We now return to the Dirichlet problem for the Laplace equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^∞, let $f \in C^\infty(\bar{\Omega})$, and let $u \in H_0^1(\Omega)$ be the unique weak solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (*)$$

i.e. $\displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega).$

Our goal is to prove that $u \in C^\infty(\bar{\Omega})$. More generally:

Theorem (T.31) Let $\Omega \subset \mathbb{R}^n$ be bounded and C^∞; let $f \in H^k(\Omega)$, $k \in \mathbb{N}_0$, and let $u \in H_0^1(\Omega)$ denote the unique weak solution of $(*)$. Then $u \in H^{k+2}(\Omega)$. Moreover, $\exists$ $C = C(\Omega, k)$ s.t.

$$\|u\|_{H^{k+2}(\Omega)} \le C \|f\|_{H^k(\Omega)}. \qquad (*)$$

Remark (R.25) If $f \in C^\infty(\bar{\Omega})$, then $f \in H^k(\Omega)$ $\forall k \in \mathbb{N}_0$, so $u \in \bigcap_{k \in \mathbb{N}_0} H^{k+2}(\Omega)$. By Sobolev embedding, Theorem (T.30), this implies that u lies in every Hölder space on $\bar{\Omega}$, so $u \in C^\infty(\bar{\Omega})$.

In the special case $k=0$, Theorem (T.31) asserts that $u \in H^2(\Omega)$. That is, the mere assumption that $\sum_{j=1}^n \partial_{x_j}^2 u \in L^2$ (and $u \in H_0^1(\Omega)$) implies $\partial_{x_i}\partial_{x_j} u \in L^2(\Omega)$ $\forall\, i,j$ individually. Here is a little calculation that shows that this is not wholly unreasonable:

if $u \in C_c^\infty(\Omega)$, then (writing $\partial_i = \partial_{x_i}$)

$$
\begin{aligned}
|\partial_i \partial_j u|^2 &= \partial_i \partial_j u \cdot \partial_i \partial_j u \\
&= \partial_i (\partial_j u \cdot \partial_i \partial_j u) - \partial_j u \cdot \partial_j \partial_i^2 u \\
&= \partial_i (\partial_j u \cdot \partial_i \partial_j u) - \partial_j (\partial_j u \cdot \partial_i^2 u) + \underbrace{\partial_j^2 u \cdot \partial_i^2 u}_{i \leftrightarrow j},
\end{aligned}
$$

so
$$\sum_{i,j=1}^n |\partial_i \partial_j u|^2 = |\Delta u|^2 + \underbrace{\sum_{i=1}^n \partial_i \Big(\sum_{j=1}^n \partial_j u \cdot \partial_i \partial_j u - \partial_i u \cdot \partial_j^2 u \Big)}_{\text{divergence term}}$$

$$\Rightarrow \int_\Omega |\nabla^2 u|^2 \, dx = \int_\Omega |\Delta u|^2 \, dx .$$

Applying this to $\partial^\alpha u$ in place of u, one gets

$$\| \nabla^2 u \|_{H^k}^2 = \| f \|_{H^k}^2 , \quad (\#)$$

very much in line with $(*)$.

Issues: (1) u in (T.31) does not have compact support in Ω.

(2) In Theorem (T.31), we do not yet know if $u \in H^2$ — in our calculation, we assumed this was true.

(3) What to do near $\partial\Omega$?

Ideas: (1) localization ($u \rightsquigarrow \chi u$)?

(2) regularization/mollification ($u \rightsquigarrow \varphi_\varepsilon * u$?)

(3) straighten out and reduce to a problem on $\mathbb{R}^n_+$?

(1) Localization. Let $\Omega' \Subset \Omega$, and let $\chi \in C_c^\infty(\Omega)$ with $\chi = 1$ on Ω'.

Set $u' = \chi u \in H_0^1(\Omega)$, then u' is a weak solution of

$$-\Delta u' = f' := f + 2\nabla\chi \cdot \nabla u + (\Delta\chi)u \in L^2(\Omega), \quad (\#)$$

$\operatorname{supp} u' \Subset \Omega$.

(2) Regularization.

Lemma (L.16) If $v \in H_0^1(\Omega)$, $\operatorname{supp} v \Subset \Omega$, is a weak solution of $-\Delta v = g \in L^2(\Omega)$, then $v \in H^2(\Omega)$ and $\|v\|_{H^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}$.

Proof Let $\varphi \in C_c^\infty(B_1(0))$, $\int_{\mathbb{R}^n} \varphi \, dx = 1$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$.

For $\varepsilon < \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$, $v_\varepsilon := v * \varphi_\varepsilon \in C_c^\infty(\Omega)$, and

$$-\Delta v_\varepsilon = -(\Delta v) * \varphi_\varepsilon = g * \varphi_\varepsilon =: g_\varepsilon \in C_c^\infty(\Omega).$$

Therefore, by #,

$$\|\nabla^2(v_\varepsilon - v_\delta)\|_{L^2(\Omega)} = \|g_\varepsilon - g_\delta\|_{L^2(\Omega)} \xrightarrow{\varepsilon,\delta\to 0} 0,$$

and also $v_\varepsilon \to v$ in $H^1(\Omega)$.

$\Rightarrow$ v_ε is a Cauchy sequence in $H^2(\Omega)$, so $v \in H^2(\Omega)$. □

Corollary (C.9) (Interior regularity.) If $u \in H_0^1(\Omega)$ is a weak solution of $-\Delta u = f \in H^k_{loc}(\Omega) \Rightarrow u \in H^{k+2}_{loc}(\Omega)$.

(In particular, this applies when $f \in H^k(\Omega)$, the conclusion still being that $u \in H^{k+2}(\Omega')$ for all $\Omega' \Subset \Omega$.)

