

NORMALLY HYPERBOLIC TRAPPING ON ASYMPTOTICALLY STATIONARY SPACETIMES

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ABSTRACT. We prove microlocal estimates at the trapped set of asymptotically Kerr spacetimes: these are spacetimes whose metrics decay inverse polynomially in time to a stationary subextremal Kerr metric. This combines two independent results. The first one is purely dynamical: we show that the stable and unstable manifolds of a decaying perturbation of a time-translation-invariant dynamical system with normally hyperbolic trapping are smooth and decay to their stationary counterparts. The second, independent, result provides microlocal estimates for operators whose null-bicharacteristic flow has a normally hyperbolic invariant manifold, under suitable nondegeneracy conditions on the stable and unstable manifolds; this includes operators on closed manifolds, as well as operators on spacetimes for which the invariant manifold lies at future infinity.

1. INTRODUCTION

This paper has two independent parts: in the first (§2), we study perturbations of dynamical systems which exhibit normally hyperbolic trapping ‘at infinity’, and in the second (§3) we prove microlocal estimates for solutions of pseudodifferential equations whose null-bicharacteristic flow has this dynamical structure. The application tying the two together—the main motivation for the present paper—concerns the study of waves on perturbations of Kerr black holes (§4).

1.1. Structure of stable and unstable manifolds. We first describe the dynamical result. Let \mathcal{X} denote a closed manifold, and let $\bar{V} \in \mathcal{V}(\mathcal{X})$ be a smooth vector field which is tangent to a submanifold Γ , which we call the *trapped set*. Suppose the \bar{V} -flow is r -normally hyperbolic at Γ for every r , see §2.4; a simple example is shown in Figure 1.1. By classical theorems of Fenichel [Fen71] and Hirsch–Pugh–Shub [HPS77], there exist smooth stable/unstable manifolds $\bar{\Gamma}^{s/u}$ near Γ to which \bar{V} is tangent. Extending this to a dynamical system on the ‘spacetime’

$$\mathcal{M} = \mathbb{R}_t \times \mathcal{X}, \quad V_0 = \frac{\partial}{\partial t} + \bar{V},$$

V_0 is tangent to the spacetime trapped set $\Gamma_0 = \mathbb{R}_t \times \bar{\Gamma}$, which has unstable/stable manifolds $\Gamma_0^{u/s} = \mathbb{R}_t \times \bar{\Gamma}^{u/s}$.

We consider perturbations of this dynamical system which decay as $t \rightarrow \infty$: denoting by $\mathcal{C}_b^\infty(\mathcal{M})$ the space of functions which are bounded together with all their derivatives along ∂_t and vector fields on \mathcal{X} , we consider

$$V = V_0 + \tilde{V}, \quad \tilde{V} \in \rho \mathcal{C}_b^\infty(\mathcal{M}; T\mathcal{X}), \quad \rho(t) = \langle t \rangle^{-\alpha}, \quad \alpha > 0. \quad (1.1)$$

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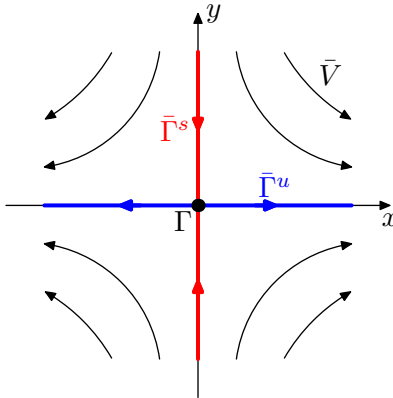


FIGURE 1.1. A simple example of a vector field on the plane with normally hyperbolic trapping: in local coordinates $(x, y) \in \mathbb{R}^2$, we take $\bar{V} = x\partial_x - y\partial_y$. The trapped set is $\Gamma = \{(0, 0)\}$, the stable manifold is $\bar{\Gamma}^s = \{(0, y) : y \in \mathbb{R}\}$, and the unstable manifold is $\bar{\Gamma}^u = \{(x, 0) : x \in \mathbb{R}\}$. One can add a neutral dimension (not shown here) and allow for higher-dimensional trapped sets, such as the trapped set $\{(0, 0, \gamma) : \gamma \in \mathbb{S}^1\}$ for the vector field $x\partial_x - y\partial_y + \partial_\gamma$ on $\mathbb{R}_{x,y}^2 \times \mathbb{S}_\gamma^1$.

Thus, \tilde{V} is a ‘spatial’ vector field whose components (in local coordinate systems on \mathcal{X}) decay to zero at the rate $\rho(t)$. Our first result concerns the existence and regularity of the perturbed stable and unstable manifolds:

Theorem 1.1. *There exist a stable manifold $\Gamma^s \subset \mathcal{M}$ and an unstable manifold $\Gamma^u \subset \mathcal{M}$ to which V is tangent, and so that $\Gamma^{s/u}$ is $\rho\mathcal{C}_b^\infty$ -close to $\Gamma_0^{s/u}$. More precisely, there exist open neighborhoods $\mathcal{U}^{s/u}$ of Γ inside of $\bar{\Gamma}^{s/u}$ such that for T large, $\Gamma^{s/u} \cap t^{-1}((T, \infty))$ is the graph over $(T, \infty) \times \mathcal{U}^{s/u}$ of a function in $\rho\mathcal{C}_b^\infty((T, \infty) \times \mathcal{U}^{s/u}; N\mathcal{U}^{s/u})$.*

See Theorem 2.6 for the full statement. Our proof of Theorem 1.1 is an application of Hadamard’s idea, called ‘graph transform’ in [HPS77], for the construction of stable and unstable manifolds, namely the repeated application of the time (-1) flow of V to Γ_0^s , which converges to Γ^s . (We can closely follow the outstanding presentation of [HPS77].) For Γ^u , one instead starts with a piece of a V -invariant manifold over $t^{-1}((T, T+2))$ and repeatedly applies the time 1 flow.¹ In fact, we will deduce Theorem 1.1 from an analogous statement about diffeomorphisms on \mathcal{X} which are normally hyperbolic at Γ , see Theorem 2.2.

Since $t \rightarrow \infty$ along integral curves of V_0 , trapping only really occurs at ‘ $t = \infty$ ’. One can make this precise by introducing a partial compactification of \mathcal{M} in which one adds a boundary $\{\tau_e = 0\} \cong \mathcal{X}$, $\tau_e = e^{-t}$; in terms of τ_e , we have $\partial_t = -\tau_e\partial_{\tau_e}$ and thus $V_0 = -\tau_e\partial_{\tau_e} + \bar{V}$. Therefore, $\bar{\Gamma} \subset \mathcal{X} \cong \tau_e^{-1}(0)$ is a V_0 -invariant set at infinity, with unstable manifold $\bar{\Gamma}^u \subset \mathcal{X} \cong \tau_e^{-1}(0)$ and stable manifold (the closure of) Γ_0^s . From this perspective, the perturbations considered here have size $1/|\log(\tau_e)|^\alpha$, i.e. are very far from differentiable in τ_e at $\tau_e = 0$. Such singular perturbations can be analyzed because of the

¹Thus, Γ^s is a ‘canonical’ object, i.e. independent of choices, whereas Γ^u is ‘noncanonical’, as it does depend on the choice of some initial V -invariant piece.

simple nature of the flow of V_0 and V in the t -variable; see also Remark 2.4. Figure 1.2 illustrates Theorem 1.1 from this point of view.

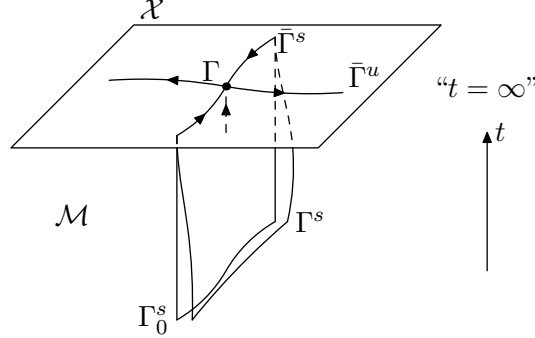


FIGURE 1.2. Illustration of Theorem 1.1 from a compactified perspective, with the (canonical) stable manifold Γ^s shown.

1.2. Propagation of singularities at trapped sets. Our second, independent, result describes the propagation of microlocal Sobolev regularity for solutions of nonelliptic pseudodifferential operators $P \in \Psi^m$ whose null-bicharacteristic flow, i.e. the flow of the Hamilton field H_p of its principal symbol p within the characteristic set $p^{-1}(0)$, has a normally hyperbolic invariant manifold $\Gamma \subset p^{-1}(0)$. We first describe this in a simple setting.

1.2.1. *Propagation on compact manifolds.* Let X be a closed manifold and suppose

$$P \in \Psi^m(X), \quad P - P^* \in \Psi^{m-2}(X),$$

is a classical ps.d.o. with principal symbol $p = \sigma(P)$ and characteristic set $\Sigma = p^{-1}(0) \cap (T^*X \setminus o)$; suppose there exists a conic, normally hyperbolic submanifold $\Gamma \subset \Sigma$ for the H_p -flow, of codimension 2 and with conic stable/unstable manifolds $\Gamma^{s/u} \subset \Sigma$ (defined in a small neighborhood of Γ) of codimension 1. Assume that the Poisson bracket of defining functions of $\Gamma^{s/u}$ inside of Σ , extended to functions on T^*X , does not vanish at Γ , and assume that in a neighborhood of Γ there exists an order function (nonvanishing and homogeneous of degree 1 in the fibers of $T^*X \setminus o$) that commutes with H_p . Then:

Theorem 1.2. *Let $v \in \mathcal{D}'(X)$, $Pv = f$, and $s \in \mathbb{R}$. Suppose $\text{WF}^{s+1}(v) \cap (\Gamma^s \setminus \Gamma) = \emptyset$ and $\text{WF}^{s-m+2}(f) \cap \Gamma = \emptyset$. Then $\text{WF}^s(v) \cap \Gamma = \emptyset$. The same conclusion remains valid when instead $\text{WF}^{s+1}(v) \cap (\Gamma^u \setminus \Gamma) = \emptyset$.*

Recall here the definition of $\text{WF}^s(u)$ in local coordinates on X : a point $(x_0, \xi_0) \in T^*\mathbb{R}^n$, $\xi_0 \neq 0$, does *not* lie in $\text{WF}^s(u)$ for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ if and only if there exist cutoff functions $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ with $\phi(x_0) \neq 0$, $\psi(\xi_0/|\xi_0|) \neq 0$, such that $\langle \xi \rangle^s \psi(\xi/|\xi|) \widehat{\phi u}(\xi) \in L^2(\mathbb{R}_\xi^n)$, where $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx$ denotes the Fourier transform.

See Example 3.5 for an explicit operator P to which Theorem 1.2 applies. One can also allow P to be a principally scalar operator acting on sections of a vector bundle, and one can furthermore allow for a nontrivial skew-adjoint part $\frac{1}{2i}(P - P^*) \in \Psi^{m-1}$ as long as its principal symbol has a suitable positive upper bound; see Remark 3.6 for details. Thus, microlocal regularity can be propagated into the trapped set, where we can control v with two derivatives less relative to elliptic estimates.

We really prove quantitative estimates for Sobolev norms of the microlocalization of v to a small neighborhood of Γ : if $B_0, B_1, G \in \Psi^0(X)$ are such that the elliptic set $\text{Ell}(B_0)$ of B_0 contains Γ , furthermore $\text{WF}'(B_0) \subset \text{Ell}(G)$, and all backwards null-bicharacteristics from $\text{WF}'(B_0)$ either tend to Γ or enter $\text{Ell}(B_1)$ in finite time, all while remaining in $\text{Ell}(G)$, then

$$\|B_0 v\|_{H^s} \leq C(\|GPv\|_{H^{s-m+2}} + \|B_1 v\|_{H^{s+1}} + \|v\|_{H^{-N}}). \quad (1.2)$$

See Figure 1.3.

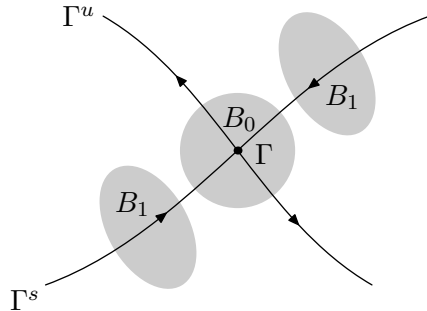


FIGURE 1.3. Illustration of Theorem 1.2: shown are the elliptic sets of B_1 (where we require a priori H^{s+1} control of u) and B_0 (where we conclude H^s regularity of u).

For $v \in C^\infty(X)$, such an estimate can be proved by a simple adaptation of the elegant semiclassical argument of Dyatlov [Dya16], which proceeds by assuming that a constant C for which this estimate holds does not exist, and reaching a contradiction using defect measures [Gér91, LP93, Tar90] and their concentration/Lipschitz properties along Γ^u . In order to *conclude* regularity of v at Γ as in Theorem 1.2 however, we proceed directly using positive commutator estimates (together with a standard regularization argument); the argument is described in §3.2 and sketched after Theorem 1.3 below. (We remark that the proof takes place entirely in the *standard* pseudodifferential calculus on X , i.e. it only utilizes quantizations of symbols in the class $S_{1,0}^m$.) An added benefit, important for applications to nonlinear problems, is the (in principle) quantitative control of the constant C on Sobolev norms of the coefficients of P . See Remarks 3.4 and 3.12.

