In $n=1$ dimension, elements of $W^{1,p}(\mathbb{R})$, $1 \le p \le \infty$, are automatically continuous by Theorem (T.18). This fails in higher dimensions.

Example (E.18) Let $\Omega = B_1(0) \subset \mathbb{R}^n$, $n \ge 2$, and consider
$$u(x) = |x|^{-\alpha} \qquad (x \in \Omega \setminus \{0\}) \quad \text{where } \alpha > 0.$$

(i) Let $p \ge 1$. We have $\int_\Omega |u|^p \, dx = \mathrm{vol}(S^{n-1}) \int_0^1 r^{-\alpha p} \, r^{n-1} dr < \infty$ iff $-\alpha p + n - 1 > -1 \iff \alpha < n/p$ $\circledast$

(ii) For which $\alpha > 0$, $p \ge 1$, $\alpha < \frac{n}{p}$ do we also have $u \in W^{1,p}(\Omega)$?

Well, for $x \ne 0$, $\frac{\partial u}{\partial x_j} = -\frac{\alpha x_j}{|x|^{\alpha+2}} \Rightarrow |\nabla u| = \frac{\alpha}{|x|^{\alpha+1}}$.

- For this to lie in $L^p(\Omega \setminus \{0\})$, need $\alpha + 1 < n/p$ (cf. $\circledast$) $\iff \alpha < \frac{n}{p} - 1$. We assume now that this holds.
- We further need to check whether the L^p-function $-\frac{\alpha x_j}{|x|^{\alpha+2}}$ is indeed the weak derivative of u; but if $\varphi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \underbrace{u}_{\in L^p \subset L^1} \frac{\partial \varphi}{\partial x_j} dx = \lim_{\varepsilon \searrow 0} \int_{\Omega \setminus B_\varepsilon(0)} u \frac{\partial \varphi}{\partial x_j} dx$$

(j th component of outward pointing normal $\nu = (\nu_1, \dots, \nu_n)$)

$$= -\lim_{\varepsilon \searrow 0} \int_{\Omega \setminus B_\varepsilon(0)} \underbrace{\frac{\partial u}{\partial x_j} \varphi}_{\in L^p \subset L^1} dx + \int_{\partial B_\varepsilon(0)} u \varphi \, \nu_j \, d\sigma$$

$$= -\int_\Omega \frac{\partial u}{\partial x_j} \varphi \, dx + R(\varepsilon),$$

where $|R(\varepsilon)| \le \|\varphi\|_{L^\infty} \mathrm{vol}(S^{n-1}) \, \varepsilon^{n-1} \, \varepsilon^{-\alpha} \xrightarrow{\varepsilon \searrow 0} 0$ since $\alpha < n-1$.

(iii) Summary: Let $\alpha > 0$. Then $|x|^{-\alpha} \in W^{1,p}(B_1(0))$ iff $\alpha < \frac{n}{p} - 1$.

(iv) In particular, $|x|^{-(n-1)} \notin W^{1,1}(B_1(0))$.

Concretely, $\frac{x}{|x|^n}$ satisfies $\operatorname{div}_{\mathbb{R}^n}\left(\frac{x}{|x|^n}\right) = c_n \delta(x)$ in the sense of distributions for some $c_n \neq 0$.

Example (E.19) Let $\{x_k\}_{k\in\mathbb{N}} \subset B_1(0)$ be countable and dense. Let $\alpha > 0$, $p \geq 1$, $\alpha < \frac{n}{p} - 1$. Then

$$u(x) := \sum_{k=1}^{\infty} 2^{-k} |x - x_k|^{-\alpha} \quad \text{converges in } W^{1,p}(B_1(0))$$

(so has some regularity), but u is unbounded on every non-empty open subset of $B_1(0)$...

Plan for the next lectures:

(I) density of spaces of smooth functions

(II) extensions of $W^{k,p}(\Omega)$-functions to $\mathbb{R}^n$

(III) boundary values of $W^{k,p}$-functions

(IV) existence of continuous or differentiable representatives

(I) Approximation by smooth functions. (Idea: perform operations on smooth functions, extend by continuity and density to $W^{k,p}$)

(Cf. Lemma (L.10), Corollary (C.3))

Lemma (L.13) Let $\varphi \in C_c^\infty(B_1(0))$, $\int \varphi \, dx = 1$, $\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)$.

Let $u \in W^{k,p}(\mathbb{R}^n)$, $1 \leq p < \infty$. Then $\varphi_\varepsilon * u \in W^{k,p}(\mathbb{R}^n)$ (and indeed $\partial^\alpha(\varphi_\varepsilon * u) = \varphi_\varepsilon * \partial^\alpha u$), and $\varphi_\varepsilon * u \xrightarrow{\varepsilon \searrow 0} u$ in $W^{k,p}(\mathbb{R}^n)$.

In the remainder of this section, we use the functions φ_ε from this Lemma.

Theorem (T.22) (Myers–Serrin, 1964) Let $\Omega \subset \mathbb{R}^n$, $1 \le p < \infty$.
Then $C^\infty(\Omega) \cap W^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ is dense.

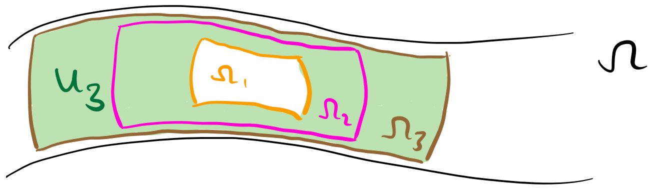
Proof Let $u \in W^{k,p}(\Omega)$.

Step 1: partition of unity.

Let $\Omega_i = \{ x \in \Omega : |x| < i, \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{i} \}$.

Set $U_i = \Omega_i \setminus \overline{\Omega_{i-2}}$ $\quad (\Omega_0, \Omega_{-1} := \emptyset)$. Then $U_i \cap U_j = \emptyset$ unless $|i-j| \le 1$.

Let $\chi_i \in C_c^\infty(U_i)$ be s.t. $\sum_{i=1}^\infty \chi_i = 1$ on Ω.



Then $\chi_i u \in W^{k,p}(\Omega)$, $\mathrm{supp}(\chi_i u) \subset U_i$.

Step 2: mollification. Fix $\delta > 0$.

Pick $\varepsilon_i > 0$ s.t.
- $\mathrm{supp}(\varphi_{\varepsilon_i} * \chi_i u) \subset \Omega_{i+1} \setminus \Omega_{i-3}$
- $\| \chi_i u - \varphi_{\varepsilon_i} * (\chi_i u) \|_{W^{k,p}} < \frac{\delta}{2^i}$.