Proof. Let $\Omega' \Subset \Omega$, and choose $\Omega_0 = \Omega' \Subset \Omega_1 \Subset \cdots \Subset \Omega_k \Subset \Omega = \Omega_{k+1}$.

Let $\chi_j \in C_c^\infty(\Omega_{j+1})$ be equal to 1 on Ω_j.

Step 0. $-\Delta(\chi_k u) = \chi_k f - 2\nabla\chi_k \cdot \nabla u - (\Delta\chi_k)u \in L^2(\Omega)$,

with $\operatorname{supp}(\chi_k u) \Subset \Omega_{k+1} = \Omega$

By Lemma (L.16), $\chi_k u \in H^2(\Omega)$.

Step $j \in \{1,\dots,k\}$ Assuming that $\chi_{k-j+1} u \in H^{j+1}(\Omega)$,

consider $-\Delta(\chi_{k-j} u) = -\Delta(\chi_{k-j}(\chi_{k-j+1} u))$

$$= \chi_{k-j} f - 2\nabla\chi_{k-j} \cdot \nabla(\chi_{k-j+1} u) - (\Delta\chi_{k-j}) \cdot \chi_{k-j+1} u =: f_{k-j} \in H^j(\Omega),$$

$\operatorname{supp}(\chi_{k-j} u) \Subset \Omega_{k-j+1} \subset \Omega$.

For $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le j$, $u_\alpha := \partial^\alpha(\chi_{k-j} u) \in H^1(\Omega)$ is a weak solution of

$$-\Delta u_\alpha = -\partial^\alpha \Delta(\chi_{k-j} u) = \partial^\alpha f_{k-j} \in L^2(\Omega),$$

$\operatorname{supp} u_\alpha \Subset \Omega$.

$\Rightarrow u_\alpha \in H^2(\Omega)$ by Lemma (L.16) $\Rightarrow \chi_{k-j} u \in H^{j+2}(\Omega)$.

Step k gives $\chi_0 u \in H^{k+2}(\Omega)$; so $u \in H^{k+2}(\Omega')$, as claimed. $\square$

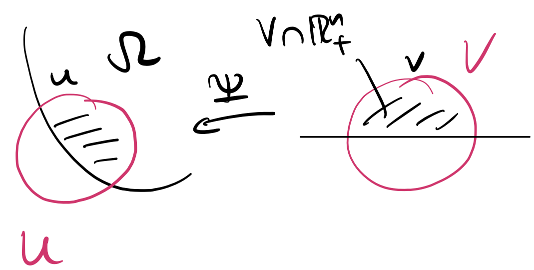
Remark (R.26) According to Corollary (C.9), the regularity of u in some open set $U \subset \Omega$ is $2+$ (regularity of $f = \Delta u$ in U).

So interior regularity of u is a local result.

(3) Straightening out $\partial\Omega$. This is much more subtle, and will lay bare a fundamental limitation of our approach thus far, which strongly relied on the fact that we are looking precisely at $\sum_{i=1}^{n} \partial_i^2$. (No constant coefficients allowed, as otherwise the proof of Lemma (L.16) fails.)

So, if $\Psi : V \to U \subset \mathbb{R}^n$ is a C^∞ diffeomorphism, with $U \subset \mathbb{R}^n$ an open neighborhood of a point in $\partial\Omega$, and $V \subset \mathbb{R}^n$,

$$\begin{cases} \Psi^{-1}(V \cap \mathbb{R}^n_+) = U \cap \Omega, \\ \Psi^{-1}(V \cap (\{0\}\times\mathbb{R}^{n-1})) = U \cap \partial\Omega, \end{cases}$$



we consider

$$v = u \circ \Psi \in H^1_0(V \cap \mathbb{R}^n_+).$$

What PDE does v solve?

Lemma (L.17) Let $u \in H^1_0(\Omega \cap U)$, $\operatorname{supp} u \Subset U$,

$$v = u \circ \Psi \in H^1_0(V \cap \mathbb{R}^n_+).$$

Then u is a weak solution of $-\Delta u = f \in L^2(\Omega \cap U)$ if and only if v is a weak solution of

$$-\Delta_g v = f \circ \Psi \in L^2(V \cap \mathbb{R}^n_+),$$

where $\Delta_g v = \sum_{i,j=1}^{n} \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j v \right)$,

$$(g^{ij})_{1\le i,j\le n} = (g_{ij})^{-1}_{1\le i,j\le n}, \quad g_{ij} = \frac{\partial\Psi}{\partial y_i} \cdot \frac{\partial\Psi}{\partial y_j} = g_{ji},$$

$|g| = \det(g_{ij})$.

Proof. Omitted. (Thought-free proof: chain rule.) $\square$

Remark (R.27) (g_{ij}) is a positive definite matrix, with smooth dependence on $y \in V \cap \mathbb{R}^n_+$, and Δ_g is the Laplace-Beltrami operator on the Riemannian manifold $(V \cap \mathbb{R}^n_+, g)$.

Thus, we need to control boundary regularity of solutions to a variable coefficient PDE (on a simpler domain).

