Let $I = (a,b) \subseteq \mathbb{R}$, $-\infty \leq a < b \leq \infty$. We shall study the spaces $W^{1,p}(I)$ in some detail.

Theorem (T.18)

(i) Let $u \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Then $\exists\ \tilde{u} \in C^0(\bar{I})$ s.t. $u = \tilde{u}$ a.e., and for $x_0, x \in I$

$$\tilde{u}(x) = \tilde{u}(x_0) + \int_{x_0}^{x} u'(t)\,dt.$$

(One thus typically replaces u by its continuous representative $\tilde{u}$.)

(ii) Conversely, if $u \in L^1_{loc}(I)$ satisfies $u(x) = u(x_0) + \int_{x_0}^{x} v(t)\,dt$ a.e. where $v \in L^1_{loc}(I)$, then $u \in W^{1,1}_{loc}(I)$, and its weak derivative is $u' = v$. If $u, u' \in L^p(I)$, then $u \in W^{1,p}(I)$.

For the proof, we need:

Lemma (L.9) Let $u \in L^1_{loc}(I)$, and suppose the distributional derivative of u vanishes; that is, $\int_a^b u\varphi'\,dx = 0\ \forall \varphi \in C_c^\infty(I)$. Then u is constant a.e.

Proof Fix $\chi \in C_c^\infty(I)$ with $\int_a^b \chi\,dx = 1$.

Let $\psi \in C_c^\infty(I)$. Then $\tilde{\psi}(x) = \psi(x) - c_0\chi(x) \in C_c^\infty(I)$ has $\int_a^b \tilde{\psi}(x)\,dx = 0$ if we take $c_0 := \int_a^b \psi\,dx$.

In this case, $\varphi(x) := \int_a^x \tilde{\psi}(t)\,dt$ defines an element $\varphi \in C_c^\infty(I)$, and we compute

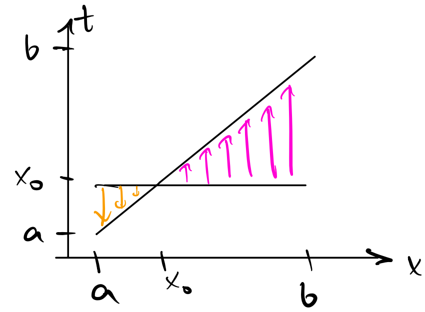
$$
\begin{aligned}
0 &= \int_a^b u\varphi'\,dx \\
&= \int_a^b u\tilde{\psi}\,dx \\
&= \int_a^b u\psi\,dx - c_0\int_a^b u\chi\,dx \\
&= \int_a^b u\psi\,dx - \int_a^b \psi\,dx\cdot c \qquad \left(c := \int_a^b u\chi\,dx\right) \\
&= \int_a^b (u-c)\psi\,dx.
\end{aligned}
$$

Since $\psi \in C_c^\infty(I)$ is arbitrary, we get $u-c=0$ a.e. by Theorem (T.2). $\square$

Proof of Theorem (T.18)

(i) · Set $v(x) = \int_{x_0}^x u'(t)\,dt$. Thus $v \in C^0(I)$; we claim that v has weak derivative u'. To see this, let $\varphi \in C_c^\infty(I)$, then

$$
\begin{aligned}
-\int_a^b v\varphi'\,dx &= -\int_a^b \left(\int_{x_0}^x u'(t)\,dt\right)\varphi'(x)\,dx \\
&= -\int_a^{x_0}\int_{x_0}^x u'(t)\varphi'(x)\,dt\,dx \\
&\quad -\int_{x_0}^b\int_{x_0}^x u'(t)\varphi'(x)\,dt\,dx \\
&= +\int_a^{x_0}\int_a^t u'(t)\varphi'(x)\,dx\,dt \\
&\quad -\int_{x_0}^b\int_t^b u'(t)\varphi'(x)\,dx\,dt \\
&= \int_a^{x_0} u'(t)\varphi(t)\,dt + \int_{x_0}^b u'(t)\varphi(t)\,dt \\
&= \int_a^b u'\varphi\,dt.
\end{aligned}
$$