Set $v = \sum_{i=1}^\infty \varphi_{\varepsilon_i} * (\chi_i u)$. At each $x \in \Omega$, only 4 summands are possibly nonzero $\Rightarrow v \in C^\infty(\Omega)$.

Moreover,
$$\|v - u\|_{W^{k,p}(\Omega)} \le \sum_{i=1}^\infty \| \varphi_{\varepsilon_i} * (\chi_i u) - \chi_i u \|_{W^{k,p}} < \delta \sum_{i=1}^\infty 2^{-i} = \delta.$$

(A similar estimate shows that the series defining v converges in $W^{k,p}(\Omega)$.) $\square$

Remark (R.18) In general, $C^\infty(\bar\Omega) = \{u|_{\bar\Omega} : u \in C^\infty(\mathbb{R}^n)\}$ is **not** dense in $W^{k,p}(\Omega)$. Simple example: $\Omega = (-1,0) \cup (0,1)$,
$u(x) = 1_{(0,1)}(x)$;

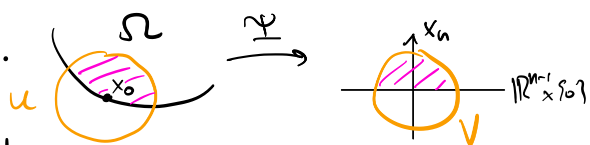
if $v \in C^\infty(\bar\Omega) = C^\infty([-1,1])$, we claim that $\|u - v\|_{W^{1,1}(\Omega)}$ cannot be small. Indeed, assuming the contrary, we get $u_k \in C^\infty(\bar\Omega)$, $u_k \to u$ in $W^{1,1}(\Omega)$, so $u_k|_{(-1,0)} \to u|_{(-1,0)} = 0$ in $W^{1,1}((-1,0)) \subset C^0([-1,0])$ and likewise $u_k|_{(0,1)} \to u|_{(0,1)} = 1$ uniformly. For large k, this requires $|u_k(0) - 0| < \frac{1}{3}$ and $|u_k(0) - 1| < \frac{1}{3}$, a contradiction.

Remark (R.18) suggests that for better density statements, we need to impose restrictions on Ω.

Definition (D.15) An open set $\Omega \subset \mathbb{R}^n$ is *of class C^k* $(k \in \mathbb{N}_0 \cup \{+\infty\})$ if $\forall x_0 \in \partial\Omega$ $\exists$ neighborhood $U \subset \mathbb{R}^n$ of x_0 and a map $\Psi \in C^k(U; \mathbb{R}^n)$ s.t. $\Psi : U \to \Psi(U) =: V \subset \mathbb{R}^n$ is a C^k-diffeomorphism (i.e. $\Psi^{-1} : V \to U$ is C^k) so that $\Psi(x_0) = 0$,

$$\Psi(U \cap \Omega) = V \cap \mathbb{R}^n_+, \qquad \mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\},$$

$$\Psi(U \cap \partial\Omega) = V \cap (\mathbb{R}^{n-1} \times \{0\}).$$



Theorem (T.23) Let $\Omega \subset \mathbb{R}^n$ be a C^1 domain with compact boundary. Then $C^\infty(\bar\Omega) \cap W^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ is dense for all $k \in \mathbb{N}_0$, $1 \le p < \infty$.
(If Ω is bounded: $C^\infty(\bar\Omega) \subset W^{k,p}(\Omega)$ is dense.)

Proof Let $u \in W^{k,p}(\Omega)$.

Case 1. If $d := \operatorname{dist}(\operatorname{supp} u, \partial\Omega) > 0$, then for $\varepsilon < d$, $\varphi_\varepsilon * u \in C^\infty(\Omega)$ vanishes near $\partial\Omega$, so $\varphi_\varepsilon * u \in C^\infty(\bar\Omega)$, and

$\varphi_\varepsilon * u \longrightarrow u$ in $W^{k,p}(\Omega)$.

Case 2. Let $x_0 \in \partial\Omega$. Then WLOG there exist $r>0$ and a C^1 function $\psi : B_r^{\mathbb{R}^{n-1}}(0) \to \mathbb{R}$ s.t. $\psi(0)=0$, $\nabla\psi(0)=0$,

$$\Omega \cap \big(\underbrace{B_r^{\mathbb{R}^{n-1}}(0)\times(-r,r)}_{=:\,\mathcal{U}}\big) = \{ (x',x_n) : |x'|<r,\ \psi(x')<x_n<r \}.$$

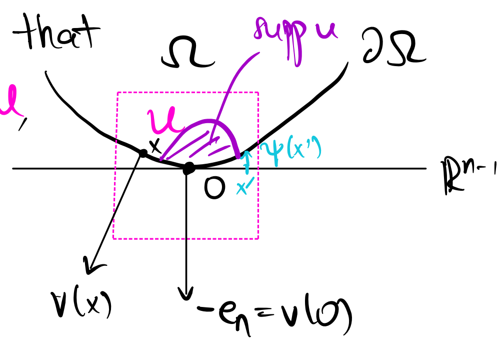
(This holds if we rotate and translate Ω; exercise: such transformations map $W^{k,p}$-functions into $W^{k,p}$-functions.)

By shrinking $r>0$, we may assume that

$(*)$ $\nu(x)\cdot\nu(0) > \frac{1}{2}$ for $x \in \partial\Omega\cap\mathcal{U}$,

where $\nu(x)$ = outward pointing unit normal,

$\nu(0) = -e_n$.



Suppose $\operatorname{supp} u \Subset \mathcal{U}$. Let $d = \operatorname{dist}(\operatorname{supp} u, \partial\mathcal{U})$.

- For (small) $\varepsilon>0$, consider the function $u_\varepsilon \in W^{k,p}(\Omega\cap\mathcal{U})$,
  $u_\varepsilon(x) := u(x+\varepsilon e_n)$ ("u shifted down by εe_n").

  Then $u_\varepsilon \xrightarrow{\varepsilon\downarrow 0} u$ in $W^{k,p}(\Omega\cap\mathcal{U})$ (exercise).
- If $x\in\mathcal{U}\cap\Omega$ and $x+\varepsilon e_n \in \operatorname{supp} u$, then
  $\operatorname{dist}(x+\varepsilon e_n, \partial\Omega\cap\mathcal{U}) \geq \varepsilon/2$ by $(*)$.