We may as well study also interior regularity again, but this time for general variable coefficient operators. The following result does it all:

Theorem (T.32). Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain, and let $g^{ij} = g^{ji}, b^i, c \in C^\infty(\bar{\Omega})$ for $1 \le i, j \le n$; assume that $\exists\, 0 < \lambda \le \Lambda$ s.t. the ellipticity condition

$$\lambda |\xi|^2 \le \sum_{i,j=1}^n g^{ij} \xi_i \xi_j \le \Lambda |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n \qquad \text{(ELL)}$$

holds. Define the operator L by

$$(Lu)(x) = -\sum_{i,j=1}^n \partial_i \left(g^{ij}(x)\, \partial_j u(x) \right) + \sum_{i=1}^n b^i(x)\, \partial_i u(x) + c(x)\, u(x).$$

- Suppose $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f \in L^2(\Omega)$, i.e.

$$\int_\Omega f v \, dx = \int_\Omega \sum_{i,j=1}^n g^{ij}(x)\, \partial_j u(x) \cdot \partial_i v(x) + \sum_{i=1}^n b^i(x) \partial_i u(x) \cdot v(x)$$

$$+\, c(x)u(x)\cdot v(x)\,dx \qquad \forall v \in H_0^1(\Omega).$$

- If $f \in H^k(\Omega)$ for some $k \in \mathbb{N}_0$, then $u \in H^{k+2}(\Omega)$, and

$$\|u\|_{H^{k+2}(\Omega)} \le C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right). \quad (*)$$

Remark (R.28) For $L = -\Delta$, so $g^{ij} = \delta_{ij}$, $b^i = c = 0$, we have

$$\|u\|^2_{L^2(\Omega)} \overset{\text{Lemma (L.8)}}{\le} C\|\nabla u\|^2_{L^2(\Omega)} = C\int_\Omega \nabla u \cdot \nabla u \, dx$$

$$= C\int_\Omega f\, u\, dx \le C\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

$\Rightarrow \|u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$, and thus $(*)$ gives

$$\|u\|_{H^{k+2}(\Omega)} \le C\|f\|_{H^k(\Omega)}.$$

This is the estimate $(*)$ in Theorem (T.31). Thus we have proved Theorem (T.31), given Theorem (T.32).

In the proof of Theorem (T.32), we will use finite difference quotients

$$D_h u = \frac{T_h u - u}{|h|}, \quad (T_h u)(x) = u(x+h), \quad 0 \ne h \in \mathbb{R}^n.$$

For $u \in H_0^1(\mathbb{R}^n)$, we have $D_h u \in H_0^1(\mathbb{R}^n)$ with $\nabla(D_h u) = D_h \nabla u$.

- Furthermore, $\|D_h u\|_{L^2} \le \|\nabla u\|_{L^2}$ (see the proof of Thm. (T.19)), and conversely if $u \in L^2$ and $D_h u$ is uniformly bounded in L^2 as $h \to 0$, then $u \in H^1$, $\nabla u = \lim_{h\to 0} D_h u$.

- We also have an "integration by parts" identity: for $u, v \in H_0^1(\mathbb{R}^n)$,

$$\int u \cdot D_{-h} v \, dx = |h|^{-1}\left[\int u(x)v(x-h)\,dx - \int u(x)v(x)\,dx\right]$$

$$= |h|^{-1}\left[\int u(x+h)v(x)\,dx - \int u(x)v(x)\,dx\right]$$

$$= \int D_h u \cdot v\, dx.$$

· Finally, a "product rule": for $u,v \in H^1_0(\mathbb{R}^n)$,

$$D_h(uv)(x) = |h|^{-1}\Big((u(x+h)-u(x))\, v(x+h) + u(x)\,(v(x+h)-v(x))\Big)$$

$$\Rightarrow D_h(uv) = D_h u \cdot \tau_h v + u\, D_h v.$$

Proof of Theorem (T.32)

Step 1: estimate for $\|u\|_{H^1_0(\Omega)}$.

Plugging $v=u$ into the definition (#) of weak solution gives

$$\int_\Omega f u\, dx = \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\,\partial_i u(x)\,\partial_j u(x)\, dx + \int_\Omega \sum_{i=1}^n b^i(x)\partial_i u(x)\cdot u(x)$$

$$+ \int_\Omega c(x)u(x)\cdot u(x)\, dx$$

$$\overset{(ELL)}{\geq} \lambda \|\nabla u\|^2_{L^2(\Omega)} - \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^2} - \|c\|_{L^\infty}\|u\|^2_{L^2}$$

$$\Rightarrow \lambda\|\nabla u\|^2_{L^2(\Omega)} \leq \|f\|_{L^2}\|u\|_{L^2} + C\|\nabla u\|_{L^2}\|u\|_{L^2} + C\|u\|^2_{L^2}$$

$$\leq \|f\|^2_{L^2} + C'\|u\|^2_{L^2} + \frac{\lambda}{2}\|\nabla u\|^2_{L^2}$$

since $xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon}y^2$ ($x,y\in\mathbb{R}$; apply to $x=\|\nabla u\|_{L^2}$, $y=\|u\|_{L^2}$)

$$\Rightarrow \|\nabla u\|_{L^2(\Omega)} \leq C(\|f\|_{L^2} + \|u\|_{L^2})$$

We can more sharply estimate $\left|\int_\Omega f u\, dx\right| \leq \|f\|_{H^{-1}(\Omega)}\|\nabla u\|_{L^2(\Omega)}$,
$H^{-1}(\Omega) := (H^1_0(\Omega))^*$, and thus obtain

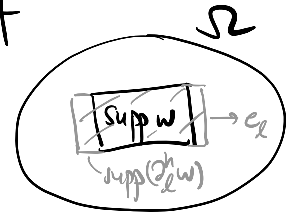
$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Step 2: interior estimate.

(i) $k=0$. Let $\chi \in C_c^\infty(\Omega)$, $d := \mathrm{dist}(\mathrm{supp}\,\chi, \partial\Omega) > 0$.

Then $w = \chi u \in H^1_0(\Omega)$ is a weak solution of

$$Lw = q := \underbrace{\chi f}_{\in L^2} - \underbrace{[L,w]}_{\substack{\text{first order partial differential} \\ \text{operator, maps } H^1 \to L^2}} u \in L^2(\Omega)$$



Instead of mollifying (which does not work anymore) we study finite difference quotients of w: for $0 < h < d$, $1 \le \ell \le n$, put

$$(\partial_\ell^h w)(x) = \frac{w(x+he_\ell) - w(x)}{h},$$ so $\partial_\ell^h w \in H^1_0(\Omega)$ still.

