THE LINEARIZED EINSTEIN EQUATIONS WITH SOURCES

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ABSTRACT. On vacuum spacetimes of general dimension, we study the linearized Einstein vacuum equations with a spatially compactly supported and (necessarily) divergence-free source. We prove that the vanishing of appropriate charges of the source, defined in terms of Killing vector fields on the spacetime, is necessary and sufficient for solvability within the class of spatially compactly supported metric perturbations. The proof combines classical results by Moncrief with the solvability theory of the linearized constraint equations with control on supports developed by Corvino–Schoen and Chruściel–Delay.

1. INTRODUCTION

Let (M, g) be a smooth connected globally hyperbolic spacetime, of dimension $n + 1$ where $n \geq 2$, which solves the Einstein vacuum equations

$$\mathrm{Ein}(g) = 0, \qquad \mathrm{Ein}(g) := \mathrm{Ric}(g) - \frac{1}{2} R_g g. \tag{1.1}$$

Here $\mathrm{Ric}(g)$ and R_g denote the Ricci and scalar curvature, respectively. Let $\Sigma \subset M$ denote a smooth spacelike Cauchy hypersurface; denote its unit normal by ν_Σ and the surface measure by $\mathrm{d}\sigma$. We study linearized perturbations of g sourced by smooth linearized stress-energy-momentum tensors $f \in \mathcal{C}^\infty_{\mathrm{sc}}(M; S^2 T^* M)$ which are *spatially compactly supported* (hence the subscript 'sc'): this means that there exists a compact subset $K \subset \Sigma$ so that $\mathrm{supp}\, f \subset \bigcup_\pm J^\pm(K)$, with $J^\pm(K) \subset M$ denoting the causal future ('+'), resp. past ('−'), of K.[1] That is, we shall study the equation

$$D_g \mathrm{Ein}(h) := \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ein}(g + sh) \Big|_{s=0} = f. \tag{1.2}$$

Recall the second Bianchi identity, which states $\delta_g \mathrm{Ein}(g) = 0$ for *all* metrics g (where δ_g is the negative divergence operator, i.e. $(\delta_g h)_\mu = -h_{\mu\nu}{}^{;\nu}$); linearizing this in g and using (1.1) gives

$$\delta_g \big(D_g \mathrm{Ein}(h) \big) = 0 \quad \forall\, h \in \mathcal{C}^\infty(M; S^2 T^* M). \tag{1.3}$$

Therefore, a *necessary* condition for the solvability of (1.2) is $\delta_g f = 0$. The main result of this note precisely determines the extent to which this condition is also *sufficient*. To state it, denote by $\mathscr{K}(M, g) \subset \mathcal{V}(M) = \mathcal{C}^\infty(M; TM)$ the (finite-dimensional) space of Killing vector fields on (M, g).

Theorem 1.1 (Main theorem, smooth version). *The map*

$$\mathcal{C}^\infty_{\mathrm{sc}}(M; S^2 T^* M) \ni f \mapsto \int_\Sigma f(\nu_\Sigma, X) \, \mathrm{d}\sigma, \qquad X \in \mathscr{K}(M, g), \tag{1.4}$$

[1]The space $\mathcal{C}^\infty_{\mathrm{sc}}(M)$ is well-defined independently of the choice of Cauchy hypersurface.

induces a linear isomorphism

$$\big\{ f \in \mathcal{C}^\infty_{\mathrm{sc}}(M; S^2 T^* M) \colon \delta_g f = 0 \big\} \big/ \big\{ D_g \mathrm{Ein}(h) \colon h \in \mathcal{C}^\infty_{\mathrm{sc}}(M; S^2 T^* M) \big\} \to \mathcal{K}(M, g)^*. \quad (1.5)$$

Here $\mathcal{K}(M, g)^* = \mathcal{L}(\mathcal{K}(M, g), \mathbb{R})$ is the dual space. This isomorphism is moreover independent of the choice of Cauchy hypersurface Σ.

In other words, $D_g \mathrm{Ein}(h) = f \in \mathcal{C}^\infty_{\mathrm{sc}} \cap \ker \delta_g$ has a solution $h \in \mathcal{C}^\infty_{\mathrm{sc}}$ if and only if all 'charges' (1.4) vanish. (In particular, when $\mathcal{K}(M, g) = \{0\}$, then $\delta_g f = 0$ is sufficient for the solvability of (1.2).) The proof of solvability utilizes the Cauchy problem for a gauge-fixed version of this equation. The Cauchy data must be chosen to satisfy the linearized constraint equations on Σ with source $\psi = f(\nu_\Sigma, \cdot)$ while being compactly supported; by results of Corvino–Schoen [CS06] and Chruściel–Delay [CD03], this is possible if and only if ψ is L^2_g-orthogonal to the kernel of the formal L^2_g-adjoint of the linearized constraints map, which can be canonically identified with $\mathcal{K}(M, g)$ by a result of Moncrief [Mon75]. We recall Moncrief's result in Proposition 2.1 and give a new perspective on its proof. For a finite regularity version of Theorem 1.1, see Theorem 3.3.

In Theorem 3.4, we show that if Σ is noncompact and one drops the support assumptions on h, there are no obstructions to solvability beyond $\delta_g f = 0$, the reason being that the aforementioned cokernel on the dual space $\mathscr{E}'(\Sigma)$ of $\mathcal{C}^\infty(\Sigma)$ is trivial. When M and Σ have more structure, e.g. if they are asymptotically flat, one can make more precise statements regarding the weights at infinity of f and h depending on which (some of) the additional obstructions given by (1.4) disappear; we shall not discuss this here.

Remark 1.2 (Cosmological constant). Our arguments go through with purely notational modifications if $\mathrm{Ein}(g) - \Lambda g = 0$ where $\Lambda \in \mathbb{R}$ is the cosmological constant. In this case, Theorem 1.1 becomes a characterization of those $f \in \mathcal{C}^\infty_{\mathrm{sc}} \cap \ker \delta_g$ for which $D_g \mathrm{Ein}(h) - \Lambda h = f$ has a solution $h \in \mathcal{C}^\infty_{\mathrm{sc}}$.