· Set now $f = u - v \in L^1_{loc}(I)$; then $\forall \varphi \in C_c^\infty(I)$,

$$\int_a^b f\varphi'\,dx = \int_a^b u\varphi'\,dx - \int_a^b v\varphi'\,dx = -\int_a^b u'\varphi\,dx + \int_a^b u'\varphi\,dx = 0$$

$\overset{\text{Lemma (L.9)}}{\Longrightarrow} f \equiv c$ a.e., so $u(x) = c + \int_{x_0}^{x} u'(t)\,dt$ a.e.

where $c = f(x_0) = u(x_0) - v(x_0) = u(x_0)$.

> More precisely, we need to pick x_0 here to be a Lebesgue point of u: $u(x_0) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(t)\,dt$.
> This implies that in $u(x) = c + v(x)$ we can average over $x \in [x_0-\varepsilon, x_0+\varepsilon]$ to get
> $u(x_0) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} u(t)\,dt = c + \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} v(t)\,dt = c + v(x_0) = c$ (since v is continuous, $v(x_0)=0$).
> $\Rightarrow c = u(x_0)$.

(ii) follows from the first part of the proof of part (i). $\square$

Remark (R.15) Functions $u : [a,b] \to \mathbb{C}$ of the form

$$u(x) = u(x_0) + \int_{x_0}^{x} f(t)\,dt \quad \text{for some } f \in L^1([a,b])$$ are called *absolutely continuous*. (An equivalent characterization is that $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for all disjoint $(x_k, y_k) \subset [a,b]$ with $\sum (y_k - x_k) < \delta$, one has $\sum |u(y_k) - u(x_k)| < \varepsilon$.) Such functions are differentiable a.e., and their pointwise derivative equals f a.e.

We aim to show that two natural definitions of differentiability agree.

Theorem (T.19) Let $1 < p \leq \infty$. Then the following are equivalent:

(i) $u \in W^{1,p}(\mathbb{R})$.

(ii) $\exists C > 0$ s.t. $\|\tau_h u - u\|_{L^p(\mathbb{R})} \leq C|h|$, where $(\tau_h u)(x) = u(x+h)$

(iii) If $p < \infty$: there exists a sequence $\{u_k\} \subset C_c^\infty(\mathbb{R})$ s.t. $u_k \to u$ in $L^p(\mathbb{R})$, and u_k' converges in $L^p(\mathbb{R})$ to some limit $v \in L^p(\mathbb{R})$ which is in fact the weak derivative of u.

The construction of the approximating sequence $\{u_k\}$ in (iii) will use the technique of mollification: let $\rho \in C_c^\infty((-1,1))$ with $\int \rho\, dx = 1$,

set $\rho_k(x) = k\,\rho(kx)$ (so $\int \rho_k\, dx = 1\ \forall k$), and set

$$(\rho_k * u)(x) = \int_{\mathbb{R}} \rho_k(y)\, u(x-y)\, dy \quad , u \in L^p(\mathbb{R}).$$

Lemma (L.10) Let $1 \leq p < \infty$. For $u \in L^p(\mathbb{R})$, we have

$\rho_k * u \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ and $\rho_k * u \xrightarrow{k\to\infty} u$ in $L^p(\mathbb{R})$.

Moreover, $\rho_k * u \in C_c^\infty(\mathbb{R})$ if supp u is compact.

Proof • Since $(\rho_k * u)(x) = \int_{\mathbb{R}} u(y)\, \rho_k(x-y)\, dy$, we get

$\rho_k * u \in C^\infty$ by differentiation under the integral sign.