Arguing informally for a moment, consider the equation that $\partial_\ell^h w$ satisfies: if $b^i = c = 0$, compute (omitting the summation sign)

$$-\partial_i\left(g^{ij}(x)\,\partial_j \partial_\ell^h w(x)\right) + \underbrace{\partial_\ell^h\Big(\underbrace{\partial_i(g^{ij}\partial_j w)}_{=f \in L^2}\Big)(x)}_{\xrightarrow{h\to 0}\ \partial_\ell q \text{ in } H^{-1}(\Omega)\ (?)}$$

$$= \partial_i\left(\partial_\ell^h g^{ij}(x)\,\partial_j w(x+he_\ell)\right)$$

$$\xrightarrow{h\to 0} \partial_i\left(\partial_\ell g^{ij}(x)\,\partial_j w(x)\right) \text{ in } H^{-1}(\Omega)\ (?),$$

so from step 1 we suspect that $\|\partial_\ell^h w\|_{H^1(\Omega)}$ remains uniformly bounded as $h \to 0$, and thus $\partial_\ell w \in H^1(\Omega)$.

This can be carried out rigorously.

In order to avoid dealing with $H^{-1}(\Omega)$, we do not apply step 1 in same way to $\partial_\ell^h w$ with $\partial_\ell^h w$ also used as a test function, but rather to w with $\partial_\ell^{-h}\partial_\ell^h w$ as a test function.

So: into

$$\int_\Omega q v \, dx = \int_\Omega g^{ij}\partial_j w \cdot \partial_i v + b^i \partial_i w \cdot v + c w \cdot v \, dx$$
$$(v \in H_0^1(\Omega)),$$

plug $v = \partial_\ell^{-h}\partial_\ell^h w$.

$$|\text{Left hand side}| = \left| \int_\Omega q \, \partial_\ell^{-h}\partial_\ell^h w \, dx \right| \leq \|q\|_{L^2} \|\nabla \partial_\ell^h w\|_{L^2}.$$

$$|\text{Right hand side}| \geq \left| \int_\Omega g^{ij}\partial_j w \; \partial_i(\partial_\ell^{-h}\partial_\ell^h w) \, dx \right|$$
$$- \left| \int_\Omega b^i \partial_i w \cdot \partial_\ell^{-h}\partial_\ell^h w \, dx \right| - \left| \int_\Omega c w \cdot \partial_\ell^{-h}\partial_\ell^h w \, dx \right|$$

$$\partial_\ell^h(g^{ij}\partial_j w) = (\partial_\ell^h g^{ij})(\underbrace{\tau_\ell^h \partial_j w}_{= \partial_j w(\cdot + h e_\ell)}) + g^{ij}\partial_\ell^h \partial_j w,$$
with $\|\partial_\ell^h g^{ij}\|_{L^\infty} \leq \|g^{ij}\|_{C^1}$

$$= \left| \int_\Omega \partial_\ell^h (g^{ij}\partial_j w) \; \partial_i(\partial_\ell^h w) \, dx \right|$$
$$- \left| \int_\Omega \partial_\ell^h (b^i \partial_i w) \cdot \partial_\ell^h w \, dx \right| - \left| \int_\Omega \partial_\ell^h (c w) \cdot \partial_\ell^h w \, dx \right|$$

$$\geq \left| \int_\Omega g^{ij} \underbrace{(\partial_\ell^h \partial_j w)}_{= \partial_j \partial_\ell^h w} \cdot \partial_i(\partial_\ell^h w) \, dx \right| - \|g^{ij}\|_{C^1} \|\nabla w\|_{L^2} \|\nabla \partial_\ell^h w\|_{L^2}$$
$$- \|b\|_{L^\infty} \|\nabla \partial_\ell^h w\|_{L^2} \|\nabla w\|_{L^2} - \|\nabla b\|_{L^\infty} \|\nabla w\|_{L^2}^2$$
$$- \|c\|_{L^\infty} \|\nabla w\|_{L^2}^2 \qquad - \|\nabla c\|_{L^\infty} \|w\|_{L^2} \|\nabla w\|_{L^2}$$

(ELL)

$$\geq \lambda \|\nabla \partial_\ell^h w\|_{L^2}^2 - C_\varepsilon \|w\|_{H^1}^2 - \varepsilon \|\nabla \partial_\ell^h w\|_{L^2}^2$$

for all $\varepsilon > 0$, similarly to step 1.

$$\Longrightarrow \|\nabla \partial_\ell^h w\|_{L^2} \leq C(\|w\|_{H^1} + \|q\|_{L^2}) \quad \forall h \in (0, d),$$

with C independent of h.

$\Rightarrow$ $\nabla w \in H^1$, so $w \in H^2$, and

$$\|w\|_{H^2} \le C(\|w\|_{H^1} + \|q\|_{L^2}) \overset{\text{Step 1}}{\le} C(\|\underset{\overset{\|}{Lw}}{q}\|_{L^2} + \|w\|_{L^2}). \quad \circledast$$

(ii) Higher k. This is analogous to the proof of Corollary (C.9).