Our motivation for solving $D_g \mathrm{Ein}(h) = f$ with nontrivial f, and understanding the obstructions to solvability, comes from perturbation theory. To give a concrete example, suppose (M, g) is a vacuum spacetime, and we wish to modify it near a timelike geodesic γ, in local Fermi normal coordinates (t, x) given by $\gamma = \{(t, 0) \colon t \in \mathbb{R}\}$, by gluing in a small black hole; in $3 + 1$ dimensions this could be a Schwarzschild black hole with mass $\epsilon > 0$, given by the metric $-\mathrm{d}t^2 + \mathrm{d}x^2 + \frac{2\epsilon}{r}(\mathrm{d}t^2 + \mathrm{d}r^2) + \mathcal{O}(\epsilon^2 r^{-2})$ where $r = |x|$. The naive ansatz $g_\epsilon = g + \chi(r)\frac{2\epsilon}{r}(\mathrm{d}t^2 + \mathrm{d}r^2)$ for the modified spacetime metric leads to $\mathrm{Ein}(g_\epsilon) = \mathrm{Ein}(g) + \epsilon f + \mathcal{O}(\epsilon^2)$ where $f = \mathcal{O}(1)$ near the gluing region $\mathrm{supp}\,\chi' \subset \{r > 0\}$ (ignoring the singularity of f at $r = 0$, which one must deal with separately), and $\delta_g f = 0$ by the second Bianchi identity for $g_\epsilon = g + \mathcal{O}(\epsilon)$. One then wishes to eliminate the error f by replacing g_ϵ by $g_\epsilon + \epsilon h$ where h solves $D_g \mathrm{Ein}(h) = -f$ (while being more regular at $r = 0$ than $\frac{2\epsilon}{r}(\mathrm{d}t^2 + \mathrm{d}r^2)$). For the details of such a gluing procedure, see [Hin23].

Prior work on solutions of the linearized Einstein equations has largely focused on solutions of the *homogeneous* equation $D_g \mathrm{Ein}(h) = 0$. Control of solutions modulo pure gauge solutions (i.e. symmetric gradients $\delta_g^* \omega$ of 1-forms ω on M, which always satisfy this equation) is one important problem, especially in the context of stability problems [RW57]. More pertinent to this work is the problem of *linearization stability* as introduced by Fischer–Marsden [FM75, FM73], namely whether such an infinitesimal deformation h can be integrated to a nonlinear solution, i.e. whether there exists a family g_s of metrics with $\mathrm{Ein}(g_s) = 0$ and $h = \frac{\mathrm{d}}{\mathrm{d}s} g_s|_{s=0}$. This is the original context of [Mon75], which

shows that linearization stability at (M, g) holds for the Einstein equations if and only if $\mathscr{K}(M, g) = \{0\}$. The necessary conditions for the existence of g_s whose linearization is h in the presence of nontrivial Killing vector fields were found in [Mon76, AMM82, FMM80].

In this context, Theorem 1.1 gives a necessary and sufficient condition on $f \in \mathcal{C}^\infty_{\mathrm{sc}} \cap \ker \delta_g$ so that there exist spacetime metrics g_s, $s \in (-1, 1)$, with $\mathrm{Ein}(g_s) = sf + \mathcal{O}(s^2)$. (Absent a general natural physical prescription of further lower order terms beyond sf, we do not study the problem of determining when a metric $\tilde{g}$ near g with, say, $\mathrm{Ein}(\tilde{g}) = sf$, s small, exists.) The problem of prescribing the *nonlinear* Ricci curvature tensor of a Lorentzian metric on a given smooth manifold has been studied by DeTurck [DeT83]; the question of the solvability of the analogue of the constraint equations with source, see [DeT83, Equations (3.1)–(3.2)], is however not discussed there.

2. Killing vector fields and the linearized constraint equations

We recall classical results by Moncrief [Mon75] and Fischer–Marsden–Moncrief [FMM80]. For any Lorentzian metric g for which Σ is spacelike, the 1-form

$$\mathrm{Ein}(g)(\nu_\Sigma, \cdot) \in \mathcal{C}^\infty(\Sigma; T^*_\Sigma M)$$

depends only on the first and second fundamental forms $\gamma(X, Y) = g(X, Y)$, $k(X, Y) = \langle \nabla_X \nu_\Sigma, Y \rangle$, $X, Y \in T\Sigma$, of Σ; this gives rise to the constraints map

$$P(\gamma, k) := \mathrm{Ein}(g)(\nu_\Sigma, \cdot) = \Big(\tfrac{1}{2} \big(R_\gamma - |k|^2_\gamma + (\mathrm{tr}_\gamma k)^2 \big), -\delta_\gamma k - \mathrm{d}\,\mathrm{tr}_\gamma k \Big). \qquad (2.1)$$

We write $D_{\gamma,k}P$ for the linearization of P at (γ, k); thus $D_{\gamma,k}P \in \mathrm{Diff}^2(\Sigma; S^2T^*\Sigma \oplus S^2T^*\Sigma; T^*_\Sigma M)$, i.e. $D_{\gamma,k}P$ is a second order differential operator mapping sections of $S^2T^*\Sigma \oplus S^2T^*\Sigma \to \Sigma$ to sections of $T^*_\Sigma M \to \Sigma$. (Here we identify $T_\Sigma M = \mathbb{R} \oplus T\Sigma$ using the orthogonal projections onto $\mathbb{R}\nu_\Sigma$ and $T\Sigma$, and likewise for the cotangent bundles.) Suppose that $\mathrm{Ein}(g) = 0$. If $h \in \mathcal{C}^\infty(M; S^2T^*M)$, and $\dot{\gamma}, \dot{k} \in \mathcal{C}^\infty(\Sigma; S^2T^*\Sigma)$ denote the induced linearized initial data at Σ (i.e. the derivatives at $s = 0$ of the initial data of $g + sh$), then linearizing (2.1) implies

$$D_g\mathrm{Ein}(h)(\nu_\Sigma, \cdot) = D_{\gamma,k}P(\dot{\gamma}, \dot{k}). \qquad (2.2)$$

Proposition 2.1 (Killing vector fields and the linearized constraints map). *(See [Mon75] and also [FMM80, Lemma 2.2].) Let (M, g) be globally hyperbolic with $\mathrm{Ein}(g) = 0$. Then the map $\mathcal{V}(M) \ni X \mapsto g(X|_\Sigma, \cdot) \in \mathcal{C}^\infty(\Sigma; T^*_\Sigma M)$ induces a linear isomorphism*

$$\mathscr{K}(M, g) \to \ker(D_{\gamma,k}P)^*. \qquad (2.3)$$

Our proof of the surjectivity of (2.3) is based on a distributional characterization of $\ker(D_{\gamma,k}P)^*$ which seems to not have been noted before; see (2.7)–(2.8). Before starting the proof in earnest, we observe that $\mathscr{K}(M, g) \ni X \mapsto X|_\Sigma \in \mathcal{C}^\infty(\Sigma; T_\Sigma M)$ is injective, in view of the following result:

Lemma 2.2 (Unique determination of Killing vector fields). *(Cf. [Mon75, Lemma 4.1].) Let (M, g) be a smooth connected N-dimensional pseudo-Riemannian manifold. Then the dimension of the space $\mathscr{K}(M)$ of Killing vector fields on (M, g) satisfies $\dim \mathscr{K}(M) \le \frac{N(N+1)}{2}$. Moreover, if $X \in \mathscr{K}(M)$ vanishes along a hypersurface $\Sigma \subset M$, then $X = 0$ on M.*

Proof. This follows from the well-known fact that X is uniquely determined by $X(p) \in T_pM$ and $\nabla X(p) \in T_p^*M \otimes T_pM$. (Since the Killing equation imposes $\frac{1}{2}N(N+1)$ linearly independent constraints on $X(p), \nabla X(p)$, the space of Killing vector fields has dimension $\leq N + N^2 - \frac{1}{2}N(N+1) = \frac{1}{2}N(N+1)$.) We recall a prolongation argument for the proof of this statement. The Killing equation

$$\langle \nabla_V X, W \rangle + \langle \nabla_W X, V \rangle = 0, \qquad V, W \in \mathcal{V}(M), \tag{2.4}$$

implies for $Z \in \mathcal{V}(M)$ the following identity, where we write '$\equiv$' for equality modulo terms involving only X and ∇X but no higher derivatives:

$$\nabla_Z(\nabla X)(V, W) \equiv \langle \nabla_Z \nabla_V X, W \rangle = Z\langle \nabla_V X, W \rangle - \langle \nabla_V X, \nabla_Z W \rangle \equiv -Z\langle \nabla_W X, V \rangle$$
$$\equiv -\langle \nabla_Z \nabla_W X, V \rangle \equiv -\langle \nabla_W \nabla_Z X, V \rangle \equiv -\nabla_W(\nabla X)(Z, V).$$

Cyclically permuting the vector fields (Z, V, W) twice more, we obtain $\nabla_Z(\nabla X)(V, W) \equiv -\nabla_Z(\nabla X)(V, W)$, and thus $\nabla_Z(\nabla X)(V, W)$ is an (explicit) expression involving only X and ∇X. Therefore, if $\alpha \colon [0,1] \to M$ is a smooth curve, then there exists a smooth bundle endomorphism F on the restriction of $TM \oplus (T^*M \otimes TM)$ to $\alpha([0,1])$ so that $Z(t) = (X, \nabla X)|_{\alpha(t)}$ satisfies the ODE

$$\frac{DZ}{\mathrm{d}t} = F(Z).$$

In particular, if $Z(0) = 0$, then $Z = 0$ on $\alpha([0,1])$, and thus $Z = 0$ on M since M is connected.

To prove the final claim, note that if V, W are vector fields on M, then (2.4) gives $V\langle X, W \rangle + W\langle X, V \rangle = 0$ at Σ (where $X = 0$). Fix V to be transversal to Σ at a point $p \in \Sigma$. For all W which are tangent to Σ, we have $W\langle X, V \rangle = W(0) = 0$ and thus $V\langle X, W \rangle = 0$, while for $W = V$ we obtain $V\langle X, V \rangle = 0$. Therefore, $\nabla X = 0$ at p, from where $X = 0$ follows from the first part of the proof. $\square$

Proof of Proposition 2.1. • *Well-definedness; injectivity.* Let $X \in \mathcal{K}(M, g)$. Given $h \in \mathcal{C}^\infty_{\mathrm{sc}}(M; S^2T^*M)$, write $\dot{\gamma}, \dot{k} \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma; S^2T^*\Sigma)$ for the linearized initial data at Σ. Since $\delta_g(D_g\mathrm{Ein}(h)) = 0$, the fact that X is Killing implies that also $D_g\mathrm{Ein}(h)(\cdot, X)$ is divergence-free. Therefore, if Σ_t, $t \in [-1, 1]$, is a smooth family of spacelike hypersurfaces, with unit normal ν_{Σ_t} and surface measure $\mathrm{d}\sigma_t$, so that Σ_t equals Σ outside a large compact set, then

$$I(t) := \int_{\Sigma_t} D_g\mathrm{Ein}(h)(\nu_{\Sigma_t}, X) \, \mathrm{d}\sigma_t$$

is independent of t and thus equals

$$I(0) = \int_\Sigma D_g\mathrm{Ein}(h)(\nu_\Sigma, X) \, \mathrm{d}\sigma = \int_\Sigma \langle D_{\gamma,k}P(\dot{\gamma}, \dot{k}), X \rangle \, \mathrm{d}\sigma = \int_\Sigma \langle (\dot{\gamma}, \dot{k}), (D_{\gamma,k}P)^*(X) \rangle \, \mathrm{d}\sigma.$$

Since we can choose $(\dot{\gamma}, \dot{k}) \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma; S^2T^*\Sigma) \oplus \mathcal{C}^\infty_{\mathrm{c}}(\Sigma; S^2T^*\Sigma)$ to be arbitrary at Σ and 0 at Σ_1, we deduce that $I(0) = 0$ for all $\dot{\gamma}, \dot{k}$, and therefore $(D_{\gamma,k}P)^*(X) = 0$. Together with Lemma 2.2, we conclude that the map (2.3) is well-defined and injective. (This argument is taken from [FMM80, Lemma 1.5 and Corollary 1.9].)