• Let $\varepsilon > 0$. Choose a continuous function with compact support

$v \in C_c^0(\mathbb{R})$ s.t. $\|u-v\|_{L^p(\mathbb{R})} < \varepsilon$. Then

$$u - \rho_k * u = (u-v) + (v - \rho_k * v) + \rho_k * (v-u). \qquad \circledast$$

* Now, for $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ we have

$$\|f * g\|_{L^p} = \left\| \int_{\mathbb{R}} f(y)\, g(x-y)\, dy \right\|_{L^p}$$

$$\leq \int_{\mathbb{R}} |f(y)|\, \underbrace{\|g(\cdot - y)\|_{L^p}}_{= \|g\|_{L^p}}\, dy$$

$$= \|f\|_{L^1} \|g\|_{L^p}.$$

So $\|\rho_k * (v-u)\|_{L^p} \leq \|\rho_k\|_{L^1} \|v-u\|_{L^p} = \|\rho\|_{L^1} \|v-u\|_{L^p} = C\varepsilon$.

* Finally, since v is uniformly continuous, $\exists\, k_0$ s.t. $|v(x)-v(y)| < \tilde{\varepsilon}$

$\forall x, y \in \mathbb{R}$, $|x-y| < \frac{1}{k_0}$. For $k \geq k_0$, this gives

$$\Rightarrow \quad |v(x) - (\rho_k * v)(x)| = \left| \int \rho_k(z)\, v(x) - \rho_k(z)\, v(x-z)\, dz \right|$$

$$\leq \int |\rho_k(z)|\, |v(x) - v(x-z)|\, dz$$

$$\leq \left(\int |g(z)|\,dz\right)\tilde{\varepsilon}$$
$$= C\tilde{\varepsilon}$$

$\Rightarrow \|v - \rho_k * v\|_{L^p} \leq C\tilde{\varepsilon}\cdot \mathcal{L}(\operatorname{supp} v)^{\frac{1}{p}}$. Take $\tilde{\varepsilon} = \frac{\varepsilon}{1+\mathcal{L}(\operatorname{supp} v)^{1/p}}$.

* All three terms in (*) are $\leq C'\varepsilon$ for all sufficiently large k, where C' only depends on ρ. Since $\varepsilon > 0$ is arbitrary, we are done. $\square$

<u>Corollary (C.3)</u> Let $1 \leq p < \infty$. Let $u \in W^{1,p}(\mathbb{R})$. Then $\exists\, u_k \in C_c^\infty(\mathbb{R})$ s.t. $u_k \to u$ in $W^{1,p}(\mathbb{R})$.

<u>Proof</u> <u>Step 1</u>: $v_k := \rho_k * u \longrightarrow u$ in $W^{1,p}(\mathbb{R})$.

<u>Well</u>, Lemma (L.10) gives $v_k \to u$ in $L^p(\mathbb{R})$. It suffices to show that v_k has a weak derivative in $L^p(\mathbb{R})$ and $v_k' = \rho_k * u'$. (Then $u' \in L^p$ implies $v_k' \to u'$ in L^p)

To this end, let $\varphi \in C_c^\infty(\mathbb{R})$, then

$$\int_{\mathbb{R}} v_k \varphi'\,dx = \int_{\mathbb{R}} (\rho_k * u)\,\varphi'\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_k(x-y)\,u(y)\,\varphi'(x)\,dx\,dy$$
$$= \int_{\mathbb{R}} u(y)\left(\int_{\mathbb{R}} \rho_k(x-y)\,\varphi'(x)\,dx\right) dy$$
$$= \int_{\mathbb{R}} u(y)\,\frac{d}{dy}\Big(\underbrace{\int_{\mathbb{R}} \rho_k(x-y)\,\varphi(x)\,dx}_{\in C_c^\infty(\mathbb{R})}\Big)\,dy$$
$$= -\iint u'(y)\,\rho_k(x-y)\,\varphi(x)\,dx\,dy$$
$$= -\int (\rho_k * u')\,\varphi\,dx.$$

Step 2: Given $u \in C^\infty(\mathbb{R}) \cap W^{1,p}(\mathbb{R})$, $\varepsilon>0$, $\exists\, v \in C_c^\infty(\mathbb{R})$ s.t. $\|u-v\|_{W^{1,p}(\mathbb{R})} < \varepsilon$.

Well, let $\varphi \in C_c^\infty((-2,2))$, $\varphi = 1$ on $[-1,1]$. Set $\varphi_k(x) = \varphi(\frac{x}{k})$.