Suppose $f \in H^1(\Omega)$, and let $\chi_1, \chi_2 \in C_c^\infty(\Omega)$, $\chi_1 = 1$ on $\Omega' \Subset \Omega$, $\chi_2 = 1$ on $\operatorname{supp} \chi_1$. We already know that $\chi_2 u \in H^2(\Omega)$. Thus $w := \chi_1 u \in H^2(\Omega)$ satisfies

$$Lw = L(\chi_1 \chi_2 u) = \underbrace{\chi_1 f}_{\in H^1} + \underbrace{[L, \chi_1]}_{\substack{1^{st}\text{ order} \\ \text{diff. op.}}} \underbrace{\chi_2 u}_{\in H^2} =: q \in H^1(\Omega).$$

$$\Rightarrow \underbrace{\partial_\ell^h q}_{\substack{\text{unif. bdd.} \\ \text{in } L^2}} = \partial_\ell^h Lw$$

$$= \partial_\ell^h(-\partial_i g^{ij} \partial_j w + b^i \partial_i w + cw)$$

$$= L(\partial_\ell^h w) - \partial_i(\underbrace{(\partial_\ell^h g^{ij})}_{\substack{\text{unif. bdd.} \\ \text{in } C^1}} \underbrace{\partial_j(\tau_\ell^h w)}_{\substack{\text{unif. bdd.} \\ \text{in } H^1}}) + \overbrace{(\partial_\ell^h b^i)}^{C^0} \overbrace{\partial_i(\tau_\ell^h w)}^{H^1} + \underbrace{(\partial_\ell^h c)}_{C^0} \underbrace{\tau_\ell^h w}_{H^2}$$

$\Rightarrow$ $L(\partial_\ell^h w) =: q_\ell^h \in L^2(\Omega)$ is uniformly bounded in $L^2(\Omega)$ as $h \to 0$, with

$$\liminf_{h \to 0} \|q_\ell^h\|_{L^2} \le C(\|q\|_{H^1} + \|w\|_{H^2}) \overset{\circledast}{\le} C'(\|q\|_{H^1} + \|w\|_{L^2})$$

So $\|\partial_\ell^h w\|_{H^2(\Omega)} \overset{\circledast}{\le} C(\|q_\ell^h\|_{L^2} + \|\partial_\ell^h w\|_{L^2})$

$$\le C''\left(\|q\|_{H^1} + \|w\|_{L^2}\right) \qquad \text{implies}$$

that $\partial_\ell^h w$ is uniformly bounded in H^2 $\Rightarrow$ $\partial_\ell w \in H^2$

$\overset{\ell=1,\dots,n}{\Longrightarrow}$ $w = \chi_1 u \in H^3(\Omega)$, $\quad \|w\|_{H^3(\Omega)} \le C\big(\|\underset{\substack{\| \\ Lw}}{q}\|_{H^1(\Omega)} + \|w\|_{L^2(\Omega)}\big)$. $\quad$ (‡)

- Higher regularity follows similarly, using (‡) for $k=2$ (instead of $(*)$ for $k=1$), etc.

Step 3. Boundary regularity, $k=0$.

- Near a point $p \in \partial\Omega$, we can straighten out Ω using a smooth diffeomorphism $\Psi : V \to U$ onto a neighborhood U of p.
- The localization $\chi u \in H^1_0(\Omega)$, $\chi \in C^\infty_c(U \cap \overline{\Omega})$, satisfies

$$L(\chi u) = q \in L^2(\Omega) \quad \text{as in Step 2 (i)},$$

and $(\chi u) \circ \Psi \in H^1_0(V \cap \mathbb{R}^n_+)$ satisfies a PDE

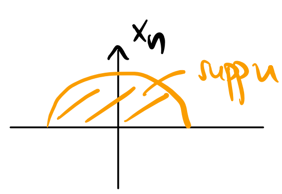
$$\tilde{L}\big((\chi u) \circ \Psi\big) = \tilde{q} \in L^2(V \cap \mathbb{R}^n_+)$$

where $\tilde{L}$ is of the same form as L, with different g^{ij}, b^i, c (but still elliptic).

- Relabeling $\tilde{L}$, $(\chi u)\circ\Psi$, $\tilde{q}$ as L, u, f, it thus suffices to study the following situation: $Q = \{(x', x_n) : |x'| < 1, |x_n| < 1\}$, $Q_+ = Q \cap \{x_n > 0\}$

$$u \in H^1_0(Q_+), \qquad Lu = f \in L^2(Q_+) \quad \text{(weakly)},$$

$$\operatorname{supp} u \Subset Q.$$



We claim that $u \in H^2(Q_+)$ (with a quantitative estimate).

(i) Tangential regularity. Exactly as in Step 2, we can consider the test function $\partial_\ell^{-h}\partial_\ell^h u$ for $\ell = 1, \dots, n-1$; note that $\partial_\ell^h : H_0^1(Q_+) \to H_0^1(Q_+)$, i.e. the Dirichlet boundary condition is preserved.

$$\Rightarrow \partial_\ell u \in H^1(Q_+), \text{ so } \quad \partial_\ell \partial_i u \in L^2(Q_+), \quad 1 \le i \le n, \ 1 \le \ell \le n-1.$$

And $M_2(u) := \max_{\substack{|\alpha|=2 \\ \alpha_n \le 1}} \|\partial^\alpha u\|_{L^2(Q_+)} \le C(\|f\|_{L^2(Q_+)} + \|u\|_{L^2(Q_+)})$. $(*)$

(ii) Normal regularity. The PDE for u expresses the distributional derivative $\partial_n^2 u$ as follows:

$$\left.\begin{aligned} f = -g^{nn}\partial_n^2 u - \sum_{\substack{i,j=1,\dots,n \\ (i,j)\neq(n,n)}} \partial_i(g^{ij}\partial_j u) - (\partial_n g^{nn})\partial_n u \\ + \sum_{i=1}^n b^i \partial_i u + cu. \end{aligned}\right\} \quad (\#)$$

