We now consider duals/adjoints of unbounded operators.

Definition (D.3) Let $A: D(A) \subset X \to Y$ be linear and densely defined. Then $A^*: D(A^*) \subset Y^* \to X^*$, with

$$D(A^*) := \{ y^* \in Y^* : \ell_{y^*} : D(A) \ni x \mapsto \langle y^*, Ax \rangle_{Y^* \times Y} := y^*(Ax) \text{ is continuous} \},$$

is defined by $A^* y^* = \bar{\ell}_{y^*} \in X^*$, the unique continuous extension of ℓ_{y^*}. We call A^* the dual of A.

Remark (R.2) (i) A^* is well-defined: $\bar{\ell}_{y^*}$ is unique since $\overline{D(A)} = X$.

(ii) For $x \in D(A)$ and $y^* \in D(A^*)$, we have

$$\langle y^*, Ax \rangle_{Y^* \times Y} = \ell_{y^*}(x) = (A^* y^*)(x)$$
$$= \langle A^* y^*, x \rangle_{X^* \times X}.$$

(Looks like integration by parts!)

Example (E.6) $\Omega \subset \mathbb{R}^n$ open, bounded; $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $X = Y = L^p(\Omega)$, so $X^* = Y^* \cong L^q(\mathbb{R})$. Consider

$$A_p : D(A_p) := C_c^\infty(\Omega) \subset X \to Y, \quad A_p u = \Delta u.$$

Claim: $A_p^* \supset A_q$.

Well, for $g \in C_c^\infty(\Omega)$ and $u \in D(A_p)$, we have

$$\ell_g(u) = \int_\Omega g \Delta u \, dx = \int_\Omega \Delta g \, u \, dx$$

$\Rightarrow |\ell_g(u)| \le \|\Delta g\|_{L^q} \|u\|_{L^p}$. Thus, ℓ_g has a continuous extension $\tilde{l}_g \in (L^p)^*$, with $\tilde{l}_g(u) = \langle \Delta g, u \rangle_{L^q \times L^p}$ for $g \in C_c^\infty(\Omega)$

$$\Rightarrow A^* g = \Delta g \quad (g \in C_c^\infty(\Omega)).$$

Theorem (7.3) Let $A : D(A) \subset X \to Y$ be densely defined.

(i) $A^* : D(A^*) \subset Y^* \to X^*$ is closed.

(ii) $A \subset B$ (i.e. $B : D(B) \subset X \to Y$ is an extension of A)

$$\Rightarrow B^* \subset A^*.$$

Proof (i) Let $y_k^* \in D(A^*)$ with $y_k^* \to y^*$ in Y^*,

$A^* y_k^* \to x^*$ in X^*.

For $x \in D(A)$, we then have

$$\begin{aligned}
\langle y^*, Ax \rangle_{Y^* \times Y} &= \lim_{k \to \infty} \langle y_k^*, Ax \rangle_{Y^* \times Y} \\
&= \lim_{k \to \infty} \langle A^* y_k^*, x \rangle_{X^* \times X} \qquad (\text{def. of } A^*) \\
&= \langle x^*, x \rangle_{X^* \times X}.
\end{aligned}$$

$\Rightarrow y^* \in D(A^*)$, $A^* y^* = x^*$.

(ii) Let $y^* \in D(B^*)$ and $x \in D(A) \subset D(B)$, then

$$\langle y^*, Ax \rangle_{Y^* \times Y} = \langle y^*, Bx \rangle_{Y^* \times Y} = \langle B^* y^*, x \rangle_{X^* \times X}.$$

$\Rightarrow y^* \in D(A^*)$, and $A^* y^* = B^* y^*$. $\square$

As in the case of bounded operators, we have a characterization of $\operatorname{ran} A$ in terms of $\ker A^*$ when $\operatorname{ran} A$ is closed:

Theorem (T.4) Let X, Y be Banach spaces, $A: D(A) \subset X \longrightarrow Y$ densely defined and closed. Then $\operatorname{ran} A \subset Y$ is closed iff
$$\operatorname{ran} A = {}^{\perp}(\ker A^*) := \{ y \in Y : y^*(y) = 0 \ \forall y^* \in \ker A^* \}.$$

Proof "$\Leftarrow$" is clear since ${}^{\perp}W \subset Y$ is closed for all $W \subset Y^*$).