1.2.2. *Uniform propagation near infinity.* For us, the main interest lies in analogues of Theorem 1.2 when the trapped set is ‘at infinity’ as explained after Theorem 1.1. We refer the reader to §3.2 for the general microlocal result. For the purposes of this introduction, we merely describe the special case of asymptotically Schwarzschild spacetimes; the more general case of asymptotically (subextremal) Kerr spacetimes is described in §4. The Schwarzschild spacetime with mass $m > 0$ is the Lorentzian manifold (with signature $(+, -, -, -)$)

$$(M^\circ, g_0), \quad M^\circ = \mathbb{R}_t \times X, \quad X = (2\mathbf{m}, \infty)_r \times \mathbb{S}^2 \subset \mathbb{R}_x^3, \\ g_0 = \left(1 - \frac{2\mathbf{m}}{r}\right) dt^2 - \left(1 - \frac{2\mathbf{m}}{r}\right)^{-1} dr^2 - r^2 \not{g},$$

with \not{g} the standard metric on \mathbb{S}^2 . Here, $2\mathbf{m}$ is the Schwarzschild radius, i.e. the radius of the event horizon of the black hole. The Schwarzschild metric is stationary: $\mathcal{L}_{\partial_t} g_0 = 0$. Define

the dual metric function $G_0(\zeta) = |\zeta|_{g_0^{-1}}^2$, $\zeta \in T^*M^\circ$, and denote the future component of the characteristic set, i.e. the collection of future light cones, by

$$\Sigma_0 = \{\zeta = \sigma dt + \xi dr + \eta \in T^*M^\circ \setminus o : G_0(\zeta) = 0, \sigma > 0\};$$

here $\sigma, \xi \in \mathbb{R}$ and $\eta \in T^*\mathbb{S}^2$. The lifted (future) null-geodesic flow of g_0 is the flow of the Hamilton vector field H_{G_0} on Σ_0 . Since all notions of interest here are conic in the fibers of T^*M° , let us pass to the cosphere bundle $S^*M^\circ = (T^*M^\circ \setminus o)/\mathbb{R}_+$, where \mathbb{R}_+ acts by dilations on the fibers; the rescaled vector field $\mathbf{H}_{G_0}|_\zeta := (g_0^{-1}(\zeta, dt))^{-1}H_{G_0}|_\zeta$, which is homogeneous of degree 0 in the fibers, descends to a vector field on Σ_0 , identified with a subset of S^*M° . The feature of interest of the \mathbf{H}_{G_0} -flow here is the existence of a smooth, conic, flow-invariant trapped set

$$\Gamma_0 = \{\zeta \in \Sigma_0 : r = 3\mathbf{m}, \xi = 0\} \equiv \mathbb{R}_t \times \Gamma \subset \Sigma_0.$$

Null-geodesics in Γ_0 never escape through the event horizon or to infinity, and instead, when projected to X , remain in the compact subset $r^{-1}(3\mathbf{m})$. The \mathbf{H}_{G_0} -flow in Σ_0 is r -normally hyperbolic for every r at Γ_0 , as described in §§2.4 and 4. For more general Kerr black holes (which depend, in addition on an angular momentum $\mathbf{a} \in \mathbb{R}$), this was first observed by Wunsch and Zworski [WZ11] for slowly rotating ($|\mathbf{a}| \ll \mathbf{m}$) Kerr black holes, and proved in the full subextremal range ($\mathbf{a} \in (-\mathbf{m}, \mathbf{m})$) by Dyatlov [Dya15a]; see also [Vas13] for the Kerr–de Sitter case. The unstable/stable manifolds, consisting of those covectors $\zeta \in \Sigma_0$ for which the backward/forward integral curve with initial condition ζ tends to Γ_0 , are analytic conic codimension 1 submanifolds. On Schwarzschild, they are explicitly given by

$$\begin{aligned} \Gamma_0^{u/s} &= \left\{ (t, r, \omega; \sigma, \xi, \eta) : r > 2\mathbf{m}, |\eta|_{g_0^{-1}}^2 = 27\mathbf{m}^2\sigma^2, \right. \\ &\quad \left. \xi = \pm \operatorname{sgn}(r - 3\mathbf{m}) \left(1 - \frac{2\mathbf{m}}{r}\right)^{-1} \sqrt{\sigma^2 - \left(1 - \frac{2\mathbf{m}}{r}\right)r^{-2}|\eta|_{g_0^{-1}}^2} \right\} \\ &\equiv \mathbb{R}_t \times \bar{\Gamma}^{u/s} \subset \Sigma_0, \end{aligned}$$

with the sign $+$, resp. $-$, for the unstable, resp. stable manifold, and the square root positive.

We consider metric perturbations $g = g_0 + \tilde{g}$ of g_0 which are asymptotically (as $t \rightarrow \infty$) stationary: analogously to (1.1), we assume

$$\tilde{g} \in \rho \mathcal{C}_b^\infty, \tag{1.3}$$

i.e. all components of \tilde{g} in (t, x) -coordinates are $\rho(t) = \langle t \rangle^{-\alpha}$ times functions in $\mathcal{C}_b^\infty(M^\circ)$. The rescaled Hamilton vector field \mathbf{H}_G of $G(\zeta) = |\zeta|_{g^{-1}}^2$ on the perturbed future part Σ of the characteristic set $G^{-1}(0) \subset S^*M^\circ$ fits (after a coordinate change in S^*M° and an additional rescaling) into the framework of Theorem 1.1, providing perturbed stable/unstable manifolds $\Gamma^{s/u}$ which are $\rho \mathcal{C}_b^\infty$ -perturbations of $\Gamma_0^{s/u}$.

We then study the propagation of weighted (in t) uniform (as $t \rightarrow \infty$) microlocal Sobolev regularity of solutions of wave equations on (M, g) . Working in $t \geq 1$, we say that (x_0, ζ_0) , with $\zeta_0 = \sigma_0 dt + \xi_0 \in \mathbb{R} dt \oplus T_{x_0}^*X$ nonzero, does not lie in $\operatorname{WF}_{\text{cu}}^{s,r}(v)$ for a distribution $v \in t^{-r}H^{-N}(\mathbb{R}_{t,x}^4)$ if there exists a cutoff $\chi = \chi(t)$, identically 1 for sufficiently large t , and cutoffs $\phi \in \mathcal{C}_c^\infty(X)$ (nonzero at x_0) and $\psi \in \mathcal{C}^\infty(\mathbb{S}^3)$ (nonzero at $\zeta_0/|\zeta_0|$, defined using the Euclidean norm on $\mathbb{R}^4 \cong \mathbb{R} dt \oplus T_{x_0}^*X$) such that

$$\langle \zeta \rangle^s \psi(\zeta/|\zeta|) \mathcal{F}(t^r \chi \phi v)(\zeta) \in L^2(\mathbb{R}_\zeta^4), \quad \zeta = (\sigma, \xi),$$

where $\mathcal{F}(f)(\sigma, \xi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}} e^{-i(\sigma t + \xi x)} f(t, x) dt dx$ is the spacetime Fourier transform. Thus, $\text{WF}_{\text{cu}}^{s,r}(v)$ captures those positions and spacetime frequencies where weighted (by $\langle \zeta \rangle^s$) amplitudes of high frequency components of $t^r v$ fail to be square integrable in spacetime.

Theorem 1.3. *Let $v \in t^{-r} H^{-N}(\mathbb{R}^4)$, $\square_g v = f$. Suppose that $\text{WF}^{s+1}(v) \cap \Gamma^s = \emptyset$ and $\text{WF}_{\text{cu}}^{s+1,r}(v) \cap (\bar{\Gamma}^s \setminus \Gamma) = \emptyset$; suppose further that $\text{WF}_{\text{cu}}^{s,r}(f) \cap \Gamma = \emptyset$. Then $\text{WF}_{\text{cu}}^{s,r}(v) \cap \Gamma = \emptyset$.*

There is an analogous statement for propagation from the unstable manifold into Γ , as well as extensions to principally scalar, non-self-adjoint (with suitable upper bound on the subprincipal symbol) operators between vector bundles. Thus, uniform microlocal regularity propagates from a punctured neighborhood of Γ within Γ^s into Γ , with a loss of two derivatives as in Theorem 1.2. Theorem 1.3 is closely related to the estimates in [WZ11, Dya15b, Dya16] for semiclassical operators; we discuss this further below.

Remark 1.4. Theorem 1.3 can be phrased more naturally on a compactification of M to a manifold with boundary

$$[0, \infty)_\tau \times \mathcal{X}, \quad \tau := t^{-1}, \quad (1.4)$$

in which case $\text{WF}_{\text{cu}}^{s,r}(v)$ is the (complete) cusp wave front set of v ; we explain these notions in §3.2.1, following [MM99, Vas18]. The trapped set Γ then lies over $\tau = 0$, while the stable manifold is the closure $\Gamma^s \sqcup \bar{\Gamma}^s$ of Γ^s , where one views $\bar{\Gamma}^s \subset \{\tau = 0\}$. As in the closed manifold setting, we prove Theorem 1.3 by means of a positive commutator argument (now employing the cusp pseudodifferential algebra) inspired by [Dya16]; this provides quantitative microlocal bounds in weighted Sobolev spaces analogous to (1.2).

The main motivation for this theorem is the study of quasilinear wave equations on Kerr spacetimes, specifically the nonlinear stability problem for the family of subextremal Kerr metrics as solutions of the Einstein vacuum equation. Recall that in the proof of black hole stability for cosmological Kerr–de Sitter black holes given by the author in joint work with Vasy [HV18b, Hin18] (and indeed in a simpler setting already in the paper [Hin16] on quasilinear wave equations on de Sitter spacetimes) we used microlocal propagation estimates to get (almost) sharp control on the regularity of solutions of linearized wave equations on asymptotically de Sitter and Kerr–de Sitter backgrounds. Thus, Theorem 1.3 can be regarded as a ‘black box’ result, which due to the microlocal nature of the assumptions and the conclusion can be concatenated with radial point estimates [Vas13, §2], real principal type propagation estimates [DH72], and microlocal elliptic estimates in order to show global regularity of solutions of wave equations on asymptotically Kerr spacetimes which are a priori known to be polynomially bounded. See [HV16, §2] for more details on this approach, and² [HV17, Figure 11] for a schematic depiction of the global structure of the null-geodesic flow in phase space on Kerr–de Sitter spacetimes which close to the black hole is the same as for subextremal Kerr black holes.

The proof of Theorem 1.3 in the more general subextremal Kerr setting depends, via Theorem 1.1, crucially on the aforementioned work [WZ11], as it strongly uses the *dynamical (normally hyperbolic) nature of the trapping*. By contrast, the special algebraic structure of the trapped set of Kerr, namely the complete integrability of the null-geodesic flow which allows for separation of variables, is *irrelevant* (except insofar as it is useful for actually proving the normal hyperbolicity); in fact, the complete integrability by itself seems to

² $\Gamma_{+/-}^-$ in the reference take the place of $\Gamma^{u/s}$ here.

be insufficient for proving this theorem, as exact control of the perturbed (and in general certainly not completely integrable) dynamics of g is strongly used in the proof.

As in Dyatlov’s work [Dya16], a key ingredient in the proof of Theorem 1.3 is that the microlocalization $\Phi^u v$ of v , where Φ^u quantizes a defining function of Γ^u , satisfies a pseudo-differential equation which effectively has a damping term at Γ . (Thus, Φ^u microlocalizes in a weak manner away from Γ^u .) For this step it is *crucial* that Γ^u be *exactly* invariant by the H_G -flow (rather than merely asymptotically so, as is the case for Γ_0^u); this is discussed after equation (3.28). Using a *quantitative* version of real principal type propagation based on a careful construction of commutants and Gårding’s inequality, this equation implies that the squared $t^{-r} H^s$ mass of v is evenly spread out along $\bar{\Gamma}^u$. On the other hand, quantitative propagation for $\square_g v = f$ and the unstable nature of the H_G -flow on $\bar{\Gamma}^u$ imply that the squared mass on the part of $\bar{\Gamma}^u$ which is distance δ to 1 away from Γ can be bounded (modulo contributions from f , and from u away from $\bar{\Gamma}^u$) by $\log(\delta^{-1})$ times the squared mass δ -close to Γ , which is $\sim \delta$ due to the uniform spreading along $\bar{\Gamma}^u$. Since $\delta \log(\delta^{-1}) \ll 1$ for small $\delta > 0$, this provides the desired bound of v near Γ .

Previously, spacetime bounds at normally hyperbolic trapping were obtained by the author in joint work with Vasy [HV14]; these took place on *exponentially* weighted (growing) function spaces, or on unweighted but mildly degenerate (at Γ) function spaces when studying symmetric operators. In the present paper, we obtain estimates for nonsymmetric operators on polynomially weighted (possibly *decaying*), or even mildly exponentially decaying function spaces, though with additional loss of regularity; see Remarks 3.11 and 4.7. The more delicate estimates we prove here were not needed in the analysis of quasilinear waves on (and the nonlinear stability of) Kerr–de Sitter spacetimes [HV16, HV18b], since the exponential decay of metric perturbations there implied that estimates on exponentially growing function spaces together with an exact analysis of the stationary (exact Kerr–de Sitter) model were sufficient to prove exponential decay to a finite sum of resonant states (mode solutions). In particular, one could in principle have used separation of variable techniques at the trapped set in those works, akin to [Dya11b, Dya12], though this would have distracted from the conceptual, namely *dynamical*, reason for having (high energy) estimates at the trapped set.

1.3. Prior and related work. Metcalfe–Tataru–Tohaneanu [MTT12] prove sharp pointwise t^{-3} decay (‘Price’s law’) for scalar waves on perturbations $g = g_0 + \tilde{g}$ of slowly rotating Kerr spacetimes (M°, g_0) where the perturbation \tilde{g} is integrable in time near the trapped set, roughly corresponding to $\rho = \langle t \rangle^{-\alpha}$ in (1.3) with $\alpha > 1$; this extends Tataru’s proof of Price’s law [Tat13] on stationary spacetimes to the case of suitable asymptotically Kerr spacetimes. The proof is perturbative, based on [TT11], and directly uses the pseudodifferential multiplier near the trapped set of the stationary slowly rotating Kerr spacetime which was constructed in [TT11] (which in turn relies on the careful choice of vector field multiplier on Schwarzschild in [MMTT10]). The integrable time decay of the metric perturbation allows one to estimate error bulk terms in integrated local energy decay estimates essentially using energy boundedness. High frequency analysis directly on the perturbed spacetime is thereby sidestepped.