All terms except $g^{nn}\partial_n^2 u$ are already known to lie in $L^2(Q_+)$. Since $\lambda \le g^{nn} \le \Lambda$ by (ELL), this shows that $\partial_n^2 u \in L^2(Q_+)$,

$$\|\partial_n^2 u\|_{L^2(Q_+)} \le C(\|f\|_{L^2(Q_+)} + M_2). \quad (*')$$

$$\Rightarrow u \in H^2(Q_+).$$

Step 4. Boundary regularity, $k \ge 1$. We again only discuss the case $k=1$, the cases $k \ge 2$ being completely analogous. So now $u \in H_0^1(Q_+)$, $Lu = f \in H^1(Q_+)$; we already know that $u \in H^2(Q_+)$. As in step 2 (ii) (only with different notation: $Lu = f$ instead of $Lw = g$), we then get:

for $1 \le \ell \le n-1$, $\quad L(\partial_\ell^h u) = f_\ell^h$, $0 < h < 1$, where $f_\ell^h \in L^2(Q_+)$ is uniformly bounded. By the estimates (*), (*)',

$\quad \|\partial_\ell^h u\|_{H^2(Q_+)}$ is uniformly bounded

$$\Rightarrow \max_{\substack{|\alpha| \le 3 \\ \alpha_n \le 2}} \|\partial^\alpha u\|_{L^2(Q_+)} < \infty.$$

Differentiating (‡) along ∂_n (in the sense of distributions) expresses $\partial_n^3 u$ in terms of $\partial^\alpha u$, $|\alpha| \le 3$, $\alpha_n \le 2$, so

$$\partial_n^3 u \in L^2(Q_+).$$

$$\Rightarrow u \in H^3(Q_+).$$

The case of higher k being analogous, we are done. $\square$

Application 1: self-adjoint realization of Δ

We can now prove Theorem (T.15).

Lemma (L.18) $\Omega \subset \mathbb{R}^n$ bounded C^∞ domain. Let $A = \Delta$ with $D(A) = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. Then

$$D(\overline{A}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Proof "$\subseteq$": If $u_k \in D(A)$, $u_k \to u$ in L^2,

$$-\Delta u_k \to f \text{ in } L^2,$$

then $\|u_k - u_\ell\|_{H^2(\Omega)} \le C \|f_k - f_\ell\|_{L^2(\Omega)}$ by Remark (R.28)

$\Rightarrow$ u_k has a limit in $H^2(\Omega)$, which must be equal to its L^2-limit u; so $u \in H^2(\Omega)$. Since also

$0 = u_k|_{\partial\Omega} \longrightarrow u|_{\partial\Omega}$ in $L^2(\Omega)$ by Theorem (T.26),

we get $u|_{\partial\Omega} = 0 \Rightarrow u \in H_0^1(\Omega)$ by Theorem (T.27).

"$\supseteq$": Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Pick $f_k \in C_c^\infty(\Omega)$ s.t.

$f_k \to f := -\Delta u$ in $L^2(\Omega)$; let $u_k \in C^\infty(\overline{\Omega})$ be the unique $H_0^1(\Omega)$-solution of $-\Delta u_k = f_k$. Then $u_k \in D(A)$,

$$\|u_k - u\|_{L^2(\Omega)} \le C \|f_k - f\|_{L^2(\Omega)} \xrightarrow{k\to\infty} 0 \Rightarrow u \in D(\overline{A}). \quad \square$$

Theorem (T.15) $\Omega \Subset \mathbb{R}^n$ C^∞ domain. Then Δ with domain $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ is self-adjoint.

Proof We only need to check that $v \in D(\Delta^*)$ implies $v \in D(\Delta)$. That is, we have: $\exists\, C > 0$ s.t.

$$\left|\int_\Omega (\Delta u) v\,dx\right| \le C\|u\|_{L^2(\Omega)} \quad \forall u \in D(\Delta),$$

or equivalently: $\exists\, g \in L^2(\Omega)$ s.t.

$$\int_\Omega (\Delta u)\cdot v\,dx = \int_\Omega u\, g\,dx \quad \forall\, u \in H^2(\Omega)\cap H^1_0(\Omega). \quad \circledast$$

Let now $\tilde v \in H^1_0(\Omega)$ be the unique solution of

$$\Delta \tilde v = g \quad \Rightarrow \quad \tilde v \in H^2(\Omega) \text{ by Theorem (T.32).}$$

We shall prove $\tilde v = v$, which would finish the proof.

Now, $\tilde v \in D(\Delta)$, and $\circledast$ and the symmetry $\Delta^* \supseteq \Delta$ give

$$\int_\Omega \Delta u\cdot v\,dx = \int_\Omega u\,\Delta\tilde v\,dx = \int_\Omega \Delta u\cdot \tilde v\,dx \quad \forall u \in H^2\cap H^1_0,$$

$$\Rightarrow \int_\Omega (v-\tilde v)\,\Delta u\,dx = 0 \quad \forall u \in H^2\cap H^1_0.$$

Take u to be the solution of

$$\begin{cases} \Delta u = v-\tilde v \in L^2(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$\Rightarrow \int_\Omega |v-\tilde v|^2 dx = 0 \Rightarrow v = \tilde v \in H^2\cap H^1_0. \qquad \square$$

Corollary (C.10) $\Omega \subseteq \mathbb{R}^n$ C^∞ domain. $\exists$ complete orthonormal basis of $L^2(\Omega)$ consisting of $u_k \in C^\infty(\overline{\Omega})$, $k \in \mathbb{N}$, which are eigenfunctions of $-\Delta$:

$$\begin{cases} -\Delta u_k = \lambda_k u_k & \text{in } \Omega \\ \quad u_k = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 < \lambda_1 \le \lambda_2 \le \cdots \longrightarrow \infty$.

Proof. Note that $-\Delta : H^2(\Omega)\cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$ is an isomorphism.

Consider $T: L^2(\Omega) \ni f \longmapsto -\Delta^{-1} f \in L^2(\Omega)$; since this factors as $L^2(\Omega) \xrightarrow{-\Delta^{-1}} H^2(\Omega)\cap H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, it is compact by Rellich's theorem.