"$\Rightarrow$" Certainly $\operatorname{ran} A \subseteq {}^{\perp}(\ker A^*)$, for if $x \in D(A)$, $y^* \in \ker A^*$, then $\langle y^*, Ax \rangle_{Y^* \times Y} = \langle A^* y^*, x \rangle_{X^* \times X} = \langle 0, x \rangle_{X^* \times X} = 0$.

Suppose now $\exists$ $y \in {}^{\perp}(\ker A^*) \setminus \operatorname{ran} A$. Since $\operatorname{ran} A = \overline{\operatorname{ran} A}$, $\exists$ $y^* \in Y^*$ s.t. $y^*|_{\operatorname{ran} A} = 0$, $y^*(y) = 1$. Thus, $\forall x \in D(A)$
$$\langle y^*, Ax \rangle_{Y^* \times Y} = 0 \Rightarrow y^* \in D(A^*) \text{ and } A^* y^* = 0.$$
So $y^* \in \ker A^*$; but $y^*(y) = 1$ contradicts $y \in {}^{\perp}(\ker A^*)$. $\square$

We now specialize to the case of operators between Hilbert spaces.

Definition (D.3') X, Y Hilbert spaces, $A: D(A) \subset X \to Y$ densely defined. Then $A^*: D(A^*) \subset Y \to X$ is defined by
$$D(A^*) = \{ y \in Y : \exists z \in X \text{ s.t. } (Ax, y) = (x, z) \ \forall x \in D(A) \},$$
$A^* y = z$. We call A^* the adjoint of A.

So $(Ax, y) = (x, A^* y)$ for $x \in D(A)$, $y \in D(A^*)$.

Remark (R.3) Recalling the (antilinear) Isomorphism $J: H \to H^*$, $J(y)(x) = (x, y)$, set $B := J \circ A^* \circ J^{-1}$; then for $y^* \in J(D(A^*))$

and $x \in D(A)$, we have

$$\begin{aligned} \langle x, By^* \rangle_{H \times H^*} &= (x, A^* J^{-1} y^*) \\ &= (Ax, J^{-1}y^*) \\ &= \langle Ax, y^* \rangle_{H \times H^*}, \end{aligned}$$

so B is the dual operator of A; and it seems acceptable that we should write A^* for the adjoint, too. (We committed essentially the same mild abuse of notation already in Example (E.6) above.)

Definition (D.4) Let $A: D(A) \subset H \to H$ be a densely defined linear operator.

(i) A is *symmetric* if $(Ax, y) = (x, Ay)$ $\forall x, y \in D(A)$.
(That is, $A \subset A^*$.)

(ii) A is *self-adjoint* if $A = A^*$.

Remark (R.4)

(i) If A is everywhere defined ($D(A) = H$) and self-adjoint, then A is bounded! (Hellinger–Toeplitz theorem, see FA I.)

(ii) If A is bounded, $D(A) = H$, and symmetric, then A is self-adjoint (since $A^* \supset A$, so $D(A^*) = H$).

(iii) If A is self-adjoint, then A is closed by Theorem (T.3)(i).

Example (E.7) $\Omega \Subset \mathbb{R}^n$ smoothly bounded. Consider

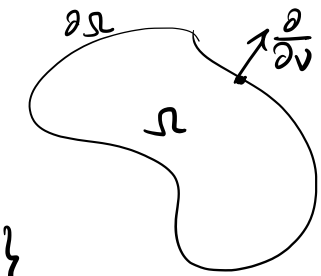
$A_\infty, A_0, A_1, A_2 = \Delta$ as unbounded operators on $L^2(\Omega)$, where

$D(A_\infty) = C_c^\infty(\Omega)$

$D(A_0) = C^2(\overline{\Omega})$

$D(A_1) = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$

$D(A_2) = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$.



Claim 1: A_∞, A_2, A_1 are symmetric; A_0 is not.