  So for $\delta < \varepsilon/2$, we can define a $C^\infty(\overline{\Omega})$-function $u_{\varepsilon,\delta}$ via
  $$u_{\varepsilon,\delta}(x) = \begin{cases} (\varphi_\delta * u_\varepsilon)(x), & x\in\overline{\Omega}\cap\mathcal{U},\\ 0, & x\notin\mathcal{U},\end{cases}$$
  using that for $x\in\Omega\cap\mathcal{U}$, the evaluation of $(\varphi_\delta * u_\varepsilon)(x)$ only uses values of u at points with distance $\geq \frac{\varepsilon}{2}-\delta>0$ from $\partial\Omega\cap\mathcal{U}$. Then $u_{\varepsilon,\delta}\xrightarrow{\delta\downarrow 0} u_\varepsilon$ in $W^{k,p}(\Omega\cap\mathcal{U})$.
- In combination, we have approximated u by $C^\infty(\overline{\Omega})$-functions

with support in $\mathcal{U}$.

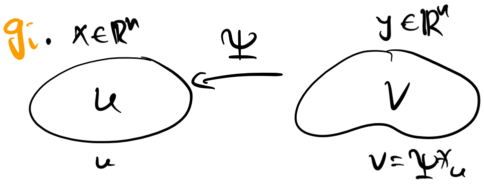
<u>Case 3.</u> General u. Use a partition of unity (applying Case 1 once, and Case 2 a finite number of times at a finite number of points in $\partial\Omega$ s.t. their neighborhoods $\mathcal{U}$ as above cover $\partial\Omega$—which is compact). $\square$

(II) Behavior of $W^{k,p}$-functions under coordinate changes; extensions

(We work on general domains.) In light of Definition (D.16), this is clearly useful.

<u>Theorem (T.24)</u> Let $U, V \subset \mathbb{R}^n$ be open, $\Psi : V \to \mathcal{U}$ a C^1 diffeomorphism, and $|d\Psi|, |d(\Psi^{-1})| \leq C < \infty$ on V and $\mathcal{U}$, respectively. For $u \in W^{1,p}(\mathcal{U})$, $1 \leq p \leq \infty$, we then have $v := u \circ \Psi \in W^{1,p}(V)$ and

$$\frac{\partial v}{\partial y_i} = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \circ \Psi\right) \frac{\partial \Psi_j}{\partial y_i} =: g_i .$$



<u>Proof</u> · <u>$1 \leq p < \infty$.</u> · We first show that $v, g_i \in L^p(\Omega)$, $i = 1, \dots, n$. But

$$\int_V |v(y)|^p dy = \int_V |u(\Psi(y))|^p dy = \int_{\mathcal{U}} |u(x)|^p \, |\det d(\Psi^{-1})(x)|^p dx$$
$$\leq C^p \|u\|_{L^p(\mathcal{U})},$$

similarly for g_i.

· Let now $u_k \in C^1(\mathcal{U}) \cap W^{1,p}(\mathcal{U})$, $u_k \xrightarrow{k\to\infty} u$ in $W^{1,p}(\mathcal{U})$ (using Theorem (T.23)). Then $v_k := u_k \circ \Psi \in C^1(V) \cap W^{1,p}(V)$ satisfies

$$\|v_k - v\|_{L^p(V)} + \|\nabla v_k - g\|_{L^p(V)} \leq C \|u_k - u\|_{W^{1,p}(V)} \xrightarrow{k\to\infty} 0 .$$

Therefore, $v \in W^{1,p}(V)$ and $\nabla v = g$.

· $p=\infty$. We only need to show $v\in W^{1,\infty}(V)$, since the formula for ∇v follows from what we have already proved in view of $W^{1,\infty}(U)\subset W^{1,1}_{loc}(U)$. Now, for $x\in V$ and small $h>0$,

$$\frac{|v(x+h)-v(x)|}{h}=\underbrace{\frac{|u(\Psi(x+h))-u(\Psi(x))|}{|\Psi(x+h)-\Psi(x)|}}_{\le \|\nabla u\|_{L^\infty}}\cdot\underbrace{\frac{|\Psi(x+h)-\Psi(x)|}{h}}_{\le C}$$
$$\le C\|\nabla u\|_{L^\infty}.$$

This implies $\nabla v\in L^\infty(V)$ by the generalization of Theorem (T.19') to general dimensions. $\square$

Much like lack of smoothness makes handling $W^{k,p}$-functions directly occasionally awkward, the presence of a boundary $\partial\Omega\neq\emptyset$ can similarly cause trouble. Thus:

Theorem (T.25) Suppose $\Omega\subset\mathbb{R}^n$ is a C^1 domain with compact boundary. Then there exists a linear extension operator E s.t.

(i) $(Eu)|_\Omega=u$.

(ii) $E: W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^n)$ is continuous.

As a preparation, we first consider a special case:

Lemma (L.14) Let $Q=\{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}: |x'|<1, |x_n|<1\}$,
$Q_\pm=Q\cap\{\pm x_n>0\}$.

Let $1\le p\le\infty$, $u\in W^{1,p}(Q_+)$. Set

$$u^*(x',x_n)=\begin{cases}u(x',x_n) & : x_n>0\\ u(x',-x_n) & : x_n<0.\end{cases}$$

Then $u^*\in W^{1,p}(\Omega)$, and $\|u^*\|_{W^{1,p}(Q)}\le C\|u\|_{W^{1,p}(Q_+)}$.

Proof. Set $g_i = \left(\frac{\partial u}{\partial x_i}\right)^*$, $1 \le i \le n-1$,

$$g_n(x', x_n) = \begin{cases} \frac{\partial u}{\partial x_n}(x', x_n) : & x_n > 0 \\ -\frac{\partial u}{\partial x_n}(x', -x_n) : & x_n < 0 \end{cases}$$

Clearly, $\|u^*\|^p_{L^p(Q)} = 2\|u\|_{L^p(Q_+)}$ $(p<\infty)$,

$$\|u^*\|_{L^\infty(Q)} = \|u\|_{L^\infty(Q_+)},$$

and analogous statements for g_i $(1 \le i \le n)$.

- It remains to prove that $\frac{\partial u^*}{\partial x_i} = g_i$.