- T is symmetric, since for $f, g \in L^2(\Omega)$ and $u, v \in H^2(\Omega)\cap H^1_0(\Omega)$ with $-\Delta u = f$, $-\Delta v = g$, we have

$$(Tf, g)_{L^2(\Omega)} = (u, -\Delta v)_{L^2(\Omega)} = (-\Delta u, v)_{L^2(\Omega)} = (f, Tg)_{L^2(\Omega)}$$

Since T is bounded, we thus conclude that

$$T = -\Delta^{-1} \text{ is a compact self-adjoint operator on } L^2(\Omega).$$

- Since $\ker T = \{0\}$, we obtain a complete ONB $\{u_k\}_{k\in\mathbb{N}}$ of $L^2(\Omega)$ consisting of eigenfunctions of T with eigenvalues

$$0 \neq \mu_k \longrightarrow 0 : \qquad Tu_k = \mu_k u_k.$$

But $Tu_k \in H^2(\Omega)\cap H^1_0(\Omega)$, so $u_k = \mu_k^{-1} Tu_k \in H^2(\Omega)\cap H^1_0(\Omega)$.

$$\Rightarrow -\Delta u_k = \mu_k^{-1} u_k \in H^2(\Omega) \Rightarrow u_k \in H^4(\Omega)$$

$$\Rightarrow -\Delta u_k = \mu_k^{-1} u_k \in H^4(\Omega) \Rightarrow u_k \in H^6(\Omega)$$

$$\Rightarrow \cdots \Rightarrow u_k \in \bigcap_{j\in\mathbb{N}} H^j(\Omega) = C^\infty(\overline{\Omega}).$$

- Finally, $-\Delta u_k = \lambda_k u_k$ (with $\lambda_k = \mu_k^{-1}$) implies

$$\lambda_k \|u_k\|^2_{L^2(\Omega)} = -\int_\Omega \Delta u_k \cdot u_k \, dx = \int_\Omega |\nabla u_k|^2 \, dx \geq 0$$

$\Rightarrow \lambda_k \geq 0$, and since $\lambda_k = \mu_k^{-1} \neq 0$, indeed $\lambda_k > 0$. $\square$

Application 2: solvability, Fredholm theory.

Fix a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

- We saw that we can always solve $\begin{cases} -\Delta u = f \in L^2(\Omega), \\ u|_{\partial\Omega} = 0, \end{cases}$

and get higher regularity (T.31).

The following is an important generalization:

Theorem (T.33) Let $g^{ij} = g^{ji} \in C^\infty(\overline{\Omega})$, $1 \le i,j \le n$, and suppose $\exists\, \Lambda \ge \lambda > 0$ s.t.

$$\lambda |\xi|^2 \le \sum_{i,j=1}^n g^{ij}(x)\, \xi_i \xi_j \le \Lambda |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Let $Lu(x) := -\sum_{i,j=1}^n \partial_i \left(g^{ij}(x)\, \partial_j u(x) \right)$.

Then $\forall\, f \in L^2(\Omega)$ $\exists!$ weak solution $u \in H_0^1(\Omega)$ of

$$\begin{cases} Lu = f \\ u|_{\partial\Omega} = 0. \end{cases}$$

If $f \in H^k(\Omega)$, then $u \in H^{k+2}(\Omega)$; $\|u\|_{H^{k+2}(\Omega)} \le C \|f\|_{H^k(\Omega)}$.

Proof • Define for $u, v \in C_c^\infty(\Omega)$

$$(u,v)_g := \int_\Omega \sum_{i,j=1}^n g^{ij}(x)\, \partial_i u(x)\, \partial_j v(x)\, dx.$$

Then $(u,u)_g \le \int_\Omega \Lambda |\nabla u(x)|^2 dx = \Lambda (u,u)_{H_0^1}$,

and likewise $\lambda (u,u)_{H_0^1} \le (u,u)_g$. So $\|\cdot\|_g = (\cdot,\cdot)_g^{\frac{1}{2}}$ is equivalent to the $H_0^1(\Omega)$-norm.

$\Rightarrow$ The completion of $\left(C_c^\infty(\Omega), \|\cdot\|_g = (\cdot,\cdot)_g^{1/2}\right)$ is $(H_0^1(\Omega), \|\cdot\|_g)$;

- Given $f \in L^2(\Omega)$, consider

$$C_c^\infty(\Omega) \ni v \longmapsto \int_\Omega f v \, dx;$$

then $\left|\int_\Omega f v \, dx\right| \le \|f\|_{L^2} \|v\|_{L^2} \le C \|f\|_{L^2} \|\nabla v\|_{L^2} \le C' \|f\| \, \|v\|_g$.

Riesz $\Rightarrow \exists!\, u \in H_0^1(\Omega)$ s.t. $\forall\, v \in H_0^1(\Omega)$:

$$\int_\Omega f v \, dx = (u,v)_g = \sum_{i,j=1}^n \int_\Omega g^{ij}(x)\, \partial_i u \, \partial_j v \, dx.$$

This means that $-\sum_{i,j=1}^n \partial_j \left(g^{ij}(x) \partial_i u(x)\right) = f(x)$ weakly.

· Higher regularity follows from Theorem (T.32), $\square$

<u>Remark (R.28)</u> The same arguments as in the proof of Theorem (T.15) above imply that L is self-adjoint with domain $H_0^1(\Omega) \cap H^2(\Omega)$; and also Corollary (C.10) remains valid for L.

· For general operators $L = -\partial_i g^{ij} \partial_j + b^i \partial^i + c$ as in Thm. (T.32), solvability and uniqueness may fail in general. (E.g. $-\Delta - \lambda_1$ is not injective in the notation of Corollary (C.10).) The following is the best one can get:

<u>Theorem (T.34)</u> (Fredholm alternative.)

