We now switch gears entirely; it will be a while before we get back to spectral theory. This is because spectral theoretic tools (e.g. the functional calculus) can in principle be used to solve partial differential equations, the verification of the assumptions of the various theorems (self-adjointness? domains?) is rather nontrivial, and involves serious PDE machinery — which we will need to develop first!

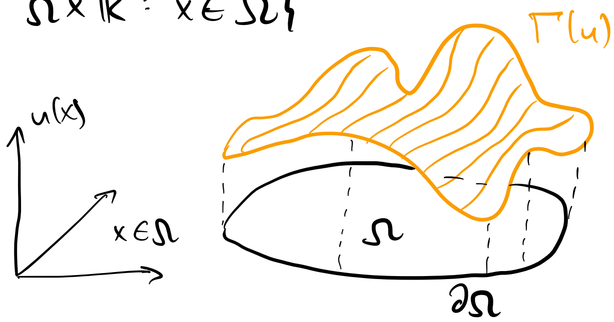
Motivation: variational problems, PDE, and solution attempts.

Consider an elastic membrane: given a connected open set $\Omega \Subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, let $u \in C^2(\overline{\Omega})$, and

$$\Gamma(u) = \{ (x, u(x)) \in \overline{\Omega} \times \mathbb{R} : x \in \overline{\Omega} \}$$

is the membrane.

- We fix its boundary to be the graph of $g \in C^2(\partial\Omega)$, i.e. $u(x) = g(x)$ for $x \in \partial\Omega$.



- The energy of the membrane is $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx$
- Physical model: given $g \in C^2(\partial\Omega)$, u should have least energy among all elements of $V(g) := \{ u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = g \}$

Questions: (i) Is $e(g) := \inf_{u \in V(g)} E(u)$ attained for some $u \in V(g)$?

(ii) If $\exists$ minimizer u of E on $V(g)$ is u unique?

(iii) How can one find / characterize u? ($\rightsquigarrow$ PDE.)

(iv) If $g \in C^\infty(\partial\Omega)$, is $u \in C^\infty(\overline{\Omega})$? (Higher regularity.)

Let us try to answer some of these questions and see how far we get:

(ii) Uniqueness of minimizers is a consequence of the strict convexity of E. Namely, if $u_1, u_2 \in V(g)$ with $E(u_1) = E(u_2) = e(g)$, then $\frac{u_1+u_2}{2} \in V(g)$, and

$$
\begin{aligned}
E\left(\frac{u_1+u_2}{2}\right) &= \frac{1}{2}\int_\Omega \frac{1}{4}|\nabla u_1 + \nabla u_2|^2 \, dx \\
&= \frac{1}{8}\int_\Omega |\nabla u_1|^2 + 2\nabla u_1 \cdot \nabla u_2 + |\nabla u_2|^2 \, dx \\
&\leq \frac{1}{8}\int_\Omega |\nabla u_1|^2 + 2|\nabla u_1||\nabla u_2| + |\nabla u_2|^2 \, dx \\
&\leq \frac{1}{8}\int_\Omega 2|\nabla u_1|^2 + 2|\nabla u_2|^2 \, dx \\
&= \frac{1}{2}\left(E(u_1) + E(u_2)\right) = e(g).
\end{aligned}
$$

$\Rightarrow \frac{u_1+u_2}{2}$ is a minimizer too $\Rightarrow$ equality holds everywhere, so $\forall x \in \Omega$, $\nabla u_1(x) = c(x)\nabla u_2(x)$ for some $c > 0$, and $|\nabla u_1(x)| = |\nabla u_2(x)|$

$\Rightarrow \nabla u_1 = \nabla u_2 \Rightarrow u_1 - u_2 = c \in \mathbb{R}$; but at $\partial\Omega$, $u_1 - u_2 = 0$,

so $u_1 = u_2$.

(iii) PDE for the minimizer $u \in V(g)$. Let $\varphi \in C_c^\infty(\Omega)$ (or just $\varphi \in V(0)$, i.e. $\varphi \in C^2(\bar\Omega)$, $\varphi|_{\partial\Omega} = 0$); then $u + s\varphi \in V(g)$ $\forall s \in \mathbb{R}$,

so $E(u) \leq E(u+s\varphi) = \frac{1}{2}\int_\Omega |\nabla u|^2 + 2s\nabla u \cdot \nabla\varphi + s^2|\nabla\varphi|^2 \, dx$

$$\overset{\frac{d}{ds}\big|_{s=0}}{\Longrightarrow} \quad \int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0. \qquad \underset{\text{clear}}{\Longleftarrow}$$

We can integrate by parts since $\varphi|_{\partial\Omega} = 0 \Rightarrow \int_\Omega \Delta u \, \varphi \, dx = 0$

Since $\varphi \in C_c^\infty(\Omega)$ is arbitrary:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad \circledast$$

(i) Existence of minimizers. Two options:

(a) try to solve the PDE $\circledast$: see below.

