Motivation

(1) Differential operators (of order ≥ 1) are not bounded operators $X \to X$ for any "interesting" or "natural" normed space.

E.g.: (i) $\frac{d}{dt}: u = u(t) \mapsto u'$ is bounded $C^1([0,1]) \to C^0([0,1])$

but not defined (let alone bounded) as a map

$C^1([0,1]) \to C^1([0,1])$.

(ii) $D_\theta^2 : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, $\mathcal{F}(D_\theta^2 u)(n) = n^2 \mathcal{F}u(n)$.

$$H^2(\mathbb{S}^1) = \{u \in L^2(\mathbb{S}^1) : \sum_{n\in\mathbb{Z}} (1+|n|)^4 |\mathcal{F}u(n)|^2 < \infty\}$$

This is not defined (or bounded) as a map $H^2 \to H^2$ or $L^2 \to L^2$ or ...

Nonetheless, the operators $\frac{d}{dt} - \lambda : C^1([0,1]) \to C^0([0,1])$ and $D_\theta^2 - \lambda : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are well-defined, and one can inquire about their mapping properties — injective, surjective, dense range, etc.? $\rightsquigarrow$ spectral theory.

(i) $(\frac{d}{dt} - \lambda)u(t) = 0 \iff u(t) = u(0)e^{\lambda t}$. So: $\frac{d}{dt} - \lambda \in L(C^1, C^0)$ has nontrivial kernel for all $\lambda \in \mathbb{C}$! (In particular, it is never invertible.)

(ii) $(D_\theta^2 - \lambda)u(\theta) = 0 \iff \lambda = n^2$ for some $n \in \mathbb{Z}$, and $u(\theta) = e^{\pm in\theta}$.

For $\lambda \notin \{n^2 : n \in \mathbb{Z}\}$, $D_\theta^2 - \lambda : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is invertible.

We will study operators such as these in large generality and understand what distinguishes them.

(2) Quantum mechanics. The Schrödinger equation for a function $\psi = \psi(t,x)$ on $\mathbb{R}^2_{t,x}$ reads

$$\begin{cases} i \frac{\partial}{\partial t} \psi = -\frac{\partial^2}{\partial x^2} \psi \\ \psi(0,x) = \psi_0(x), \qquad \psi_0 \in L^2(\mathbb{R}), \ \int_{\mathbb{R}} |\psi_0(x)|^2 dx = 1. \end{cases}$$

Solution is formally given by $\psi(t,x) = e^{it\frac{\partial^2}{\partial x^2}} \psi_0(x)$.

What is this? Looks like functional calculus for an unbounded operator $\left(\frac{\partial^2}{\partial x^2}\right)$. We will make sense of this, and relate the self-adjointness of $\frac{\partial^2}{\partial x^2}$ to the unitarity of $\psi_0 \mapsto \psi(t,-)$ (i.e. $\int_{\mathbb{R}} |\psi(t,x)|^2 dx = 1 \quad \forall t$).

First task: definition and basic properties of unbounded operators.

Definition (D.1) Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed vector spaces.

Let $A : D(A) \to Y$ be a linear operator, where $D(A) \subseteq X$ is a linear subspace (the domain of A).

(i) A is an unbounded operator if $A : (D(A), \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is not bounded (i.e. $\nexists\, C < \infty$ s.t. $\|Ax\|_Y \leq C\|x\|_X \ \forall x \in D(A)$).

(ii) A is densely defined if $\overline{D(A)} = X$ (i.e. $D(A) \subset X$ is a dense subspace).

(iii) A is closed if its graph $\Gamma_A = \{(x, Ax) : x \in D(A)\}$ is a closed subspace of $X \times Y$.

Recall: (i) If $A \in L(X,Y)$ (so $A: D(A) := X \to Y$ is bounded), then Γ_A is closed. If X, Y are Banach spaces the converse holds as well (Closed Graph Theorem).

(ii) If $A: D(A) \subset X \to Y$ is densely defined and bounded, then A has a unique continuous extension $\bar{A}: X \to Y$
($\bar{A}x = \lim_{j\to\infty} Ax_j$, where $x \in X$ and $x_j \in D(A)$, $x_j \xrightarrow{j\to\infty} x$).