There has been relatively little work on quasilinear wave equations on Kerr spacetimes. Klainerman–Szeftel [KS17] proved the nonlinear stability of Schwarzschild spacetimes under axially symmetric polarized perturbations, with integrable decay rates of the metric

perturbation. Lindblad–Tohaneanu [LT18] prove a general existence result for scalar quasi-linear wave equations where the metric (depending on the solution) decays to a Schwarzschild metric. For results on semilinear wave equations, based on vector field methods or Strichartz estimates, see [DR05, BS06, Luk13, LMS⁺14] and references therein; note that for semilinear equations, the metric is *fixed*.

The study of waves on Schwarzschild and Kerr spacetimes has a long history, starting with [Wal79, KW87]; Tataru [Tat13] and Dafermos–Rodnianski–Shlapentokh–Rothman [DRSR16] proved sharp polynomial decay for scalar waves on all subextremal Kerr spacetimes, see also [BS06, FKS06, TT11, DSS11, DSS12, Toh12, Luk13, AB15a, AB15b], and the recent [DHR19b, DHR19a, FS16, ABBM19, HHV20] for results concerning the linear stability of Schwarzschild and Kerr black holes.

Aretakis [Are12] obtained results for axisymmetric waves on extremal Kerr spacetimes. The trapped set extremal Kerr spacetimes is not normally hyperbolic (see [Dya15a, Figure 2(a)]), hence our results do not apply; upon restriction to axial symmetry however, the trapped set is effectively like that on Schwarzschild spacetimes, as the part of the trapped set on extremal Kerr (with mass m) which has azimuthal frequency 0 lies only over the sphere $r = (1 + \sqrt{2})m$, see [Are12, §1.2.2].

Resonance expansions for scalar waves on Kerr–de Sitter spacetimes were proved by Dyatlov [Dya11b, Dya11a, Dya12] and Vasy [Vas13], following Bony–Häfner [BH08] and building on the work of Sá Barreto–Zworski [SBZ97]; see also [MSBV14]. Many of these results rely on delicate separation of variables techniques at the trapped set; the work [WZ11] was the first to utilize the (stable under perturbations!) dynamical nature of the flow directly. For tensor-valued waves in the presence of trapping, see [Hin17, HV18a, HV18b, Hin18].

Most prior results on microlocal estimates at normally hyperbolic trapping take place in the semiclassical setting, which is closely related, via the Fourier transform in time, to estimates for stationary (time-translation-invariant) problems. This is the context of [WZ11, Dya16, DZ13] as well as the fine analysis of resonances associated with normally hyperbolically trapped sets by Dyatlov [Dya15b]; see Gérard–Sjöstrand [GS87], Christianson [Chr07] for the case of isolated hyperbolic orbits, and Ikawa [Ika83] and Gérard [G88] for scattering by several convex obstacles. Nonnenmacher–Zworski [NZ13] study estimates at trapped sets when the stable and unstable normal bundles have low regularity. Bony–Fujiie–Ramond–Zerzeri [BFRZ18] present a detailed analysis of resonances generated by trapped sets with homoclinic trajectories. We also mention the work by Bony–Burq–Ramond [BBR10] who prove that a loss of semiclassical control (powers of h^{-1}), which heuristically corresponds to a loss of control in the Sobolev regularity sense in (1.2),³ *does* occur in the presence of trapping; see also Ralston [Ral69]. We refer the reader to the excellent review articles [Zwo17, Wun12] as well as [DZ19, §6.3] for further historical background and references in these directions.

1.4. Structure of the paper.

- In §2, we prove the dynamical results on perturbations of dynamical systems with normally hyperbolic invariant sets, in particular proving Theorem 1.1.

³We encourage the reader to compare [HV16, Theorem 4.7] with the estimate (1.2).

- In §3, we establish the microlocal propagation results, Theorem 1.2 and the pseudodifferential generalization of Theorem 1.3.
- In §4, we combine the first two parts, thus obtaining a description of trapping on asymptotically Kerr spacetimes, and the accompanying microlocal estimates of Theorem 1.3.

We stress that §2 and §3 are independent and can be read in any order.

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2. STABLE AND UNSTABLE MANIFOLDS

In this first, dynamical, part of the paper, we closely follow the arguments and notation of [HPS77].

2.1. Function spaces and Lipschitz jets. Denote by

$$0 < \rho \in C^\infty(\mathbb{R})$$

a weight which is monotonically decreasing with $\lim_{t \rightarrow \infty} \rho(t) = 0$, and so that

$$|\rho^{(k)}| \leq C_k \rho \quad \forall k \in \mathbb{N}_0. \quad (2.1)$$

For example, one can take $\rho(t) = \langle t \rangle^{-\alpha}$ or $e^{-\alpha t}$ with $\alpha > 0$ fixed. (These are natural choices for asymptotically Kerr, resp. Kerr–de Sitter spaces.) Let \mathcal{X} denote a closed manifold⁴ and put

$$\mathcal{M} = \mathbb{R}_t \times \mathcal{X}.$$

Denoting by $\mathcal{C}_b^0(\mathcal{M})$ the space of bounded continuous functions, we define

$$\begin{aligned} \mathcal{C}_b^r(\mathcal{M}) := \{u \in \mathcal{C}_b^0(\mathcal{M}) : V_1 \dots V_k u \in \mathcal{C}_b^0(\mathcal{M}) \quad \forall 1 \leq k \leq r, \\ V_i \in \mathcal{V}(\mathcal{X}) \cup \{\partial_t\}, \quad i = 1, \dots, k\}, \end{aligned} \quad (2.2)$$

and $\mathcal{C}_b^\infty(\mathcal{M}) := \bigcap_{r \in \mathbb{N}} \mathcal{C}_b^r(\mathcal{M})$. The weighted analogues are

$$\rho \mathcal{C}_b^r(\mathcal{M}) := \{u : \mathcal{M} \rightarrow \mathbb{R} : u/\rho \in \mathcal{C}_b^r(\mathcal{M})\}, \quad r \in \mathbb{N}_0 \cup \{\infty\}.$$

Note that by (2.1), differentiation along ∂_t or $V \in \mathcal{V}(\mathcal{X})$ gives continuous maps $\rho \mathcal{C}_b^r \rightarrow \rho \mathcal{C}_b^{r-1}$ for all $r \geq 1$. For open sets $\mathcal{U} \subset \mathcal{M}$, we denote the space of restrictions to \mathcal{U} by

$$\rho \mathcal{C}_b^r(\mathcal{U}) = \{u|_{\mathcal{U}} : u \in \rho \mathcal{C}_b^r(\mathcal{M})\}.$$

⁴All our arguments below will be local near the closed positive codimensional submanifold Γ ; the structure of \mathcal{X} away from Γ will be irrelevant.

If $E \rightarrow \mathcal{X}$ is a vector bundle and

$$\pi_{\mathcal{X}}: \mathcal{M} \rightarrow \mathcal{X} \quad (2.3)$$

denotes the projection, we similarly define $\mathcal{C}^\infty(\mathcal{M}; \pi_{\mathcal{X}}^*E)$ and $\rho\mathcal{C}^\infty(\mathcal{M}; \pi_{\mathcal{X}}^*E)$ using local trivializations of E or, equivalently, using a connection to differentiate sections of E along vector fields on \mathcal{X} .

We recall from [HPS77, §3] the useful notion of *Lipschitz jets*:

Definition 2.1. Let (X, d_X) and (Y, d_Y) denote two metric spaces, and fix a point $x \in X$.

- (1) We say that the continuous maps $g_1, g_2: X \rightarrow Y$, defined near x , are *tangent at x* iff $g_1(x) = g_2(x)$ and

$$d_x(g_1, g_2) := \limsup_{x' \rightarrow x} \frac{d_Y(g_1(x'), g_2(x'))}{d_X(x, x')} = 0.$$

- (2) The *Lipschitz jet* of a continuous function $g: X \rightarrow Y$ at x , denoted $J_x g$, is the equivalence class of g modulo tangency at x , i.e. the set of all continuous maps defined near x which are tangent to g at x .
- (3) The set of Lipschitz jets of maps carrying x into y is denoted

$$J(X, x; Y, y) = \{J_x g: g \text{ is a continuous map } X \rightarrow Y, g(x) = y\}.$$

- (4) For $j_1, j_2 \in J(X, x; Y, y)$, we define $d(j_1, j_2) := d_x(g_1, g_2)$, where g_i is a representative of j_i ; this is independent of the choices of representatives.
- (5) Denoting by y the constant map $X \rightarrow Y$, $x' \mapsto y$, we define by

$$L_x g := d(J_x g, J_x y) \in [0, \infty]$$

the *Lipschitz constant of g at x* .

- (6) We define the space of bounded Lipschitz jets by

$$J^b(X, x; Y, y) := \{j \in J(X, x; Y, y): d(j, J_x y) < \infty\}.$$

If X, Y are differentiable manifolds, we set

$$J^d(X, x; Y, y) := \{j \in J^b(X, x; Y, y): j \text{ has a differentiable representative}\}.$$

If X, Y are Banach spaces, we recall from [HPS77, Theorem (3.3)] that $J^b(X, 0; Y, 0)$, equipped with the norm $|j| := d(j, 0)$, is a Banach space, and $J^d(X, 0; Y, 0)$ is a closed subspace.

We record that if $a \in \rho\mathcal{C}_b^1(\mathbb{R} \times \mathbb{R}^n)$, then there exists $C > 0$ such that

$$\begin{aligned} |a(t, x) - a(t', x')| &\leq C(\rho_+(t, t')|t - t'| + \rho_-(t, t')|x - x'|), \\ \rho_+(t, t') &:= \max\{\rho(t), \rho(t')\}, \quad \rho_-(t, t') := \min\{\rho(t), \rho(t')\}; \end{aligned} \quad (2.4)$$

in fact, we can take $C = \sup_{(t, x)} (L_{(t, x)}(a)/\rho(t))$.

2.2. An invariant section theorem. As an illustration of the relevant techniques, and as a technical tool for later, we prove existence and higher regularity of invariant sections for fiber contractions; this is a version of [HPS77, Theorem 3.5].

Theorem 2.2. *Let $r \in \mathbb{N}$. Let \mathcal{X} be a closed Riemannian manifold, and let \mathcal{X}_1 be an open subset. Let $\bar{f}: \bar{\mathcal{X}}_1 \rightarrow \mathcal{X}$ be a smooth map which is a diffeomorphism onto its image, and suppose \bar{f} overflows \mathcal{X}_1 : $\bar{f}(\bar{\mathcal{X}}_1) \supset \mathcal{X}_1$. Let moreover $\mathcal{E} \rightarrow \mathcal{X}$ be a vector bundle equipped with a fiber metric, and let $\bar{\ell}: \bar{\mathcal{E}}|_{\mathcal{X}_1} \rightarrow \mathcal{E}$ be a fiber bundle map⁵ covering \bar{f} . Let*

$$\alpha_x := \|D_{\bar{f}(x)}\bar{f}^{-1}\|, \quad k_x := \sup_{e \in \bar{\mathcal{E}}_x} \|D_e(\bar{\ell}|_{\bar{\mathcal{E}}_x})\|, \quad x \in \mathcal{X}_1,$$

denote the base contraction and fiber expansion, respectively, and suppose that

$$\sup_{x \in \mathcal{X}_1} k_x, \quad \sup_{x \in \mathcal{X}_1} k_x \alpha_x^r < 1. \quad (2.5)$$

Let $\bar{\sigma}: \mathcal{X}_1 \rightarrow \bar{\mathcal{E}}|_{\mathcal{X}_1}$ denote the unique $\bar{\ell}$ -invariant section, i.e. $\bar{\ell}(\bar{\sigma}(\mathcal{X}_1)) \cap \mathcal{E}|_{\mathcal{X}_1} = \bar{\sigma}(\mathcal{X}_1)$, which is of class C^r .⁶

Let $\mathcal{M} = \mathbb{R}_t \times \mathcal{X}$ and $\mathcal{M}_1 = \mathbb{R}_t \times \mathcal{X}_1$, and put $f_0(t, x) = (t - 1, \bar{f}(x))$ for $(t, x) \in \mathcal{M}_1$. With $\mathcal{E}_0 := \pi_{\mathcal{X}}^* \bar{\mathcal{E}}$ denoting the pullback bundle, $\pi_{\mathcal{X}}: \mathcal{M} \rightarrow \mathcal{X}$ being the projection, extend $\bar{\ell}$ to the map $\ell_0: (t, x, e) \mapsto (t - 1, \bar{\ell}(x, e))$ which covers f_0 . The section $\sigma_0(t, x) = (t, \bar{\sigma}(x))$ is a stationary and invariant section for ℓ_0 .

Let next $f: \mathcal{M}_1 \rightarrow \mathcal{M}$ and $\ell: \mathcal{E}_0|_{\mathcal{M}_1} \rightarrow \mathcal{E}_0$ be $\rho\mathcal{C}_b^r$ -perturbations of f_0 and ℓ_0 , defined for $t > t_0$ with $t_0 \in \mathbb{R}$ fixed, with ℓ covering f . That is, fix a finite cover of $\bar{\mathcal{X}}_1$ by coordinate systems, and fix trivializations of $\bar{\mathcal{E}}$ over these; write $\ell(t, x, e) = (f(t, x), \check{\ell}(t, x, e))$ in local coordinates, with $\check{\ell}(t, x, e)$ valued in the fibers of \mathcal{E}_0 , likewise for ℓ_0 ; then

$$\begin{aligned} f(t, x) - f_0(t, x) &= \tilde{f}(t, x), & \tilde{f}(t, x) &\in \rho\mathcal{C}_b^r, \\ \check{\ell}(t, x, e) - \check{\ell}_0(t, x, e) &= \tilde{\ell}(t, x, e), & \tilde{\ell}(t, x, e) &\in \rho\mathcal{C}_b^r. \end{aligned}$$

Furthermore, assume $\pi_T \tilde{f}(t, x) = 0$ where $\pi_T: \mathbb{R} \times \mathcal{X} \rightarrow \mathbb{R}$ projects onto the first factor.

