In our second approach to the Dirichlet problem, we saw the utility of thinking of $\frac{d^2}{dx^2}u(x) = f(x)$ (on $\Omega = (a,b) \in \mathbb{R}$) weakly.

Even simpler: $\quad \frac{du}{dx} = f \implies \forall \varphi \in C_c^\infty(\Omega), \int_a^b u \frac{d\varphi}{dx}\,dx = -\int_a^b f\varphi\,dx.$

<u>Definition (D.12)</u> Let $\Omega \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(\Omega)$.

(i) The distributional (partial) derivative $\frac{\partial u}{\partial x_j}$ is the linear map

$$\frac{\partial u}{\partial x_j} : C_c^\infty(\Omega) \ni \varphi \longmapsto -\int_\Omega u \frac{\partial \varphi}{\partial x_j}\,dx.$$

(i') We say that u has a weak derivative $\frac{\partial u}{\partial x_j} \in \mathcal{F}$ where $\mathcal{F} \subset L^1_{loc}$ (typically $\mathcal{F} = L^p_{loc}, L^p, \ldots$) if $\exists\, g_j \in \mathcal{F}$ so that

$$-\int_\Omega u \frac{\partial \varphi}{\partial x_j}\,dx = \int_\Omega g_j \varphi\,dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

(ii) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, write $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. Then $\partial^\alpha u$ is the linear map

$$C_c^\infty(\Omega) \ni \varphi \longmapsto (-1)^{\overset{\sum_{j=1}^n \alpha_j}{\overset{\shortparallel}{|\alpha|}}} \int_\Omega u\, \partial^\alpha \varphi\,dx.$$

(ii') u has a weak derivative $\partial^\alpha u \in \mathcal{F}$ if $\exists\, g_\alpha \in \mathcal{F}$ s.t.

$$(-1)^{|\alpha|}\int_\Omega u\, \partial^\alpha \varphi\,dx = \int_\Omega g_\alpha \varphi\,dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

<u>Remark (R.13)</u> (i) Weak derivatives, when they exist, are unique by Theorem (T.2).

(ii) If $u \in C^1(\Omega)$, then the weak derivative $\frac{\partial u}{\partial x_j}$ is the ordinary derivative (so in $C^0(\Omega)$). Similarly for $\partial^\alpha u$ when $u \in C^{|\alpha|}(\Omega)$.

Example (E.17) (i) Let $u(x) = |x|$ on $\mathbb{R}$. Then $u'(x) = \begin{cases} -1; & x<0 \\ 1; & x>0 \end{cases}$ (as an L^1_{loc}-function), since for $\varphi \in C_c^\infty(\mathbb{R})$,

$$-\int_{\mathbb{R}} u\varphi'\,dx = \int_{-\infty}^0 x\,\varphi'(x)\,dx - \int_0^\infty x\varphi'(x)\,dx$$

$$= \underbrace{x\varphi\Big|_{-\infty}^0 - x\varphi\Big|_0^\infty}_{=0} - \int_{-\infty}^0 \varphi(x)\,dx + \int_0^\infty \varphi(x)\,dx.$$

So $u' \in L^\infty(\mathbb{R})$ even, though $u \notin C^1(\mathbb{R})$.

(ii) Not all L^1_{loc}-functions have weak derivatives. E.g. the Heaviside function $u(x) = H(x) := \begin{cases} 0, & x<0 \\ 1, & x>1 \end{cases} \in L^1_{loc}(\mathbb{R})$ has distributional derivative

$$C_c^\infty(\mathbb{R}) \ni \varphi \longmapsto -\int_{\mathbb{R}} u\varphi'\,dx = -\int_0^\infty \varphi'(x)\,dx = \varphi(0).$$

(The Dirac distribution, a.k.a. "δ-function".)

But for no $g \in L^1_{loc}(\Omega)$ is $\int_{\mathbb{R}} g\varphi\,dx = \varphi(0)\ \forall \varphi \in C_c^\infty(\mathbb{R})$.

(Exercise.)

(iii) If $u \in L^1_{loc}(\mathbb{R})$ is a.e. differentiable with derivative $u'(x) = f(x)$ in $L^1_{loc}(\mathbb{R})$, one cannot conclude that f is the weak derivative of u. Example: Let $\mu =$ Cantor measure. Then $u(x) := \mu([0,x])$ defines a continuous function $u \in C^0(\mathbb{R})$ whose distributional derivative is "$d\mu$": $\varphi \longmapsto \int \varphi\,d\mu$. (Exercise.)

But $u' = 0\ \forall x \notin$ Cantor set!

(iv) Let $v \in L^2(\Omega)$, $\Omega \Subset \mathbb{R}^n$, and $\exists\, C$ s.t.

$$\left|\int_\Omega v\,\Delta\varphi\,dx\right| \le C\,\|\varphi\|_{L^2(\Omega)} \quad \forall \varphi \in C^2(\Omega),\ \varphi|_{\partial\Omega} = 0.$$

(i.e. $v \in D(A^*)$ in the notation of Approach 1 earlier)

By Riesz, $\exists\, f \in L^2(\Omega)$ s.t.

$$\int_\Omega v\, \Delta\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

This means: Δv (sum of distributional derivatives) is represented by the L^2-function f. Write: $\Delta v = f \in L^2(\Omega)$.

