We shall now put the theory of Sobolev spaces $W^{1,p}(I)$, $I \subset \mathbb{R}$, to use to study simple 2^{nd} order ODEs on I using the 2 approaches outlined earlier.

Solutions via the Riesz representation theorem ("APPROACH 2").

Let $I = (a,b) \subset \mathbb{R}$, $-\infty < a < b < \infty$.

Let $f \in C^0(\bar{I})$ and consider

$$\begin{cases} u'' = f & \text{in } I \\ u(a) = u(b) = 0. \end{cases} \quad (*)$$

Using Riesz, $\exists!$ $u \in H_0^1(I)$ solving $(*)$ weakly, in that

$$\int_I u'v'\,dx = -\int_I f v\,dx \quad \forall v \in H_0^1(I). \quad (+)$$

Proposition (P.8) The weak solution u satisfies $u \in C^2(\bar{I})$ and is a classical solution of $(*)$.

Proof. By $(+)$, $u' \in L^2(I)$ has weak derivative

$$(u')' = f \in C^0(\bar{I}) \subset L^2(I)$$

$\Rightarrow u' \in W^{1,2}(I) \Rightarrow u'(x) = u'(x_0) + \int_{x_0}^{x} \underbrace{f(t)}_{\text{continuous}}\,dt$ lies in $C^1(\bar{I}) \Rightarrow u \in C^2(\bar{I})$ since $u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\,dt$.

• Boundary conditions: since $C_c^\infty(I) \subset H_0^1(I)$ is dense, and $H_0^1(I) \subset C^0(\bar{I})$ by Theorem (T.20), every $u \in H_0^1(I)$ vanishes at a, b (since this is true for $u \in C_c^\infty(I)$). $\square$

Solutions via self-adjointness arguments ("APPROACH 1").

Let again $I=(a,b)$, $-\infty<a<b<\infty$.

We define $A=\frac{d^2}{dx^2}$ on $D(A)=\{u\in C^2(\bar{I}): u(a)=u(b)=0\}$.

We aim to prove Theorem (T.15) ($\bar{A}$ is self-adjoint). First, we identify the domain of $\bar{A}$. We need:

Lemma (L.11) $H_0^1(I)=\{u\in H^1(I): u(a)=u(b)=0\}$.

(This was our original definition of $H_0^1(I)$, so really this lemma is about showing that the original definition is consistent with Definition (D.13)).

Proof "$\subseteq$" Easy.

"$\supseteq$". Given $u\in H^1(I)$, $u(a)=u(b)=0$, let $v(x)=\begin{cases} u(x), & x\in I \\ 0, & x\notin I.\end{cases}$

For $\varphi\in C_c^\infty(\mathbb{R})$,

$$\int \varphi' v\,dx=\int_I \varphi' u\,dx$$
$$=\int_a^b (\varphi u)'-\varphi u'\,dx$$
$$=\underbrace{\big(\varphi(b)u(b)-\varphi(a)u(a)\big)}_{=0}-\int_a^b \varphi u'\,dx;$$

so v has a weak derivative $v'\in L^2(\mathbb{R})$ defined by

$$v'(x)=\begin{cases} u'(x), & x\in I \\ 0, & x\notin I\end{cases}. \Rightarrow v\in H^1(\mathbb{R}),\ \operatorname{supp} v\subset \bar{I}.$$

• WLOG, $I=(-1,1)$. Let $v_k(x)=v\left(\frac{x}{1-2/k}\right)$, then

$v_k\in H^1(\mathbb{R})$, $v_k\xrightarrow{k\to\infty} v$ in $H^1(\mathbb{R})$, $\operatorname{supp} v_k\subset(-1+\frac{1}{k},1-\frac{1}{k})$.

We can then approximate v_k in $H^1(\mathbb{R})$ by

$$v_{k,\ell} := \rho_{\ell^{-1}} * v_k \in C_c^\infty\Big(\big(-1+\tfrac{1}{k}-\tfrac{1}{\ell},\, 1-\tfrac{1}{k}+\tfrac{1}{\ell}\big)\Big) \subset C_c^\infty(I) \quad (\ell \geq k)$$

as in Corollary (C.3). So v is the limit of a sequence $v_{k,\ell_k} \in C_c^\infty(I)$ in $H^1(\mathbb{R})$. This proves $u = v\big|_I \in H_0^1(I)$. $\square$

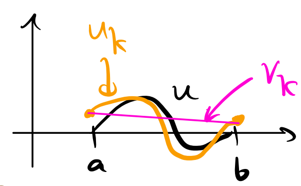
Lemma (L.12) $D(\bar{A}) = H^2(I) \cap H_0^1(I)$.

