We return to Corollary (C.10) and ask: what can one say about the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ of $-\Delta$ as an operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded and of class C^∞?

Theorem (T.35) (Weyl's Law) Let $\Omega \Subset \mathbb{R}^n$ be C^∞, let $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ denote the Dirichlet eigenvalues of $-\Delta$ on Ω, and let

$$N(T) = \#\{ i : \lambda_i \le T \}$$

denote the eigenvalue counting function. Then

$$\lim_{T \to \infty} \frac{N(T)}{T^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega),$$

where $\omega_n = \mathcal{L}^n(B_1(0))$, $\mathcal{L}^n$ = n-dimensional Lebesgue measure.

It is important that we can work in less regular domains.

Lemma (L.19) Let $\Omega \subset \mathbb{R}^n$ be bounded. Let

$$\sigma(-\Delta) = \{ \lambda \in \mathbb{C} : \exists\, u \in H^1_0(\Omega) \text{ s.t. } -\Delta u = \lambda u \}.$$

(i) $\sigma(-\Delta) = \{\lambda_1, \lambda_2, \dots\}$ where $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$, and $\exists$ complete ONB $\{u_1, u_2, \dots\} \subset L^2(\Omega)$ with $u_j \in H^1_0(\Omega)$, $-\Delta u_j = \lambda_j u_j$.

(ii) We have the min–max characterization

$$\lambda_k = \inf_{\substack{V \subset H^1_0(\Omega) \\ \dim V = k}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} .$$

This largely generalizes Corollary (C.10), except it is not true anymore that the eigenfunctions u_k lie in $C^\infty(\bar{\Omega})$ (or even in $H^2(\Omega)$).

Proof Step 1. We define $A : H^1_0(\Omega) \to H^{-1}(\Omega) := (H^1_0(\Omega))^*$ by

$$(Au)(v) := (\nabla u, \nabla v)_{L^2(\Omega)} .$$

To understand what this is, note that there is a continuous embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ given by

$$L^2(\Omega) \ni f \mapsto \left(H^1_0(\Omega) \ni v \mapsto (f,v)_{L^2(\Omega)} \right);$$

and then $Au = f$ — an equation in $H^{-1}(\Omega)$ — means

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega),$$

i.e. $-\Delta u = f$ weakly.

1.1. A is an isomorphism.

Indeed, since $(u,v)_{H^1_0} = (Au)(v)$, this is precisely the content of the Riesz representation theorem for $H^1_0(\Omega)$.

1.2. We compose $A^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ with embeddings from/into $L^2(\Omega)$ and get the operator

$$\begin{tikzcd} T: L^2(\Omega) \arrow[r] \arrow[d, hook] & L^2(\Omega). \\ H^{-1}(\Omega) \arrow[r, "A^{-1}"] & H^1_0(\Omega) \arrow[u] \end{tikzcd}$$

The inclusion map $H^1_0(\Omega) \to L^2(\Omega)$ is compact (exercise — this does not require any regularity on $\partial\Omega$, unlike Corollary (C.8)). Therefore, T is a compact operator.

One can also easily check (exercise) that T is symmetric.

Finally, T is positive: if $f \in L^2(\Omega)$, then

$$(Tf, f)_{L^2(\Omega)} = (u, f)_{L^2(\Omega)} \qquad \text{(where } u := Tf \in H^1_0(\Omega) \text{ solves } -\Delta u = f\text{)}$$

$$= (\nabla u, \nabla u)_{L^2(\Omega)}$$

$$\geq 0,$$

with equality iff $u = 0$ in $H^1_0(\Omega)$ and thus $f = 0$.

Step 2. (i) follows from the spectral theorem for compact self-adjoint operators, with the λ_j = reciprocals of the eigenvalues of T (which are all >0).

(ii) follows either from a similar characterization of the eigenvalues of T (see FA I), or directly using the complete ONB of eigenfunctions of $-\Delta$ (exercise). $\square$

Example (E.21) $\Omega = (0, L)^n$, $L > 0$.

(i) **Claim**: for $k \in \mathbb{N}^n$, let
$$u_k(x) = \left(\tfrac{2}{L}\right)^{n/2} \prod_{j=1}^{n} \sin\left(\tfrac{\pi}{L} k_j x_j\right).$$
Then $u_k \in H_0^1(\Omega)$, $-\Delta u_k = \left(\tfrac{\pi}{L}\right)^2 |k|^2 u_k$, and $\{u_k\}_{k \in \mathbb{N}^n}$ is a complete ONB of $L^2(\Omega)$.