Check: Note that $A_\infty \subsetneq A_2 \subsetneq A_1 \subsetneq A_0$. For $u, v \in D(A_1)$,

we have
$$\int_\Omega u\, \Delta v\, dx = \underbrace{\int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, d\sigma}_{=0} - \int_\Omega \nabla u \cdot \nabla v\, dx$$
$$= -\overbrace{\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\, d\sigma}^{=0} + \int_\Omega \Delta u\, v\, dx = \int_\Omega \Delta u\, v\, dx.$$

The integration by parts produces boundary terms for general $u, v \in D(A_0)$ (E.g. when $u|_{\partial\Omega} = 1$, $\frac{\partial u}{\partial\nu}|_{\partial\Omega} = 0$,

$v|_{\partial\Omega} = 0$, $\frac{\partial v}{\partial \nu}|_{\partial\Omega} = 1$,

we have $\int_\Omega u\, \Delta v\, dx - \int_\Omega \Delta u\, v\, dx = \int_{\partial\Omega} 1 \cdot 1\, d\sigma$

$= \mathrm{vol}(\partial\Omega)$.)

Claim 2: $A_2^* \supsetneq A_2$, i.e. A_2 is a nontrivial extension of A_2.

Indeed, we have $\int_\Omega u\, \Delta v\, dx = \int_\Omega \Delta u\, v\, dx$ also for $u \in D(A_0)$, $v \in D(A_2)$; so $D(A_0) \subset D(A_2^*)$, and the claim follows from $D(A_2) \subsetneq D(A_0)$.

— Self-adjoint extensions of A_∞ (or A_0, A_1) will be found later, as will the closures of A_∞, A_2, A_1, A_0.

This example shows that symmetry (or self-adjointness) of a (differential) operator depend sensitively on the choice of domain.

In order to give a more concrete example, we need:

Definition (D.5) (i) We say that $u \in L^2((0,1))$ has a weak derivative $v \in L^2((0,1))$ if $\forall \varphi \in C_c^\infty((0,1))$,

$$\int_0^1 u\varphi' \, dx = -\int_0^1 v\varphi \, dx.$$

In this case, we write $v = u'$.

(ii) We set $H^1((0,1)) := \{u \in L^2((0,1)) : u' \in L^2((0,1))\}$, with norm $\|u\|_{H^1}^2 := \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$.

Remark (R.5) (i) By Theorem (T.2), u' is unique (if it exists).

(ii) $H^1((0,1))$ is a Hilbert space. $C^\infty([0,1]) \subset H^1((0,1))$ is dense, and $H^1((0,1)) \subset C^0([0,1])$.

(iii) For $u, v \in H^1((0,1))$, we have $uv \in H^1((0,1))$ and $(uv)' = u'v + uv'$.

$$\Rightarrow \int_0^1 u'v \, dx = uv\Big|_0^1 - \int_0^1 uv' \, dx.$$

(We will prove (ii) & (iii) later.)

Example (E.8) Let $A_\infty : C_c^\infty((0,1)) \subset L^2([0,1]) \longrightarrow L^2([0,1])$, $A_\infty u = i\frac{d}{dt}u$. We consider various extensions of A_∞.

(i) Let $A_0 = i\frac{d}{dt}$ with domain $D(A_0) := H^1((0,1))$,

$A_2 = i\frac{d}{dt}$ with domain $D(A_2) := \{u \in H^1((0,1)) : u(0) = u(1) = 0\}$.

Claim: $A_0^* = A_2$ and $A_2^* = A_0$. In particular, A_0, A_2 are closed; and since $A_0 \supsetneqq A_2$, A_2 is symmetric but A_0 is not; and neither A_0 nor A_2 is self-adjoint.

Check: • Since for $u \in D(A_0)$ and $v \in D(A_2)$ we have

$$\int_0^1 iu' \cdot \bar{v}\, dt = -i\int_0^1 u\, \overline{v'}\, dt = \int_0^1 u \cdot \overline{iv'}\, dt,$$

we get $A_0^* \supset A_2$ and $A_2^* \supset A_0$.

• Let $u \in D(A_0^*)$; we want to show $u \in D(A_2)$.

