\Rightarrow u = 0. \quad \square$$

Corollary (C.11) Let $u_0, u_1 \in H^2(\mathbb{R}^n)$, and set

$$u(t) = \cos(t\sqrt{-\Delta+V})\,u_0 + \frac{\sin(t\sqrt{-\Delta+V})}{\sqrt{-\Delta+V}}\,u_1.$$

Then $u \in C^2(\mathbb{R}; L^2(\mathbb{R}^n)) \cap C^0(\mathbb{R}; H^2(\mathbb{R}^n))$, and

$$\begin{cases} (\partial_t^2 - \Delta + V)\,u(t,x) = 0 \\ u(0,\cdot) = u_0 \\ \partial_t u(0,\cdot) = u_1. \end{cases}$$

Proof An application of the functional calculus for $-\Delta + V$.

(Exercise.) $\square$

Remark (R.31) One can show: $u_0 \in H^k(\mathbb{R}^n)$, $u_1 \in H^{k-1}(\mathbb{R}^n)$

$$\Rightarrow \quad u \in C^0(\mathbb{R}; H^k(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{k-1}(\mathbb{R}^n)) \cap C^2(\mathbb{R}; H^{k-2}(\mathbb{R}^n)).$$

Suppose $V \geq 0$. Then formally differentiating the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + V(x)\,|u(t,x)|^2 \, dx$$

gives
$$E'(t) = \int_{\mathbb{R}^n} \partial_t u \cdot \partial_t^2 u + \nabla_x u \cdot \partial_t \nabla_x u + V u \cdot \partial_t u \, dx$$

$$\overset{\text{integrate by parts}}{=} \int_{\mathbb{R}^n} (\partial_t^2 u - \Delta u + V u)\cdot \partial_t u \, dx$$

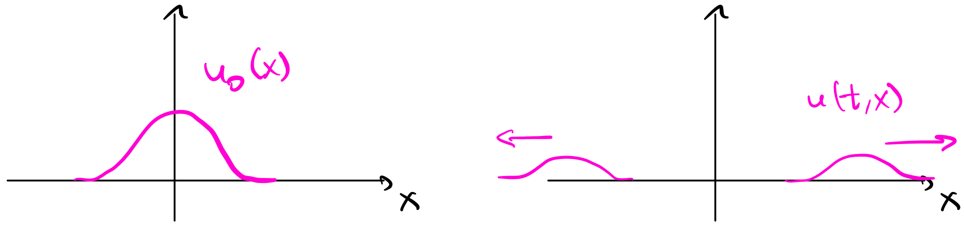
$$= 0.$$

That is, energy is conserved.

Example (E.23) $n=1$, $V=0$: the solution of

$$\begin{cases} (-\partial_t^2 + \partial_x^2)\, u(t,x) = 0 \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = 0 \end{cases}$$

is $\quad u(t,x) = \frac{1}{2}\left(u_0(x-t) + u_0(x+t) \right)$



There are explicit solutions of this free wave equation on $\mathbb{R}^n$ for all n.

• A qualitatively similar picture holds for solutions of

$$(\partial_t^2 - \Delta + V)\, u(t,x) = 0, \qquad u(0,\cdot), \partial_t u(0,\cdot) \in C_c^\infty(\mathbb{R}^3),$$

for general $V \in C_c^\infty(\mathbb{R}^3)$, $V \geq 0$, except $u(t,x)$ is not 0 in $|x| < R_0$ for $t > t(R_0)$, but rather exponentially small (i.e. $|u(t,x)| \leq C e^{-\alpha t}$, $|x| < R_0$, $t > t(R_0)$, for some $\alpha > 0$).

• Interesting: energy is conserved, but leaves every bounded region $\{x \in \mathbb{R}^n : |x| < R_0\}$ of space!

Theorem (T.37) Let $V \in C_c^\infty(\mathbb{R}^3)$, $V \geq 0$. Let $u_0, u_1 \in C_c^\infty(\mathbb{R}^3)$, and let $u = u(t,x)$ be the solution of

$$\begin{cases} (\partial_t^2 - \Delta + V)\, u = 0 \\ \quad u(0,x) = u_0(x) \\ \quad \partial_t u(0,x) = u_1(x). \end{cases}$$

Then $\forall\, A \in \mathbb{R}$, $\exists\, J \in \mathbb{N}_0$, $\lambda_1, \dots, \lambda_J \in \mathbb{C}$, $\operatorname{Im} \lambda_j < 0$,

$a_1, \dots, a_J \in C^\infty(\mathbb{R}^3)$

so that $$u(t,x) = \sum_{j=1}^{J} e^{-i\lambda_j t} a_j(x) + E_A(t,x), \qquad (\circledast)$$

where $|E_A(t,x)| \leq C e^{-At}$, $\quad |x| < R_0$, $t \geq 0$, $C = C(R_0, A)$.

The numbers λ_j only depend on V, and $(-\Delta + V - \lambda_j^2)\, a_j = 0$.

Remark (R.32) (i) In $\circledast$, we are assuming that all λ_j are "simple"; without this assumption, there are terms $\sum_{k=0}^{k(j)} e^{-i\lambda_j t} t^k a_{jk}(x)$.

(ii) Note that $\circledast$ looks quite similar to an eigenfunction expansion of solutions to the wave equation; but the λ_j are not real! Moreover, since ⊛ only states exponential decay in compact regions of space, this is not a contradiction to global-in-space energy conservation.