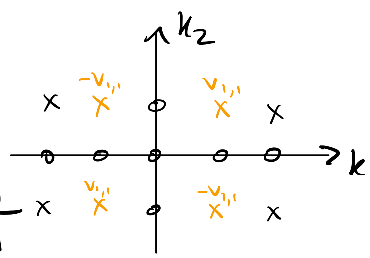
Indeed, only the final claim is not clear. Extend u_k to an odd function $\tilde{u}_k$ on $\tilde{\Omega} := (-L, L)^n$, i.e.
$$\tilde{u}_k(x_1, \dots, -x_i, \dots, x_n) = -\tilde{u}_k(x_1, \dots, x_i, \dots, x_n) \quad \forall i$$
— that is, $\tilde{u}_k(x) = \left(\tfrac{2}{L}\right)^{n/2} \prod_{j=1}^{n} \sin\left(\tfrac{\pi}{L} k_j x_j\right)$ simply.

- Every L^2-function on $\tilde{\Omega}$ can be expanded into Fourier series,
$$v = \sum_{k \in \mathbb{Z}^n} v_k \tilde{e}_k, \qquad (v_k)_{k \in \mathbb{Z}^n} \in \ell^2, \quad v_k = \int_{\tilde{\Omega}} v(x) \overline{\tilde{e}_k(x)}\, dx,$$
$$\tilde{e}_k(x) = (2L)^{-\frac{n}{2}} e^{\frac{i\pi}{L} x \cdot k}.$$

If now $u \in L^2(\Omega)$, let $v \in L^2(\tilde{\Omega})$ be its odd extension to $\tilde{\Omega}$. Let $k = (k_1, \dots, k_n)$.

- If $k_i = 0$ for some i, then $\tilde{e}_k(\dots, x_i, \dots) = \tilde{e}_k(\dots, -x_i, \dots)$ is even in x_i, but v is odd in x_i, so $v_k = 0$.
- If $k_1, \dots, k_n \neq 0$, and $k^\# = (\pm k_1, \dots, \pm k_n)$, then
$$v_{k^\#} = (-1)^{\text{number of } - \text{ signs}}\, v_k$$

These 2^n terms contribute to the Fourier series of v a total of



$$\sum_{k^\#=(\pm k_1,\dots,\pm k_n)} v_{k^\#}\, \tilde{e}_{k^\#}(x) = v_k \sum_{k_\#=(\pm k_1,\dots,\pm k_n)} (-1)^{\#(-\text{signs})}\, \tilde{e}_{k^\#}(x)$$

$$= v_k \sum_{k'=(\pm k_1,\dots,\pm k_{n-1})} (-1)^{\#(-\text{signs})} \left(\tilde{e}_{(k',k_n)}(x) - \tilde{e}_{(k',-k_n)}(x)\right)$$

$x=(x',x_n)$

$$= v_k \sum_{k'=(\pm k_1,\dots,\pm k_{n-1})} (-1)^{\#(-\text{signs})}\, 2i\,(2L)^{-\frac{n}{2}} e^{\frac{i\pi}{L}x'\cdot k'} \sin\left(\tfrac{\pi}{L}x_n k_n\right)$$

$$= \cdots = (2i)^n (2L)^{-\frac{n}{2}} v_k \prod_{j=1}^n \sin\left(\tfrac{\pi}{L}x_j k_j\right)$$

$$= v_k\, i^n\, \tilde{u}_k(x).$$

Therefore, $v = \sum_{k\in\mathbb{N}^n} i^n v_k \tilde{u}_k$ in $L^2(\tilde{\Omega})$,

and thus also $u = \sum_{k\in\mathbb{N}^n} i^n v_k u_k$ in $L^2(\Omega)$, as required.

(ii) We conclude that the set of Dirichlet eigenvalues of $-\Delta$ on $\Omega=(0,L)^n$ is

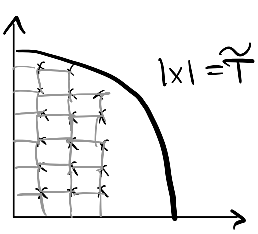
$$S = \left\{ \left(\tfrac{\pi}{L}\right)^2 |k|^2 : k\in\mathbb{N}^n \right\}.$$

$\left(\frac{\pi}{L}\right)^2 |k|^2 \le T$

$\Leftrightarrow |k| \le \frac{L}{\pi}\sqrt{T}$

(iii) Claim. Weyl's law holds on $\Omega=(0,L)^n$.