* By definition of $D(A_0^*)$, $\exists\, C$ s.t. $\forall v \in C_c^\infty((0,1)) \subset D(A_0)$,

$$\left| \int_0^1 u\, \overline{v'}\, dt \right| = |(u, A_0 v)| \leq C\|v\|_{L^2}.$$

Riesz representation theorem $\Rightarrow \exists\, w \in L^2([0,1])$ s.t.

$$\int_0^1 u\overline{v'}\, dt = \int_0^1 w\bar{v}\, dt \quad \forall v \in C_c^\infty((0,1)); \text{ i.e. } u \in H^1, u' = -w.$$

Moreover, $(A_0^* u, v) = (u, A_0 v) = \int_0^1 u\, \overline{iv'}\, dt = \int_0^1 iu' \bar{v}\, dt$,

and hence $A_0^* u = iu'$.

* For general $v \in D(A_0)$, get boundary terms when $u \in H^1$:

$$(u, A_0 v) = \int_0^1 u \cdot \overline{iv'}\, dt = iu(1)\overline{v(1)} - iu(0)\overline{v(0)} + \int_0^1 iu' \bar{v}\, dt$$

$\|$ def. of A_0^*

$$(A_0^* u, v) = \int_0^1 iu' \bar{v}\, dt.$$

$\Rightarrow u(1)\overline{v(1)} - u(0)\overline{v(0)} = 0 \quad \forall u \in D(A_0^*), v \in D(A_0).$

Plugging in $v \in C^\infty([0,1]) \subset D(A_0)$ with $v(0)=0$ $v(1)=1$

gives $u(1)=0$, and $v(0)=1$, $v(1)=0$ gives $u(0)=0$.

* Altogether, we have proved $u \in D(A_2)$. ——

(ii) Let $A_1 = i\frac{d}{dt}$ with domain $D(A_1) = \{u \in H^1((0,1)) : u(0)=u(1)\}$.

Then A_1 is self-adjoint, i.e. $A_1 = A_1^*$. (Exercise.)

We now discuss the ramifications of choices of domains for spectral theory.

Definition (D.6) H Hilbert space, $A : D(A) \subset H \to H$.

Then $\rho(A) = \{ z \in \mathbb{C} : \; z - A : D(A) \to H$ is bijective with bounded inverse$\}$,

$\sigma(A) = \mathbb{C} \setminus \rho(A)$.

$\sigma_p(A) = \{ z \in \mathbb{C} : \; z - A : D(A) \to H$ is not injective$\}$;

$\sigma_c(A) = \{ z \in \mathbb{C} : \; z - A$ is injective with dense range $\neq H\}$;

$\sigma_r(A) = \sigma(A) \setminus (\sigma_p(A) \cup \sigma_c(A))$.

By Theorem (T.1), $(z-A)^{-1} : H \to D(A) \subset H$ is automatically bounded when A is closed.

Example (E.8, continued) $i\frac{d}{dt}$ on $D(A_0) = H^1((0,1))$,

$D(A_1) = \{u \in H^1 : u(0) = u(1)\}$,

$D(A_2) = \{u \in H^1 : u(0) = u(1) = 0\}$.

Claim 1: $\sigma(A_0) = \sigma(A_2) = \mathbb{C}$ (even though A_2 is symmetric!)

Well, • if $z \in \mathbb{C}$, then $u(t) = e^{-izt} \in C^\infty([0,1]) \subset H^1((0,1)) = D(A_0)$,

$i\frac{d}{dt} u = zu \implies u \in \ker(z - A_0)$, so $z \in \sigma_p(A_0)$.

• $(\ker(z-A_0))^\perp \supset \operatorname{ran}(z-A_0)^* = \operatorname{ran}(\bar{z} - A_2)$, so

by what we have already shown, $\bar{z} \in \sigma(A_2)$.

(In fact, $\sigma_p(\bar{z}-A_2) = \emptyset$ and so $\bar{z} \in \sigma_r(A_2)$.) $\square$

Claim 2: $\sigma(A_1) = \sigma_p(A_1) = 2\pi\mathbb{Z} \subset \mathbb{R}$.