• *Surjectivity.* Suppose that $\omega_{(0)} \in \mathcal{C}^\infty(\Sigma; T^*_\Sigma M)$ satisfies

$$(D_{\gamma,k}P)^*(\omega_{(0)}) = 0. \tag{2.5}$$

Write δ_g^* for the symmetric gradient on 1-forms; this is related to the Lie derivative via $\delta_g^*\omega = \frac{1}{2}\mathcal{L}_{\omega^\sharp}g$, where we use the musical isomorphism for g. The task is thus to show that there exists a solution $\omega \in \mathcal{C}^\infty(M;T^*M)$ of the Killing equation $\delta_g^*\omega = 0$ which satisfies $\omega|_\Sigma = \omega_{(0)}$. At Σ, the requirements $\omega = \omega_{(0)}$ and $0 = 2(\delta_g^*\omega)(\nu_\Sigma,V) = \nabla_{\nu_\Sigma}\omega(V) + \nabla_V\omega(\nu_\Sigma)$ for $V = \nu_\Sigma$ and $V \in T\Sigma$ uniquely determine the 1-jet $\omega_{(1)}$ of ω at Σ. Write $\mathsf{G}_g = I - \frac{1}{2}g\operatorname{tr}_g$; then we may extend $\omega_{(1)}$ to a 1-form on M by solving the wave equation

$$\delta_g\mathsf{G}_g\delta_g^*\omega = 0$$

on M, with the Cauchy data of ω matching $\omega_{(1)}$, by means of [BGP07, Theorem 3.2.11]; here we use that $\delta_g\mathsf{G}_g\delta_g^*$ equals $\frac{1}{2}\Box_g$ plus lower order terms. Let now

$$\pi := \delta_g^*\omega;$$

thus $\pi|_\Sigma = 0 \in \mathcal{C}^\infty(\Sigma;S^2T_\Sigma^*M)$ and $\delta_g\mathsf{G}_g\pi = 0$ globally. Our aim is to show $\pi = 0$. Note that since $\operatorname{Ein}(g) = 0$ and $D_g\operatorname{Ein} = \mathsf{G}_g \circ D_g\operatorname{Ric}$, we have

$$0 = D_g\operatorname{Ein}(\delta_g^*\omega) = \mathsf{G}_gD_g\operatorname{Ric}(\pi) = \mathsf{G}_g\Big(\frac{1}{2}\Box_g - \delta_g^*\delta_g\mathsf{G}_g + \mathscr{R}_g\Big)\pi = \frac{1}{2}\mathsf{G}_g(\Box_g + 2\mathscr{R}_g)\pi,$$

see [GL91, Equation (2.4)], where $\mathscr{R}_g$ is a 0-th order operator involving the curvature of g. In view of this hyperbolic equation for π, we only need to show that

$$\text{the 1-jet of } \pi \text{ vanishes at } \Sigma \tag{2.6}$$

to conclude the proof. This is where the equation satisfied by $\omega_{(0)}$ will enter.

To wit, (2.5) is equivalent to the following statement: for all $h \in \mathcal{C}_{\mathrm{sc}}^\infty(M;S^2T^*M)$, and its induced initial data $\dot\gamma, \dot k \in \mathcal{C}_{\mathrm{c}}^\infty(\Sigma;S^2T^*\Sigma)$ at Σ,[2]

$$\begin{aligned}
0 &= \int_\Sigma \langle(\dot\gamma,\dot k),(D_{\gamma,k}P)^*\omega_{(0)}\rangle\,\mathrm{d}\sigma &&= \int_\Sigma \langle D_{\gamma,k}P(\dot\gamma,\dot k),\omega_{(0)}\rangle\,\mathrm{d}\sigma \\
&= \int_\Sigma D_g\operatorname{Ein}(h)(\nu_\Sigma,(\omega_{(0)})^\sharp)\,\mathrm{d}\sigma &&= \int_\Sigma \langle D_g\operatorname{Ein}(h),\nu_\Sigma^\flat \otimes_s \omega_{(0)}\rangle\,\mathrm{d}\sigma \\
&= \langle D_g\operatorname{Ein}(h),(\nu_\Sigma^\flat \otimes_s \omega_{(0)})\delta(\Sigma)\rangle.
\end{aligned}$$

Here we use the distributional pairing on M in the final expression; and $\delta(\Sigma)$ is the distribution which integrates a test function over Σ with the volume density $\mathrm{d}\sigma$ (and hence $(\nu_\Sigma^\flat \otimes_s \omega_{(0)})\delta(\Sigma)$ is an $S^2T_\Sigma^*M$-valued distribution). With $(D_g\operatorname{Ein})^* = D_g\operatorname{Ein}$ denoting the formal L_g^2-adjoint, this is further equivalent to

$$0 = (D_g\operatorname{Ein})^*\big((\nu_\Sigma^\flat \otimes_s \omega_{(0)})\delta(\Sigma)\big) = D_g\operatorname{Ein}(\mathrm{d}H \otimes_s \omega_{(0)}) \tag{2.7}$$

where $H \in L_{\mathrm{loc}}^1(M)$ is 1, resp. 0 in the causal past $J^-(\Sigma)$, resp. future of Σ. The relevant distributional identity here is $\mathrm{d}H = \nu_\Sigma^\flat\delta(\Sigma)$; this is proved by computing, for a compactly supported test vector field $X \in \mathcal{C}_{\mathrm{c}}^\infty(M;TM)$,

$$\langle\mathrm{d}H,X\rangle = \langle H,-\operatorname{div}X\rangle = -\int_{J^-(\Sigma)}\operatorname{div}X\,\mathrm{d}g = \int_\Sigma\langle X,\nu_\Sigma\rangle\,\mathrm{d}\sigma = \langle X,\nu_\Sigma^\flat\delta(\Sigma)\rangle,$$

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[2]We recall, see e.g. [Hin23, Remark 4.1], that for all $\dot\gamma,\dot k \in \mathcal{C}^\infty(\Sigma;S^2T^*\Sigma)$ there exists $h \in \mathcal{C}^\infty(M;S^2T^*M)$ which induced the data $\dot\gamma,\dot k$. The construction in the reference also implies that one can take h to be spatially compactly supported when $\dot\gamma,\dot k$ are compactly supported.