Then there exists an ℓ -invariant section $\sigma_\ell: \mathcal{M}_1 \rightarrow \mathcal{E}_0$, defined for $t > t_0$, that is, $\ell(\sigma_\ell(\mathcal{M}_1 \cap \{t > t_0 + 1\})) \cap \mathcal{E}_0|_{\mathcal{M}_1} = \sigma_\ell(\mathcal{M}_1 \cap \{t > t_0\})$, such that

$$\sigma_\ell - \sigma_0 \in \rho\mathcal{C}_b^r(\mathcal{M}_1; \mathcal{E}_0).$$

This section is unique among sections σ_ℓ with $\sigma_\ell - \sigma_0 \in \rho\mathcal{C}_b^0$.

For a linear transformation A between two normed vector spaces, we put

$$m(A) := \inf_{\|x\|=1} \|Ax\|.$$

Proof of Theorem 2.2. We shall follow [HPS77, §3] closely; the strategy is to show that the perturbation \tilde{f} does not destroy the contraction properties of the stationary model f_0 . By replacing $\bar{\mathcal{E}}$ with a vector bundle $\bar{\mathcal{E}} \oplus \bar{\mathcal{E}}'$ for suitable $\bar{\mathcal{E}}' \rightarrow \mathcal{X}$, we may assume that $\bar{\mathcal{E}}$ is trivial with typical fiber denoted E ; we extend $\bar{\ell}$ by mapping $(x, e \oplus e') \mapsto \bar{\ell}(x, e) \oplus 0$, similarly for ℓ_0, ℓ ; note that an invariant section is necessarily valued in $\mathcal{E} \oplus 0$. It suffices to construct σ as a section over

$$\mathcal{M}_1(\epsilon) := \{(t, x) \in \mathcal{M}_1 : t > \epsilon^{-1}\}$$

⁵That is, we do not require it to be fiber-linear.

⁶This is the content of the first part of [HPS77, Theorem 3.5].

for any small $\epsilon > 0$, as σ in $t > t_0$ can be reconstructed from this by repeated application of ℓ . At first, we will seek the invariant section in the space

$$\Sigma^0(\epsilon) := \{\text{sections } \sigma: \mathcal{M}_1(\epsilon) \rightarrow \mathcal{E}_0: |\sigma(t, x) - \sigma_0(t, x)| \leq C_\Sigma \rho(t)\},$$

where C_Σ will be specified later, see (2.9). We then wish to consider the map $\ell_\#$ on $\Sigma^0(\epsilon)$, defined by

$$\ell_\# \sigma(t, x) = \ell \sigma f^{-1}(t, x), \quad (t, x) \in \mathcal{M}_1(\epsilon);$$

this is a ‘graph transform’ of σ : the graph of $\ell_\# \sigma$ is the image of the graph of σ under ℓ . We first need to check that f^{-1} is well-defined and maps $\mathcal{M}_1(\epsilon)$ into itself.

Step 1: control of f^{-1} . First, we show (using $\tilde{f} \in \rho \mathcal{C}_b^1$) that $f|_{\mathcal{M}_1(\epsilon)}$ is injective for small $\epsilon > 0$. Indeed, $f(t, x) = f(t', x')$ requires $t' = t$; but then, in geodesic coordinates centered at $x \in \mathcal{X}_1$ and $\tilde{f}(x)$, working in a geodesic ball around x with small radius ϵ_0 (which we will take small, independently of ϵ), and letting $\mu_x := m(D_x \tilde{f}) = \alpha_{\tilde{f}^{-1}(x)}^{-1} > 0$,

$$\begin{aligned} |f(t, x) - f(t, x')| &\geq |f_0(t, x) - f_0(t, x')| + |\tilde{f}(t, x) - \tilde{f}(t, x')| \\ &\geq (\mu_x - o(1) - \tilde{C}\rho(t))|x - x'|, \quad \epsilon_0 \rightarrow 0, \end{aligned}$$

where \tilde{C} is the Lipschitz constant of $\rho^{-1}\tilde{f}$. This implies injectivity of f restricted to small geodesic balls in \mathcal{X} provided t is sufficiently large; the injectivity of f on the compact set $\overline{\mathcal{X}_1}$ then implies the injectivity of f on $\mathcal{M}_1(\epsilon)$ for $\epsilon > 0$ small.

Given t large and $y \in \overline{\mathcal{X}_1}$, we can then solve $f(t+1, x) = (t, y)$ with $x \in \mathcal{X}_1$. Indeed, working in local coordinates, the map

$$x \mapsto \bar{f}^{-1}(y - \pi_{\mathcal{X}} \tilde{f}(t+1, x)), \quad (2.6)$$

for x in an ϵ_0 -neighborhood of $\bar{f}^{-1}(y)$, is well-defined for large t (thus small \tilde{f}) by the overflow assumption on \bar{f} , and is a contraction, again since \tilde{f} and its Lipschitz constant decay as $t \rightarrow \infty$. From this construction, we infer that the inverse map

$$g: \mathcal{M}_1(\epsilon) \rightarrow \mathcal{M}_1(\epsilon), \quad g(t, y) = f^{-1}(t, y) = (t+1, x),$$

satisfies Lipschitz bounds

$$\begin{aligned} |\pi_{\mathcal{X}} g(t, y) - \pi_{\mathcal{X}} g(t, y')| &\leq (\alpha_{\bar{f}^{-1}(y)} + o(1) + \tilde{C}\rho(t))|y - y'| \\ &\leq (\alpha_{\bar{f}^{-1}(y)} + o(1))|y - y'|, \quad \epsilon \rightarrow 0, \end{aligned} \quad (2.7)$$

when $|y - y'| < \epsilon$ and $t > \epsilon^{-1}$. Writing $g_0(t, y) = f_0^{-1}(t, y) = (t+1, \bar{f}^{-1}(y))$, we also note that

$$|g(t, y) - g_0(t, y)| \leq \tilde{C}\rho(t) \quad (2.8)$$

using the description of $\pi_{\mathcal{X}} g(t, y)$ as the fixed point of (2.6).

Step 2: existence and uniqueness in $\rho \mathcal{C}_b^0$. This uses the $r = 1$ assumptions. Denote by $\pi_E: \mathcal{E}_0 \rightarrow E$ the projection onto the fiber, and write sections of \mathcal{E}_0 as

$$\sigma(t, x) = (t, x, \check{\sigma}(t, x)), \quad \check{\sigma} = \pi_E \sigma.$$

Equip $\Sigma^0(\epsilon)$ with the (complete) \mathcal{C}^0 metric. For $x = \pi_{\mathcal{X}} g(t, y)$ and $x_0 = \pi_{\mathcal{X}} g_0(t, y)$, and using (2.8), we thus have

$$\begin{aligned} &|\ell_\# \sigma(t, y) - \sigma_0(t, y)| \\ &= |\ell_\# \sigma(t, y) - ((\ell_0)_\# \sigma_0)(t, y)| \end{aligned}$$

$$\begin{aligned}
&= |\ell(g(t, y), \check{\sigma}(g(t, y))) - \ell_0(g_0(t, y), \check{\sigma}_0(g_0(t, y)))| \\
&\leq |(\check{\ell} - \check{\ell}_0)(g(t, y), \check{\sigma}(g(t, y)))| + |\check{\ell}_0(g(t, y), \check{\sigma}(g(t, y))) - \check{\ell}_0(g_0(t, y), \check{\sigma}_0(g_0(t, y)))| \\
&\quad + |\check{\ell}_0(g(t, y), \check{\sigma}_0(g_0(t, y))) - \check{\ell}_0(g_0(t, y), \check{\sigma}_0(g_0(t, y)))| \\
&\quad + |\check{\ell}_0(g_0(t, y), \check{\sigma}_0(g_0(t, y))) - \check{\ell}_0(g_0(t, y), \check{\sigma}_0(g_0(t, y)))| \\
&\leq \tilde{C}_1 \rho(t) + k_x |\check{\sigma}(g(t, y)) - \check{\sigma}_0(g_0(t, y))| + C_0 \tilde{C} \rho(t) + k_{x_0} C_0 \tilde{C} \rho(t) \\
&\leq C_\Sigma \rho(t),
\end{aligned}$$

where \tilde{C}_1 , \tilde{C} , resp. C_0 , are upper bounds for $\tilde{\ell}$, the Lipschitz constant of \tilde{f} (via (2.8)), resp. ℓ_0 , $\check{\sigma}_0$; the final inequality requires $\tilde{C}_1 + k_x C_\Sigma + C_0 \tilde{C}(1 + k_{x_0}) \leq C_\Sigma$, which can be arranged to hold for all x by fixing

$$C_\Sigma > \frac{\tilde{C}_1 + C_0 \tilde{C}(1 + K)}{1 - K}, \quad K := \sup_{x \in \mathcal{X}_1} k_x. \quad (2.9)$$

Thus $\ell_\# \sigma \in \Sigma^0(\epsilon)$. We further estimate for $\sigma, \sigma' \in \Sigma^0(\epsilon)$:

$$|\ell_\# \sigma(t, y) - \ell_\# \sigma'(t, y)| \leq (k_y + o(1)) |\sigma(t + 1, y) - \sigma'(t + 1, y)|, \quad \epsilon \rightarrow 0.$$

The contraction mapping principle implies the existence of a unique invariant section of $\ell_\#$,

$$\sigma_\ell \in \Sigma^0(\epsilon) \subset \sigma_0 + \rho \mathcal{C}_b^0.$$

Step 3: Lipschitz regularity ($r = 1$). Using the assumptions for $r = 1$, we next prove that σ_ℓ is Lipschitz; this uses the estimate (2.7). Consider the space of sections

$$\Sigma(\epsilon) := \{\sigma \in \Sigma^0(\epsilon) : L_{(t,x)}(\check{\sigma} - \check{\sigma}_0) \leq C_\Sigma \rho(t)\}. \quad (2.10)$$

Here, the Lipschitz constant is defined using the triviality of $\bar{\mathcal{E}}$ by

$$L_{(t,x)}(\check{\sigma}) := \limsup_{(s,y) \rightarrow (t,x)} \frac{|\check{\sigma}(s, y) - \check{\sigma}(t, x)|_{\bar{\mathcal{E}}_x}}{|s - t| + d(y, x)}, \quad (2.11)$$

where d is a Riemannian distance function on \mathcal{X} . We contend that $\ell_\#(\sigma) \in \Sigma(\epsilon)$ for $\sigma \in \Sigma(\epsilon)$. Indeed, for $(t + 1, x) = g(t, y)$, we have the bound

$$\begin{aligned}
L_{(t,y)}(\pi_E(\ell_\# \sigma - \sigma_0)) &\leq L_{(t,y)}(\pi_E(\ell \sigma - \ell_0 \sigma)g) + L_{(t,y)}(\pi_E(\ell_0 \sigma - \ell_0 \sigma_0)g) \\
&\leq (L_{(t,y)}(\tilde{\ell})L_{(t+1,x)}(\check{\sigma}) + L_{(t,y)}(\pi_E \ell_0)L_{(t+1,x)}(\check{\sigma} - \check{\sigma}_0))L_{(t,x)}(g) \\
&\leq (\tilde{C} \rho(t)(C_0 + C_\Sigma \rho(t)) + k_{\bar{f}^{-1}(y)} C_\Sigma \rho(t + 1))(\alpha_{\bar{f}^{-1}(y)} + o(1)) \\
&\leq (k_x \alpha_x + o(1))C_\Sigma \rho(t), \quad \epsilon \rightarrow 0
\end{aligned}$$

where in the last step we used $|\bar{f}^{-1}(y) - x| \leq \tilde{C} \rho(t)$ from (2.8), and also that $\tilde{C} C_0 + k_x C_\Sigma < C_\Sigma$ by our choice of Σ . In view of assumption (2.5), this proves our contention. Applying the contraction mapping principle produces a unique $\ell_\#$ -invariant section in $\Sigma(\epsilon)$, which must be equal to that constructed in the previous step. Thus,

$$\sigma_\ell \in \Sigma(\epsilon).$$

Step 4: pointwise differentiability ($r = 1$). As in [HPS77, §3], consider the bundle $J^b \rightarrow f(\mathcal{M}_1(\epsilon)) =: \mathcal{M}_2(\epsilon)$, where the base is equipped with the discrete topology;⁷ the fibers are

$$J_{(t,x)}^b = \{J_{(t,x)}\sigma \in J^b(\mathcal{M}_2(\epsilon), (t, x); \mathcal{E}_0, \sigma_\ell(t, x)) : \sigma \in \Sigma(\epsilon)\},$$

where we define $\sigma_\ell = \sigma$ on $\mathcal{M}_1(\epsilon)$ and $\sigma_\ell(t, x) = \ell(\sigma(g(t, x)))$ for $(t, x) \in \mathcal{M}_2(\epsilon) \setminus \mathcal{M}_1(\epsilon)$, and membership in $\Sigma(\epsilon)$ near such (t, x) is defined by the bounds in (2.10). The bundle map ℓ induces a bundle map $J\ell$ on $J^b|_{\mathcal{M}_1(\epsilon)}$, covering f , by

$$J\ell(J_{(t,x)}\sigma) = J_{f(t,x)}(\ell_\# \sigma).$$

This is well-defined by the estimates from the previous step. It is also a fiber contraction:

$$\begin{aligned} |J\ell(J_{(t,x)}\sigma) - J\ell(J_{(t,x)}\sigma')| &= L_{f(t,x)}(\pi_E \ell \sigma g - \pi_E \ell \sigma' g) \\ &\leq L_{\sigma_\ell(t,x)}(\pi_E \ell)L_{(t,x)}(\sigma - \sigma')L_{f(t,x)}(g) \\ &\leq (k_x + \tilde{C}\rho(t))(\alpha_{\bar{f}-1}\pi_{\mathcal{X}}f(t,x) + o(1))L_{(t,x)}(\sigma - \sigma') \\ &\leq (k_x\alpha_x + o(1))L_{(t,x)}(\sigma - \sigma'), \quad \epsilon \rightarrow 0, \end{aligned}$$

by (2.7). The contraction mapping principle produces a unique bounded $J\ell$ -invariant section $\sigma_{J\ell}$ of J . On the other hand, $J\sigma_\ell$ is a bounded section of J^b which by construction is $J\ell$ -invariant, hence

$$J\sigma_\ell = \sigma_{J\ell}.$$

Now, $J\ell$ preserves the closed subbundle $J^d \rightarrow \mathcal{M}_2(\epsilon)$ of Lipschitz jets of differentiable sections, as g is differentiable whenever f is. Thus, $\sigma_{J\ell}$ is necessarily a section of J^d , proving the pointwise differentiability of σ_ℓ .