Indeed, • for $k \in \mathbb{Z}$, $u_k(t) := e^{-2\pi i k t} \in D(A_1)$, $(2\pi k - A_1)u_k = 0$

$\implies 2\pi k \in \sigma_p(A_1)$. $\qquad (e^{-2\pi i k} = e^0 = 1)$

• For $z \in \mathbb{C} \setminus 2\pi\mathbb{Z}$, $\ker(z - A_1) = \{0\}$.

• For $z \in \mathbb{C}$, $f \in L^2((0,1))$, all solutions of $(z-A_1)u = f$

are given by $u(t) = e^{-izt} u(0) + \int_0^t i e^{iz(t-s)} f(s)\, ds$.

(The R.H.S. indeed defines an element of $H^1((0,1))$.)

If $z \notin 2\pi\mathbb{Z}$, i.e. $e^{iz} \neq 1$, we have

$$u(0) \overset{?}{=} u(1) = e^{-iz} u(0) + \int_0^1 i e^{iz(1-s)} f(s)\, ds$$

for the unique choice $u(0) = \dfrac{\int_0^1 i e^{iz(1-s)} f(s)\, ds}{1 - e^{-iz}}$.

$\implies z - A_1 : D(A_1) \to L^2((0,1))$ is bijective, and thus

has a continuous inverse, when $z \notin 2\pi\mathbb{Z}$.

The crucial difference between symmetry and (the stronger notion of) self-adjointness is captured by the following result:

Theorem (T.5) Let $A : D(A) \subset H \to H$ be symmetric. Then the following are equivalent:

(i) $A = A^*$ (i.e. A is self-adjoint)

(ii) $\sigma(A) \subset \mathbb{R}$

(iii) $\exists\, z_-, z_+ \in \rho(A)$ with $\operatorname{Im} z_- < 0$, $\operatorname{Im} z_+ > 0$.

(iv) $A+i : D(A) \to H$ and $A-i : D(A) \to H$ are invertible.

Characterization (iv) is typically the most convenient to check: when A is a differential operator, say, it amounts to a solvability and uniqueness statement (with H and $D(A)$ typically L^2 and a Sobolev space).

For the proof of Theorem (T.5), we need:

Lemma (L.3) (i) If $A \subset A^*$, then $\forall z \in \mathbb{C}$

$$\|(z-A)u\| \geq |\operatorname{Im} z|\,\|u\| \quad \forall u \in D(A). \quad (*)$$

(ii) If $z - A : D(A) \to H$ is moreover surjective and $z \in \mathbb{C} \setminus \mathbb{R}$, then $z \in \rho(A)$ and $\|(z-A)^{-1}\|_{L(H)} \leq \frac{1}{|\operatorname{Im} z|}$.

Proof (i) $(*)$ follows from $\operatorname{Im}\big((z-A)u, u\big) = (\operatorname{Im} z)(u,u) - \underbrace{\frac{(Au,u)-(u,Au)}{2i}}_{=0\ (A \subset A^*)}$

$$\Rightarrow |\operatorname{Im} z|\,\|u\|^2 \leq \|(z-A)u\| \cdot \|u\|.$$

(ii) $(*)$ implies that $z-A$ is bijective, so $\exists\,(z-A)^{-1} : H \to D(A)$. The boundedness follows from $(*)$ since for $u \in H$,

$$|\operatorname{Im} z|\,\|(z-A)^{-1}u\| \leq \|(z-A)(z-A)^{-1}u\| = \|u\|. \qquad \square$$

Proof of Theorem (T.5)

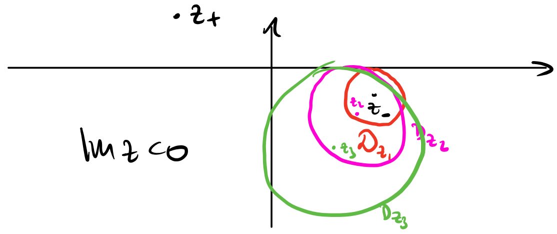
(iv) $\Rightarrow$ (iii): trivial.