Indeed, $N(T) = \#\left(\mathbb{N}^n \cap \underbrace{\overline{B_{\tilde{T}}(0)}}_{=\{x\in\mathbb{R}^n :\ |x|\le\tilde{T}\}}\right)$, $\quad \tilde{T} = \frac{L}{\pi} T^{\frac{1}{2}}$



This the Lebesgue measure of the union of the gray boxes, so

$$N(T) \le \mathcal{L}^n\left([0,\infty)^n \cap \overline{B_{\tilde{T}}(0)}\right)$$
$$= 2^{-n}\mathcal{L}^n\left(\overline{B_{\tilde{T}}(0)}\right)$$
$$= 2^{-n}\tilde{T}^n \omega_n \qquad \left(\omega_n = \mathcal{L}^n(\overline{B_1(0)})\right)$$

$$= \left(\frac{L}{2\pi}\right)^n T^{n/2} \omega_n .$$

On the other hand, $[0,\infty)^n \times \overline{B_{\tilde{T}-\sqrt{n}}}(0) \subset$ union of gray boxes,

so $N(T) \geq 2^{-n} (\tilde{T}-\sqrt{n})^n \omega_n = \underbrace{\left(\frac{L}{2\pi}\right)^n T^{\frac{n}{2}} \omega_n}_{\text{from } \tilde{T}^n} \underbrace{\left(1-\frac{\pi\sqrt{n}}{L\sqrt{T}}\right)^n}_{\left(\frac{\tilde{T}-\sqrt{n}}{\tilde{T}}\right)^n}$

$$\Rightarrow \quad \frac{N(T)}{T^{\frac{n}{2}}} = \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega) \cdot \left(1 + O(T^{-\frac{1}{2}})\right)$$

$$\xrightarrow{T\to\infty} \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega).$$

We next have the following monotonicity result for eigenvalues:

Lemma (L.20) Let $\lambda_k(\Omega)$ denote the k^{th} eigenvalue ($k \geq 1$, counting with multiplicity) of $-\Delta$ on $\Omega \Subset \mathbb{R}^n$.

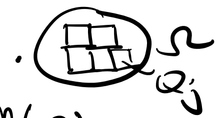
If $\Omega \subset \Omega'$, then $\lambda_k(\Omega') \leq \lambda_k(\Omega)$ $\forall k \in \mathbb{N}$.

Proof This follows directly from Lemma (L.19) (ii): if $V \subset H_0^1(\Omega)$, $\dim V = k$, then extending the elements of V by 0 to Ω' gives a subspace $V \subset H_0^1(\Omega')$. So $\lambda_k(\Omega)$ is an infimum over a smaller set (of linear subspaces of $H_0^1(\Omega')$ of dimension k) than $\lambda_k(\Omega')$, so $\lambda_k(\Omega) \geq \lambda_k(\Omega')$. $\square$

We can now already prove one half of Weyl's law.

Proof that $\liminf\limits_{T\to\infty} \frac{N(T)}{T^{n/2}} \geq \frac{\omega_n}{(2\pi)^n}\mathcal{L}^n(\Omega)$.

Given $\varepsilon>0$, we can choose $L>0$ so small that there exists a disjoint union $\bigcup\limits_{j=1}^{J} Q_j \subset \Omega$ of cubes $Q_j = q_j + Q$, $Q = (0,L)^n$, s.t.



$$\mathcal{L}^n(\Omega) - \varepsilon < \mathcal{L}^n\Big(\bigcup_{j=1}^{J} Q_j\Big) = J\,\mathcal{L}^n(Q) < \mathcal{L}^n(\Omega).$$

Write $\lambda_1^L \leq \lambda_2^L \leq \cdots$ for the Dirichlet eigenvalues of Q_j (see Example (E.21)), and

$\lambda_1^0 \leq \lambda_2^0 \leq \cdots$ for the sequence $\lambda_1^L, \dots, \lambda_1^L$ (J times), $\lambda_2^L, \dots, \lambda_2^L$ (J times), $\dots$

We claim that $\lambda_k^0 \geq \lambda_k(\Omega)$ for all $k \in \mathbb{N}$.

