We recall the statement:

Theorem (T.7) (Riesz–Markov.) Let X be a compact Hausdorff space, and let $\lambda \in (C^0(X))^*$ be a positive linear functional. Then there exists a unique Radon measure μ on X s.t.

$$\lambda(f) = \int_X f \, d\mu, \qquad f \in C^0(X). \qquad (*)$$

Conversely, for every Radon measure μ, $(*)$ defines a positive linear functional on $C^0(X)$.

We consider first the question of uniqueness.

Lemma (L.7) If $\lambda \in (C^0(X))^*$ is given by $(*)$ for a Radon measure μ, then for $U \subset X$ open,

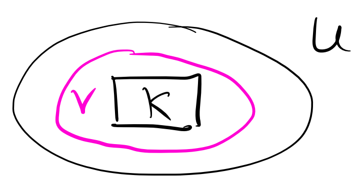
$$\mu(U) = \sup\{\lambda(f) : f \in C^0(X),\ 0 \le f \le 1,\ \operatorname{supp} f \subset U\}.$$

Proof "$\ge$" For $f \in C^0(X)$ with $0 \le f \le 1$, $\operatorname{supp} f \subset U$, we have $f(x) \le 1_U(x)$ $\forall x \in X$. Integrating over X gives

$$\lambda(f) = \int_X f \, d\mu \le \int_X 1_U(x) \, d\mu = \mu(U).$$

"$\le$" Let $K \subset U$ be a compact subset. Fix an open set $V \subset U$ with $K \subset V$ and $\overline{V} \subset U$. Let $f \in C^0(X)$ be

s.t. $\begin{cases} f = 1 & \text{on } K \\ f = 0 & \text{on } X \setminus V \\ 0 \le f \le 1. \end{cases}$



(Such f exist by the Tietze Extension Theorem.)

Then $1_K \leq f$ on X, so upon integration $\mu(K) \leq \lambda(f)$,

$\Rightarrow$ $\mu(K) \leq \sup\{\lambda(f) : f \in C^0(X),\ 0 \leq f \leq 1,\ \operatorname{supp} f \subset U\}$.

Taking the sup over all $K \Subset U$ and using that μ is inner regular gives $\mu(U) \leq \sup\{\lambda(f) : \cdots\}$. $\square$

Proof of uniqueness of μ in Theorem (T.7)

By Lemma (L.7), μ is determined on open subsets of X by λ. Since μ is outer regular, we can also compute, for $K \subset X$,

$$\mu(K) = \inf\{\mu(U) : U \supset K \text{ open}\}.$$

Finally, if $E \subset X$ is Borel measurable, then since μ is inner regular, also

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$$

is now uniquely determined. $\square$

For the proof of existence of μ, we of course start by defining, for $U \subset X$ open,

$$m(U) := \sup\{\lambda(f) : f \in C^0(X),\ 0 \leq f \leq 1,\ \operatorname{supp} f \subset U\}. \quad ⊕$$

We want to upgrade this to a Radon measure using the following result:

Proposition (P.3) Let $m : \mathcal{O} = \{U \subset X \text{ open}\} \to [0, \infty)$ be a function with the following properties:

(A) $U \subseteq V \Rightarrow m(U) \leq m(V)$ (monotonicity)

(B) $U_1, U_2, \ldots \in \mathcal{O} \Rightarrow m(\bigcup_j U_j) \leq \sum_j m(U_j)$ (subadditivity)

(C) $U, V \in \mathcal{O}$, $U \cap V = \emptyset \Rightarrow m(U \cup V) = m(U) + m(V)$

(D) $m(U) = \sup\{m(V) : V \in \mathcal{O}, \overline{V} \subset U\}$

(a form of inner regularity).

Then m can be extended uniquely to a Radon measure μ defined on all Borel sets in X.

Proof Uniqueness of μ follows by the same arguments as above.

For existence, define the set function

$$\mu^*(A) := \inf\{m(U) : U \in \mathcal{O}, U \supseteq A\} \qquad (A \subset X \text{ subset}).$$

Claim 1: μ^* is an outer measure.

Check: * $\mu^*(\emptyset) = m(\emptyset) = 0$ (since $m(\emptyset) = m(\emptyset \cup \emptyset) \overset{(C)}{=} m(\emptyset) + m(\emptyset)$ $\Rightarrow m(\emptyset) = 0$).

* $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$ directly from the definition.

* $A_1, A_2, \ldots \subset X$; we need to show $\mu^*(\bigcup_j A_j) \leq \sum_j \mu^*(A_j)$ ⊛

Let $\varepsilon > 0$. Pick $U_j \in \mathcal{O}$, $U_j \supset A_j$, $m(U_j) < \mu^*(A_j) + \frac{\varepsilon}{2^j}$.

Then $\mu^*(\bigcup_j A_j) \leq m(\bigcup_j U_j) \overset{(B)}{\leq} \sum_j m(U_j)$

$$< \sum_j (\mu^*(A_j) + \varepsilon 2^{-j})$$

$$= \sum_j \mu^*(A_j) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives $\circledast$.

Claim 2: $\mu^*(U) = m(U)$ for $U \in \mathcal{O}$ (obvious),

every $U \in \mathcal{O}$ is measurable: $\mu^*(A) \overset{\oplus}{=} \mu^*(A \cap U) + \mu^*(A \cap \underset{X \setminus U}{\underset{\shortparallel}{U^c}})$ $\forall A \subseteq X$.