A full proof of Theorem (T.37) is quite involved. See e.g. Dyatlov–Zworski, or my lecture notes "A minicourse on scattering theory" on my webpage. Here, I merely want to indicate where the $\lambda_j \in \mathbb{C}$ come from — how do we get a discrete set of these so-called resonances in $\{\operatorname{Im}\lambda<0\}$ from the self-adjoint operator $-\Delta+V$?!

Proposition (P.9) $0 \le V \in C_c^\infty(\mathbb{R}^3)$. For $\lambda\in\mathbb{C}$, $\operatorname{Im}\lambda>0$, set $R_V(\lambda) := (-\Delta+V-\lambda^2)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ (using (T.36), (L.22)). Let $\chi \in C_c^\infty(\mathbb{R}^3)$. Then the cutoff resolvent

$$\chi\, R_V(\lambda)\, \chi : L^2(\mathbb{R}^3) \longrightarrow H^2(\mathbb{R}^3)$$

extends from $\operatorname{Im}\lambda>0$ to a meromorphic family of operators in $\lambda\in\mathbb{C}$.

The $\lambda_j \in \mathbb{C}$ in Theorem (T.37) are the poles of $\chi R_V(\lambda)\chi$ if we fix $\chi \in C_c^\infty(\mathbb{R}^3)$ s.t. $\chi = 1$ on $\operatorname{supp} V$.

Lemma (L.23) For $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > 0$, set $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ (the free resolvent). Then $\chi R_0(\lambda)\chi : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ extends to a holomorphic family of operators in $\lambda \in \mathbb{C}$ for all $\chi \in C_c^\infty(\mathbb{R}^3)$.

Proof $(R_0(\lambda)f)(x) = \int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} f(y)\,dy$, $\operatorname{Im}\lambda > 0$. (exercise)

We thus need to show that

$\chi R_0(\lambda)\chi$, with integral kernel $:= \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\chi(x)\chi(y)$,

defines a holomorphic family of bounded operators $L^2 \to H^2$.

This can be checked explicitly; using elliptic regularity is useful for this purpose (exercise). $\square$

Proof of Proposition (P.9) WLOG, $\chi = 1$ near $\operatorname{supp} V$. For $\operatorname{Im}\lambda > 0$, write

$$-\Delta + V - \lambda^2 = (I + V(-\Delta-\lambda^2)^{-1}) \circ (-\Delta - \lambda^2) \qquad (\ast)$$

(equality of operators $H^2 \to L^2$); since $-\Delta + V - \lambda^2$ and $-\Delta - \lambda^2$ are invertible, so is

$$I + V \circ (-\Delta - \lambda^2)^{-1} = I + V \circ R_0(\lambda) : L^2 \to L^2.$$

Write further

$$I + V R_0(\lambda) = I + V R_0(\lambda)(1-\chi) + V R_0(\lambda)\chi$$

$$\overset{(1-\chi)V = 0}{=} (I + V R_0(\lambda)(1-\chi)) \circ (I + V R_0(\lambda)\chi).$$

But $I + V R_0(\lambda)(1-\chi) : L^2 \to L^2$ is invertible with

$$(I + VR_0(\lambda)(1-\chi))^{-1} = I - VR_0(\lambda)(1-\chi),$$

so also $I + VR_0(\lambda)\chi : L^2 \to L^2$ is invertible.

(*) now implies (still for $\operatorname{Im} \lambda > 0$)

$$R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1} = R_0(\lambda)(I + VR_0(\lambda))^{-1}$$

$$= R_0(\lambda)(I + VR_0(\lambda)\chi)^{-1}(I - VR_0(\lambda)(1-\chi))$$

$$\Rightarrow \chi R_V(\lambda)\chi = \chi R_0(\lambda)(I + VR_0(\lambda)\chi)^{-1} \underbrace{(\chi - VR_0(\lambda)(1-\chi)\chi)}_{L^2(\mathbb{R}^3) \to L^2_c(\mathbb{R}^3)}. \quad (\#)$$

compact support

- It suffices to show that

$$(I + VR_0(\lambda)\chi)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

has a meromorphic continuation from $\operatorname{Im} \lambda > 0$ to $\lambda \in \mathbb{C}$. $(\ddagger)$

Indeed, we use that

$(I + VR_0(\lambda)\chi)^{-1} : L^2_c(\mathbb{R}^3) \to L^2_c(\mathbb{R}^3)$: if $f \in L^2_c(\mathbb{R}^3)$ and $u = (I + VR_0(\lambda)\chi)^{-1} f$, then $u + \underbrace{VR_0(\lambda)(\chi u)}_{\in H^2_c(\mathbb{R}^3) \text{ since } V \in C^\infty_c(\mathbb{R}^3)} = f$

$\Rightarrow u = f - VR_0(\lambda)(\chi u)$ has compact support indeed.

$\Rightarrow$ In $(\#)$, we have expressed $R_V(\lambda)$ as a composition of holomorphic and meromorphic families of operators.

- To show $(\ddagger)$, we note that

$$VR_0(\lambda)\chi : L^2(\mathbb{R}^3) \xrightarrow{\chi\cdot} L^2_c(\mathbb{R}^3) \xrightarrow{R_0(\lambda)} H^2_{loc}(\mathbb{R}^3) \xrightarrow{V} H^2_c(\mathbb{R}^3) \xhookrightarrow{\text{compact}} L^2(\mathbb{R}^3)$$

is compact $\Rightarrow$ $A(\lambda) := I + VR_0(\lambda)\chi$ is a holomorphic family of Fredholm operators on $L^2(\mathbb{R}^3)$; and $A(\lambda)$ is invertible for $\operatorname{Im}\lambda > 0$. By the analytic Fredholm theorem, the claim follows $\square$