(iii) $\Rightarrow$ (ii): if $z_0 \in \rho(A)$, then also

$\mathcal{D}_{z_0} = \{ z \in \mathbb{C} : |z - z_0| < \operatorname{Im} z_0 \} \subset \rho(A).$

(Indeed, $z - A = (z - z_0) + (z_0 - A) = (I + (z - z_0) R_{z_0})(z_0 - A)$
where $R_{z_0} = (z_0 - A)^{-1} : H \to D(A)$ is bounded by $\frac{1}{|\operatorname{Im} z_0|}$.
For $z \in \mathcal{D}_{z_0}$, $(z - A)^{-1} = (z_0 - A)^{-1} \sum_{j=0}^{\infty} (-1)^j \big(\underbrace{(z - z_0) R_{z_0}}_{\|\cdot\| < 1}\big)^j.$)

Starting with $z_0 = z_-$, we can cover all of $\{ z \in \mathbb{C} : \operatorname{Im} z < 0 \}$ with such disks. $\Rightarrow \rho(A) \subset \{ \operatorname{Im} z < 0 \}$. Similarly for $z_0 = z_+$.



(ii) $\Rightarrow$ (i). Let $u \in D(A^*)$. Since $A + i$ is surjective, $\exists\, u_0 \in D(A)$ s.t. $(A^* + i) u = (A + i) u_0 \overset{A \subset A_*}{=} (A_* + i) u_0$; therefore, $(A^* + i)(u - u_0) = 0$.

For all $v \in D(A)$, this implies

$$0 = \big((A^* + i)(u - u_0), v \big) = \big(u - u_0, (A - i) v \big),$$

If we take v s.t. $(A - i) v = u - u_0$ (using the surjectivity of $A - i$), this gives $0 = \|u - u_0\|^2 \Rightarrow u = u_0 \in D(A)$.

- We have proved $D(A^*) \subset D(A)$. But $D(A) \subset D(A^*)$ by symmetry $\Rightarrow D(A) = D(A^*)$.

(i) $\Rightarrow$ (iv) . $\operatorname{ran}(A \pm i)$ is closed: if $f_k = (A \pm i) u_k \to f \in H$, where $u_k \in D(A)$, then by Lemma (L.2)(i)

$\|u_k - u_\ell\| \le \|f_k - f_\ell\|$, so u_k converges to some $u \in H$.

Since $A \pm i = (A \mp i)^*$ is closed, we have

$$(u,f) = \lim_{k\to\infty} (u_k, f_k) \in \Gamma_{A\pm i} \Rightarrow (A \pm i)u = f.$$

- Lemma (L.2)(i) implies that $A \pm i$ is injective.
- By Theorem (T.4), $\operatorname{ran}(A \pm i) = (\ker (A \mp i)^*)^\perp = H$; so $A \pm i : D(A) \to H$ is invertible indeed. $\square$

As an example, one can use this result (specifically, characterization (iv)) to check Example (E.8)(ii).

Example (E.9) $H = L^2(S^1)$, $A = D_\theta^2 : D(A) = H^2(S^1) \subset H \longrightarrow H$.

This is self-adjoint, and $\sigma(A) = \sigma_p(A) = \{n^2 : n \in \mathbb{Z}\}$.

(Exercise.)

Here is another class of examples:

Proposition (P.2) Let (M,μ) be a finite measure space. Let $f : M \to \mathbb{R}$ be measurable. Define $T_f : D(T_f) \subset L^2(M, d\mu) \to L^2(M, d\mu)$ by

$$T_f u(x) = f(x)u(x), \text{ where } D(T_f) = \{u \in L^2(M,d\mu) : fu \in L^2(M,d\mu)\}.$$

Then T_f is self-adjoint. Moreover,

$$\sigma(T_f) = \operatorname{ess\,ran} f = \{x \in \mathbb{R} : \mu(f^{-1}((x-\varepsilon, x+\varepsilon)) > 0 \ \forall \varepsilon > 0\}.$$

Proof
- T_f is symmetric: for $u, v \in D(T_f)$,

$$(T_f u, v) = \int_M fu\,\bar{v}\,d\mu = \int_M u\,\overline{fv}\,d\mu = (u, T_f v).$$

- $T_f \pm i$ is invertible: given $g \in L^2(M,\mu)$, we have

$u_\pm := \dfrac{g}{f \pm i} \in D(T_f)$ since $\left|\dfrac{1}{f \pm i}\right| \le 1$ (so $u_\pm \in L^2$) and

$\left|\frac{f}{f\pm i}\right| \le 1$ (so $f u_\pm \in L^2$).