using the divergence theorem in the third equality. Finally, we record the distributional identity
$$\mathrm{d}H \otimes_s \omega_{(0)} = \mathrm{d}H \otimes_s \omega = \delta_g^*(H\omega) - H\delta_g^*\omega,$$
where the first equality follows from the fact that the product of $\delta(\Sigma)$ with the 1-form $\omega - \omega_{(0)}$, which vanishes at Σ, is 0 as a distribution on M. Together with the identity $D_g\mathrm{Ein}(\delta_g^*\omega) = 0$ (valid for all ω), we conclude from (2.7) that (2.5) is equivalent to the distributional equation $0 = D_g\mathrm{Ein}(H\delta_g^*\omega) = [D_g\mathrm{Ein}, H](\delta_g^*\omega)$, or equivalently (upon applying G_g^{-1} to $[D_g\mathrm{Ein}, H](\delta_g^*\omega) = 0$)

$$[D_g\mathrm{Ric}, H]\pi = 0, \qquad \pi = \delta_g^*\omega. \tag{2.8}$$

Expanding this equation (using $2D_g\mathrm{Ric} = \Box_g - 2\delta_g^*\delta_g\mathsf{G}_g + 2\mathscr{R}_g$) and using $\pi|_\Sigma = 0$ gives

$$0 = [\Box_g, H]\pi - 2[\delta_g^*\delta_g\mathsf{G}_g, H]\pi = [\Box_g, H]\pi - 2[\delta_g^*, H](\delta_g\mathsf{G}_g\pi) - 2\delta_g^*([\delta_g\mathsf{G}_g, H]\pi),$$

with the final term on the right being zero since $[\delta_g\mathsf{G}_g, H]\pi = -(\mathsf{G}_g\pi)(\nabla H, \cdot) = 0$ (this being the product of a δ-distribution at Σ with a tensor vanishing at Σ). In abstract index notation, we thus have

$$-g^{\mu\nu}(H_{;\mu\nu}\pi_{\kappa\lambda} + H_{;\mu}\pi_{\kappa\lambda;\nu} + H_{;\nu}\pi_{\kappa\lambda;\mu}) - H_{;\kappa}(\delta_g\mathsf{G}_g\pi)_\lambda - H_{;\lambda}(\delta_g\mathsf{G}_g\pi)_\kappa = 0.$$

Since $H_{;\mu}\pi_{\kappa\lambda} = 0$, the first two terms in parentheses cancel. Using $\delta_g\mathsf{G}_g\pi = \delta_g\pi + \frac{1}{2}\mathrm{d}\,\mathrm{tr}_g\,\pi = \frac{1}{2}\mathrm{d}\,\mathrm{tr}_g\,\pi$ yields the equation $-H_{;\mu}\pi_{\kappa\lambda}{}^{;\mu} - \frac{1}{2}(H_{;\kappa}\pi_\mu{}^\mu{}_{;\lambda} + H_{;\lambda}\pi_\mu{}^\mu{}_{;\kappa}) = 0$. Choosing coordinates $z^0, \ldots, z^n$ so that $\Sigma = \{z^0 = 0\}$ and $\mathrm{d}z^0$ is a unit covector, this gives

$$0 = \pi_{\kappa\lambda;0} - \frac{1}{2}(\delta_{0\kappa}\pi_\mu{}^\mu{}_{;\lambda} + \delta_{0\lambda}\pi_\mu{}^\mu{}_{;\kappa}) = \pi_{\kappa\lambda;0} + \delta_{0\kappa}\delta_{0\lambda}(\pi_{00;0} - \pi_m{}^m{}_{;0})$$

at Σ, where m runs from 1 to n; here we use that π and therefore also its spatial covariant derivatives vanish at Σ. For $\kappa = i \geq 1$, $\lambda \geq 0$, this gives $\pi_{i\lambda;0} = 0$. For $\kappa = \lambda = 0$ then, we get $\pi_{00;0} = 0$. This shows (2.6) and finishes the proof. $\qquad\square$

3. Proof of the main result; variants

The independence of the map (1.4) of the choice of Cauchy hypersurface Σ in (1.4) follows from the divergence theorem, since $\delta_g f = 0$ and $X \in \mathscr{K}(M, g)$ implies that $f(X, \cdot)$ is divergence-free. The vanishing of $\int_\Sigma f(\nu_\Sigma, X)\,\mathrm{d}\sigma$ for $f = D_g\mathrm{Ein}(h)$ was shown in the first step of the proof of Proposition 2.1. Thus, (1.5) is well-defined.

For the proof of Theorem 1.1, we require the following result:

Theorem 3.1 (Solvability of the linearized constraints)**.** *Let* (Σ, γ) *be a smooth connected Riemannian manifold, and let* $k \in \mathcal{C}^\infty(\Sigma; S^2 T^*\Sigma)$.

(1) (Smooth version.) *Suppose* $\psi \in \mathcal{C}_c^\infty(\Sigma; \underline{\mathbb{R}} \oplus T^*\Sigma)$ *satisfies* $\langle \psi, f^* \rangle = 0$ *for all* $f^* \in \ker(D_{\gamma,k}P)^* \subset \mathcal{C}^\infty(\Sigma; \underline{\mathbb{R}} \oplus T^*\Sigma)$. *Then there exist* $\dot\gamma, \dot k \in \mathcal{C}_c^\infty(\Sigma; S^2 T^*\Sigma)$ *so that*

$$D_{\gamma,k}P(\dot\gamma, \dot k) = \psi. \tag{3.1}$$

(2) (Finite regularity version.) *Let* $s \geq 0$. *If* $\psi \in H_c^s(\Sigma; \underline{\mathbb{R}} \oplus T^*\Sigma)$ *is* L^2-*orthogonal to* $\ker(D_{\gamma,k}P)^*$, *then one can find* $\dot\gamma \in H_c^{s+2}(\Sigma; S^2 T^*\Sigma)$ *and* $\dot k \in H_c^{s+1}(\Sigma; S^2 T^*\Sigma)$ *solving* (3.1)*.*

Proof. The existence of regular solutions with controlled support is due to Corvino–Schoen [CS06], with the existence of smooth solutions proved by Chruściel–Delay [CD03]; see [Del12] for general results of this flavor. The key point is that $(D_{\gamma,k}P)^*$ is an overdetermined (Douglis–Nirenberg) elliptic partial differential operator for which a priori estimates hold on spaces of tensors defined on a smoothly bounded open precompact subset $\mathcal{U} \subset \Sigma$, containing $\operatorname{supp} \psi$, which allow for exponential growth at $\partial\mathcal{U}$. By duality, this gives the solvability of $D_{\gamma,k}P(\dot{\gamma}, \dot{k}) = \psi$ provided ψ is L^2_g-orthogonal to the cokernel, with $\dot{\gamma}, \dot{k}$ exponentially decaying at $\partial\mathcal{U}$; their extension by 0 to $\Sigma \setminus \mathcal{U}$ furnishes the desired solution.