(b) direct method of calculus of variations. $C^2(\overline{\Omega})$ is not reflexive; $E(u)$ has "little to do" with $C^2(\overline{\Omega})$... Will need to extend the class of u we work with (to $H^1_0(\Omega)$).

(iv) Higher regularity of u: via regularity theory for the PDE $\circledast$.

We briefly discuss 2 natural approaches to solving the PDE $\circledast$. Let $u_0 \in C^2(\overline{\Omega})$ be any function with $u_0|_{\partial\Omega} = g$. Write $u = u_0 + v$, then we want
$$\begin{cases} \Delta v = f := -\Delta u_0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Ignoring the precise nature of the function spaces in which f here lives, we call $v \rightsquigarrow u$ and study now
$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad \oplus$$

APPROACH 1: solvability via functional analysis (domains, adjoints).

Idea: Regard $\oplus$ as the equation $Au = f$ where $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is the Laplacian; $D(A)$ encodes the Dirichlet boundary condition $u|_{\partial\Omega} = 0$.

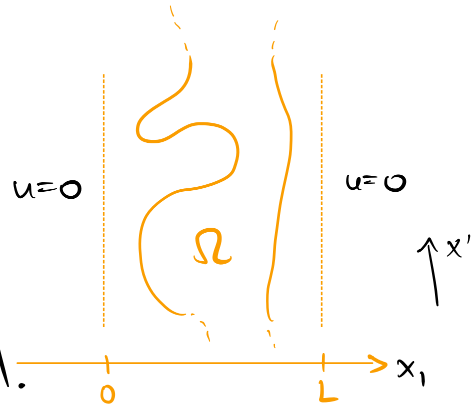
Execution: Let $D(A) := \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$,
$$Au := \Delta u \quad (u \in D(A)).$$

We already checked that A is symmetric (Example (E.7)), so $A \subset A^* \Rightarrow A$ is closable; the closure $\bar{A} \subseteq A^*$ is still symmetric (exercise).

Proposition (P.5) $\operatorname{ran} \bar{A} \subset L^2(\Omega)$ is closed, and $\ker \bar{A} = \{0\}$.

Lemma (L.8) (Poincaré inequality.) Let $\Omega \subset [0,L] \times \mathbb{R}^{n-1}$ be open. Then $\forall u \in C_c^\infty(\Omega)$,

$$\int_\Omega |u|^2 \, dx \le L^2 \int_\Omega |\nabla u|^2 \, dx.$$



Intuition: If u goes from 0 back to 0 over an interval of length L, then $|u|$ cannot exceed $L|\nabla u|$.

Proof of Lemma (L.8) For $x = (x_1, x') \in \Omega$,

$$|u(x_1, x')|^2 = \left| \int_0^{x_1} \frac{\partial u}{\partial x_1}(s, x')\, ds \right|^2 \le \left(\int_0^{x_1} \left| \frac{\partial u}{\partial x_1}(s,x') \right| ds \right)^2$$

$$\le L \int_0^L \left| \frac{\partial u}{\partial x_1}(s,x') \right|^2 ds$$

by Cauchy-Schwarz. Integrate over $(x_1, x') \in \Omega$

$$\Rightarrow \|u\|_{L^2(\Omega)} \le \int_0^L \int_{\mathbb{R}^{n-1}} L \int_0^L \left| \frac{\partial u}{\partial x_1}(s,x') \right|^2 ds \, dx' \, dx_1$$

$$= L^2 \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega)} \qquad \square$$

Proof of Proposition (P.5).

* *Injectivity:* If $u \in \ker \bar{A}$, choose $u_k \in D(A)$ s.t.

$$u_k \to u, \quad A u_k = \Delta u_k \to \bar{A} u = 0.$$

Then $\|u_k\|^2_{L^2(\Omega)} \overset{\text{Lemma (L.8)}}{\le} C \|\nabla u_k\|^2_{L^2(\Omega)} = C \int_\Omega \nabla u_k \cdot \nabla u_k \, dx$

$$\overset{\text{I.B.P.}}{=} C\int_\Omega u_k(-\Delta u_k)\,dx$$

$$\leq C\|u_k\|_{L^2(\Omega)}\|\Delta u_k\|_{L^2(\Omega)}.$$

$$\Rightarrow \|u_k\|_{L^2} \leq C\|\Delta u_k\|_{L^2} \xrightarrow{k\to\infty} 0$$

$$\Rightarrow u = \lim_{k\to\infty} u_k = 0.$$

* Closed range: Let $u_k \in D(\bar{A})$, $\bar{A}u_k = f_k \xrightarrow{k\to\infty} f \in L^2(\Omega)$.
Replacing u_k by good approximations (in the graph norm) by elements of $D(A)$, we may assume $u_k \in D(A)$; so

$$\begin{cases} u_k \in C^2(\bar{\Omega}), \\ u_k|_{\partial\Omega} = 0 \\ \Delta u_k = f_k \longrightarrow f. \end{cases}$$

$$\Rightarrow \|u_k - u_\ell\|^2_{L^2(\Omega)} \overset{\text{Lemma (L.8)}}{\leq} C\|\nabla(u_k-u_\ell)\|^2_{L^2(\Omega)}$$

$$= C\int_\Omega (u_k-u_\ell)(-\Delta(u_k-u_\ell))\,dx$$

$$= C\left|\int_\Omega (u_k-u_\ell)(f_\ell-f_k)\,dx\right|$$

$$\leq C\|u_k-u_\ell\|_{L^2(\Omega)}\|f_k-f_\ell\|_{L^2(\Omega)}$$

$$\Rightarrow \|u_k-u_\ell\|_{L^2(\Omega)} \leq C\|f_k-f_\ell\|_{L^2(\Omega)} \xrightarrow{k,\ell\to\infty} 0.$$