Well, let $\varphi \in C_c^\infty(Q)$. Then for $1 \le i \le n-1$,

$$\int_Q u^* \frac{\partial \varphi}{\partial x_i}\, dx = \int_{Q_+} u \frac{\partial \psi}{\partial x_i}\, dx \quad \text{where } \psi(x', x_n) = \varphi(x', x_n) + \varphi(x', -x_n).$$

- If $\varphi(x', x_n) = 0$ for $|x_n| < \delta$, then $\psi \in C_c^\infty(Q_+)$, and we get

$$\int_Q u^* \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{Q_+} \frac{\partial u}{\partial x_i}\psi\, dx = -\int_Q g_i \varphi\, dx.$$

- For general φ, we can replace φ in this calculation by $\eta(kx_n)\varphi(x', x_n)$ where $\eta \in C^\infty(\mathbb{R})$ is 1 on $\mathbb{R} \setminus (-2,2)$,
0 on $[-1,1]$,
in the limit $k \to \infty$ (using the Dominated Convergence theorem); note that $\frac{\partial}{\partial x_i}(\eta\varphi) = \eta \frac{\partial \varphi}{\partial x_i}$.

- Lastly, for $i=n$,

$$\int_Q u^* \frac{\partial \varphi}{\partial x_n}\, dx = \int_{Q_+} u(x', x_n)\left(\frac{\partial \varphi}{\partial x_n}(x', x_n) + \frac{\partial \varphi}{\partial x_n}(x', -x_n)\right) dx$$

$$= \int_{Q_+} u \frac{\partial \rho}{\partial x_n}\, dx, \qquad \rho(x', x_n) = \varphi(x', x_n) - \varphi(x', -x_n).$$

Note that $\rho(x', 0) = 0$. If $\rho(x', x_n) = 0$ for $|x_n| < \delta$, then

$\rho \in C_c^\infty(Q_+)$, so

$$\int_Q u^* \frac{\partial \varphi}{\partial x_n}\,dx = -\int_{Q_+} \frac{\partial u}{\partial x_n}\rho\,dx = -\int_Q g_n \varphi\,dx.$$

In general,

$$\int_{Q_+} u \frac{\partial \rho}{\partial x_n}\,dx = \lim_{k\to\infty}\left(\int_{Q_+} u \frac{\partial(\eta(kx_n)\rho)}{\partial x_n}\,dx - \int_{Q_+} u \frac{\partial(\eta(kx_n))}{x_n}\rho\,dx\right)$$

$$= \lim_{k\to\infty}\Bigg(\underbrace{-\int_{Q_+} \frac{\partial u}{\partial x_n}\eta(kx_n)\rho\,dx}_{\xrightarrow{k\to\infty} -\int_{Q_+}\frac{\partial u}{\partial x_n}\rho\,dx \;=\; -\int_Q g_n\varphi\,dx} - \underbrace{\int_{Q_+} u\,k\eta'(kx_n)\rho\,dx}_{=:\,\mathrm{Err}(k)}\Bigg),$$

where $|\mathrm{Err}(k)| \le \int_{Q_+} |u|\,\underbrace{|kx_n\,\eta'(kx_n)|}_{\substack{\le \sup_{t\in\mathbb{R}}|t\eta'(t)|<\infty,\\ =0 \text{ unless } |x_n|\le \frac{2}{k}}}\,\underbrace{\left|\frac{\rho(x',x_n)}{x_n}\right|}_{\substack{=\left|\int_0^1 \frac{\partial\rho}{\partial x_n}(x',tx_n)\,dt\right|,\\ \text{uniformly bounded on } Q_+}}\,dx$

$$\le C\int_{|x_n|\le\frac{2}{k}} |u(x',x_n)|\,dx \xrightarrow{k\to\infty} 0. \qquad \square$$

Proof of Theorem (T.25). Since $\partial\Omega$ is compact, $\exists$ open sets $U_1,\dots,U_N\subset\mathbb{R}^n$ with $\partial\Omega\subset\bigcup_{k=1}^N U_k$ and C^1 diffeomorphisms $\Psi_k: Q\longrightarrow U_k$ with

$$\Psi_k(Q_+) = U_k\cap\Omega,$$
$$\Psi_k(Q_0) = U_k\cap\partial\Omega,$$

where

$$Q = \{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} : |x'|<1,\ |x_n|<1\},$$
$$Q_+ = Q\cap\{x_n>0\},$$
$$Q_0 = Q\cap\{x_n=0\}.$$

Pick $U_0 \subset \mathbb{R}^n$ open, $\operatorname{dist}(U_0, \partial\Omega) > 0$, s.t. $\Omega \subset \bigcup_{k=0}^{N} \Omega_k$.

Let $\{\varphi_k\}_{k=0,\dots,N}$ be a C^∞ partition of unity subordinate to $\{\Omega_k\}$.

We have $u = \varphi_0 u + \sum_{k=1}^{N} \varphi_k u$.

- <u>Claim:</u> $v_0(x) = \begin{cases} \varphi_0 u(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$ defines a $W^{1,p}(\mathbb{R}^n)$ extension of $\varphi_0 u$.

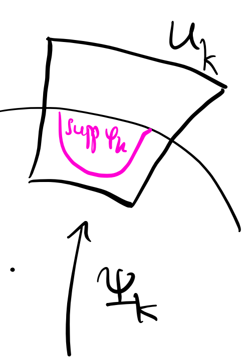
<u>Indeed</u>, we claim that $\frac{\partial v_0}{\partial x_j} = \varphi_0 \frac{\partial u}{\partial x_j} + \frac{\partial \varphi_0}{\partial x_j} u$ (the R.H.S lying in $L^p(\mathbb{R}^n)$ indeed).