The finite regularity statement proved in [CS06, §4.3], or [Hin22, Proposition 4.15(1), Footnote 23] (where for present purposes we work away from $\hat{\rho} = 0$ and thus may take $\hat{\rho} = 1$, $w_\circ = 1$ in the notation of the reference) produces for $\psi \in H^{s-1}_c(\Sigma) \oplus H^s_c(\Sigma; T^*\Sigma)$ a solution $(\dot{\gamma}, \dot{k}) \in H^{s+1}_c(\Sigma; T^*\Sigma) \oplus H^{s+1}_c(\Sigma; T^*\Sigma)$ of (3.1): in the notation of [Hin22], one takes $\mathcal{U}$ to contain $\operatorname{supp} \psi$, and then [Hin22, Footnote 23] (with $\beta > 0$) produces a solution $(\dot{\gamma}, \dot{k}) \in e^{-\beta/\rho_2} H^{s+1}_{00}(\overline{\mathcal{U}})$ where $\rho_2 \in \mathcal{C}^\infty(\overline{\mathcal{U}})$ is positive in $\mathcal{U}$ and vanishes simply at $\partial\mathcal{U}$; and the extension of $(\dot{\gamma}, \dot{k})$ by 0 is of class $H^{s+1}_c(\Sigma)$. (This follows for $s \in \mathbb{N}$ by direct differentiation, and for general $s \geq 0$ by complex interpolation.)

The regularity orders in this statement are not quite sufficient for later purposes,[3] and therefore we indicate the changes required to obtain the presently required result. Namely, analogously to [Hin22] we set

$$\mathcal{L}_{\gamma,k} := e^{\beta/\rho_2} \circ w_2 D_{\gamma,k}P \circ e^{-\beta/\rho_2} \in (\operatorname{Diff}_{00}^{t_j+s_i}(\overline{\mathcal{U}}))_{i,j=1,2}, \qquad s_1 = 2, \; s_2 = 1, \; t_1 = t_2 = 0,$$

where $w_2 = \operatorname{diag}(\rho_2^4, \rho_2^2)$. But now we consider the operator

$$\tilde{\mathcal{L}} := \mathcal{L}_{\gamma,k} \mathcal{D} \mathcal{L}_{\gamma,k}^* \in \operatorname{Diff}_{00}^4(\overline{\mathcal{U}}), \qquad \mathcal{D} := \operatorname{diag}(1, \rho_2^2(\nabla^*\nabla + 1)\rho_2^2).$$

Denote the (i,j) component of $\mathcal{L}_{\gamma,k}$ by $\mathcal{L}_{ij}$. Using that $\mathcal{L}_{12}$ is, in fact, of order 0, not 2 (see [Hin22, Equation (4.4)]), one finds that the off-diagonal terms of $\mathcal{L}_{\gamma,k} \mathcal{D} \mathcal{L}_{\gamma,k}^*$ are of order 3, while the diagonal terms are the elliptic operators $\mathcal{L}_{11}\mathcal{L}_{11}^*$, $\mathcal{L}_{22}\rho_2^2(\nabla^*\nabla + 1)\rho_2^2\mathcal{L}_{22}^* \in \operatorname{Diff}_{00}^4$. By elliptic regularity in the 00-setting as in the reference,

$$\tilde{\mathcal{L}} \colon H^{s+4}_{00}(\overline{\mathcal{U}}; \mathbb{R} \oplus T^*\Sigma) \to H^s_{00}(\overline{\mathcal{U}}; \mathbb{R} \oplus T^*\Sigma)$$

is therefore Fredholm (in fact, for all $s \in \mathbb{R}$). We can characterize its cokernel: for $q^* \in \ker_{H_{00}^{-s}} \tilde{\mathcal{L}}^*$, we have $q^* \in H_{00}^\infty$ by elliptic regularity, and thus we can integrate by parts to find $\langle \tilde{\mathcal{L}}q^*, q^* \rangle_{L^2} = \langle \mathcal{D}\mathcal{L}_{\gamma,k}^*q^*, \mathcal{L}_{\gamma,k}^*q^* \rangle_{L^2} = 0$. Writing $\mathcal{L}_{\gamma,k}^*q^* = (a, b)$, this means $\|a\|^2 + \|\nabla\rho_2^2 b\|^2_{L^2} + \|\rho_2^2 b\|^2_{L^2} = 0$ and thus $a = b = 0$; this means that $e^{\beta/\rho_2}w_2 q^* \in \ker(D_{\gamma,k}P)^*$, and we conclude that

$$\ker_{H_{00}^{-s}} \tilde{\mathcal{L}}^* = e^{-\beta/\rho_2} w_2^{-1} \ker(D_{\gamma,k}P)^*. \tag{3.2}$$

Let now $\psi \in H^s_c$ be given, with ψ orthogonal to $\ker(D_{\gamma,k}P)^*$. Fix $\mathcal{U}$ containing $\operatorname{supp} \psi$, and set $\tilde{\psi} = e^{\beta/\rho_2}w_2\psi \in H^s_{00}(\overline{\mathcal{U}})$; we can then find a solution $\tilde{q} \in H^{s+4}_{00}(\overline{\mathcal{U}})$ of $\tilde{\mathcal{L}}\tilde{q} = \tilde{\psi}$ since $\tilde{\psi}$ is L^2-orthogonal to (3.2) by assumption on ψ. But then

$$(\dot{\gamma}, \dot{k}) := e^{-\beta/\rho_2}\mathcal{D}e^{-\beta/\rho_2}(D_{\gamma,k}P)^*w_2 e^{\beta/\rho_2}\tilde{q} \in e^{-\beta/\rho_2}H^{s+2}_{00} \oplus e^{-\beta/\rho_2}H^{s+1}_{00}$$

solves $D_{\gamma,k}P(\dot{\gamma}, \dot{k}) = \psi$ on $\mathcal{U}$. As before, the extensions by 0 of $\dot{\gamma}, \dot{k}$ are the desired solutions on Σ. $\qquad \square$

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[3]See the proof of Theorem 3.3: it would cause a partial loss of regularity in that h_0, h_1 there would both be of class H^{s+1}_c only, ultimately leading to the solution h there to only be of class H^s_{sc}, assuming $s \geq 1$.