Example (E.1) $X = Y = C^0([0,1])$, with sup norm ($\|u\|_{C^0} = \sup_{t\in[0,1]} |u(t)|$).
$A = \frac{d}{dt}: D(A) := C^1([0,1]) \to Y$.

Claim: A is (i) unbounded, (ii) densely defined, (iii) closed.

Proof: (i) Consider $u_k(t) = t^k$, $k = 0,1,2,\ldots$. Then
$$\|u_k\|_X = \|u_k\|_{C^0} = 1, \quad \text{but } \|Au_k\|_Y = \|u_k'\|_{C^0} = k.$$
$(u_k'(t) = kt^{k-1})$

So $\nexists\, C < \infty$ s.t. $\|Au_k\|_{C^0} \le C\|u_k\|_{C^0}$ $\forall k$.

(ii) $C^1([0,1]) \subset C^0([0,1])$ is dense indeed. (The space of polynomials is dense in $C^0([0,1])$ even!)

(iii) Let $u_k \in D(A)$, $k \in \mathbb{N}$, and suppose
$$\begin{cases} u_k \to u \in X = C^0([0,1]) \\ Au_k = u_k' \to v \in Y = C^0([0,1]) \end{cases} \quad \text{as } k \to \infty.$$

(That is, $\underbrace{(u_k, Au_k)}_{\Gamma_A} \to (u,v)$ in $X \times Y$.) We need to show: $(u,v) \in \Gamma_A$,
i.e. $u \in D(A) = C^1([0,1])$, $v = Au = u'$. $\Big\}\circledast$

Well, $u_k(t) = u_k(0) + \int_0^t u_k'(s)\,ds$ $\forall k$; taking $k \to \infty$ gives

$$u(t) = u(0) + \int_0^t v(s)\, ds, \text{ which implies } (*). \qquad \square$$

An important reason why closed operators are important is:

Theorem (T.1) Let X, Y be Banach spaces, and suppose $A: D(A) \subset X \to Y$ is linear, closed, and bijective. Then $A^{-1}: Y \to D(A) \subset X$ is bounded.

Proof The projection $\pi_Y : \Gamma_A \to Y$ is a continuous bijection of Banach spaces (since $\Gamma_A \subset X \times Y$ is complete, being a closed subspace of a Banach space). By the Open Mapping Theorem,

$$B := \pi_X \circ \pi_Y^{-1} \in L(Y, X),$$

where $\pi_X : \Gamma_A \to X$ is the projection. But

$$A \underbrace{By}_{= x \text{ s.t. } (x,y) \in \Gamma_A} = Ax \underset{\Uparrow}{=} y \quad \text{and} \quad B(Ax) = \underset{\uparrow \, (x, Ax) \in \Gamma_A!}{x}, \text{ so } B = A^{-1}.$$

Note finally that $\pi_X : \Gamma_A \to D(A)$, so $B: H \to D(A)$. $\square$

Example (E.2) $A_0 = \frac{d}{dt} : D(A_0) \subset X \to Y$ where $X = Y = C^0([0,1])$ and $D(A_0) = \{u \in C^1([0,1]) : u(0) = 0\}$. Then A_0 satisfies the assumptions of Theorem (T.1); and

$$(A_0^{-1} f)(t) = \int_0^t f(s)\, ds, \text{ so } A_0^{-1} : C^0([0,1]) \to D(A) \subset C^0([0,1])$$

is indeed continuous.

Definition (D.2) (i) An *extension* of a linear operator $A : D(A) \subset X \to Y$ is a linear operator $A_1 : D(A_1) \subset X \to Y$ s.t. $D(A_1) \supseteq D(A)$ and $A_1|_{D(A)} = A$. Notation: $A \subset A_1$.

(ii) A linear operator $A : D(A) \subset X \to Y$ is *closable* if it has a closed extension.