To see this, we compute for $\psi \in C_c^\infty(\mathbb{R}^n)$

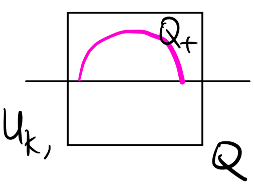
$$\begin{aligned}
\int_{\mathbb{R}^n} v_0 \frac{\partial \psi}{\partial x_j}\, dx &= \int_{\mathbb{R}^n} \varphi_0 u \frac{\partial \psi}{\partial x_j} \\
&= \int_\Omega u \frac{\partial(\overbrace{\varphi_0 \psi}^{\in C_c^\infty(\Omega)})}{\partial x_j} - u\psi \frac{\partial \varphi_0}{\partial x_j}\, dx \\
&= -\int_\Omega \left(\frac{\partial u}{\partial x_j} \varphi_0 + u \frac{\partial \varphi_0}{\partial x_j}\right) \psi\, dx \\
&= -\int_{\mathbb{R}^n} \left(\frac{\partial u}{\partial x_j} \varphi_0 + u \frac{\partial \varphi_0}{\partial x_j}\right) \psi\, dx.
\end{aligned}$$

- To extend $\varphi_k u$, $1 \le k \le N$, we set

$$v_k(x) = \begin{cases} \left((\varphi_k u \circ \Psi_k)^* \circ \Psi_k^{-1}\right)(x), & x \in U_k \\ 0, & x \notin U_k. \end{cases}$$



Since $\varphi_k u \circ \Psi_k \in W^{1,p}(Q_+)$ by Theorem (T.24), Lemma (L.14) shows that $v_k|_{U_k} \in W^{1,p}(U_k)$; but since $\operatorname{supp} v_k \Subset U_k$, the extension of $v_k|_{U_k}$ by 0 (which is precisely v_k) lies in



$W^{1,p}(\mathbb{R}^n)$

- Altogether, $Eu := \sum_{k=0}^{N} v_k \in W^{1,p}(\mathbb{R}^n)$ defines the desired extension. $\square$

(III) Boundary values

Unlike in the case $n=1$, for $n \geq 2$ elements of $H^1_0(\Omega)$ need not be continuous (or, more precisely, have continuous representatives). Thus, the meaning of "$u|_{\partial\Omega} = 0$" is not immediately clear. The following result clarifies the situation:

Theorem (T.26) (Trace theorem.) Let $\Omega \subset \mathbb{R}^n$ be open, of class C^1, with compact boundary $\partial\Omega$. Then the map

$$C^\infty(\bar{\Omega}) \ni u \longmapsto u|_{\partial\Omega} \in C^0(\partial\Omega)$$

extends by density and continuity to a bounded linear map

$$W^{1,p}(\Omega) \ni u \longmapsto u|_{\partial\Omega} \in L^p(\partial\Omega) \qquad (1 \leq p < \infty).$$

The map $W^{1,\infty}(\Omega) \ni u \longmapsto u|_{\partial\Omega} \in L^\infty(\partial\Omega)$ (where u is the unique locally Lipschitz representative of u) is bounded as well.

Remark (R.19) $\partial\Omega$ is a compact C^1 submanifold of $\mathbb{R}^n$: the local charts $\Psi: Q \to U \subset \mathbb{R}^n$, $\Psi(Q_+) = U \cap \Omega$, $\Psi(Q_0) = U \cap \partial\Omega$, restrict to charts $\Psi|_{Q_0}: Q_0 \to U \cap \partial\Omega$ of $\partial\Omega$ giving $\partial\Omega$ the claimed structure. We can then define $L^p(\partial\Omega)$ using $L^p(Q_0) = L^p(B_1(0))$ and a partition of unity. Different choices of charts and $\underset{\mathbb{R}^{n-1}}{\cap}$

partitions of unity give the same space and equivalent norms.

As usual, the key step in the proof of Theorem (T.26) is the analogous local result on the half Q_+ of the cube Q:

Lemma (L.15) For $u \in W^{1,p}(Q_+)$, $u|_{Q_0} \in L^p(Q_0)$ is well-defined (in the sense explained in Theorem (T.26)).

Proof Step 1: estimate for $u \in C^\infty(Q_+) \cap W^{1,p}(Q_+)$, $1 \le p < \infty$.

Well, $u(x',0) = u(x',x_n) - \int_0^{x_n} \frac{\partial u}{\partial x_n}(x',t)\,dt$;

integrating this over $x_n \in (0,1)$ gives

$$|u(x',0)| \le \int_0^1 |u(x',x_n)|\,dx_n + \int_0^1 (1-x_n)\left|\frac{\partial u}{\partial x_n}(x',x_n)\right| dx_n .$$

$$\Rightarrow \|u(\cdot,0)\|_{L^p(Q_0)} \le \int_0^1 \|u(\cdot,x_n)\|_{L^p(Q_0)}\,dx_n + \int_0^1 \|\nabla u(\cdot,x_n)\|_{L^p(Q_0)}\,dx_n$$

$$\le \|u\|_{L^p(Q_+)} + \|\nabla u\|_{L^p(Q_+)}$$

$$= \|u\|_{W^{1,p}(Q_+)} .$$

Step 2: density argument. Given $u \in W^{1,p}(Q_+)$, select $u_k \in C^\infty \cap W^{1,p}(Q_+)$ s.t. $u_k \to u$ in $W^{1,p}(Q_+)$; then $\{u_k|_{Q_0}\}$ is a Cauchy sequence in $L^p(Q_0)$, and $u|_{Q_0} := \lim u_k|_{Q_0}$ is independent of the approximating sequence; and $\|u|_{Q_0}\|_{L^p} \le \|\nabla u\|_{L^p(Q_+)}$.

Step 3: $p=\infty$. $\|u|_{Q_0}\|_{L^\infty} = \lim_{p\to\infty} \|u|_{Q_0}\|_{L^p} \le \lim_{p\to\infty} \|u\|_{L^p(Q_+)} = \|u\|_{L^\infty(Q_+)}$.

$\square$

Proof of Theorem (T.26). Reduce to Lemma (L.15) using C^1 coordinate charts near points in $\partial\Omega$. (Details left as an exercise.) $\square$

Remark (R.20) For the weak solution $u \in H^1_0(\Omega)$ of $\Delta u = f \in L^2(\Omega)$ ($\Omega \Subset \mathbb{R}^n$ with C^1 boundary), we deduce that $u|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined — and equal to 0 a.e. since $v|_{\partial\Omega} = 0$ for $v \in H^1_0(\Omega)$ (since this holds for v in the dense subspace $C^\infty_c(\Omega)$).

Remark (R.21) The trace map $W^{1,p}(\Omega) \ni u \longmapsto u|_{\partial\Omega} \in L^p(\partial\Omega)$ is not surjective unless $p = \infty$. Rather, $W^{1,p}(\Omega) \twoheadrightarrow$ "$W^{1-\frac{1}{p},p}(\partial\Omega)$".
Using the Fourier transform description of $H^s(\mathbb{R}^n)$, one can rather easily show that $H^s(\mathbb{R}^n) \ni u \longmapsto u|_{\{x^n=0\}} \in H^{s-\frac{1}{2}}(\mathbb{R}^n)$ is well-defined when $s > \frac{1}{2}$, continuous, and surjective (try to prove this!).