We claim that $\varphi_k u \; (\in C_c^\infty(\mathbb{R})) \longrightarrow u$ in $W^{1,p}(\mathbb{R})$

- $$\|\varphi_k u - u\|_{L^p(\mathbb{R})} = \int_{\{|x|>k\}} |(\varphi_k(x)-1)\, u(x)|^p \, dx$$
$$\leq C \int_{\{|x|>k\}} |u(x)|^p \, dx \xrightarrow{k\to\infty} 0.$$

- $(\varphi_k u)' = \varphi_k u' + \varphi_k' u$, and $\varphi_k u' \longrightarrow u'$ in L^p,
while $\varphi_k' u \to 0$ in L^p since
$$|\varphi_k' u| \begin{cases} = 0, & |x| \leq k \\ \leq C|u(x)|, & |x|>k. \end{cases} \qquad \square$$

This proves that (i) $\Rightarrow$ (iii) in Thm (T.19) for $1 \leq p < \infty$.

Proof of Thm (T.19)

(iii) $\overset{(1\leq p\leq\infty)}{\Longrightarrow}$ (i): The assumptions give $\forall \varphi \in C_c^\infty(\mathbb{R})$

$$-\int u_k \varphi' \, dx \xrightarrow{k\to\infty} \int v \varphi \, dx$$
$$\downarrow$$
$$-\int u \varphi' dx = \int u' \varphi \, dx,$$

so $v = u'$; and so $u_k \longrightarrow u$, $u_k' \to u'$ in $L^p(\mathbb{R})$.

Since $W^{1,p}(\Omega)$ is complete, this gives $u \in W^{1,p}(\mathbb{R})$.

(i) $\overset{(1\leq p\leq\infty)}{\Longrightarrow}$ (ii): If $u \in W^{1,p}(\mathbb{R})$, then $\quad T_h u(x) - u(x) = \int_0^h u'(x+t)\, dt$,

and therefore, for $p<\infty$,

$$\|\tau_h u - u\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} \left| \int_0^h u'(x+t)\, dt \right|^p dx$$
$$\leq \int_{\mathbb{R}} \left(\int_0^h |u'(x+t)|^p\, dt\right)\left(\int_0^h 1\, dt\right)^{p-1} dx$$
$$= h^{p-1} \int_0^h \int_{\mathbb{R}} |u'(x+t)|^p\, dx\, dt$$
$$= h^p \|u'\|^p_{L^p(\mathbb{R})}.$$

For $p=\infty$, $\|\tau_h u - u\|_{L^\infty} \leq \int_0^h |u'(x+t)|\,dt \leq h\|u'\|_{L^\infty}$.

(ii) $\overset{1\leq p<\infty}{\Longrightarrow}$ (i): Let $\varphi \in C_c^\infty(\mathbb{R})$. Then for $f(h) := h^{-1}\int_{\mathbb{R}} (u(x+h)-u(x))\,\varphi(x)\,dx$,

- $|f(h)| \leq h^{-1}\|\tau_h u - u\|_{L^p}\|\varphi\|_{L^q} \leq C\|\varphi\|_{L^q}$ $\left(q=\frac{p}{p-1}\right)$;
- $f(h) = \int_{\mathbb{R}} u(y)\,(\varphi(y-h)-\varphi(y))\,dy \xrightarrow{h\to 0} -\int_{\mathbb{R}} u(y)\,\varphi'(y)\,dy$.

Therefore, $\left|\int u\varphi'\,dx\right| \leq C\|\varphi\|_{L^q}$ $\forall \varphi \in C_c^\infty(\mathbb{R})$.