Example (E.3) $A = \frac{d}{dt} : C^1([0,1]) \subset C^0([0,1]) \longrightarrow C^0([0,1])$ is an extension of $A_0 = \frac{d}{dt} : \{u \in C^1([0,1]) : u(0) = 0\} \subset C^0 \to C^0$.

Lemma (L.1) $A : D(A) \subset X \to Y$ is closable iff the closure $\overline{\Gamma_A}$ of $\Gamma_A \subset X \times Y$ is a *linear graph*, i.e. $\left((0,y) \in \overline{\Gamma_A} \Rightarrow y = 0\right)$. In this case, the corresponding operator $\overline{A} : D(\overline{A}) \subset X \to Y$ with $\Gamma_{\overline{A}} = \overline{\Gamma_A}$ is the smallest closed extension of A, called the *closure* of A.

Remark (R.1) $\overline{\Gamma_A}$ is a linear subspace; it is a linear graph iff $(x, y_1), (x, y_2) \in \overline{\Gamma_A} \Rightarrow y_1 = y_2$. ("$\Rightarrow$": $\overline{\Gamma_A} \ni (x,y_1) - (x,y_2) = (0, y_1 - y_2)$ implies $y_1 - y_2 = 0$. "$\Leftarrow$": $(0, \underset{y_1}{0}) \in \overline{\Gamma_A}$, $(0, \underset{y_2}{y}) \in \overline{\Gamma_A}$ implies $0 = y$.)

Proof of Lemma (L.1) "$\Rightarrow$" If $A_1 : D(A_1) \subset X \to Y$ is a closed extension of A, then $\Gamma_{A_1} \subset X \times Y$ is a closed set containing Γ_A, and therefore $\Gamma_{A_1} \supset \overline{\Gamma_A}$. Since Γ_{A_1} is a linear graph, so is $\overline{\Gamma_A}$.

Thus, $\overline{A} : D(\overline{A}) \subset X \to Y$ where $D(\overline{A}) = \pi_X(\overline{\Gamma_A})$ and

$\overline{A}x = y$ where $(x,y) \in \overline{\Gamma_A}$ is well-defined, closed, and the smallest closed extension of A. $\square$

"$\Leftarrow$" Obvious.

Warning Typically, $D(\bar{A}) \subsetneq \overline{D(A)}$! For instance, $A = \frac{d}{dt} : C^1 \subset C^0 \to C^0$ is closed, so $A = \bar{A}$, but $D(\bar{A}) = D(A) = C^1 \neq \overline{D(A)} = C^0$.

Lemma (L.2) $A : D(A) \subset X \to Y$ is closable iff for all sequences $(x_k, y_k)_{k \in \mathbb{N}} \subset \Gamma_A$ with $x_k \to 0$ and $Ax_k = y_k \to y$, $k \to \infty$, we have $y = 0$.

Proof Since $X \times Y$ is a metric space, the stated condition is equivalent to $\bar{\Gamma}_A \ni (0, y) \Rightarrow y = 0$. $\square$

Example (E.4)

(i) $A_0 : D(A_0) := C^\infty([0,1]) \subset C^0([0,1]) \to C^0([0,1])$.

Claim: $D(\bar{A}_0) = C^1([0,1])$.

Indeed, $A_1 : D(A_1) := C^1([0,1]) \subset C^0 \to C^0$ is closed (Example (E.1)) and extends A_0; so $D(\bar{A}_0) \subset D(A_1)$.

Conversely, given $u \in C^1([0,1])$, pick$^{\circledast}$ $u_k \in C^\infty([0,1])$ s.t. $u_k \to u$ in C^1; then $u_k' \to u'$ in C^0, so $(u_k, u_k') \in \Gamma_A$ converges to $(u, v) \in \Gamma_{A_1}$. Therefore, $A_1 = \bar{A}_0$. $\square$

($^{\circledast}$ e.g. write $u(t) = u(0) + \int_0^t u'(s)\,ds$ and approximate $u' \in C^0$ uniformly by a sequence of polynomials.)

(ii) If $A: D(A) \subset X \to Y$ is continuous, then A is closable.