Step 5: $\rho\mathcal{C}_b^1$ -regularity. Consider the bundle $L \rightarrow \mathcal{M}_2(\epsilon)$, where the base carries its standard topology again, and the fibers of L are spaces of linear maps:

$$L_{(t,x)} = \mathcal{L}(T_{(t,x)}\mathcal{M}, (\mathcal{E}_0)_{(t,x)}).$$

We shall work in the subbundle

$$\mathcal{B}_{(t,x)} = \{P \in L_{(t,x)} : \|P - D_{(t,x)}\sigma_0\| \leq C_\Sigma\rho(t)\},$$

on which we have a bundle map $L\ell$ which acts on $P \in L_{(t,x)}$ by

$$\text{graph}(L\ell(P)) = (D_{\sigma(t,x)}\ell)(\text{graph}(P))$$

where on the right, we write, using the triviality of \mathcal{E}_0 ,

$$\text{graph}(P) := \{\xi + P\xi : \xi \in T_{(t,x)}\mathcal{M}\} \subset T_{(t,x,\check{\sigma}(t,x))}\mathcal{E}_0,$$

and similarly on the left. The previous estimates show that $L\ell$, which covers the base map f , contracts the fibers by $k_x\alpha_x + o(1)$, hence, using the contraction mapping principle as before, there exists a unique bounded invariant section of $L\ell$, denoted

$$\sigma_{L\ell} : \mathcal{M}_1 \rightarrow \mathcal{B}.$$

Note that $L\ell$ is \mathcal{C}^0 since ℓ is \mathcal{C}^1 . Therefore, the section $\sigma_{L\ell}$ is necessarily equal to the unique continuous invariant section. The (a priori discontinuous) section $D\sigma_\ell$ of \mathcal{B} must be equal to $\sigma_{L\ell}$ and is therefore continuous, which gives

$$\sigma_\ell - \sigma_0 \in \rho\mathcal{C}_b^1$$

⁷This circumvents difficulties in giving J^b the structure of a Banach bundle over $\mathcal{M}_2(\epsilon)$ with its usual topology; see also the proof of [HPS77, Theorem (3.5)].

under the $r = 1$ assumptions.

Step 6: $\rho\mathcal{C}_b^r$ -regularity. We argue inductively. Thus, assume $\sigma_\ell - \sigma_0 \in \rho\mathcal{C}_b^{r-1}$, $r \geq 2$. Consider again the map $L\ell$ from the previous step: under the $\rho\mathcal{C}_b^r$ -assumptions of the theorem, $L\ell$ is a $\rho\mathcal{C}_b^{r-1}$ -perturbation of its stationary analogue $L\ell_0$, and it contracts fibers by $k_x\alpha_x + o(1)$. Therefore, it satisfies the $\rho\mathcal{C}_b^{r-1}$ -assumptions of the theorem, which implies that $\sigma_{L\ell} - \sigma_{L\ell_0} \in \rho\mathcal{C}_b^{r-1}$. Therefore, $\sigma_\ell - \sigma_0 \in \rho\mathcal{C}_b^r$, finishing the inductive step. \square

2.3. Stable/unstable manifold theorem for maps. We now turn to the main dynamical result of this paper. Let \mathcal{X} denote a closed n -dimensional ($n \geq 1$) manifold. We make the following assumptions:

- (I.1) Γ is a closed C^∞ submanifold of \mathcal{X} ;
- (I.2) $\bar{f}: \mathcal{X} \rightarrow \mathcal{X}$ is a C^∞ map so that $\bar{f}(\Gamma) = \Gamma$, and there exists an open neighborhood \mathcal{U} of Γ so that $\bar{f}: \mathcal{U} \rightarrow \bar{f}(\mathcal{U})$ is a diffeomorphism;
- (I.3) there is a C^∞ bundle splitting

$$T_\Gamma\mathcal{X} = T\Gamma \oplus \bar{N}^u \oplus \bar{N}^s \quad (2.12)$$

which is preserved by the linearization $D\bar{f}$ at Γ . Denote

$$\Gamma_p\bar{f} := D_p\bar{f}|_{T_p\Gamma}, \quad \bar{N}_p^{u/s}\bar{f} := D_p\bar{f}|_{\bar{N}_p^{u/s}};$$

- (I.4) for all $r \in \mathbb{N}$, the map \bar{f} is *immediately relatively r -normally hyperbolic* at Γ in the sense of [HPS77, Definition 2]. That is, for all r , there exist fiber metrics on the summands in (2.12) with respect to which

$$m(\bar{N}_p^u\bar{f}) > \|\Gamma_p\bar{f}\|^k, \quad \|\bar{N}_p^s\bar{f}\| < m(\Gamma_p\bar{f})^k, \quad \forall p \in \Gamma, \quad 0 \leq k \leq r. \quad (2.13)$$

By [HPS77, Theorem 4.1], these assumptions imply that in a neighborhood of Γ , there exist stable (s) and unstable (u) manifolds

$$\bar{\Gamma}^s, \bar{\Gamma}^u \subset \mathcal{X}, \quad T_\Gamma\bar{\Gamma}^{u/s} = T\Gamma \oplus \bar{N}^{u/s},$$

which are C^∞ and locally invariant under \bar{f} , namely $\bar{f}(\bar{\Gamma}^s) \subset \bar{\Gamma}^s$ and $\bar{f}(\bar{\Gamma}^u) \supset \bar{\Gamma}^u$.

On the spacetime

$$\mathcal{M} := \mathbb{R}_t \times \mathcal{X}, \quad (2.14)$$

define then the *stationary model* $f_0: \mathcal{M} \rightarrow \mathcal{M}$ by⁸

$$f_0(t, x) := (t - 1, \bar{f}(x)). \quad (2.15)$$

As perturbations of f_0 , we consider smooth maps

$$f: \mathcal{M} \rightarrow \mathcal{M},$$

defined for $t > t_0 \in \mathbb{R}$, with the following properties:

- (II.1) let $\pi_T: \mathcal{M} \rightarrow \mathbb{R}_t$, $(t, x) \mapsto t$, denote the projection, then $\pi_T(f(t, x)) = t - 1$;

⁸One can equally well consider time translations $t \mapsto t + 1$ instead, as done in §1, in which case unstable and stable manifolds exchange roles in Theorem 2.3 below. We consider (2.15) for simpler comparison with [HPS77, §4] since in this case the unstable manifold is canonical, and it is the unstable manifold which was explicitly discussed in the reference.

(II.2) f is a $\rho\mathcal{C}_b^r$ -perturbation of f_0 for every r . That is, fixing a Riemannian metric on \mathcal{X} and denoting by \exp its exponential map, we have

$$f(t, x) = (t - 1, \exp_{\bar{f}(x)} \tilde{V}(t, \bar{f}(x))), \quad x \in \mathcal{U},$$

where $\tilde{V}(t, -) \in \mathcal{V}(\mathcal{X})$, $\tilde{V} \in \rho\mathcal{C}_b^\infty(\{t > t_0\}; T\mathcal{X})$, is a t -dependent vector field on \mathcal{X} .

Let us denote by

$$\Gamma_0^{u/s} := \mathbb{R}_t \times \bar{\Gamma}^{u/s} \quad (2.16)$$

the stationary spacetime extension of the unstable and stable manifolds, and $\Gamma_0 := \mathbb{R}_t \times \Gamma$.

Theorem 2.3. *Under the assumptions (I.1)–(I.4) and (II.1)–(II.2), there exists a submanifold $\Gamma^u \subset \mathcal{M}$ with the following properties, parts (1) and (2) of which are illustrated in Figure 2.1:*

(1) Γ^u is f -invariant in the sense that

$$f(\Gamma^u) \cap \{t > t_0\} = \Gamma^u \cap \{t > t_0\}; \quad (2.17)$$

(2) Γ^u approaches Γ_0^u as $t \rightarrow \infty$ in a $\rho\mathcal{C}_b^\infty$ sense. This means: let $\bar{\Gamma}^u(\epsilon)$ denote an ϵ -neighborhood of Γ within $\bar{\Gamma}^u$, and fix a \mathcal{C}^∞ tubular neighborhood of $\bar{\Gamma}^u(\epsilon)$ in \mathcal{X} . Extend this t -independently to a tubular neighborhood of $\Gamma_0^u(\epsilon)$ in \mathcal{M} . Then, for small $\epsilon > 0$, the unstable manifold Γ^u is the graph of a function in the space $\rho\mathcal{C}_b^\infty(\Gamma_0^u(\epsilon); N\Gamma_0^u)$;

(3) Γ^u is the unique manifold satisfying (2.17) within the class of manifolds approaching Γ_0^u in a $\rho\mathcal{C}_b^1$ sense.

Furthermore:

(4) there exists a (nonunique) manifold $\Gamma^s \subset \mathcal{M}$ approaching Γ_0^s as $t \rightarrow \infty$ in a $\rho\mathcal{C}_b^\infty$ sense, that is, Γ^s is the graph of a function in the space $\rho\mathcal{C}_b^\infty(\Gamma_0^s(\epsilon); N\Gamma_0^s)$ (defined using the stationary extension of a tubular neighborhood of $\bar{\Gamma}^s$ analogously to (2)) so that $f(\Gamma^s) \cap \{t > t_0\} \subset \Gamma^s$.

In these statements, the regularity of Γ^u and Γ^s is $\rho\mathcal{C}_b^r$ if we relax assumption (II.2) by only assuming $\rho\mathcal{C}_b^r$ -regularity, for some fixed $r \geq 1$, of \tilde{V} .⁹

Assuming that $\bar{\Gamma}^u$ is orientable, an equivalent formulation of (2) is the following:

(2') let $\bar{\varphi}^u \in \mathcal{C}^\infty(\mathcal{X})$ denote a defining function of $\bar{\Gamma}^u$, that is, $(\bar{\varphi}^u)^{-1}(0) = \bar{\Gamma}^u$, and $d\bar{\varphi}^u \neq 0$ on $\bar{\Gamma}^u$. Then there exists a function $\tilde{\varphi}^u \in \rho\mathcal{C}_b^\infty(\mathcal{M})$ such that

$$\varphi^u(t, x) := \bar{\varphi}^u(x) + \tilde{\varphi}^u(t, x) \quad (2.18)$$

is a defining function of Γ^u .

Similarly, claim (4) can be restated using defining functions when $\bar{\Gamma}^s$ is orientable:

⁹One can likewise relax assumptions (I.3) and (I.4) by assuming that the bundle splitting (2.12) is merely continuous, and only assuming r -normal hyperbolicity for r fixed, in which case Γ , $\bar{\Gamma}^{u/s}$ are merely \mathcal{C}^r . This adds only minor technical complications to the proof, which can be handled by smoothing techniques as in [HPS77, §4]. We do not state the theorem in this generality, as it will not be needed in our application.

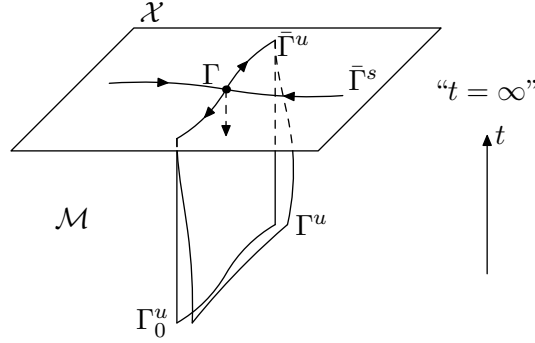


FIGURE 2.1. Illustration of Theorem 2.3, parts (1) and (2). The stationary model f_0 (or \bar{f}) should be thought of as living at “ $t = \infty$ ”, see Remark 2.4. Its invariant (Γ), stable (Γ^s), and unstable ($\bar{\Gamma}^u$) manifolds are shown, as well as the stationary extension Γ_0^u . The unstable manifold for f is a $\rho\mathcal{C}_b^\infty$ -graph over Γ_0^u . Also drawn is the arrow indicating the action of f in the time variable.

- (4') let $\bar{\varphi}^s \in \mathcal{C}^\infty(\mathcal{X})$ denote a defining function of $\bar{\Gamma}^s$. There exists a function $\tilde{\varphi}^s \in \rho\mathcal{C}_b^\infty(\mathcal{M})$ such that

$$\varphi^s(t, x) := \bar{\varphi}^s(x) + \tilde{\varphi}^s(t, x) \quad (2.19)$$

is a defining function of a manifold Γ^s as in (4).