Thus, $\varphi \longmapsto -\int u\varphi'\,dx$ extends to a continuous linear functional on $L^q \Rightarrow \exists\, v \in L^p(\mathbb{R})$ s.t.

$$-\int u\varphi'\,dx = \int v\varphi\,dx \quad \forall \varphi \in C_c^\infty(\mathbb{R})$$

$\Rightarrow u \in W^{1,p}(\mathbb{R})$ with $u' = v$. $\square$

Remark (R.16) (i) Theorem (T.19) says for $p=\infty$ that

$$W^{1,\infty}(\mathbb{R}) = \mathrm{Lip}(\mathbb{R}) \equiv C^{0,1}(\mathbb{R}) := \Big\{ u:\mathbb{R}\to\mathbb{C} : \|u\|_\infty = \sup|u| < \infty,$$
$$[u]_{C^{0,1}} = \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|} < \infty \Big\}.$$

(ii) If $p=\infty$ and $u_k \in C_c^\infty(\mathbb{R})$ is a Cauchy sequence in $W^{1,\infty}(\mathbb{R})$, then $u = \lim_{k\to\infty} u_k \in W^{1,\infty}(\mathbb{R})$ lies in $C^1(\mathbb{R}) = \{u:\mathbb{R}\to\mathbb{C}:$

$\|u\|_\infty < \infty$, $\|u'\|_\infty < \infty$, u' continuous$\}$. Thus, $C_c^\infty(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R})$ is not dense (just like it is not dense in $L^\infty(\mathbb{R})$ either).

(iii) $u(x) = \mathbb{1}_{[0,1]}(x)$ satisfies condition (ii) in Thm. (T.19), but $u \notin W^{1,1}(\mathbb{R})$ (since the distributional derivative of u is $\delta(x) - \delta(x-1) \notin L^1(\mathbb{R})$).

The next characterization is the basis upon which one can conveniently define Sobolev spaces with general orders $s \in \mathbb{R}$; we will see applications of this later in the discussion of trace theorems.

Theorem (T.20) Let $k \in \mathbb{N}_0, u \in L^2(\mathbb{R})$. Then $u \in H^k(\mathbb{R})$ if and only if $\int_{\mathbb{R}} (1+|\xi|)^{2k} |\hat{u}(\xi)|^2 \, d\xi < \infty$, where $\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\, dx$ is the Fourier transform of u. Moreover, the norms $\|u\|_{H^k(\mathbb{R})}$ and $\|(1+|\cdot|)^k \hat{u}\|_{L^2(\mathbb{R})}$ are equivalent.

Remark (R.17) (i) $\hat{u}$ is defined as a tempered distribution ($\hat{u} \in \mathcal{S}'(\mathbb{R})$) if u is itself a tempered distribution. Thus, $u \in \mathcal{S}'(\mathbb{R})$ lies in $H^k(\mathbb{R})$ if $\hat{u} \in \mathcal{S}'(\mathbb{R})$ is in fact in $L^2_{loc}(\mathbb{R})$ and $\int (1+|\xi|)^{2k} |\hat{u}(\xi)|^2 d\xi < \infty$.

(ii) One can then define, for $s \in \mathbb{R}$,

$$H^s(\mathbb{R}) = \left\{ u \in \mathcal{S}'(\mathbb{R}) : \hat{u} \in L^2_{loc}(\mathbb{R}), \int (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi < \infty \right\}.$$

For $s \geq 0$, one can replace $\mathcal{S}'(\mathbb{R})$ by $L^2(\mathbb{R})$. For $s \in \mathbb{R}$, this is the completion of $C_c^\infty(\mathbb{R})$ w.r.t. $\|u\|_{H^s} := \|(1+|\cdot|)^s \hat{u}\|_{L^2}$.

Proof of Theorem (T.20). Set $\tilde{H}^k(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : (1+|\cdot|)^k \hat{u} \in L^2\}$ with norm $\|u\|_{\tilde{H}^k} := \|(1+|\cdot|)^k \hat{u}\|_{L^2}$. We want to show that $\tilde{H}^k(\mathbb{R}) = H^k(\mathbb{R})$.