Indeed, $x_k \to 0$, $Ax_k \to y$ implies

$$\|y\| = \lim_{k\to\infty} \|Ax_k\| \leq \|A\| \cdot \liminf_{k\to\infty} \|x_k\| = 0, \text{ so } y=0.$$

(Exercise: if Y is complete, then $D(\bar{A}) = \overline{D(A)}$.)

(iii) There do exist operators that are not closeable, e.g.

$$A: D(A) = \{u \in C^0(\mathbb{R}) : \operatorname{supp} u \Subset \mathbb{R}\} \subset L^2(\mathbb{R}) \to \mathbb{R},$$

$$D(A) \ni u \mapsto \int_{-\infty}^{\infty} u(t)\,dt \in \mathbb{R}.$$

(Here, $\operatorname{supp} u = \overline{\{x \in \mathbb{R} : u(x) \neq 0\}}$.) $\rightsquigarrow$ Exercise.

(iv) Recall $C^\infty(S^1) = C^\infty_{per}([0,2\pi])$

$$= \{u \in C^\infty([0,2\pi]) : u^{(k)}(0) = u^{(k)}(2\pi)\ \forall k \in \mathbb{N}_0\}.$$

Let $A = D_\theta^2 : D(D_\theta^2) := C^\infty(S^1) \subset L^2(S^1) \to L^2(S^1)$.

<u>Claim</u> A is closable, and $D(\bar{A}) = H^2(S^1)$.

<u>Proof</u> Consider $B = \mathcal{F} \circ A \circ \mathcal{F}^{-1}$ instead (and recall that $\mathcal{F}: L^2(S^1) \to \ell^2(\mathbb{Z})$ is an isomorphism), with

$$D(B) = \mathcal{F}(D(A)) = s(\mathbb{Z}) = \{a = (a_n)_{n\in\mathbb{Z}} : |a_n| \leq C_k (1+|n|)^{-k}\ \forall k\},$$

$$B: D(B) \subset \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}),$$

$$(Ba)_n = n^2 a_n.$$

We need to show:

$$D(\bar{B}) = h^2(\mathbb{Z}) = \{a = (a_n)_{n\in\mathbb{Z}} : \sum (1+|n|)^4 |a_n|^2 < \infty\}.$$

<u>Well</u>, if $a \in h^2(\mathbb{Z})$, then $a^{(N)} = (a_n^{(N)}) \in s(\mathbb{Z}) = D(B)$,

where

$$a_n^{(N)} = \begin{cases} a_n, & |n| \le N \\ 0, & |n| > N. \end{cases}$$

Then $a^{(N)} \to a$ in $h^2(\mathbb{Z})$ (proof: $\sum_{|n|>N} (1+|n|)^4 |a_n|^2 \xrightarrow{N\to\infty} 0$)

and thus $Ba^{(N)} = (n^2 a_n^{(N)})_n \longrightarrow (n^2 a)_n$ in $\ell^2(\mathbb{Z})$;

so $(a^{(N)}, Ba^{(N)}) \longrightarrow ((a_n), (n^2 a_n))$ in $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$,

proving that $D(\overline{B}) \supset h^2(\mathbb{Z})$.

- Conversely, $h^2(\mathbb{Z}) \ni a \longmapsto (n^2 a_n)_n \in \ell^2(\mathbb{Z})$, as an unbounded operator on ℓ^2 with domain $h^2(\mathbb{Z})$, is closed (Exercise), and extends $B \Rightarrow D(\overline{B}) \subset h^2(\mathbb{Z})$. $\square$

Example (E.4) (iv) is a first demonstration of how Sobolev spaces such as $H^2(S^1)$ arise naturally in the analysis of differential operators on L^2. A major goal of FA II is the analysis of linear partial differential operators. For now, a simple result:

Example (E.5) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Write

$$\Delta u(x) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x) \quad \text{(Laplace operator, "Laplacian").}$$

Let $C_c^\infty(\Omega) = \{u \in C^\infty(\Omega) : \operatorname{supp} u \Subset \Omega\}$.

Claim: $\Delta : C_c^\infty(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ is closable.