Indeed, by Lemma (L.19),

$$\lambda_k^0 = \inf_{\substack{V \subset H_0^1(\bigcup_{j=1}^J Q_j)\\ \dim V = k}} \ \sup_{\substack{u\in V\\ u\neq 0}} \frac{\|\nabla u\|^2}{\|u\|^2}$$

$$\geq \inf_{\substack{V \subset H_0^1(\Omega)\\ \dim V = k}} \ \sup_{\substack{u\in V\\ u\neq 0}} \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_k(\Omega).$$

$$\Rightarrow N(T) = \#\{k : \lambda_k(\Omega) \leq T\}$$
$$\geq \#\{k : \lambda_k^0 \leq T\}$$
$$= J\cdot \#\{k : \lambda_k^L \leq T\}$$
$$= J\,\frac{\omega_n}{(2\pi)^n}\,\mathcal{L}^n(Q)\,T^{\frac{n}{2}}(1+o(1))$$

$$> \frac{\omega_n}{(2\pi)^n}\left(\mathcal{L}^n(\Omega)-\varepsilon\right) T^{n/2}(1+o(1)), \quad T\to\infty.$$

$$\Longrightarrow \liminf_{T\to\infty} \frac{N(T)}{T^{n/2}} \geq \frac{\omega_n}{(2\pi)^n}\left(\mathcal{L}^n(\Omega)-\varepsilon\right).$$

Since $\varepsilon>0$ is arbitrary, we are done. $\square$

The idea here was to use only test functions u in the min-max characterization of $\lambda_k(\Omega)$ which are H^1_0-functions on each cube Q_j to get an upper bound on $\lambda_k(\Omega)$. That this gives a sharp asymptotic lower bound for $N(T)$ (equivalently, a sharp asymptotic upper bound on $\lambda_k(\Omega)$ — exercise) is not obvious; we shall prove this (i.e. the other inequality required for a proof of Weyl's law) by getting a lower bound on $\lambda_k(\Omega)$ by using a larger space of test functions than $H^1_0(\Omega)$.

We have few choices apart from considering (e.g. for $\Omega=(0,L)^n$)

$$\nu_k := \inf_{\substack{V\subset H^1(\Omega)\\ \dim V=k}} \sup_{\substack{u\in V\\ u\neq 0}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

What is this? We have the following analogue of Lemma (L.19):

Lemma (L.21) Let $\Omega \Subset \mathbb{R}^n$. Assume $H^1(\Omega)\longrightarrow L^2(\Omega)$ is compact.

$$\sigma_N(-\Delta) = \{\lambda \in \mathbb{C} : \exists\, u \in H^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_{L^2(\Omega)} = \lambda (u,v)_{L^2(\Omega)}\ \forall v \in H^1(\Omega)\}.$$

(i) $\sigma_N(-\Delta) = \{\lambda_1, \lambda_2, \dots\}$ where $0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$, and $\exists$ complete ONB $\{u_1, u_2, \dots\} \subset L^2(\Omega)$ with $u_j \in H^1(\Omega)$,

$$(\nabla u_j, \nabla v) = \lambda_j (u_j, v) \quad \forall v \in H^1(\Omega).$$

(ii) We have the min-max characterization $\lambda_k = \nu_k$.

Proof. We set $B : H^1(\Omega) \to (H^1(\Omega))^*$,

$$(Bu)(v) = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2} = (u,v)_{H^1(\Omega)}.$$

(So $Bu = -\Delta u + u$ when $u \in H^2(\Omega)$.) This is an isomorphism. By assumption,

$$T : L^2(\Omega) \hookrightarrow (H^1(\Omega))^* \xrightarrow{B^{-1}} H^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact. Its symmetry is a simple calculation: let $f, g \in L^2(\Omega)$, and set $u = Tf$, $v = Tg \in H^1(\Omega)$; then $Bv = g$ (regarded as an element of $(H^1(\Omega))^*$ via $H^1(\Omega) \ni u \mapsto (g,u)_{L^2(\Omega)}$), and therefore

$$(Tf, g)_{L^2(\Omega)} = (u, g)_{L^2(\Omega)} = (Bv)(u) = (u,v)_{H^1(\Omega)}$$
$$= (v,u)_{H^1(\Omega)} = \dots = (f, Tg)_{L^2(\Omega)}$$

(working with real-valued functions here).