Traces lead to the following very pleasant characterization of $W^{1,p}_0 \subset W^{1,p}$:

Theorem (T.27) If $\Omega \subset \mathbb{R}^n$ is of class C^1 and has compact boundary, then $W^{1,p}_0(\Omega) = \{ u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = 0 \in L^p(\partial\Omega) \}$.

Proof Exercise. $\square$

(IV). Embedding theorems

We finish our discussion of general properties of Sobolev spaces by relating them to other function spaces, specifically other L^p- and

Hölder spaces.

Recall: • Let $\alpha \in (0,1]$, $\Omega \subset \mathbb{R}^n$. Then

$$C^{0,\alpha}(\Omega) = \{u \in C^0(\overline{\Omega}) : [u]_{C^{0,\alpha}} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\},$$

$$C^{k,\alpha}(\Omega) = \{u \in C^k(\overline{\Omega}) : \partial^\beta u \in C^{0,\alpha}(\Omega) \text{ for all } \beta \in \mathbb{N}_0^n, |\beta| \le k\}.$$

These spaces are Banach spaces, with $\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{C^0} + [u]_{C^{0,\alpha}}$ and $\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{|\beta|\le k} \|\partial^\beta u\|_{C^{0,\alpha}(\Omega)}$.

• Suppose $\Omega \Subset \mathbb{R}^n$, $0 < \alpha < \beta \le 1$. Then the embedding

$$C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \text{ is compact.}$$

(Indeed, if $\{u_k\} \subset C^{0,\beta}(\Omega)$ is bounded, then WLOG $u_k \to u$ in $C^0(\overline{\Omega})$ by Arzelà-Ascoli, since $|u_k(x)-u_k(y)| \le C|x-y|^\beta$, $C := \sup_k [u_k]_{C^{0,\beta}}$. But then

$$\frac{|(u_k-u_\ell)(x)-(u_k-u_\ell)(y)|}{|x-y|^\alpha} = \left(\frac{|(u_k-u_\ell)(x)-(u_k-u_\ell)(y)|}{|x-y|^\beta}\right)^{\alpha/\beta} \cdot 2\|u_k-u_\ell\|_{C^0}^{1-\frac{\alpha}{\beta}}$$

$$\le C\|u_k-u_\ell\|^{\overbrace{1-\frac{\alpha}{\beta}}^{>0}} \xrightarrow{k,\ell\to\infty} 0.$$

Since $C^{0,\alpha}(\Omega)$ is complete, $u = \lim_{k\to\infty} u_k \in C^{0,\alpha}(\Omega)$. $\square$)

First, we generalize Theorem (T.21) to higher dimensions.

Theorem (T.28) (Morrey's inequality) Let $n < p \le \infty$, $\alpha = 1 - \frac{n}{p} \in (0,1]$.

Then $\exists\, C > 0$ s.t.

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

(That is, every $u \in W^{1,p}(\mathbb{R}^n)$ has a Hölder-α-continuous representative, and the estimate holds.)

Proof Step 1. We estimate the average variation of u near a point $x \in \mathbb{R}^n$ in terms of the L^p-norm of ∇u, as follows: for $t \geq 0$, $\omega \in S^{n-1}$, $|u(x+t\omega)-u(x)| \leq \int_0^t |\nabla u(x+s\omega)|\, ds$

$$\Rightarrow \int_{S^{n-1}} |u(x+t\omega)-u(x)|\, d\omega \leq \int_0^t \int_{S^{n-1}} |\nabla u(x+s\omega)|\, d\omega\, ds$$

$$\xrightarrow{\omega = \frac{\alpha}{s}} = \int_0^t \int_{\partial B_s(0)} \frac{|\nabla u(x+\alpha)|}{s^{n-1}}\, d\alpha\, ds$$

$$\xrightarrow{\alpha = y-x \ (\& \ |\alpha| = s)} = \int_{B_t(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy$$

$$\Rightarrow \int_{\partial B_t(x)} |u(y)-u(x)|\, d\sigma(y) = t^{n-1} \int_{S^{n-1}} |u(x+t\omega)-u(x)|\, d\omega$$

$$\leq t^{n-1} \int_{B_t(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy.$$

Integrate this in t from 0 to $r>0$ to get

$$\int_{B_r(x)} |u(y)-u(x)|\, dy \leq \frac{r^n}{n} \int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy.$$

$$\Rightarrow r^{-n} \int_{B_r(x)} |u(y)-u(x)|\, dy \leq C \left(\int_{B_r(x)} |\nabla u(y)|^p\, dy \right)^{\frac{1}{p}} \times \underbrace{\left(\int_{B_r(x)} |y-x|^{-(n-1)\frac{p}{p-1}}\, dy \right)^{\frac{p-1}{p}}}_{= \left(\int_0^r R^{-(n-1)\frac{p}{p-1}} R^{n-1}\, dR \right)^{\frac{p-1}{p}} = C\, r^{1-\frac{n}{p}} \ (\text{using } p>n)}$$

So we have proved:

$$r^{-n} \int_{B_r(x)} |u(y)-u(x)|\, dy \leq C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B_r(x))} \qquad (*)$$

$$(\leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{ when } r \leq 1).$$

Step 2: pointwise bound.

$$|u(x)| \le |u(x)-u(y)| + |u(y)|.$$

Average over $y \in B_1(x)$ to get

$$|u(x)| \le C\left(\int_{B_1(x)} |u(x)-u(y)|\,dy + \int_{B_1(x)} |u(y)|\,dy\right)$$

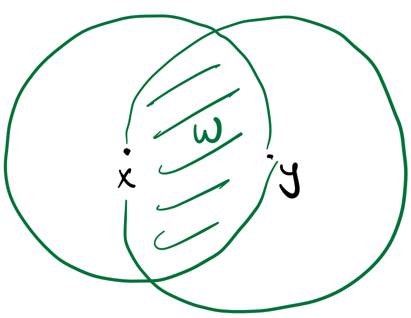
$$\overset{\circledast}{\le} C\left(\|\nabla u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right)$$

$$= C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

(More precisely, this is valid at Lebesgue points x of u, similarly to the proof of Theorem (T.18).)

This estimate implies that $\|u\|_{C^0} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$; therefore, every $u \in W^{1,p}(\mathbb{R}^n)$ has a continuous representative.

Step 3: Hölder bound. Let $x, y \in \mathbb{R}^n$, $r = |x-y|$.