Proof of Theorem 1.1. • *Injectivity of* (1.5). Suppose $f \in \mathcal{C}^\infty_{sc}(M; S^2 T^* M)$ satisfies $\delta_g f = 0$ and $\int_\Sigma f(\nu_\Sigma, X) \, d\sigma = 0$ for all $X \in \mathscr{K}(M, g)$. By Proposition 2.1, $f(\nu_\Sigma, \cdot) \in \mathcal{C}^\infty_c(\Sigma; T^*_\Sigma M)$ is orthogonal to $\ker_{\mathcal{C}^\infty(\Sigma; T^*_\Sigma M)}(D_{\gamma, k} P)^*$.

Choose $\dot{\gamma}, \dot{k} \in \mathcal{C}^\infty_c(\Sigma; S^2 T^* \Sigma)$ according to Theorem 3.1 with $\psi = f(\nu_\Sigma, \cdot)$. Pick then $h_0, h_1 \in \mathcal{C}^\infty_c(\Sigma; S^2 T^*_\Sigma M)$ so that $\dot{\gamma}, \dot{k}$ are the linearized initial data corresponding to any metric perturbation $\tilde{h} \in \mathcal{C}^\infty_{sc}(M; S^2 T^* M)$ with Cauchy data $h_0 = \tilde{h}|_\Sigma$, $h_1 = \nabla_{\nu_\Sigma} \tilde{h}$. Let $\theta \in \mathcal{C}^\infty_c(M; T^* M)$ be any extension of $\theta_{(0)} := \delta_g \mathsf{G}_g \tilde{h}|_\Sigma \in \mathcal{C}^\infty_c(\Sigma; T^*_\Sigma M)$. We then solve the initial value problem for the gauge-fixed linearized Einstein vacuum equation

$$D_g \mathrm{Ric}(h) + \delta^*_g(\delta_g \mathsf{G}_g h - \theta) = \mathsf{G}^{-1}_g f, \qquad (h|_\Sigma, \nabla_{\nu_\Sigma} h) = (h_0, h_1) \tag{3.3}$$

using [BGP07, Theorem 3.2.11]; the solution h satisfies $h \in \mathcal{C}^\infty_{sc}(M; S^2 T^* M)$ by finite speed of propagation. If we set $\eta := \delta_g \mathsf{G}_g h - \theta$, then we have

$$\eta|_\Sigma = (\delta_g \mathsf{G}_g \tilde{h} - \theta)|_\Sigma = 0 \tag{3.4}$$

by definition of θ. Moreover, applying G_g to (3.3) and recalling (2.2), we obtain

$$(\mathsf{G}_g \delta^*_g \eta)(\nu_\Sigma, \cdot) = f(\nu_\Sigma, \cdot) - D_g \mathrm{Ein}(h)(\nu_\Sigma, \cdot) = 0.$$

Together with (3.4), this implies $\nabla_{\nu_\Sigma} \eta = 0$. Finally, applying $\delta_g \mathsf{G}_g$ to (3.3) and using the linearized second Bianchi identity together with $\delta_g f = 0$ gives the hyperbolic equation

$$\delta_g \mathsf{G}_g \delta^*_g \eta = 0$$

for η, whence $\eta = 0$ on M and thus $D_g \mathrm{Ric}(h) = \mathsf{G}^{-1}_g f$. Therefore, $f = D_g \mathrm{Ein}(h)$ projects to 0 in the quotient space (1.5).

• *Surjectivity of* (1.5). Given $\lambda \in \mathscr{K}(M, g)^*$, there exists a 1-form $\psi \in \mathcal{C}^\infty_c(\Sigma; T^*_\Sigma M)$ with $\langle \psi, X|_\Sigma \rangle = \lambda(X)$ for all $X \in \mathscr{K}(M, g)$. Indeed, it suffices to arrange this for X in a basis of $\mathscr{K}(M, g)$, in which case it follows from the linear independence of the restrictions of the basis elements to Σ (a consequence of Lemma 2.2). Pick then any $f_{(0)} \in \mathcal{C}^\infty_c(\Sigma; S^2 T^*_\Sigma M)$ with $f_{(0)}(\nu_\Sigma, \cdot) = \psi$, and let $\tilde{f} \in \mathcal{C}^\infty_c(M; S^2 T^* M)$ denote any compactly supported symmetric 2-tensor with $\tilde{f} = f_{(0)}$ at Σ. We claim that there exists $\omega \in \mathcal{C}^\infty_{sc}(M; T^* M)$ with $(\omega|_\Sigma, \nabla_{\nu_\Sigma} \omega) = 0$ so that

$$f := \tilde{f} + \mathsf{G}_g \delta^*_g \omega \in \ker \delta_g.$$

Indeed, we simply solve the initial value problem

$$\delta_g \mathsf{G}_g \delta^*_g \omega = -\delta_g \tilde{f} \in \mathcal{C}^\infty_c(M; S^2 T^* M), \qquad (\omega|_\Sigma, \nabla_{\nu_\Sigma} \omega) = (0, 0),$$

using [BGP07, Theorem 3.2.11]. Since $\mathsf{G}_g \delta^*_g \omega$ vanishes at Σ for such ω, we have

$$\int_\Sigma f(\nu_\Sigma, X) \, d\sigma = \int_\Sigma f_{(0)}(\nu_\Sigma, X) \, d\sigma = \int_\Sigma \langle \psi, X \rangle \, d\sigma = \lambda(X)$$

for all $X \in \mathscr{K}(M, g)$ still, and the surjectivity of (1.5) follows. This completes the proof of Theorem 1.1. $\qquad\square$

Remark 3.2 (Equivalence to solving the linearized constraints). Not only did Theorem 3.1 play a key role in the argument; one can conversely *deduce* Theorem 3.1 from Theorem 1.1. Indeed, given ψ satisfying the assumptions of Theorem 3.1, one constructs $f \in \ker \delta_g \cap \mathcal{C}^\infty_{sc}(M; S^2 T^* M)$ with $f(\nu_\Sigma, \cdot) = \psi$ as in the last part of the above proof. Theorem 1.1 produces $h \in \mathcal{C}^\infty_{sc}(M; S^2 T^* M)$ with $D_g \mathrm{Ein}(h) = f$; evaluating this on Σ at $(\nu_\Sigma, \cdot)$ gives $D_{\gamma, k} P(\dot{\gamma}, \dot{k}) = \psi$ where $\dot{\gamma}, \dot{k} \in \mathcal{C}^\infty_c(\Sigma; S^2 T^* \Sigma)$ are the linearized initial data induced by h.