Indeed, suppose $u_k \in C_c^\infty(\Omega)$ is such that

$$u_k \longrightarrow 0 \text{ in } L^2(\Omega) \quad \text{and} \quad f_k := \Delta u_k \longrightarrow f \text{ in } L^2(\Omega).$$

Then for all $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega f_k \varphi \, dx = \int_\Omega \Delta u_k \cdot \varphi \, dx \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{=} \int_\Omega u_k \cdot \Delta\varphi \, dx \xrightarrow{k\to\infty} 0$$

$$\downarrow$$

$$\int_\Omega f\varphi \, dx.$$

Therefore, $\int_\Omega f\varphi \, dx = 0 \ \forall \varphi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega) \subset L^2(\Omega)$ is dense, this gives $f = 0$. Use Lemma (L.2) to conclude. $\square$

The domain of the closure will be identified with

$$H_0^2(\Omega) = D(\overline{\Delta}) \qquad \text{(later..)}$$

Generalizing this example, consider for $1 \le p \le \infty$

$$A: C_c^\infty(\Omega) \subset L^p(\Omega) \to L^p(\Omega),$$

$$Au(x) = \sum_{|\alpha| \le N} a_\alpha(x) \partial^\alpha u(x), \text{ where } \alpha \in \mathbb{N}_0^n, \ \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

$$|\alpha| = \sum_{j=1}^n \alpha_j, \text{ and } a_\alpha \in C^N(\overline{\Omega}).$$

<u>Proposition (P.1)</u> A is closable.

<u>Proof</u> If $u_k \in C_c^\infty(\Omega)$, $u_k \to 0$ in $L^p(\Omega)$,

$f_k := Au_k \to f$ in $L^p(\Omega)$,

then for $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega f_k \varphi \, dx = \int_\Omega \sum_{|\alpha| \le N} a_\alpha \partial^\alpha u_k \cdot \varphi \, dx$$

$$= \int_\Omega u_k \cdot \sum_{|\alpha| \le N} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \varphi) \, dx.$$

Letting $k \to \infty$ gives $\int_\Omega f\varphi \, dx = 0 \ (\forall \varphi \in C_c^\infty(\Omega))$.

This implies $f = 0$ a.e. (see Theorem (T.2) below) $\square$

Theorem (T.2) ("Fundamental Lemma of the calculus of variations")

Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1_{loc}(\Omega)$ (i.e. $1_K f \in L^1$ for all $K \Subset \Omega$). Suppose that $\int_\Omega f\varphi\, dx = 0 \ \forall \varphi \in C_c^\infty(\Omega)$. Then $f = 0$ a.e.

Proof Recall the Lebesgue differentiation theorem: for a.e. $x_0 \in \Omega$,

$$\lim_{r \searrow 0} \frac{1}{\underset{\substack{\uparrow \\ \text{Lebesgue measure}}}{\mu}(B_r(x_0))} \int |f(x) - f(x_0)|\, dx = 0.$$

Consider such a "Lebesgue point" $x_0 \in \Omega$. Let $\varphi \in C_c^\infty(B_1(0))$ with $\int \varphi\, dx = 1$, and set $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\frac{x - x_0}{\varepsilon}\right)$; then $\varphi_\varepsilon \in C_c^\infty(\Omega)$ for small $\varepsilon > 0$, and $\int_\Omega \varphi_\varepsilon\, dx = 1$. Thus,

$$0 = \int_\Omega f(x)\varphi_\varepsilon(x)\, dx = \underbrace{\int_\Omega (f(x) - f(x_0))\varphi_\varepsilon(x)\, dx}_{} + f(x_0),$$

where we can estimate

$$|\cdot| \le \int_{B_\varepsilon(x_0)} |f(x) - f(x_0)| \cdot \underbrace{C\varepsilon^{-n}}_{= \frac{\text{const}}{\mu(B_\varepsilon(x_0))}} dx \xrightarrow{\varepsilon \searrow 0} 0.$$

Therefore, $f(x_0) = 0 \Rightarrow f = 0$ a.e. $\square$