Set $W = B_r(x) \cap B_r(y)$. Then

$$|u(x)-u(y)| \le \frac{1}{\mathrm{vol}(W)}\left(\int_W |u(x)-u(z)|\,dz + \int_W |u(z)-u(y)|\,dz\right), \quad (\#)$$

and

$$\frac{1}{\mathrm{vol}(W)}\int_W |u(x)-u(z)|\,dz \le \frac{C}{r^n}\int_{B_r(x)} |u(x)-u(z)|\,dz \overset{\circledast}{\le} C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

$\frac{\mathrm{vol}(W)}{r^n}$ = r-independent constant $\in (0,\infty)$

$(\#) \Rightarrow |u(x)-u(y)| \le C|x-y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$, as desired. $\square$

Corollary (C.7) Let $\Omega \subset \mathbb{R}^n$ be of class C^1 with compact boundary. Let $p > n$, $\alpha := 1 - \frac{n}{p}$. Then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$.

(More precisely, every $u \in W^{1,p}(\Omega)$ has a Hölder-α continuous representative. And $\exists\, C$ s.t. $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}$ $\forall u \in W^{1,p}(\Omega)$, where u on the left is the Hölder continuous representative.)

Proof Theorem (T.25) produces $u^* \in W^{1,p}(\mathbb{R}^n)$ with $u^*|_\Omega = u$ and $\|u^*\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$. Theorem (T.28) then implies the corollary. $\square$

Remark (R.22) In the case $p = n \geq 2$, we do not have $W^{1,n}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$, or even $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. An example on $B_{1/e}(0) \subset \mathbb{R}^n$ is $u(x) = \log\log\left(\frac{1}{|x|}\right)$ (with weak gradient satisfying $|\nabla u(x)| = \frac{1}{|x||\log|x||} \in L^n(B_{1/e}(0))$).

Second, when $1 \leq p < n$, we do not have $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$, but we do have "improved integrability":

Theorem (T.29) (Gagliardo–Nirenberg.) Let $1 \leq p < n$, and set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (so $p^* = \frac{np}{n-p} \in (p, \infty)$). Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$, and $\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}$. ⊛

Thus, one weak derivative is worth "$\frac{1}{n}$ degrees of $\frac{1}{\text{exponent of Lebesgue space}}$".

Remark (R.23) The exponent p^* is the unique one for which the estimate ⊛ has any chance of holding (even for $u \in C_c^\infty(\mathbb{R}^n)$).

Indeed, if $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$, then

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(\lambda x)|^{p^*} dx\right)^{1/p^*} = \left(\lambda^{-n} \int_{\mathbb{R}^n} |u(y)|^{p^*} dy\right)^{1/p^*}$$

$$= \lambda^{-\frac{n}{p^*}} \|u\|_{L^{p^*}(\mathbb{R}^n)},$$

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \|\lambda (\nabla u)(\lambda \cdot)\|_{L^p(\mathbb{R}^n)} = \lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

So ⊛ for u_λ, with $0 \neq u \in C_c^\infty(\mathbb{R}^n)$ fixed, forces

$$\lambda^{-\frac{n}{p^*}} \leq C' \lambda^{1-\frac{n}{p}} \quad \forall \lambda > 0 \iff 1 + \frac{n}{p^*} - \frac{n}{p} = 0$$

$$\iff p^* = \frac{np}{n-p}.$$

Proof of Theorem (T.29) It suffices to consider $u \in C_c^\infty(\mathbb{R}^n)$.

Case 1. $p = 1$. (So $p^* = \frac{n}{n-1}$.) Write $x = (x_1, \dots, x_n)$, let $1 \leq i \leq n$, $x_i' := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and estimate

$$|u(x)| = \left| \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \, dt \right|$$

$$\leq \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)| \, dt =: f_i(x_i').$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n |f_i(x_i')|^{\frac{1}{n-1}}$$

independent of x_1

$$\Rightarrow \int_{\mathbb{R}} |u(x_1, x_1')|^{\frac{n}{n-1}} dx_1 \leq f_1(x_1')^{\frac{1}{n-1}} \int_{\mathbb{R}} \underbrace{f_2(x_2')^{\frac{1}{n-1}} \cdots f_n(x_n')^{\frac{1}{n-1}}}_{(n-1) \text{ factors}} dx_1$$

Hölder inequality
$$\leq f_1(x_1')^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} f_2(x_2') \, dx_1\right)^{\frac{1}{n-1}} \cdots \left(\int_{\mathbb{R}} f_n(x_n') \, dx_1\right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 \right) dx_2$$

$$\le \left(\int_{\mathbb{R}} f_2(x_2') \, dx_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} f_1(x_1')^{\frac{1}{n-1}} \prod_{j=3}^{n} \left(\int_{\mathbb{R}} f_j(x_j') dx_1 \right)^{\frac{1}{n-1}} dx_2$$

independent of x_2

Hölder's inequality

$$\le \left(\int_{\mathbb{R}} f_2(x_2') \, dx_1 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} f_1(x_1') \, dx_2 \right)^{\frac{1}{n-1}} \prod_{j=3}^{n} \left(\int_{\mathbb{R}^2} f_j(x_j') \, dx_1 dx_2 \right)^{\frac{1}{n-1}}$$

$$\Rightarrow \cdots \Rightarrow \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \le \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j(x_j') \, dx_j' \right)^{\frac{1}{n-1}}$$

$$\le \left(\int_{\mathbb{R}^n} |\nabla u(x)| \, dx \right)^{\frac{n}{n-1}}$$

by definition of f_j. This means $\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|\nabla u\|_{L^1(\mathbb{R}^n)}$, i.e. the estimate (*) for $p=1$.

Case 2. $1<p<n$. For some $t>1$ to be determined, let $v(x) := |u(x)|^t$. Since $G(y) := |y|^t \in C^1(\mathbb{R})$, we have $v \in W^{1,s}(\mathbb{R}^n)$ for all $1 \le s \le \infty$ by the chain rule (exercise), with $\nabla v = t|u|^{t-1} \nabla u$; therefore,

$$\|v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \|u\|^t_{L^{\frac{tn}{n-1}}}$$

$$\le \|\nabla v\|_{L^1(\mathbb{R}^n)} = t \, \| |u|^{t-1} |\nabla u| \|_{L^1(\mathbb{R}^n)}$$

$$= t \int_{\mathbb{R}^n} |u|^{t-1} |\nabla u| \, dx$$

$$\le t \, \|\nabla u\|_{L^p(\mathbb{R}^n)} \|u\|^{t-1}_{L^{(t-1)q}(\mathbb{R}^n)}, \quad q = \frac{p}{p-1}.$$

Select t s.t. $\frac{tn}{n-1} = (t-1)q = (t-1)\frac{p}{p-1}$, so $t = \frac{p(n-1)}{n-p} > 1$.