Proof "$\supseteq$" Given $u \in H^2(I) \cap H_0^1(I)$, pick (using Theorem (T.19')(iii)) $u_k \in C^\infty(\bar{I})$ with $u_k \to u$ in $H^2(I)$. Since $H^2(I) \subset C^0(\bar{I})$, Lemma (L.11) gives $u_k(a), u_k(b) \xrightarrow{k\to\infty} 0$, so

$$v_k(x) := u_k(a) + \frac{x-a}{b-a}\big(u_k(b) - u_k(a)\big) \xrightarrow{k\to\infty} 0 \text{ in } H^2(I)$$

$\Rightarrow$ $\tilde{u}_k := u_k - v_k \longrightarrow u$ in $H^2(I)$,

$\tilde{u}_k(a) = \tilde{u}_k(b) = 0$,



so $\tilde{u}_k \in D(A)$. Since $\tilde{u}_k \to u$ in $L^2(I)$,

$\tilde{u}_k'' \to u''$ in $L^2(I)$,

we get $u \in D(\bar{A})$.

"$\subseteq$" Let $u \in D(\bar{A})$, so $\exists \{u_k\} \subset D(A) \overset{\text{Lemma (L.11)}}{=} C^2(\bar{I}) \cap H_0^1(I)$ s.t.

$$\begin{cases} u = \lim\limits_{k\to\infty} u_k \text{ in } L^2(I), \\ \exists \lim\limits_{k\to\infty} A u_k = \lim\limits_{k\to\infty} u_k'' =: v \text{ in } L^2(I). \end{cases}$$

To conclude, we only need to show that u_k' converges in $L^2(I)$.

But this follows from

$$\|u_k' - u_\ell'\|^2_{L^2(I)} = \int_I |u_k' - u_\ell'|^2 \, dx$$

$$= -\int_I (u_k'' - u_\ell'')(u_k - u_\ell) \, dx$$

$$\leq \|u_k'' - u_\ell''\|_{L^2(I)} \|u_k - u_\ell\|_{L^2(I)}$$

$$\xrightarrow{k,\ell\to\infty} 0. \qquad \square$$

<u>Proof of Theorem (T.15) in $n=1$ dimension</u>: $\bar{A}$ is self-adjoint.

If $v \in L^2(I)$ satisfies $v \in D(\bar{A}^*)$, i.e. $\exists\, C$ s.t.

$$|(v, \bar{A}u)_{L^2(I)}| \leq C \|u\|_{L^2(I)} \quad \forall\, u \in D(\bar{A}), \quad \#$$

then we must show that $v \in D(\bar{A})$.

<u>Well</u>, # implies the existence of $g \in L^2(I)$ s.t.

$$(v, \bar{A}u) = (g, u) \quad \forall u \in D(\bar{A}).$$

In particular, $\quad (v, \varphi'') = (g, \varphi) \quad \forall \varphi \in C_c^\infty(I).$

Let $G(x) = \int_a^x g(t)\, dt$, so $G \in H^1(I)$ and

$$(v, \varphi'') = (G', \varphi) = -(G, \varphi') \quad \forall \varphi \in C_c^\infty(I). \quad \circledast$$

Formally, $(v, \varphi'') = -(v', \varphi')$, which suggests that $v' = G - c$, $c \in \mathbb{R}$, (similarly to Lemma (L.9)).

<u>Claim</u>: $v \in H^2(I)$. <u>To prove this</u>, let $\chi \in C_c^\infty(I)$, $\int \chi\, dx = 1$; given $\psi \in C_c^\infty(I)$, we then have $\varphi(x) := \int_a^x \left(\psi(t) - \chi(t) \int_I \psi\, ds\right) dt \in C_c^\infty(I)$,

$$\varphi' = \psi - \chi \int_I \psi\, dt,$$

$$\varphi'' = \psi' - \chi' \int_I \psi\, dt,$$

and therefore from $\circledast$

$$0 = (v, \varphi'') + (G, \varphi')$$

$$= \int_I v(x) \left(\psi'(x) - \chi'(x) \int_I \psi \, dt \right) + G(x) \left(\psi(x) - \chi(x) \int_I \psi \, dt \right) dx$$

$$= \int_I v\psi' + G\psi - \psi \underbrace{\left(\int v\chi' + G\chi \, dt \right)}_{=: c \in \mathbb{R}} dx$$

$$= \int_I v\psi' + (G-c)\psi \, dx.$$

Since $\psi \in C_c^\infty(I)$ is arbitrary, this implies that $v \in H^1(I)$ with $v' = G - c \in H^1(I)$, so in fact even $v \in H^2(I)$ with $v'' = G' = g$; so $\overline{A}^* v = g = v''$.

Claim: $v(a) = v(b) = 0$. Indeed, for $u \in D(A)$ we have

$$0 = \int_I v u'' - g u \, dx = v u' \Big|_a^b - \int_I v' u' + g u \, dx$$

$$= \left(v(b) u'(b) - v(a) u'(a) \right) - \underbrace{v' u \Big|_a^b}_{=0} + \int \underbrace{(v'' - g)}_{=0} u \, dx$$

$$= v(b) u'(b) - v(a) u'(a).$$

Plug in $u \in D(A)$ with $u'(a) = 1, u'(b) = 0 \Rightarrow v(a) = 0$;
$u'(a) = 0, u'(b) = 1 \Rightarrow v(b) = 0$.

By Lemma (L.11), this gives $v \in H_0^1(I)$, finishing the proof. $\square$