Definition (D.13) (Sobolev spaces.) $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$, $k \in \mathbb{N}_0$.

(i) $W^{k,p}(\Omega) = \{ u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \ \forall \alpha \in \mathbb{N}_0^n, |\alpha| \le k \}$,

with norm $\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^p(\Omega)}$.

(ii) $p < \infty$: $W_0^{k,p}(\Omega) =$ closure of $C_c^\infty(\Omega)$ in $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$.

(iii) For $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$ and

$W_0^{k,2}(\Omega) = H_0^k(\Omega)$;

and we use the (equivalent) norm

$$\|u\|_{H^k(\Omega)} := \Big(\sum_{|\alpha| \le k} \|\partial^\alpha u\|^2_{L^2(\Omega)} \Big)^{1/2}$$

(which comes from the inner product $(u,v)_{H^k(\Omega)} = \sum_{|\alpha| \le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$);

Of course, $W^{0,p}(\Omega) = W_0^{0,p}(\Omega) = L^p(\Omega)$.

Remark (R.14) By Lemma (L.8), if $\Omega \Subset \mathbb{R}^n$, there exists $C < \infty$ s.t. $\frac{1}{C}\|u\|_{H_0^1(\Omega)} \le \|\nabla u\|_{L^2(\Omega)} \le C\|u\|_{H_0^1(\Omega)}$ $\forall u \in C_c^\infty(\Omega)$.

The same is true for $W_0^{1,p}$, $1 \le p < \infty$, i.e.

$$\frac{1}{C}\|u\|_{W_0^{1,p}(\Omega)} \le \|\nabla u\|_{L^p(\Omega)} \le C\|u\|_{W_0^{1,p}(\Omega)} \quad \forall u \in C_c^\infty(\Omega).$$

Indeed, if $\Omega \subset [0,L] \times \mathbb{R}^{n-1}$, then

$$\begin{aligned}\|u\|_{L^p(\Omega)}^p = \int_\Omega |u|^p\,dx &\le \int_\Omega \left| \int_0^L \left|\frac{\partial u}{\partial x_1}\right| dx_1 \right|^p dx \\ &\le \int_\Omega \left(\int_0^L \left|\frac{\partial u}{\partial x_1}\right|^p dx_1\right)\left(\int_0^L 1\,dx_1\right)^{p-1} dx \\ &\le L^p \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^p dx \\ &\le L^p \|\nabla u\|_{L^p(\Omega)}^p.\end{aligned}$$

<u>Theorem (T.17)</u> Let $k \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$. Then $W^{k,p}(\Omega)$ is...

(i) complete for $1 \le p \le \infty$,

(ii) separable for $1 \le p < \infty$,

(iii) reflexive for $1 < p < \infty$.

<u>Proof</u> These statements hold for $L^p(\Omega)$, i.e. $k=0$. We only consider the case $k=1$. Define the isometric map

$$\begin{aligned} i: W^{1,p}(\Omega) &\longrightarrow (L^p(\Omega))^{n+1}, \\ u &\longmapsto \left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right). \end{aligned}$$

(i) If $\{u_k\} \subset W^{1,p}(\Omega)$ is a Cauchy sequence, then so is $i(u_k)$, and therefore $\exists$

$$u = \lim_{k\to\infty} u_k \in L^p(\Omega),$$
$$v_j = \lim_{k\to\infty} \frac{\partial u_k}{\partial x_j} \in L^p(\Omega).$$

<u>Claim</u>: $u \in W^{1,p}(\Omega)$, $\frac{\partial u}{\partial x_j} = v_j$.

<u>Well</u>, if $\varphi \in C_c^\infty(\Omega)$, then

$$-\int_\Omega u \frac{\partial \varphi}{\partial x_j} dx = -\lim_{k\to\infty} \int_\Omega u_k \frac{\partial \varphi}{\partial x_j} dx$$
$$= \lim_{k\to\infty} \int_\Omega \frac{\partial u_k}{\partial x_j} \varphi \, dx$$
$$= \int_\Omega v_j \varphi \, dx.$$

(ii) $L^p(\Omega)$ is separable for $1 \le p < \infty$, therefore so is $(L^p(\Omega))^{n+1}$ and thus also its subspace $i(W^{1,p}(\Omega))$. Since i is an isometry, this implies that $W^{1,p}(\Omega)$ is separable.

(iii) By (i), $i(W^{1,p}(\Omega)) \subset (L^p(\Omega))^{n+1}$ is closed. For $1 < p < \infty$, $L^p(\Omega)$ is reflexive, and thus so is $(L^p(\Omega))^{n+1} \Rightarrow i(W^{1,p}(\Omega))$ is reflexive. Since i is an isometry, $W^{1,p}(\Omega)$ is reflexive. $\square$

<u>Corollary (C.3)</u> $H^k(\Omega)$ is a Hilbert space with inner product
$$(u,v)_{H^k(\Omega)} := \sum_{|\alpha| \le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.$$

<u>Definition (D.14)</u> Let $\Omega \subset \mathbb{R}^n$ be open. We write $W^{k,p}_{loc}(\Omega)$ for the space of all $u \in L^p_{loc}(\Omega)$ s.t. $\partial^\alpha u \in L^p_{loc}(\Omega)$ $\forall |\alpha| \le k$.