Then $\frac{tn}{n-1} = \frac{np}{n-p} = p^*$, and the proof is complete. $\square$

Remark (R.24) While in the case $p=n$ we cannot have an estimate of the form ⊛ for any $p^* < \infty$ by Remark (R.23), and since $W^{1,n}(\mathbb{R}^n) \not\hookrightarrow L^\infty(\mathbb{R}^n)$ by Remark (R.22), we have to settle for less; one can show that $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ $\forall\, q < \infty$.

Corollary (C.8) Let $\Omega \subset \mathbb{R}^n$ be C^1 with compact boundary, $1 \le p < n$, $p^* = \frac{np}{n-p}$.
Then $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. If Ω is bounded, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $\forall\, 1 \le q < p^*$ is compact. (Rellich compactness theorem.)

Proof · $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ follows via extension to $W^{1,p}(\mathbb{R}^n)$ and application of Theorem (T.29) (cf. the proof of Corollary (C.7)).

· Compactness: (i) let $\mathcal{F} \subset W^{1,p}(\Omega)$ be bounded. Then for $h \in \mathbb{R}^n$, $|h| < 1$, we have

$$\sup_{\substack{v = Eu \\ u \in \mathcal{F}}} \| \tau_h v - v \|_{L^p(\mathbb{R}^n)} \le C|h| \sup_{\substack{v = Eu \\ u \in \mathcal{F}}} \| \nabla v \|_{L^p}$$

$$\le C' |h| \sup_{u \in \mathcal{F}} \|u\|_{W^{1,p}} .$$

$\Rightarrow \mathcal{F} \subset L^p(\Omega)$ is precompact by the Fréchet–Kolmogorov theorem.

(ii) For general $q < p^*$, suppose $\{u_k\} \subset W^{1,p}(\Omega)$ is bounded; WLOG $u_k \xrightarrow{k\to\infty} u \in L^p(\Omega)$, and thus $u_k \longrightarrow u$ in $L^{p'}(\Omega)$

$\forall p' \le p$ since Ω is bounded. For $p < q < p^*$, pick $\alpha \in (0,1)$ with $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$, then

$$\|u_k - u\|_{L^q(\Omega)} \le \underbrace{\|u_k - u\|^{\alpha}_{L^p(\Omega)}}_{\to 0} \underbrace{\|u_k - u\|^{1-\alpha}_{L^{p^*}(\Omega)}}_{\text{bounded}} \longrightarrow 0. \qquad \square$$

Finally, having treated $W^{k,p}$-spaces with $k=1$, the case of $k \geq 2$ follows by simple inductive means:

Theorem (T.30) (Sobolev embedding theorem.) Let $\Omega \subset \mathbb{R}^n$ be C^1. Let $k \in \mathbb{N}$, $1 \le p \le \infty$.

(i) If $kp < n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, and if Ω is bounded $W^{k,p}(\Omega) \hookrightarrow L^{q'}(\Omega)$, $1 \le q' < q$, is compact.

(ii) If $k = \frac{n}{p} + l + \alpha$, $l \in \mathbb{N}_0$, $0 < \alpha < 1$, then

$$W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\bar{\Omega}).$$

If Ω is bounded, this embeds compactly into $C^{l,\alpha'}(\bar{\Omega})$ $\forall \alpha' < \alpha$.

(We omit the discussion of the case $k - \frac{n}{p} \in \mathbb{N}_0$ here.)

This result provides the link between weak solutions of PDE (lying in some Sobolev space) and classical solutions (in C^l or Hölder spaces).

Example (E.20) (i) $s > \frac{n}{2} \Rightarrow H^s(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$. This can also be proved using the characterization in Theorem (T.20): for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$|u(x)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \right|$$
$$\le C \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s}{2}} (1+|\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)|\, d\xi$$
$$\le C \left(\int (1+|\xi|^2)^{-s} d\xi \right)^{\frac{1}{2}} \left(\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$
$$= C' \|u\|_{H^s(\mathbb{R}^n)}$$

since $\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi < \infty$ for $s > \frac{n}{2}$.

(ii) $s = \frac{n}{2} + k + \alpha$, $k \in \mathbb{N}_0$, $0 < \alpha < 1$

$$\Rightarrow H^s(\mathbb{R}^n) \hookrightarrow C^{k,\alpha}(\mathbb{R}^n).$$

(Exercise: prove this using the characterization of $H^s(\mathbb{R}^n)$ via the Fourier transform.)

Proof of Theorem (T.30)

(i). By Corollary (C.8), for $|\alpha| \le k-1$ we have

$$\partial^\alpha u \in W^{1,p}(\Omega) \subset L^{p_1}(\Omega), \quad \frac{1}{p_1} = \frac{1}{p} - \frac{1}{n},$$

so $W^{k,p}(\Omega) \hookrightarrow W^{k-1,p_1}(\Omega)$.

- Similarly $W^{k-1,p_1}(\Omega) \hookrightarrow W^{k-2,p_2}(\Omega) \hookrightarrow \cdots \hookrightarrow W^{0,p_k}(\Omega) = L^{p_k}(\Omega)$

where $\frac{1}{p_j} = \frac{1}{p_{j-1}} - \frac{1}{n}$, so $\frac{1}{p_k} = \frac{1}{p} - \frac{k}{n} \Rightarrow p_k = q$.

- For bounded Ω, $W^{1,p_{k-1}}(\Omega) \hookrightarrow L^{q'}(\Omega)$, $1 \le q' < q$, is compact, therefore $W^{k,p}(\Omega) \hookrightarrow L^{q'}(\Omega)$ is compact, being the composition of linear maps and a compact map.

(ii) Part (i) gives

$$W^{k,p}(\Omega) \hookrightarrow W^{\ell+1,q}(\Omega) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{k-\ell-1}{n} = \frac{1-\alpha}{n} > 0.$$

Then note that $\frac{1}{q} - \frac{1}{n} = -\frac{\alpha}{n} < 0$, so $q > n$, and therefore $W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ by Corollary (C.7)

$$\Rightarrow W^{\ell+1,q}(\Omega) \hookrightarrow C^{\ell,\alpha}(\overline{\Omega}).$$ $\square$