(i) If $u \in \mathcal{S}(\mathbb{R})$ (Schwartz space: $\sup_{x\in\mathbb{R}} (1+|x|)^j \left|\frac{d^k}{dx^k} u(x)\right| < \infty \ \forall j,k$), then $\widehat{u'}(\xi) = i\xi \hat{u}(\xi)$, so since $(1+|\xi|)^{2k} \leq C \sum_{j=0}^{k} |\xi|^{2j}$,

$$\|u\|_{\tilde{H}^k}^2 \leq C \int \sum_{j=0}^{k} |\xi^j \hat{u}(\xi)|^2 \, d\xi$$

$$= C \sum_{j=0}^{k} \|\widehat{u^{(j)}}\|_{L^2}^2$$

$$= C \sum_{j=0}^{k} \|u^{(j)}\|_{L^2}^2 \qquad \left(\text{since } u \mapsto \hat{u} \text{ is an isometric isomorphism } L^2(\mathbb{R}) \to L^2(\mathbb{R})\right)$$

$$= C \|u\|_{H^k}^2.$$

Since $\sum_{j=0}^{k} |\xi|^{2j} \leq C(1+|\xi|)^{2k}$, we similarly have $\|u\|_{H^k}^2 \leq C\|u\|_{\tilde{H}^k}^2$.

So on $\mathcal{S}(\mathbb{R})$, $\|\cdot\|_{H^k}$ and $\|\cdot\|_{\tilde{H}^k}$ are equivalent norms.

(ii) Now, $\mathcal{S}(\mathbb{R}) \subset H^k(\mathbb{R})$ is dense by Theorem (T.19)(iii).

Likewise, $\mathcal{S}(\mathbb{R}) \subset \tilde{H}^k(\mathbb{R})$ is dense since

$$\{\hat{u} : u \in \mathcal{S}(\mathbb{R})\} = \mathcal{S}(\mathbb{R}) \subset \{v \in L^2_{loc}(\mathbb{R}) : (1+|\cdot|)^k v \in L^2(\mathbb{R})\}$$

is dense; and $\tilde{H}^k(\mathbb{R})$ is complete.

$\Rightarrow$ $H^k(\mathbb{R})$, $\tilde{H}^k(\mathbb{R})$ are the completions of $\mathcal{S}(\mathbb{R})$ w.r.t. equivalent norms, and therefore they are equal. $\square$

We next generalize Theorem (T.19) to Sobolev spaces on $I=(a,b)$, $-\infty \le a < b \le \infty$. The strategy is to extend elements of $W^{1,p}(I)$ to all of $\mathbb{R}$ in a controlled manner and thereby reduce to the case $I=\mathbb{R}$ we have already treated.

Proposition (P.6). Let $1 \le p \le \infty$, $I=(a,b) \subseteq \mathbb{R}$. Then there exists a continuous linear extension operator $E: W^{1,p}(I) \to W^{1,p}(\mathbb{R})$:

(i) $(Eu)_I = u$;

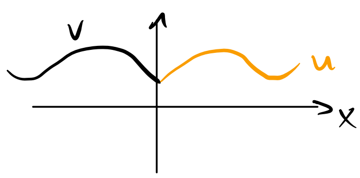
(ii) $\|Eu\|_{L^p(\mathbb{R})} \le C \|u\|_{L^p(\mathbb{R})}$;

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R})} \le C \|u\|_{W^{1,p}(\mathbb{R})}$.

Proof If $I=\mathbb{R}$: nothing to do. Case 1: I half-infinite. Case 2: I bounded.

Case 1. $I=\mathbb{R}_+=(0,\infty)$. For $u \in W^{1,p}(\mathbb{R}_+)$, set

$$v(x)=\begin{cases} u(x), & x>0 \\ u(-x), & x\le 0 \end{cases}, \qquad w(x)=\begin{cases} u'(x), & x>0 \\ -u'(-x), & x<0. \end{cases}$$



(v is continuous at $x=0$.) Then

$$\|v\|_{L^p(\mathbb{R})} \le C \|u\|_{L^p(\mathbb{R}_+)},$$
$$\|w\|_{L^p(\mathbb{R})} \le C \|u'\|_{L^p(\mathbb{R}_+)}.$$

We claim that w is the weak derivative of v. Indeed, by Theorem (T.18),

$$v(x)=\begin{cases} x>0: & u(x)=u(0)+\int_0^x u'(t)\,dt = v(0)+\int_0^x w(t)\,dt \\ x<0: & u(-x)=u(0)+\int_0^{-x} u'(t)\,dt = u(0)+\int_0^x -u'(-t)\,dt \\ & = v(0)+\int_0^x w(t)\,dt. \end{cases}$$

So $v'=w$. (See also the proof of Theorem (T.18)(ii).)