Since $(T_f \pm i) f_\pm = g$, $T_f \pm i$ is surjective.

- If $z \notin \operatorname{ess\,ran} f$, then $\exists\, \varepsilon > 0$ s.t. $|z - f(x)| \ge \varepsilon$ a.e. $x \in M$.

Thus, $L^2 \ni g \mapsto \frac{g}{z-f} \in T_f$ is an inverse of $z - T_f$. (Note here:

$\left|\frac{g}{z-f}\right| \le \varepsilon^{-1}|g| \in L^2$, and $\left|f\frac{g}{z-f}\right| \le \left|z\frac{g}{z-f}\right| + \left|(z-f)\frac{g}{z-f}\right|$
$\le z\,\varepsilon^{-1}|g| + |g| \in L^2$.)

- If $z \in \operatorname{ess\,ran} f$, let $\varepsilon > 0$, pick $A \subset f^{-1}((z-\varepsilon, z+\varepsilon))$ with $\mu(A) \in (0,\infty)$, and consider $g = \frac{1_A}{\sqrt{\mu(A)}}$. Then $\|g\|_{L^2} = 1$. But for $x \in A$,

$\left|\frac{g}{z-f}\right| \ge \frac{1}{\sqrt{\mu(A)}\,\varepsilon}$, so $\left\|\frac{g}{z-f}\right\|_{L^2} \ge \varepsilon^{-1}$. So either $\frac{g}{z-f} \notin L^2$

— in which case $z - T_f$ is not surjective; or $\frac{g}{z-f} \in L^2$ — in which case $(z-T_f)^{-1} : L^2 \to D(T_f)$, if it existed, would have

$\|(z-T_f)^{-1} g\| \ge \varepsilon^{-1}\|g\|$, i.e. would be unbounded, in contradiction to Theorem (T.1). $\square$

Example (E.10) $(M,\mu) = (\mathbb{R}, \text{finite Borel measure})$, $f(x) = x$. In this case,

$$\sigma(T_f) = \operatorname{supp}\mu = \mathbb{R} \setminus \bigcup_{\substack{U \subset \mathbb{R} \text{ open} \\ \mu(U) = 0}} U.$$

Remark (R.6) One can also prove the self-adjointness of T_f directly.

Let $v \in D(T_f^*)$. Set $\chi_N(x) = \begin{cases} 1, & |f(x)| \le N \\ 0 & \text{otherwise.} \end{cases}$ Then:

$\|T_f^* v\|_{L^2} = \lim_{N\to\infty} \|\chi_N T_f^* v\|_{L^2}$ (monotone convergence theorem)

$$= \lim_{N\to\infty} \sup_{\substack{u\in L^2 \\ \|u\|=1}} (u, \chi_N T_f^* v)$$

$$= \lim_{N\to\infty} \sup_{\substack{u\in L^2 \\ \|u\|=1}} (\chi_N u, T_f^* v)$$

$$= \lim_{N\to\infty} \sup_{\substack{u\in L^2 \\ \|u\|=1}} (\underbrace{T_f \chi_N u}_{= f\chi_N u}, v)$$

(note: $\chi_N u \in \mathcal{D}(T_f)$ since $|T_f \chi_N u| \leq N|u| \in L^2$)

$$= \lim_{N\to\infty} \sup_{\substack{u\in L^2 \\ \|u\|=1}} (u, \chi_N f v)$$

$$= \lim_{N\to\infty} \|\chi_N f v\|.$$

$\Rightarrow fv \in L^2$, so $v \in \mathcal{D}(T_f)$. $\square$

The spectral theorem, to which we now turn, shows that all self-adjoint operators are unitarily equivalent to T_f for some measurable f on some finite measure space (M, μ).