Remark 2.4. Introducing a coordinate $\tau_e := e^{-t}$ as in the introduction, the action of f_0 in the time variable is given by $\tau_e \mapsto e\tau_e$, hence f_0 induces a map on $[0, \infty)_{\tau_e} \times \mathcal{X}$ for which $\{0\} \times \bar{\Gamma}^s$ is the stable manifold and $[0, \infty) \times \bar{\Gamma}^u$ the unstable manifold. (While Γ_0^u has an invariant interpretation as the interior of the unstable manifold of $\{0\} \times \bar{\Gamma}$, the manifold Γ_0^s has no such interpretation, and we only introduce it for convenience.) The perturbations considered here, which have decay rates such as $\rho(t) = \langle t \rangle^{-\alpha} = \langle \log \tau_e \rangle^{-\alpha}$ and are thus very singular in τ_e , can be analyzed because of the particular t -dependence of f . Namely, $\pi_T f$ is a family (depending on t) of smooth small perturbations of \bar{f} ; one can show that this is already sufficient to guarantee the existence of a family of manifolds $\Gamma^u(t) \subset \mathcal{X}$ which remain close to $\bar{\Gamma}^u$ for all t and are invariant in the sense that $f(t, \Gamma^u(t)) \subset \Gamma^u(t-1)$. The point here is that under our stronger assumptions on f , we can obtain more precise control on Γ^u .

Proof of Theorem 2.3. We closely follow the constructions and arguments of [HPS77, §4] and explain the required extensions and modifications. The first, main, part of the proof concerns claims (1)–(3) about Γ^u ; in the second part, we prove claim (4) using a simple adaptation of the methods from the first part. As in the reference, the construction of Γ^u proceeds in several steps:

- After some preliminary simplifications, we will find Γ^u as the graph of a section σ via a fixed point argument using a graph transform à la Hadamard, which was already used in §2.2. Here, starting e.g. with the candidate Γ_0^u , we replace the current candidate by its image under f , which stretches it out along Γ_0^u and flattens it in the stable directions. (The perturbative part of f is damped by the main part, f_0 , in the stable directions.) In steps 1 and 2, we show that this is a well-defined procedure on candidate sections which are small sections of the normal bundle Γ_0^u .

- In step 3, we prove that this graph transform is a contraction on a space of Lipschitz sections decaying (together with its pointwise Lipschitz constants) at rate ρ , thus furnishing Γ^u as a Lipschitz submanifold ρ -close to Γ_0^u .¹⁰
- In step 4, we improve this to pointwise differentiability using the contraction mapping principle applied to a graph transform on a priori discontinuous families of Lipschitz jets. The key point, as in the proof of Theorem 2.2, is that the graph transform preserves jets of differentiable sections, which implies the pointwise differentiability of Γ^u .
- In step 5, we show in a similar manner (considering the graph transform acting on tangent planes based at points of Γ^u , on the one hand on a priori discontinuous, on the other hand on continuous sections) that the tangent distribution of Γ^u must be continuous.
- Higher regularity is proved in step 6 using an inductive argument in which one improves the regularity of the tangent bundle of Γ^u .

To start, fix a Riemannian metric on \mathcal{X} which restricts to that of assumption (I.4) (with $r = 1$) at Γ . For $\epsilon > 0$ small, denote by $\bar{\Gamma}^u(\epsilon)$ the ϵ -neighborhood of Γ within $\bar{\Gamma}^u$. Extend the bundle $\bar{N}^s \rightarrow \Gamma$ of stable tangent directions and its fiber metric to a \mathcal{C}^∞ vector bundle $\bar{\mathcal{S}}' \rightarrow \bar{\Gamma}^u(\epsilon)$. Using the exponential map on \mathcal{X} , embed an ϵ -neighborhood, $\bar{\mathcal{S}}'(\epsilon)$, of the zero section diffeomorphically into \mathcal{X} . For convenience, we fix a vector bundle $\bar{\mathcal{S}}'' \rightarrow \bar{\Gamma}^u(\epsilon)$ such that $\bar{\mathcal{S}} := \bar{\mathcal{S}}' \oplus \bar{\mathcal{S}}''$ is trivial, i.e. $\bar{\mathcal{S}} \cong \bar{\Gamma}^u(\epsilon) \times S$ for some fixed vector space S . Choose the fiber metric on $\bar{\mathcal{S}}$ to be the direct sum of that on $\bar{\mathcal{S}}'$ and any fixed metric on $\bar{\mathcal{S}}''$. We extend \bar{f} to a map on $\bar{\mathcal{S}}(\epsilon)$ by setting $\bar{f}(s' \oplus s'') := \bar{f}(s') \oplus 0$; we likewise extend f to a map on $\mathbb{R}_t \times \bar{\mathcal{S}}(\epsilon)$ by setting

$$f(t, s' \oplus s'') := f(t, s') \oplus 0, \quad (2.20)$$

and similarly for f_0 . We let $\mathcal{S} := \mathbb{R}_t \times \bar{\mathcal{S}}$.

Let $C_\Sigma > 0$ denote a constant, which will be specified in the course of the proof and only depends on Lipschitz properties of f , see (2.36). For small $\epsilon > 0$, we set

$$\begin{aligned} T(\epsilon) &:= \{t \in \mathbb{R} : \rho(t) < \epsilon^2\}, \\ \Gamma_0^u(\epsilon) &:= T(\epsilon) \times \bar{\Gamma}^u(\epsilon), \quad \mathcal{S}(\epsilon) := T(\epsilon) \times \bar{\mathcal{S}}(\epsilon), \end{aligned} \quad (2.21)$$

in particular $\mathcal{S}(\epsilon) \cong \Gamma_0^u(\epsilon) \times S$ is trivial; we denote the projection by

$$\pi_0^U : \mathcal{S}(\epsilon) \rightarrow \Gamma_0^u(\epsilon),$$

and $\bar{\pi}^U := \pi_{\mathcal{X}} \pi_0^U$, where $\pi_{\mathcal{X}} : \mathcal{M} \rightarrow \mathcal{X}$ is the projection. (We use the capital letter ‘U’ to emphasize that this projects onto more than the unstable tangent directions \bar{N}^u , which is only *part* of the tangent bundle of $\bar{\Gamma}^u$ as it does not include $T\Gamma$.) Denote points on $\bar{\Gamma}^u$ by x^U .

Part 1: proof of claims (1)–(3). We shall find $\Gamma^u \subset \mathcal{S}(\epsilon)$ as the image of a section in the space

$$\Sigma(\epsilon) := \left\{ \text{sections } \sigma : \Gamma_0^u(\epsilon) \rightarrow \mathcal{S}(\epsilon) : |\sigma(t, x^U)| \leq C_\Sigma \rho(t), L_{(t, x^U)}(\check{\sigma}) \leq C_\Sigma \rho(t) \right\}, \quad (2.22)$$

¹⁰As a simple toy example, one may keep the following in mind: Γ is a point, $\bar{\Gamma}^u$ and $\bar{\Gamma}^s$ (with coordinates x^U and s , respectively) are 1-dimensional, $\bar{f}(x^U, s) = (2x^U, \frac{1}{2}s)$, so $f_0(t, x^U, s) = (t-1, 2x^U, \frac{1}{2}s)$, and the perturbed map is $f(t, x^U, s) = (t-1, 2x^U, \frac{1}{2}s + \rho(t))$. The graph transform, denoted f_\sharp in (2.23), maps the section $(t, x^U) \mapsto \sigma(t, x^U) = (t, x^U, \check{\sigma}(t, x^U))$ into $f_\sharp \sigma(t, x^U) = (t, x^U, \frac{1}{2}\check{\sigma}(t+1, x^U/2) + \rho(t+1))$. Acting on sections of size $\mathcal{O}(\rho)$, the unique fixed point of this transform is the section $\sigma_f(t, x^U) = \sum_{j=0}^{\infty} 2^{-j} \rho(t+1+j)$.

where we define the fiber value map $\check{\sigma}: \Gamma_0^u(\epsilon) \rightarrow S$ by $\sigma(t, x^U) = (t, x^U, \check{\sigma}(t, x^U))$; the Lipschitz constant is defined analogously to (2.11) using the triviality of $\bar{\mathcal{S}}$. The space $\Sigma(\epsilon)$, equipped with the \mathcal{C}_b^0 metric, is complete. The idea is to define a graph transform on $\Sigma(\epsilon)$ by mapping σ to a section $f_{\#}\sigma$ whose graph is the image of the graph of σ under f : we wish to take

$$f_{\#}\sigma := f\sigma g, \quad (2.23)$$

where $g: \bar{\Gamma}^u(\epsilon) \rightarrow \bar{\Gamma}^u(\epsilon)$ is a right inverse of $\pi_0^U f\sigma$. We shall see that this is a contraction for two reasons: first, f expands, thus g contracts, the base $\Gamma_0^u(\epsilon)$, and second, f contracts the fibers. Here, the behavior of f is dominated by f_0 , while the perturbation only affects these statements by a decaying (as $t \rightarrow \infty$) amount.

Step 1: construction of g . We first show that the right inverse g is well-defined when $\epsilon > 0$ is sufficiently small. We introduce local coordinates on $\bar{\Gamma}^u(\epsilon_0)$, $\epsilon_0 > 0$ small and fixed, using the Riemannian exponential map $\exp^{\bar{\Gamma}^u}$ as follows: for $p \in \Gamma$ and small $v^U = (v, n) \in T_p\bar{\Gamma}^u = T_p\Gamma \oplus \bar{N}_p^u$, we let

$$\bar{\chi}_p(v^U) := \exp_p^{\bar{\Gamma}^u}(v^U + h_p(v^U)), \quad (2.24)$$

where $h_p: T_p\bar{\Gamma}^u \rightarrow T_p\bar{\Gamma}^u$ vanishes quadratically at 0 and is chosen such that for $v \in T_p\Gamma$, we have $\exp_p^{\bar{\Gamma}^u}(v + h_p(v)) = \exp_p^{\Gamma}(v) \in \Gamma$, thus straightening out Γ . Using the triviality of $\bar{\mathcal{S}}$, this induces local coordinates $(v^U, s) \in T_p\bar{\Gamma}^u \oplus S$ on $\bar{\mathcal{S}}$ via

$$\bar{e}_p(v^U, s) := (\bar{\chi}_p(v^U), s). \quad (2.25)$$

The local coordinate representation of \bar{f} ,

$$\bar{f}_p := \bar{e}_{\bar{f}(p)}^{-1} \bar{f} \bar{e}_p: T_p\bar{\Gamma}^u \times S \rightarrow T_{\bar{f}(p)}\bar{\Gamma}^u \times S,$$

can be written as $\bar{f}_p = D_p\bar{f} + \bar{r}_p$, where \bar{r}_p vanishes quadratically at 0. Since \bar{f} preserves Γ as well as $\bar{\Gamma}^u$, \bar{f}_p preserves the origin of $\bar{N}^u \oplus S$ as well as of S (considered as subbundles of $T\Gamma \oplus \bar{N}^u \oplus S$). Moreover,

$$D_p\bar{f} = \begin{pmatrix} \Gamma_p\bar{f} \oplus \bar{N}_p^u\bar{f} & 0 \\ 0 & D_p\bar{f}|_{\bar{\mathcal{S}}_p} \end{pmatrix}. \quad (2.26)$$

The charts (2.24)–(2.25) have natural spacetime extensions χ_p (giving coordinates on Γ_0^u) and e_p (giving coordinates on \mathcal{S}):

$$\chi_p(t, v^U) := (t, \bar{\chi}_p(v^U)), \quad e_p(t, v^U, s) := (t, \bar{e}_p(v^U, s)).$$

Writing $(f_0)_p = e_{\bar{f}(p)}^{-1} f_0 e_p$ and $f_p = e_{\bar{f}(p)}^{-1} f e_p$ (note that we use the chart centered at $\bar{f}(p)$ even for the perturbed map f), we have $(f_0)_p(t, v^U, s) = (t-1, \bar{f}_p(v^U, s))$, while the perturbation

$$\tilde{f}_p := f_p - (f_0)_p, \quad \pi_T \tilde{f}_p(t, v^U, s) \equiv 0,$$

satisfies

$$\tilde{f}_p \in \rho \mathcal{C}_b^\infty$$

as a map $\{t > t_0\} \times \mathcal{W} \rightarrow \{0\} \times T_{\bar{f}(p)}\bar{\Gamma}^u \times S$ where \mathcal{W} is a neighborhood of the origin in $T_p\bar{\Gamma}^u \times S$, as follows from assumptions (II.1)–(II.2) (increasing t_0 if necessary to stay within the coordinate patches e_p and $e_{\bar{f}(p)}$).