We may thus set $Eu := v$.

<u>Case 2</u>: bounded I, without loss of generality $I = (0,1)$.

Fix $\chi \in C^\infty(\mathbb{R})$ with $\chi = 1$ on $(-\infty, \frac{1}{3}]$,
$\chi = 0$ on $[\frac{2}{3}, \infty)$.

Given $u \in W^{1,p}(I)$, $u_L := \chi u \in L^p(I)$
$u_R := (1-\chi)u \in L^p(I)$.

- <u>Claim</u>: $u_L \in W^{1,p}(I)$ and $u_L' = \chi u' + \chi' u$

<u>Indeed</u>, for $\varphi \in C_c^\infty(I)$,

$$\begin{aligned}
&\int_I u_L \varphi' + (\chi u' + \chi' u)\varphi \, dx \\
&= \int_I u\chi\varphi' + u'\chi\varphi + u\chi'\varphi \, dx \\
&= \int_I u\,(\chi\varphi)' + u'\chi\varphi \, dx = 0 \quad \text{since } \chi\varphi \in C_c^\infty(I).
\end{aligned}$$

- Thus, $u_L \in W^{1,p}((0,\infty))$ (via extension by 0 on $[1,\infty)$), and Case 1 produces $v_L \in W^{1,p}(\mathbb{R})$ with

$$\|v_L\|_{L^p} \le C\|u_L\|_{L^p} \le C'\|u\|_{L^p},$$

$$\begin{aligned}
\|v_L'\|_{L^p} &\le \|\chi u'\|_{L^p} + \|\chi' u\|_{L^p} \le C\left(\|u'\|_{L^p} + \|u\|_{L^p}\right) \\
&= C\|u\|_{W^{1,p}(I)}.
\end{aligned}$$

- Arguing similarly for u_R gives $v_R \in W^{1,p}(\mathbb{R})$, $v_R|_{(-\infty,1)} = u_R$.
- Set $E(u) = v_L + v_R$. $\square$

Corollary (C.4) If $u \in W^{1,p}(I)$, $1 \leq p < \infty$, $\exists \{u_k\} \in C_c^\infty(\mathbb{R})$ s.t. $u_k|_I \longrightarrow u$ in $W^{1,p}(I)$.

Proof Apply Corollary (C.3) to $Eu \in W^{1,p}(\mathbb{R})$. $\square$

We can now generalize Theorem (T.19) as follows:

Theorem (T.19') Let $1 < p \leq \infty$. Then the following are equivalent:

(i) $u \in W^{1,p}(I)$.

(ii) $\exists\, C > 0$ s.t. $\forall\, I' \Subset I$, $0 < h < \operatorname{dist}(I', \partial I)$,

$$\| \tau_h u - u \|_{L^p(I')} \leq C|h|.$$

(iii) If $p < \infty$: there exists a sequence $\{u_k\} \subset C_c^\infty(\mathbb{R})$ s.t. $u_k|_I \longrightarrow u$ in $L^p(I)$, and $u_k'|_I$ converges in $L^p(I)$ to some limit $v \in L^p(I)$ which is in fact the weak derivative of u. $\square$

Proof Exercise.

Theorem (T.21) (Sobolev embedding.)

Let $1 \leq p \leq \infty$. Set $\alpha = 1 - \frac{1}{p}$. Then $W^{1,p}(I) \subset C^{0,\alpha}(\bar{I})$, where

$$C^{0,\alpha}(\bar{I}) = \left\{ u \in C^0(\bar{I}) : [u]_{C^{0,\alpha}(\bar{I})} = \sup_{\substack{x,y \in \bar{I} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

In particular, $W^{1,p}(I) \subset L^\infty(I)$.