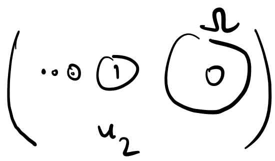
- T is positive: $(Tf, f) = (u,u)_{H^1(\Omega)} > 0$ unless $u = 0$.
- Get $0 < \mu_1 \le \mu_2 \le \dots \to \infty$, complete ONB $\{u_k\}_{k \in \mathbb{N}} \subset L^2(\Omega)$,

$$Tu_k = \mu_k u_k \iff u_k = \mu_k B u_k \iff Bu_k = \mu_k^{-1} u_k$$
$$\iff (\nabla u_k, \nabla v)_{L^2(\Omega)} = \left(\tfrac{1}{\mu_k} - 1\right)(u_k, v)_{L^2(\Omega)} \quad \forall v \in H^1(\Omega).$$

$\Rightarrow \lambda_k = \frac{1}{\mu_k} - 1.$

- To show that $\lambda_k \geq 0$, note that $(\nabla u, \nabla v) = \lambda (u, v)$ $\forall\, v \in H^1(\Omega)$ implies (for $v = u$) $\|\nabla u\|^2 = \lambda \|u\|^2$, and hence $\lambda \geq 0$ indeed. $\square$

Remark (R.29) (i) For $\Omega \subset \mathbb{R}^n$, $H^1(\Omega) \hookrightarrow L^2(\Omega)$ may fail to be compact. Example: $\Omega = \bigcup_{k=10}^{\infty} B_{\frac{1}{k^3}}(\frac{1}{k}, 0, \ldots, 0)$. (Disjoint union.)



Let $u_j = \begin{cases} j^{3n/2} & \text{on } B_{1/j^3}(\frac{1}{j}, 0, \ldots, 0), \\ 0 & \text{otherwise.} \end{cases}$

Then all derivatives of u_j vanish, so

$$\|u_j\|^2_{H^m(\Omega)} = \|u_j\|^2_{L^2(\Omega)} = \int_{B_{\frac{1}{k^3}}} (k^{3n/2})^2 \, dx = \omega_n$$

is bounded, yet $\|u_j - u_\ell\|^2_{L^2(\Omega)} = 2\omega_n$ for all $j \neq \ell$.
So $\{u_j\}$ does not have a convergent subsequence in $L^2(\Omega)$.

(ii) For $\Omega = (0, L)^n$, the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. (Exercise.) For Ω of class C^1, compactness of the inclusion follows from Corollary (C.8).

Now, what are the Neumann eigenvalues $\nu_1, \nu_2, \ldots$ of $-\Delta$ on $\Omega \subset \mathbb{R}^n$ as in Lemma (L.21)? If $u \in H^1(\Omega)$, $-\Delta u = f$ (in the sense that $(\nabla u, \nabla v) = (f, v)$ $\forall\, v \in H^1(\Omega)$), suppose that Ω is C^∞, and $u, f \in C^\infty(\bar{\Omega})$. Then for $v \in C^\infty(\bar{\Omega})$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v \, d\sigma - \int_\Omega \Delta u \cdot v \, dx$$

$$\stackrel{!}{=} \int_\Omega f \cdot v \, dx,$$

so $$\int_\Omega (\Delta u + f) \cdot v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot v \, d\sigma.$$

Plugging in $v \in C_c^\infty(\Omega)$, we conclude that $-\Delta u = f$ in Ω.

Thus $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma = 0 \; \forall v \in C^\infty(\bar{\Omega}) \Rightarrow \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0.$

Therefore, we are in fact working with a weak formulation of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark (R.30) One can develop a regularity theory analogous to Theorem (T.32) also for this problem.

In particular, the u_k from Lemma (L.21) are weak solutions of the eigenvalue problem
$$\begin{cases} -\Delta u_k = \nu_k u_k \\ \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Example (E.22) Consider $\Omega = (0, L)^n$. Then $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (exercise).

(i) Claim: for $k \in \mathbb{N}_0^n$, let
$$u_k(x) = \left(\frac{2}{L}\right)^{\frac{n}{2}} \prod_{j=1}^n \cos\left(\frac{\pi}{L} k_j x_j\right) \in H^1(\Omega).$$

Then $u_k \in H^1(\Omega)$ solves $-\Delta u_k = \left(\frac{\pi}{L}\right)^2 |k|^2 u_k$ weakly, and $\{u_k\}_{k \in \mathbb{N}_0^n}$ is a complete ONB of $L^2(\Omega)$.

Indeed, note that $\frac{\partial u_k}{\partial x_i} = 0$ at $x_i = 0, L$ for $i = 1, \dots, n$; this implies via an integration by parts that u_k is a weak solution of the Neumann eigenvalue equation indeed. The completeness of $\{u_k\}$ follows by a variation of the argument used in Example (E.21).