Let now $\sigma \in \Sigma(\epsilon)$. Write $\sigma_p := e_p^{-1} \sigma \chi_p: T(\epsilon) \times T_p\bar{\Gamma}^u \rightarrow T(\epsilon) \times T_p\bar{\Gamma}^u \times S$, which has the form $\sigma_p(t, v^U) = (t, v^U, \check{\sigma}_p(t, v^U))$, and satisfies the bounds in (2.22) (with respect to the

fixed fiber metric on S coming from $\bar{\mathcal{S}}_p$), with the right hand sides multiplied by $1 + \mathcal{O}(\epsilon)$ (since the norm on the fibers of \mathcal{S} may change smoothly away from p). We then have

$$\chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p = \pi_0^U e_{\bar{f}(p)}^{-1} f e_p e_p^{-1} \sigma \chi_p = \pi_0^U f_p \sigma_p = L + (0, \bar{\pi}^U \tilde{L}),$$

where the main and remainder terms (for present purposes) are, respectively,

$$L(t, v^U) = (t - 1, D_p \bar{f}(v^U)), \quad \tilde{L}(t, v^U) = D_p \bar{f} \circ (0, \check{\sigma}_p(t, v^U)) + \bar{r}_p(v^U, \check{\sigma}_p) + \tilde{f}_p \sigma_p(t, v^U). \quad (2.27)$$

(Thus, $L + (0, \tilde{L})$ is the local coordinate representation of $f\sigma$.) Fix λ, μ and ω, ω' (depending on p) so that

$$0 < \mu < \underline{\mu} := m(\Gamma_p \bar{f}), \quad 1 < \lambda < \underline{\lambda} := m(\bar{N}_p^u \bar{f}), \quad 0 < \omega < \omega' < 1, \quad \lambda \omega > 1.$$

We claim that, for $\epsilon_0 > 0$ fixed and small, we have, for all sufficiently small $\epsilon > 0$:

$$\chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p \text{ is injective on } T(\epsilon) \times (T_p \Gamma(\epsilon_0) \oplus \bar{N}_p^u(\omega' \epsilon)), \quad (2.28)$$

$$\chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p (T(\epsilon) \times (T_p \Gamma(\omega \epsilon) \oplus \bar{N}_p^u(\omega \epsilon))) \supset T(\epsilon) \times (T_{\bar{f}(p)} \Gamma(\mu \omega \epsilon) \oplus \bar{N}_{\bar{f}(p)}^u(\lambda \omega \epsilon)). \quad (2.29)$$

(Note that if \tilde{L} were identically zero, this would be clear.) Now, only points (t, v^U) with the same t -values can possibly map to the same point under $L + (0, \bar{\pi}^U \tilde{L})$, hence (2.28) follows from (2.26) (which implies that $\bar{\pi}^U$ annihilates the first term of \tilde{L}) and

$$\begin{aligned} & |\chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p(t, v^U) - \chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p(t, w^U)| \\ & \geq |D_p \bar{f}(v^U - w^U)| - |\bar{r}_p(v^U, \check{\sigma}_p(t, v^U)) - \bar{r}_p(w^U, \check{\sigma}_p(t, w^U))| \\ & \quad - |\bar{\pi}^U \tilde{f}_p(t, v^U, \check{\sigma}_p(t, v^U)) - \bar{\pi}^U \tilde{f}_p(t, w^U, \check{\sigma}_p(t, w^U))| \\ & \geq (\underline{\mu} - C_r(\epsilon_0 + \epsilon)(1 + C_\Sigma) - \tilde{C}\rho(t)(C_\Sigma \rho(t) + 1))(1 - \mathcal{O}(\epsilon_0 + \epsilon))|v^U - w^U| \\ & \geq (\underline{\mu} - o(1))|v^U - w^U|, \quad \epsilon_0, \epsilon \rightarrow 0, \end{aligned} \quad (2.30)$$

where \tilde{C} is greater than the Lipschitz constant of $\rho^{-1} \tilde{f}_p$, and C_r bounds the \mathcal{C}^2 norm of \bar{r}_p ; we may fix \tilde{C}, C_r uniformly for $p \in \Gamma$. The factor $1 - \mathcal{O}(\epsilon_0 + \epsilon)$ accounts for the fact that the norms used in the definition of Lipschitz constants may vary away from p .

To prove (2.29), note that $|\bar{\pi}^U \tilde{L}(t, v^U)| \leq C_r \epsilon^2 + \tilde{C}\rho(t)$ for $(t, v^U) \in T(\epsilon) \times (T_p \Gamma(\omega \epsilon) \oplus \bar{N}_p^u(\omega \epsilon))$. Let now $t \in T(\epsilon)$, and consider $y^U = (y, n) \in T_{\bar{f}(p)} \Gamma \oplus \bar{N}_{\bar{f}(p)}^u$ satisfying (slightly increasing C_r and \tilde{C} to accommodate the $(1 - \mathcal{O}(\epsilon_0 + \epsilon))$ factor in (2.30), now with $\epsilon_0 = \epsilon$)

$$|y| \leq (\underline{\mu} - \delta)\omega \epsilon - \underline{\mu}(C_r \epsilon^2 + \tilde{C}\rho(t + 1)), \quad |n| \leq (\underline{\lambda} - \delta)\omega \epsilon - \underline{\lambda}(C_r \epsilon^2 + \tilde{C}\rho(t + 1)), \quad (2.31)$$

where $0 < \delta < \min(\underline{\mu} - \mu, \underline{\lambda} - \lambda)$. Using the form (2.26) of $D_p \bar{f}$, we see that

$$G: v^U \mapsto (D_p \bar{f})^{-1}(y^U - \bar{\pi}^U \tilde{L}(t + 1, v^U)) \quad (2.32)$$

maps $T_p \Gamma(\omega \epsilon) \oplus \bar{N}_p^u(\omega \epsilon)$ into itself for small $\epsilon > 0$ since $\|(\Gamma_p \bar{f})^{-1}\| \leq \underline{\mu}^{-1}$ and $\|(\bar{N}_p^u \bar{f})^{-1}\| \leq \underline{\lambda}^{-1}$. It also is a contraction, since, much as for (2.30),

$$|G(v^U) - G(w^U)| \leq \underline{\mu}^{-1}(2\epsilon C_r + \tilde{C}\rho(t + 1)(C_\Sigma \rho(t + 1) + 1))|v^U - w^U| = o(1)|v^U - w^U|$$

as $\epsilon \rightarrow 0$. Letting v^U denote the unique fixed point of G , we have

$$\chi_{\bar{f}(p)}^{-1} \pi_0^U f \sigma \chi_p(t + 1, v^U) = (t, y^U). \quad (2.33)$$

Choosing ϵ sufficiently small and recalling from (2.21) that $\rho(t+1) < \epsilon^2$, the second summand on the right in (2.29) is contained in the set of (t, y, n) satisfying (2.31) and $t \in T(\epsilon)$, hence we have proved (2.29).

We define

$$g: \mathcal{S}(\epsilon) \rightarrow T(\epsilon) \times \bigcup_{p \in \Gamma} \chi_p(T_p \Gamma(\omega\epsilon) \oplus \bar{N}_p^u(\omega\epsilon))$$

to be the map which in the $\chi_p, \chi_{\bar{f}(p)}$ charts is given by $(t, y^U) \mapsto (t+1, v^U)$ with v^U as in (2.33). In these charts, we record (using (2.32)) that for $(t+1, v^U) = g(t, y^U)$ and $(t'+1, v'^U) = g(t', y'^U)$, and $\rho_{\pm} = \rho_{\pm}(t+1, t'+1)$,

$$\begin{aligned} |v^U - v'^U| &\leq \underline{\mu}^{-1} \left(|y^U - y'^U| + (2\epsilon C_r + \tilde{C}\rho_-(C_{\Sigma}\rho_- + 1)) |v^U - v'^U| \right. \\ &\quad \left. + (2\epsilon C_r C_{\Sigma} + \tilde{C}(C_{\Sigma}\rho_+ + 1)) \rho_+ |t - t'| \right); \end{aligned}$$

this uses the Lipschitz estimate (2.4) applied to \tilde{f}_p as well as the Lipschitz bounds on σ . Hence

$$|g(t, y^U) - g(t', y'^U)| \leq (\underline{\mu}^{-1} + o(1)) |y^U - y'^U| + (1 + o(1)) |t - t'|, \quad \epsilon \rightarrow 0. \quad (2.34)$$

Step 2: mapping properties of f_{\sharp} . For $\sigma \in \Sigma(\epsilon)$, we can now define $f_{\sharp}\sigma = f\sigma g$ as a section $\Gamma_0^u(\epsilon) \rightarrow \mathcal{S}$. We proceed to verify $f_{\sharp}\sigma \in \Sigma(\epsilon)$. Let us work in local coordinates as above, and let $(t+1, v^U) = g(t, y^U)$. Using (2.27), and writing π_S for the projection onto the fiber, the length of $(\pi_S f_{\sharp}\sigma)(t, y^U) \in \mathcal{S}_{(t, y^U)} = S$ is bounded by

$$\begin{aligned} |(\pi_S f_{\sharp}\sigma)(t, y^U)| &\leq |\pi_S D_p \bar{f}(v^U, \check{\sigma}_p(t+1, v^U))| + |\pi_S \bar{r}_p(v^U, \check{\sigma}_p(t+1, v^U))| \\ &\quad + |\pi_S \tilde{f}_p(t+1, v^U, \check{\sigma}_p(t+1, v^U))| \\ &\leq \|\bar{N}_p^s \bar{f}\| C_{\Sigma}\rho(t+1) + \epsilon C_r C_{\Sigma}\rho(t+1) + \tilde{C}\rho(t+1), \end{aligned} \quad (2.35)$$

where we used crucially that \bar{f} preserves $\bar{\Gamma}^u$ in the estimate of the term involving \bar{r}_p , as this gives $\pi_S \bar{r}_p(v^U, 0) = 0$. Since $\|\bar{N}_p^s \bar{f}\| < 1$, we may fix

$$C_{\Sigma} > \frac{\tilde{C}}{\inf_{p \in \Gamma} (\min(1, m(\Gamma_p \bar{f})) - \|\bar{N}_p^s \bar{f}\|)}. \quad (2.36)$$

For small $\epsilon > 0$, this implies the desired estimate

$$|\pi_S f_{\sharp}\sigma| \leq C_{\Sigma}\rho. \quad (2.37)$$

In order to verify the Lipschitz condition of $\Sigma(\epsilon)$ for $f_{\sharp}\sigma$, we estimate, using (2.34),

$$\begin{aligned} &L_{(t, y^U)}(\pi_S f_{\sharp}\sigma) \\ &\leq L_{(t+1, v^U)}(\pi_S f\sigma) L_{(t, y^U)}(g) \\ &\leq (\|\bar{N}_{v^U}^s \bar{f}\| C_{\Sigma}\rho(t+1) + \epsilon C_r (C_{\Sigma}\rho(t+1) + 1) + \tilde{C}\rho(t+1) (C_{\Sigma}\rho(t+1) + 1)) \\ &\quad \times \max(\underline{\mu}^{-1}, 1) (1 + o(1)) \\ &\leq C_{\Sigma}\rho(t) \end{aligned} \quad (2.38)$$

for small $\epsilon > 0$ due to our choice (2.36).

Step 3: f_{\sharp} is a contraction. We next show that f_{\sharp} is a contraction on $\Sigma(\epsilon)$ equipped with the (complete!) C_b^0 metric. To this end, let $\sigma, \sigma' \in \Sigma(\epsilon)$ and denote by g, g' the right inverses

of $\pi_0^U f\sigma$, $\pi_0^U f\sigma'$. Recalling the definition of \tilde{L} from (2.27), let us make the dependence on σ explicit by writing \tilde{L}_σ . Using the construction of g , resp. g' , via the fixed point argument involving (2.32), with \tilde{L} replaced by \tilde{L}_σ , resp. $\tilde{L}_{\sigma'}$, we estimate at a point (t, y^U) , with y^U ϵ -close to $p \in \Gamma$, and using the local coordinates as above:

$$\begin{aligned} |g - g'| &\leq \underline{\mu}^{-1} |\bar{\pi}^U \tilde{L}_\sigma(g) - \bar{\pi}^U \tilde{L}_{\sigma'}(g')| \\ &\leq |\bar{\pi}^U \bar{r}_p(\bar{\pi}^U g, \check{\sigma}_p(t+1, \bar{\pi}^U g)) - \bar{\pi}^U \bar{r}_p(\bar{\pi}^U g', \check{\sigma}'_p(t+1, \bar{\pi}^U g'))| \\ &\quad + |\bar{\pi}^U \tilde{f}_p(t+1, \bar{\pi}^U g, \check{\sigma}_p(t+1, \bar{\pi}^U g)) - \bar{\pi}^U \tilde{f}_p(t+1, \bar{\pi}^U g', \check{\sigma}'_p(t+1, \bar{\pi}^U g'))| \quad (2.39) \\ &\leq (\epsilon C_r + \tilde{C}\rho(t+1)(1 + C_\Sigma \rho(t+1))) |\bar{\pi}^U g - \bar{\pi}^U g'| \\ &\quad + (\epsilon C_r + \tilde{C}\rho(t+1)) |\check{\sigma}_p(t+1, \bar{\pi}^U g) - \check{\sigma}'_p(t+1, \bar{\pi}^U g)|; \end{aligned}$$

the first term on the right can be absorbed into the left hand side, and we thus obtain, in our local coordinates,

$$|g - g'| \leq o(1) |\sigma - \sigma'|, \quad \epsilon \rightarrow 0. \quad (2.40)$$

We use this to estimate at (t, y^U) , using the Lipschitz estimate (2.38):

$$\begin{aligned} |\pi_S f_\# \sigma - \pi_S f_\# \sigma'| &\leq |\pi_S f \sigma g - \pi_S f \sigma' g| + |\pi_S f \sigma' g - \pi_S f \sigma' g'| \\ &\leq (\|\bar{N}_p^s \bar{f}\| + \epsilon C_r + \tilde{C}\rho(t+1)) |\sigma - \sigma'| + C_\Sigma \rho(t+1) \cdot o(1) |\sigma - \sigma'| \\ &\leq \theta |\sigma - \sigma'| \end{aligned}$$

where we fix θ such that $\sup_{p \in \Gamma} \|\bar{N}_p^s \bar{f}\| < \theta < 1$, and use $\epsilon = \epsilon_1$ with $\epsilon_1 > 0$ sufficiently small. By the contraction mapping principle, this implies the existence of an $f_\#$ -invariant section

$$\sigma_f \in \Sigma(\epsilon_1), \quad f_\# \sigma_f = \sigma_f.$$

Note that σ_f automatically takes values in $\mathcal{S}' \oplus 0$ in view of the definition (2.20), hence the image of σ_f ,

$$\Gamma^u := \sigma_f(\Gamma_0^u(\epsilon_1)),$$

is a Lipschitz submanifold of \mathcal{M} which is ρ -close to Γ_0^u .

The uniqueness claim (3) is an immediate consequence of our construction and the contraction mapping principle.