Proof For $x, y \in I$ we have

$$u(x) = u(y) + \int_y^x u'(t)\, dt. \quad (*)$$

- Fix $I' \Subset I$ with $|I'| \leq 1$ and average ⊛ over $y \in I'$; then for $x \in I'$

$$|u(x)| \leq \frac{1}{|I'|}\int_{I'} |u(y)|\,dy + \frac{1}{|I'|}\int_{I'} \int_x^y |u'(t)|\,dt\,dy$$

$$\leq \|u\|_{L^1(I')} + \|u'\|_{L^1(I')}$$

$$\leq C\left(\|u\|_{L^p(I')} + \|u'\|_{L^p(I')}\right)$$

$$\leq C\,\|u\|_{W^{1,p}(I)}\,,$$ where C is independent of x.

- To control $[u]_{C^{0,\alpha}(I)}$, we estimate for $x, y \in I$

$$|u(x) - u(y)| \leq \int_x^y |u'(t)|\,dt$$

$$\leq \begin{cases} p=1: & \leq \|u'\|_{L^1(I)} \leq \|u\|_{W^{1,1}(I)} \\ p=\infty: & \leq |y-x|\,\|u'\|_{L^\infty(I)} \leq |y-x|\,\|u\|_{W^{1,\infty}(I)} \\ 1<p<\infty: & \leq \left(\int_x^y |u'(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_x^y 1^q\,dt\right)^{\frac{1}{q}} \quad \left(q = \frac{p}{p-1}\right) \end{cases}$$

$$\leq \|u\|_{W^{1,p}(I)} \underbrace{|y-x|^{\frac{p-1}{p}}}_{=|y-x|^\alpha}.$$

□

Corollary (C.5) If $I \subset \mathbb{R}$ is unbounded, $1 \leq p < \infty$, and $u \in W^{1,p}(I)$, then $|u(x)| \longrightarrow 0$ as $|x| \to \infty$, $x \in I$.

Proof Let $\varepsilon > 0$. Pick $v \in C_c^\infty(\mathbb{R})$ s.t. $\|u - v|_I\|_{W^{1,p}(I)} < \varepsilon$.

Then $|u(x)| \leq |v(x)| + \|u - v\|_{L^\infty}$

$$\leq 0 + C\varepsilon$$

for all sufficiently large $x \in I$.

□

Corollary (C.6) (Product rule.) Let $I \subset \mathbb{R}$, $1 \le p \le \infty$, $u, v \in W^{1,p}(I)$.

Then $uv \in W^{1,p}(I)$, $(uv)' = u'v + uv'$, and

$$\|uv\|_{W^{1,p}(I)} \le C\left(\|u\|_{L^\infty(I)} \|v\|_{W^{1,p}(I)} + \|u\|_{W^{1,p}(I)} \|v\|_{L^\infty(I)}\right)$$

$$\le C \|u\|_{W^{1,p}(I)} \|v\|_{W^{1,p}(I)}.$$

Proof $uv \in L^p(I)$ since $u, v \in L^\infty(I)$ by Theorem (T.21).

Once we show that $u'v + uv'$ is a weak derivative of uv, the estimate follows.

Case 1: $p < \infty$. Pick $u_k, v_k \in C_c^\infty(\mathbb{R})$ s.t. $u_k|_I \to u$ and $v_k|_I \to v$ in $W^{1,p}(I)$ (and thus also in $L^\infty(I)$). Then

$$u_k v_k \longrightarrow uv \text{ in } L^p(I),$$

$$(u_k v_k)' = u_k' v_k + u_k v_k' \longrightarrow u'v + uv' \text{ in } L^p(I).$$

Since $W^{1,p}(I)$ is complete, we are done.

Case 2: $p = \infty$. Let $\varphi \in C_c^\infty(I)$, then for $\operatorname{supp} \varphi \subset I' \Subset I$,

$$-\int_I uv\varphi'\,dx = -\int_{I'} uv\varphi'\,dx$$

$u|_{I'}, v|_{I'} \in W^{1,1}(I')$

$$= \int_{I'} (u'v + uv')\varphi\,dx.$$

$\Rightarrow$ uv has weak derivative $u'v + uv' \in L^\infty(I)$. $\square$