(ii) We conclude that the set of Neumann eigenvalues of $-\Delta$ on $\Omega = (0,L)^n$ is

$$S = \left\{ \left(\frac{\pi}{L}\right)^2 |k|^2 : k \in \mathbb{N}_0^n \right\}.$$

(iii) Claim. Weyl's law holds on $\Omega = (0,L)^n$ with Neumann boundary conditions; i.e.

$$\lim_{T \to \infty} \frac{\#\{ v \in S : |v| \leq T \}}{T^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega).$$

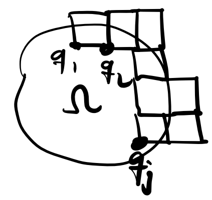
Proof via a minor modification of the argument in Example (E.21).

We can now already prove the other half of Weyl's law.

Proof that $\limsup_{T \to \infty} \frac{N(T)}{T^{n/2}} \leq \frac{\omega_n}{(2\pi)^n} \mathcal{L}^n(\Omega)$.

($\Rightarrow$ Theorem (T.35).)

- Given $\varepsilon > 0$, we choose $L > 0$ so small that there exist distinct points $q_1, \dots, q_J \in L \cdot \mathbb{Z}^n$ s.t. $\Omega \subset \bigcup_{j=1}^{J} (\overline{Q} + q_j)$ where $\overline{Q} = [0,L]^n$, and $\mathcal{L}^n(\Omega) \le J \cdot \mathcal{L}^n(\overline{Q}) < \mathcal{L}^n(\Omega) + \varepsilon$.



- Write $0 \le \nu_1 \le \nu_2 \le \cdots \to \infty$ for the Neumann eigenvalues of Q, and
$$0 \le \lambda_1^N \le \lambda_2^N \le \cdots \to \infty$$
for the sequence $\underbrace{\nu_1, \dots, \nu_1}_{J \text{ times}}, \underbrace{\nu_2, \dots, \nu_2}_{J \text{ times}}, \dots$.

- We claim that $\lambda_k(\Omega) \ge \lambda_k^N$ for all $k \in \mathbb{N}$.

Indeed, by Lemma (L.19),
$$\lambda_k(\Omega) = \inf_{\substack{V < H_1^0(\Omega) \\ \dim V = k}} \ \sup_{\substack{u \in V \\ u \ne 0}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}$$

if $u \in H_1^0(\Omega)$, then $u|_{Q+q_j} \in H^1(Q+q_j)$

$$\ge \inf_{\substack{V < L^2(\mathbb{R}^n),\ \dim V = k \\ \forall j:\ V|_{Q+q_j} \in H^1(Q+q_j)}} \ \sup_{\substack{u \in V \\ \|u\|_{L^2(\Omega)} = 1}} \|\nabla u\|^2_{L^2(\Omega)}$$

$$= \lambda_k^N$$

The final equality follows by writing $u \in V$ as
$$u(x) = \sum_{j=1}^{J} 1_{Q_j} \sum \underbrace{c_{j\ell}}_{\in \mathbb{C}} \cdot u_\ell(x - q_j), \qquad \sum_{j,\ell} |c_{j\ell}|^2 = 1,$$

ONB of $H^1(Q)$ from Example (E.22)

with
$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{j=1}^{J} |c_{j\ell}|^2 \nu_\ell$$
and following the arguments in the proof of the min-max formula from here.

$$
\begin{aligned}
\Rightarrow \quad N(T) &= \#\{k : \lambda_k(\Omega) \leq T\} \\
&\leq \#\{k : \lambda_k^N \leq T\} \\
&= J \cdot \#\{k : \nu_k \leq T\} \\
&= J \, \frac{\omega_n}{(2\pi)^n} \, \mathcal{L}^n(Q) \, T^{\frac{n}{2}} (1 + o(1)) \\
&< \frac{\omega_n}{(2\pi)^n} \left(\mathcal{L}^n(\Omega) + \varepsilon\right) T^{\frac{n}{2}} (1 + o(1)), \quad T \to \infty .
\end{aligned}
$$

$$
\Rightarrow \quad \limsup_{T \to \infty} \frac{N(T)}{T^{n/2}} \leq \frac{\omega_n}{(2\pi)^n} \left(\mathcal{L}^n(\Omega) - \varepsilon\right).
$$

Since $\varepsilon > 0$ is arbitrary, we are done. $\square$