Check: Only need to prove "$\geq$" (since "$\leq$" follows from sub-additivity). Let $\varepsilon > 0$, and pick $V \in \mathcal{O}$, $V \supset A$, s.t.

$$m(V) < \mu^*(A) + \varepsilon.$$

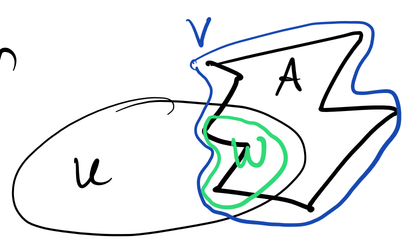
Will show: $m(V) \geq m(V \cap U) + \mu^*(V \cap U^c)$ $\boxplus$.

(This implies $\mu^*(A) > m(V) - \varepsilon$
$\geq m(V \cap U) + \mu^*(V \cap U^c) - \varepsilon$
$\geq \mu^*(A \cap U) + \mu^*(A \cap U^c) - \varepsilon$,

so upon letting $\varepsilon \downarrow 0$ gives $\oplus$.)

To show $\boxplus$, choose (using (D)) $W \in \mathcal{O}$ s.t. $\overline{W} \subset V \cap U$ is compact and $m(V \cap U) < m(W) + \varepsilon$ (where $\varepsilon > 0$ is fixed).

$\Rightarrow$ W, $V \cap \overline{W}^c$ are disjoint open subsets of V, so



$$m(W) + m(V \cap \overline{W}^c) \overset{(C)}{=} m(W \cup (V \cap \overline{W}^c)) \overset{(A)}{\leq} m(V).$$

$\Rightarrow m(V \cap U) + \mu^*(V \cap U^c) < (m(W) + \varepsilon) + \mu^*(V \cap U^c)$

$\leq (m(W) + \varepsilon) + \mu^*(V \cap \overline{W}^c)$

$$\leq (m(W)+\varepsilon) + m(V \cap \overline{W}^c)$$
$$\leq (m(W)+\varepsilon) + (m(V) - m(W))$$
$$= m(V) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ proves (#).

Conclusion of the proof. By the Carathéodory Extension Theorem, the restriction of μ^* to the σ-algebra of μ^*-measurable sets (which thus includes the Borel σ-algebra) is a measure μ with $\mu(U) = m(U)$ $\forall\, U \in \mathcal{O}$.

- μ is outer regular by construction, and finite on compact sets.
- μ is inner regular by the following argument. Let $\varepsilon>0$. If $E \subset X$ is measurable, then by outer regularity, $\exists\, U \in \mathcal{O}$, $U \supset X \setminus E$, s.t.
$$\mu(U) < \mu(X \setminus E) + \varepsilon.$$
So $K = X \setminus U$ is compact, $K \subset E$, and
$$\mu(K) = \mu(X) - \mu(U) > \mu(X) - \mu(X \setminus E) - \varepsilon = \mu(E) - \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we are done. $\square$

Proof of existence of μ in Theorem (T.7)

For the function $m : \mathcal{O} = \{U \subset X \text{ open}\} \to [0,\infty)$ defined by $(\dagger)$, we shall verify the properties (A)–(D) from Proposition (P.3).

(A): obvious.

(B): Given $U_1, U_2, \dots \in \mathcal{O}$, set $U = \bigcup_j U_j$ and let $f \in C^0(X)$, $0 \le f \le 1$, $\operatorname{supp} f \subset U$. There exist $n \in \mathbb{N}$ and a partition of unity of X: $\phi_1, \dots, \phi_n \in C^0(X)$ with

$$0 \le \phi_j \le 1, \quad \operatorname{supp} \phi_j \subset U_j, \quad \sum_{j=1}^n \phi_j = 1 \text{ on } \operatorname{supp} f.$$

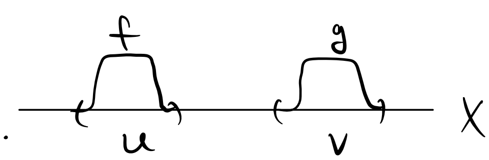
$\Rightarrow f = \sum_{j=1}^n \phi_j f$, with $\operatorname{supp} \phi_j f \subset U_j$, so

$$\lambda(f) = \sum_{j=1}^n \lambda(\phi_j f) \le \sum_{j=1}^n m(U_j) \le \sum_j m(U_j).$$

Take sup over all f to get $m(\bigcup U_j) \le \sum m(U_j)$.

(C): For $U, V \in \mathcal{O}$, $U \cap V = \emptyset$, have $m(U \cup V) \le m(U) + m(V)$ by (B).

To prove "$\ge$", let $f, g \in C^0(X)$, $0 \le f, g \le 1$, $\operatorname{supp} f \subset U$, $\operatorname{supp} g \subset V$. Then $0 \le f+g \le 1$ (since $U \cap V = \emptyset$) and $\operatorname{supp}(f+g) \subset U \cup V$



$\Rightarrow \lambda(f) + \lambda(g) = \lambda(f+g) \le m(U \cup V)$.

Take sup over f, g to conclude.

(D): If $f \in C^0(X)$, $0 \le f \le 1$, $\operatorname{supp} f \subset U$, then (since $\operatorname{supp} f$ is closed) $\exists\, V \subset U$ open s.t.

$$\operatorname{supp} f \subset V \subset \overline{V} \subset U.$$

$$\Rightarrow \lambda(f) \le m(V)$$

$$\le \sup\{m(W) : W \in \mathcal{O}, \overline{W} \subset U\}.$$

Take sup over f to conclude. $\square$