Let $g^{ij} = g^{ji}, b^i, c \in C^\infty(\bar\Omega)$ be as in Theorem (T.32).

Let $L = -\partial_i g^{ij} \partial_j + b^i \partial_i + c$ and

$${}^tL := -\partial_i g^{ij} \partial_j - b^i \partial_i + (c - \partial_i b^i) \quad \text{(formal adjoint).}$$

(i) Let $N = \{u \in C^\infty(\bar\Omega) : u|_{\partial\Omega} = 0,\ Lu = 0\}$,

$${}^tN = \{v \in C^\infty(\bar\Omega) : v|_{\partial\Omega} = 0,\ {}^tLv = 0\}.$$

Then $\dim N = \dim {}^tN < \infty$.

(ii) Let $f \in L^2(\Omega)$. Then $\exists u \in H^1_0(\Omega)$, $Lu = f$, iff $(f, v)_{L^2(\Omega)} = 0 \ \forall v \in {}^tN$. Any two solutions u differ by an element of N. Higher regularity holds ($f \in H^k \Rightarrow u \in H^{k+2}$).

Proof • Write $L = L_{b,c}$ for clarity. Then

$$L_{0,0} : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$$

is invertible by Theorem (T.33). Moreover,

$$L_{b,c} - L_{0,0} = b^i \partial_i + c \ : \ H^2(\Omega) \underset{\substack{\text{inclusion} \\ \text{(compact!)}}}{\hookrightarrow} H^1(\Omega) \underset{u \mapsto b^i \partial_i u + c}{\longrightarrow} L^2(\Omega)$$

is a compact operator

$$\Rightarrow L_{b,c} = L_{0,0} + (L_{b,c} - L_{0,0}) : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega) \quad \oplus$$

is a Fredholm operator of index 0.

• Null space. For $u \in H^2(\Omega) \cap H^1_0(\Omega)$, $L_{b,c} u = 0$, we get $u \in C^\infty(\bar{\Omega})$ from Theorem (T.32) $\Rightarrow \ker_{H^2 \cap H^1_0} L_{b,c} = N$, and this is finite-dimensional by $\oplus$.

• Cokernel.- If $v \in {}^tN$, then for $u \in H^2(\Omega) \cap H^1_0(\Omega)$,

$0 = (u, {}^tLv) = (Lu, v)$, so $v \in (\operatorname{ran} L)^\perp$. $\Rightarrow {}^tN \subseteq (\operatorname{ran} L)^\perp$.

- The converse is more subtle. We would like to show that $v \in L^2(\Omega)$, $(Lu, v) = 0 \ \forall u \in H^2 \cap H^1_0$ implies $v \in H^1_0$ and ${}^tLv = 0$. (Then $v \in C^\infty(\bar{\Omega})$, so $v \in {}^tN$.)

We argue in a roundabout way:

Step 1: invertible perturbation: $\exists z \in \mathbb{R}$ s.t. $L_{b,c}+z : H^2 \cap H^1_0 \to L^2$ is invertible.

Indeed, $L_{b,c}+z$ has index 0; and for real $z \gg 1$, it is injective since $(L_{b,c}+z)u=0$, $u \in H^1_0 \cap H^2$, implies that

$$
\begin{aligned}
0 &= \left((L_{b,c}+z)u, u\right)_{L^2(\Omega)} \\
&= \sum_{i,j=1}^{n} \int_\Omega g^{ij}(x)\partial_i u(x) \partial_j u(x)\, dx + z\|u\|^2_{L^2(\Omega)} \\
&\quad + \left(\sum b^i \partial_i u, u\right)_{L^2} + (cu,u)_{L^2} \\
&\geq \lambda \|\nabla u\|^2_{L^2} + z\|u\|^2_{L^2} - C\|\nabla u\|_{L^2}\|u\|_{L^2} \\
&\qquad - C'\|u\|^2_{L^2} \\
&\geq \frac{\lambda}{2}\|\nabla u\|^2 + (z - C'')\|u\|^2_{L^2}
\end{aligned}
$$

where C'' only depends on λ, b^i, c. So if $z > C''$, this implies $u=0$.

We can also make sure that ${}^tL+z$ is invertible as well.

Step 2: rewrite $Lu=f$ as

$$
f = L(L+z)^{-1}(L+z)u = \left(I - z(L+z)^{-1}\right)(L+z)u.
$$

Since $L+z$ is surjective (onto $L^2(\Omega)$), we conclude that

$$
\begin{aligned}
(\operatorname{ran} L)^\perp &= \left(\operatorname{ran}\left(I - z(L+z)^{-1}\right)\right)^\perp \\
&= \ker_{L^2(\Omega)} \left(I - z(L+z)^{-1}\right)^*. \quad (*)
\end{aligned}
$$

(i) Claim: $((L+z)^{-1})^* = ({}^tL+z)^{-1}$.

Indeed, if $\varphi, \psi \in L^2(\Omega)$, then $\psi = ({}^tL+z)v$ for some $v \in H^2 \cap H^1_0$

$$\Rightarrow \big(\underbrace{(L+z)^{-1}\varphi}_{=:u \in H^2 \cap H^1_0}, \psi \big) = \big(u, ({}^tL+z)v \big) = \big((L+z)u, v \big) = (\varphi, ({}^tL+z)^{-1}\psi).$$

(ii) Returning to ⊛, if $v \in L^2(\Omega)$, $(I - z({}^tL+z)^{-1})v = 0$,

then $v = z({}^tL+z)^{-1}v \in H^2(\Omega) \cap H^1_0(\Omega)$,

and thus upon applying ${}^tL+z$:

$$({}^tL+z)v = 0.$$

As we said before, this gives $v \in C^\infty(\overline{\Omega}) \Rightarrow v \in {}^tN$.

Step 3. Since L has index 0, $\dim N = \dim {}^tN$. $\square$