So $\exists \lim_{k\to\infty} u_k = u \in L^2(\Omega)$.

But $Au_k = \bar{A}u_k \to f$; so $(u_k, Au_k) \longrightarrow (u,f)$ in $\Gamma_{\bar{A}}$, and thus $f = \bar{A}u$. $\square$

Finally, by Theorem (T.4), $\operatorname{ran}\bar{A} = (\ker \bar{A}^*)^\perp$.

Theorem (T.15) $\bar{A}$ is self-adjoint.

This is hard, and we will need to prepare well to prove it.

Granted Theorem (T.15), we get $\operatorname{ran}\bar{A} = (\ker \bar{A})^{\perp} = L^2(\Omega)$ from Proposition (P.5).

$\Rightarrow \bar{A}: D(\bar{A}) \longrightarrow L^2(\Omega)$ is invertible.

Theorem (T.16) $\forall f \in L^2(\Omega)\ \exists!\ u \in D(\bar{A})$ s.t. $\bar{A}u = f$.

Open questions: What is $D(\bar{A})$? In what sense does $u \in D(\bar{A})$ satisfy the Dirichlet boundary condition? How to prove Theorem (T.15)? ($\rightarrow$ Sobolev spaces, boundary traces of Sobolev functions).

APPROACH 2: Riesz representation theorem.

The starting point is the "weak formulation" of the PDE ⊕:

if $u \in C^2(\bar{\Omega})$ is a solution and $v \in C_c^\infty(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \cdot v \, dx = -\int_\Omega f v \, dx. \qquad ⊕$$

Regard the L.H.S. as an inner product, with associated norm

$$\|u\|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla u|^2 dx.$$

By Lemma (L.8), $\|u\|^2_{L^2(\Omega)} \leq C^2 \|u\|^2_{H_0^1(\Omega)}$ for $u \in C_c^\infty(\Omega)$,

so $\|\cdot\|_{H_0^1(\Omega)}$ is a norm on $C_c^\infty(\Omega)$.

Moreover, the R.H.S. of ⊕ is continuous in the $H_0^1(\Omega)$-norm:

$$\left| \int_\Omega f v \, dx \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

To apply Riesz, need Hilbert spaces:

Definition (D.11) $H^1_0(\Omega) :=$ closure of $C^\infty_c(\Omega)$ w.r.t. $\|\cdot\|_{H^1_0(\Omega)}$.

This is a Hilbert space with $H^1_0(\Omega) \subset L^2(\Omega)$ and inner product

$$(u,v)_{H^1_0} = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

where $\nabla u = \lim_{k\to\infty} \nabla u_k \in L^2(\Omega)$ if $C^\infty_c(\Omega) \ni u_k \longrightarrow u$ in $H^1_0(\Omega)$.

Theorem (T.16') Let $f \in L^2(\Omega)$. Then $\exists!$ $u \in H^1_0(\Omega)$ s.th.

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega f v \, dx \quad \forall v \in C^\infty_c(\Omega).$$

We say that u is a weak solution of $\#$.

Proof $H^1_0(\Omega) \ni v \longmapsto -\int_\Omega fv \, dx \in \mathbb{R}$ defines an element of $H^1_0(\Omega)^*$, which is equal to $(u,v)_{H^1_0(\Omega)}$ for some unique $u \in H^1_0(\Omega)$ by Riesz. $\square$

Open questions: Is u, for $f \in C^\infty(\overline{\Omega})$, also a classical (i.e. $C^2(\overline{\Omega})$, or even $C^\infty(\overline{\Omega})$) solution of $\#$?

In what sense does $u \in H^1_0(\Omega)$ satisfy the Dirichlet boundary condition?

Plan for the next $N \gg 1$ lectures:

- define, analyze Sobolev spaces such as $H^1_0(\Omega)$
- solvability and regularity for general elliptic PDEs

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x) u(x) = f(x) & (*) \\ \text{boundary conditions for } u, \end{cases}$$

where $(a_{ij})_{i,j=1}^{n}$ is symmetric, and $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq c|\xi|^2$, $x \in \Omega$, for all $\xi \in \mathbb{R}^n$, where $c > 0$ ("ellipticity" of $(*)$).

$$\left(\text{Laplace equation: } \begin{cases} (a_{ij}) = \text{identity matrix} \\ b_i = 0 \\ c = 0. \end{cases} \right)$$