Theorem 3.3 (Finite regularity). *Let $s \geq 0$. Suppose that $f \in H^s_{\mathrm{sc}}(M; S^2T^*M)$ (i.e. f lies in $H^s_{\mathrm{loc}}(M; S^2T^*M)$ and has spatially compact support) satisfies $\delta_g f = 0$, and $\int_\Sigma f(\nu_\Sigma, X)\, \mathrm{d}\sigma = 0$ for all $X \in \mathscr{K}(M, g)$.[4] Then there exists $h \in H^{s+1}_{\mathrm{sc}}(M; S^2T^*M)$ with $D_g\mathrm{Ein}(h) = f$.*

Together with the surjectivity of (1.5), this shows that (1.4) induces an isomorphism

$$\{f \in H^s_{\mathrm{sc}}\colon \delta_g f = 0\}/\{D_g\mathrm{Ein}(h)\colon h \in H^{s+1}_{\mathrm{sc}} \text{ with } D_g\mathrm{Ein}(h) \in H^s_{\mathrm{sc}}\} \to \mathscr{K}(M, g)^*.$$

Proof of Theorem 3.3. Foliating a neighborhood of Σ by level sets $\Sigma_\tau := t^{-1}(\tau)$ (which we identify with Σ), $\tau \in (-1, 1)$, of a time function $t \in \mathcal{C}^\infty(M)$, we have, a fortiori, $f \in L^1_{\mathrm{loc}}((-1, 1), H^s_c(\Sigma; S^2T^*_\Sigma M))$, whence $f|_{t=\tau} \in H^s_c$ for almost all τ. (This uses $s \geq 0$.) Fix such a $\tau \in (-1, 1)$. Using the finite regularity version of Theorem 3.1, one can then find $\dot\gamma \in H^{s+2}_c(\Sigma; S^2T^*\Sigma)$, $\dot k \in H^{s+1}_c(\Sigma; S^2T^*\Sigma)$, so that $D_{\gamma,k}P(\dot\gamma, \dot k) = f(\nu_{\Sigma_\tau}, \cdot)$ at Σ_τ, and thus Cauchy data $h_0 \in H^{s+2}_c$, $h_1 \in H^{s+1}_c$ inducing $\dot\gamma, \dot k$. We can then compute $\theta_{(0)} \in H^{s+1}_c(\Sigma; T^*_\Sigma M)$ as in the proof of Theorem 1.1, extend it to be constant in t followed by cutting it off to a neighborhood of $t = \tau$ to obtain $\theta \in \mathcal{C}^\infty_c((-1, 1); H^{s+1}_c)$. The equation (3.3) we then solve for h has spatially compactly supported forcing in $L^1_{\mathrm{loc}}H^s_c$ and thus a spatially compactly supported solution of class $\mathcal{C}^0 H^{s+1}_c \cap \mathcal{C}^1 H^s_c$. The arguments leading to $D_g\mathrm{Ein}(h) = f$ then go through without changes since $\eta = \delta_g\mathsf{G}_g h - \theta$ and $\mathsf{G}_g\delta^*_g\eta(\nu_{\Sigma_\tau}, \cdot)$ vanish at Σ_τ. Finally, $h \in H^{s+1}_{\mathrm{sc}}$ follows by standard hyperbolic theory [Hör07, §23.2]. $\square$

Theorem 3.4 (Solvability without support conditions). *Suppose the Cauchy hypersurface Σ inside the globally hyperbolic spacetime (M, g) with $\mathrm{Ein}(g) = 0$ is noncompact. Let*

$$f \in \mathcal{C}^\infty(M; S^2T^*M), \qquad \delta_g f = 0.$$

Then there exists $h \in \mathcal{C}^\infty(M; S^2T^*M)$ with $D_g\mathrm{Ein}(h) = f$.

Thus, there are no obstructions to solvability if we drop the support assumptions on h, and one can even drop the support assumptions on the source term f. (Even for $f \in \mathcal{C}^\infty_{\mathrm{sc}}(M; S^2T^*M)$, the solution h produced below is typically large at infinity.)

Proof of Theorem 3.4. We need to prove the existence of $\dot\gamma, \dot k \in \mathcal{C}^\infty(\Sigma; S^2T^*\Sigma)$ so that $D_{\gamma,k}P(\dot\gamma, \dot k) = f(\nu_\Sigma, \cdot)$. Once this is done, the arguments using the gauge-fixed equation (3.3) apply verbatim to produce a solution of $D_g\mathrm{Ein}(h) = f$. The existence of $(\dot\gamma, \dot k)$ follows from [Trè67, Theorem 37.2] once we show that the adjoint operator

$$(D_{\gamma,k}P)^*\colon \mathscr{E}'(\Sigma; \mathbb{R} \oplus T^*\Sigma) \to \mathscr{E}'(\Sigma; S^2T^*\Sigma \oplus S^2T^*\Sigma)$$

has trivial kernel and weak* closed range. The injectivity follows from the fact that an element of the kernel vanishes outside some compact set and is thus identically zero by unique continuation, as follows from the relationship of $\ker(D_{\gamma,k}P)^*$ with Killing vector fields on (M, g) and Lemma 2.2 (or directly on the level of $(D_{\gamma,k}P)^*$ via [Hin22, Lemma 4.3]). The weak* closed range property follows, via the characterization [Trè67, Theorem 37.1], from the fact that $(D_{\gamma,k}P)^*$ is overdetermined (Douglis–Nirenberg) elliptic, which indeed implies that for all $s \in \mathbb{R}$ and compact $K \subset \Sigma$, the operator $(D_{\gamma,k}P)^*\colon \dot{H}^{s+2}(K) \oplus \dot{H}^{s+1}(K; T^*_K\Sigma) \to$

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[4]In coordinates $(z^0, \ldots, z^n)$ in which $\mathrm{d}z^0$ is a unit conormal at $\Sigma = \{z^0 = 0\}$, we have $f_{0\mu;0} = f_{j\mu;}{}^j$. Therefore $f(\nu_\Sigma, \cdot) \in H^{s-\frac{1}{2}}_c(\Sigma; T^*_\Sigma M)$ is well-defined for $s > -\frac{1}{2}$.

$\dot{H}^s(K; S^2 T_K^* \Sigma \oplus S^2 T_K^* \Sigma)$ has closed range. (Here $\dot{H}^s(K)$ denotes the subspace of $H^s_{\mathrm{loc}}(\Sigma)$ consisting of distributions with support in K.) $\hfill\square$

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