Step 4: pointwise differentiability. We next improve the Lipschitz regularity of Γ^u to the pointwise existence of tangent planes. We again argue using Lipschitz jets. Thus, consider $\epsilon \in (0, \epsilon_1)$ such that $\Gamma_0^u(\epsilon_1) \supset \pi_0^U f \sigma_f \Gamma_0^u(\epsilon)$, and equip $\Gamma_0^u(\epsilon_1)$ with the discrete topology. Define the fiber bundle $\mathcal{D}^b \rightarrow \Gamma_0^u(\epsilon_1)$ whose fiber at (t, x^U) is

$$\mathcal{D}_{(t, x^U)}^b = \{J_{(t, x^U)} \sigma \in J^b(\Gamma_0^u(\epsilon_1), (t, x^U); \mathcal{S}, \sigma_f(t, x^U)) : \sigma \in \Sigma(\epsilon_1)\}. \quad (2.41)$$

This is a subbundle of the vector bundle J^b whose fiber over (t, x^U) is equal to all of $J^b(\Gamma_0^u(\epsilon_1), (t, x^U); \mathcal{S}, \sigma_f(t, x^U))$. The map f induces a natural bundle map $Jf: \mathcal{D}^b|_{\Gamma_0^u(\epsilon)} \rightarrow \mathcal{D}^b$ covering the map $(t, x^U) \mapsto (t-1, x_1^U) = \pi_0^U f \sigma_f(t, x^U)$, defined by

$$Jf(J_{(t, x^U)} \sigma) := J_{(t-1, x_1^U)}(f_\# \sigma) = J_{(t-1, x_1^U)}(f \sigma g),$$

where g is the right inverse, defined near $(t-1, x_1^U)$, of $\pi_0^U f \sigma$ constructed above. Note that since σ is tangent to σ_f at (t, x^U) , we have $g(t-1, x_1^U) = (t, x^U)$, so the Lipschitz jet is well-defined. The membership in $\mathcal{D}_{(t-1, x_1^U)}^b$ follows from the estimates (2.37) and (2.38).

We assert that Jf is a fiber contraction. Let $\sigma, \sigma' \in \Sigma(\epsilon_1)$ denote two local sections near (t, x^U) with $\sigma(t, x^U) = \sigma'(t, x^U) = \sigma_f(t, x^U)$, and let g, g' denote the local right inverses of $\pi_0^U f \sigma, \pi_0^U f \sigma'$, defined near $(t-1, x_1^U)$. Then in local coordinates as above, and with $\check{\sigma} = \pi_S \sigma, \check{\sigma}' = \pi_S \sigma'$, we estimate using (2.34):

$$\begin{aligned}
& L_{(t-1, x_1^U)}(\pi_S f \sigma g - \pi_S f \sigma' g') \\
& \leq L_{(t-1, x_1^U)}(\pi_S f \sigma g - \pi_S f \sigma' g) + L_{(t-1, x_1^U)}(\pi_S f \sigma' g - \pi_S f \sigma' g') \\
& \leq L_{\sigma_f(t, x^U)}(\pi_S f)(L_{(t, x^U)}(\check{\sigma} - \check{\sigma}')L_{(t-1, x_1^U)}(g) + L_{(t, x^U)}(\check{\sigma}')L_{(t-1, x_1^U)}(g - g')) \\
& \leq (\|\bar{N}_{\check{f}(x^U)}^s \check{f}\| + o(1))(L_{(t, x^U)}(\check{\sigma} - \check{\sigma}')(\max(\underline{\mu}^{-1}, 1) + o(1)) + o(1) \cdot o(1)L_{(t, x^U)}(\check{\sigma} - \check{\sigma}'))
\end{aligned} \tag{2.42}$$

as $\epsilon \rightarrow 0$; the estimate on $L_{(t-1, x_1^U)}(g - g')$ follows much like the estimates (2.39)–(2.40). By the normal hyperbolicity of \check{f} , we have $\|\bar{N}_{\check{f}(x^U)}^s \check{f}\| < \min(1, \underline{\mu})$, hence this proves our contention. Consider then the map $J_{\#}f$ on sections of $\mathcal{D}^b|_{\Gamma_0^u(\epsilon)}$, defined by mapping a section $\{J_{(t, x^U)}\sigma^{(t, x^U)}\}$ to $(t, x^U) \mapsto Jf(J_{g(t, x^U)}\sigma^{g(t, x^U)})$; this is the analogue of the map $f_{\#}$ above.¹¹ This is a contraction and hence has a unique fixed point, the section σ_{Jf} of $\mathcal{D}^b|_{\Gamma_0^u(\epsilon)}$.

Since Jf (meaning: its composition with restriction of sections of \mathcal{D}^b to $\Gamma_0^u(\epsilon)$) has the invariant bounded, but a priori discontinuous, section $J\sigma_f$, we have

$$J\sigma_f = \sigma_{Jf}.$$

On the other hand, Jf maps *differentiable* jets into differentiable jets since f is differentiable, and g is differentiable when σ is. That is, Jf preserves the closed subbundle \mathcal{D}^d , defined like (2.41) but using J^d and differentiable sections σ ; the unique bounded Jf -invariant section thus necessarily lies in J^d , which proves the pointwise differentiability of σ_f . (Strictly speaking, this only holds over $\Gamma_0^u(\epsilon)$ rather than $\Gamma_0^u(\epsilon_1)$; but the image $\sigma_f(\Gamma_0^u(\epsilon_1))$ is obtained from $\sigma_f(\Gamma_0^u(\epsilon))$ by repeated application of f , hence the former inherits the regularity of the latter.)

Step 5: $\rho\mathcal{C}_b^1$ -regularity. We improve the pointwise differentiability to continuous differentiability: we shall show that $\sigma_f \in \rho\mathcal{C}_b^1(\Gamma_0^u; S)$, using arguments similar to those employed in the proof of Theorem 2.2. To wit, consider the bundle $L \rightarrow \Gamma_0^u$ (with the base equipped with its standard topology coming from \mathcal{M}) with fibers equal to spaces of linear maps,

$$L_{(t, x^U)} = \mathcal{L}(T_{(t, x^U)}\Gamma_0^u, \mathcal{S}_{(t, x^U)}).$$

Consider then the smooth disc subbundle $\mathcal{B} \rightarrow \Gamma_0^u(\epsilon)$ with fibers

$$\mathcal{B}_{(t, x^U)} = \{P \in L_{(t, x^U)} : \|P\| \leq C_{\Sigma}\rho(t)\}.$$

Since $\mathcal{S} \rightarrow \Gamma_0^u$ is trivial, we can identify $T_{(t, x^U, s)}\mathcal{S} = T_{(t, x^U)}\Gamma_0^u \oplus S$. Define then the bundle map Lf on \mathcal{B} as mapping $L_{(t, x^U)} \ni P \mapsto Lf(P) \in L_{\pi_0^U f \sigma_f(t, x^U)}$ such that

$$\text{graph}(Lf(P)) = (D_{\sigma_f(t, x^U)}f)(\text{graph}(P)), \tag{2.43}$$

where on the right,

$$\text{graph}(P) := \{\xi + P\xi : \xi \in T_{(t, x^U)}\Gamma_0^u\} \subset T_{(t, x^U, \check{\sigma}_f(t, x^U))}\mathcal{S},$$

¹¹We could have worked with $J_{\#}f$ from the beginning of this step of the proof; however, the present arguments are more easily extended to the inductive argument for proving higher regularity below.

and similarly on the left.

By the same estimates as in the previous step, Lf is a well-defined fiber contraction. One invariant section of Lf is given by the (a priori discontinuous) tangent bundle of Γ^u , i.e. the pointwise derivative $(t, x^U) \mapsto D_{(t, x^U)}\sigma_f$. On the other hand, Lf is continuous since σ_f is, hence it preserves the space of continuous sections (which is L^∞ -closed in the space of sections of \mathcal{B} , which are bounded by definition). Thus, Lf has a continuous invariant section, which is the unique bounded section, and therefore must agree with $D\sigma_f$. This gives

$$\sigma_f \in \rho\mathcal{C}_b^1.$$

Step 6: $\rho\mathcal{C}_b^r$ -regularity. To prove higher regularity, we proceed inductively. Thus, assume that $\sigma_f \in \rho\mathcal{C}_b^{r-1}$, $r \geq 2$. Consider again the map Lf in (2.43), acting on the \mathcal{C}^∞ fiber bundle $\mathcal{B} \rightarrow \Gamma_0^u$. In view of assumption (II.2),¹² the map Lf is $\rho\mathcal{C}_b^{r-1}$ -close to Lf_0 , where the latter is defined by (2.43) with f replaced by f_0 , and $\sigma_{f_0}(t, x^U) = (t, x^U, 0)$. Indeed, the image of σ_{f_0} is the unstable manifold Γ_0^u for f_0 ; note similarly that $D\sigma_{f_0}$ is the invariant section of Lf_0 , and is smooth and stationary, i.e. t -invariant. Furthermore, Lf_0 is covered by the base map $\pi_0^U f_0 \sigma_{f_0}$, for whose inverse g_{f_0} the Lipschitz constant of $x^U \mapsto \bar{\pi}^U g_{f_0}(t=0, x^U)$ is $m(\Gamma_p \bar{f})^{-1} + o(1)$, $\epsilon \rightarrow 0$, in an ϵ -neighborhood of $\bar{f}(p)$ by (a simplified version of) (2.34). On the other hand, $L_0 f$ contracts fibers by the factor $\sup_{p \in \Gamma} \|\bar{N}_p^s \bar{f}\| \max(m(\Gamma_p \bar{f})^{-1}, 1) + o(1)$ by (2.42). In view of the r -normal hyperbolicity of \bar{f} , Lf therefore satisfies the $\rho\mathcal{C}_b^{r-1}$ -assumptions of Theorem 2.2. We conclude that $D\sigma_f \in \rho\mathcal{C}_b^{r-1}$, hence $\sigma_f \in \rho\mathcal{C}_b^r$, finishing the inductive step.

This finishes the proof of claims (1)–(3) of the theorem.

Part 2: proof of claim (4) of the theorem. This proceeds along somewhat similar lines, with the roles of Γ_0^u , \bar{N}^s , f now being played by Γ_0^s , \bar{N}^u , f^{-1} . Note here that the inverse map f^{-1} satisfies analogous conditions to f , namely, it is $\rho\mathcal{C}_b^r$ -close to f_0^{-1} for all r , though now $\pi_T f^{-1}(t, x) = t + 1$ steps forward in time.

The main difference to the unstable manifold theorem is that there is no unique f -invariant spacetime extension Γ^s of the stable manifold $\bar{\Gamma}^s$ at Γ . We construct one possible Γ^s as follows. For $I \subset \mathbb{R}$, let $\Gamma_0^s(I) := \Gamma_0^s \cap t^{-1}(I)$; further, let $t_0(\epsilon)$ denote the value of t for which $\rho(t) = \epsilon$. For ϵ small, define the \mathcal{C}^∞ submanifolds

$$\begin{aligned} \Gamma^s(t_0(\epsilon), t_0(\epsilon) + \tfrac{1}{2}) &:= \Gamma_0^s(t_0(\epsilon), t_0(\epsilon) + \tfrac{1}{2}), \\ \Gamma^s(t_0(\epsilon) + 1, t_0(\epsilon) + \tfrac{3}{2}) &:= f^{-1}(\Gamma^s(t_0(\epsilon), t_0(\epsilon) + \tfrac{1}{2})), \end{aligned}$$

which are graphs of smooth local sections (uniformly bounded in $\rho\mathcal{C}_b^r$ for all r as $\epsilon \rightarrow 0$) of the vector bundle U , which we define to be an extension of \bar{N}^u from Γ to Γ^s and then, by stationarity, to Γ_0^s ; if necessary, we take the direct sum with another stationary vector bundle to obtain a trivial vector bundle, as done above for \mathcal{S} . We shall henceforth assume that U is trivial. Define Γ_\bullet^s to be the graph of a local section σ_\bullet of U over $\Gamma_0^s((t_0(\epsilon), t_0(\epsilon) + \frac{3}{2}))$ such that it is equal to $\Gamma^s(t_0(\epsilon), t_0(\epsilon) + \frac{1}{2})$ and $\Gamma^s(t_0(\epsilon) + 1, t_0(\epsilon) + \frac{3}{2})$ over the respective slabs in Γ_0^s ; we may arrange that σ_\bullet has $\rho\mathcal{C}_b^r$ bounds for all r , with the bound depending on the $\rho\mathcal{C}_b^r$ bounds on the perturbation of f_0 in assumption (II.2), and so that moreover the $\rho\mathcal{C}_b^r$ norms of σ_\bullet are uniformly bounded as $\epsilon \rightarrow 0$. See Figure 2.2.

¹²The $\rho\mathcal{C}_b^r$ -regularity version is sufficient at this point.

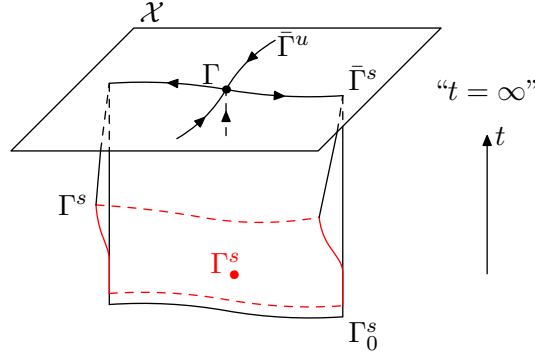


FIGURE 2.2. Construction of a spacetime extension of the stable manifold: starting from a prescribed part Γ_\bullet^s , we iteratively apply f^{-1} to obtain Γ^s . The arrows at “ $t = \infty$ ” indicate the behavior of the stationary model f_0^{-1} .

We then define

$$\Gamma^s := \bigcup_{n \in \mathbb{N}_0} f^{-n} \Gamma_\bullet^s,$$

where $f^{-n} = f^{-1} \circ \dots \circ f^{-1}$ (n -fold composition); by construction of Γ_\bullet^s , this is the graph of a smooth section $\sigma: \Gamma_0^s \rightarrow U$ defined in $t > t_0(\epsilon)$. Our goal is to show that $\sigma \in \rho C_b^\infty$. It is reasonable to expect this to be the case: the map \bar{f}^{-1} expands in the direction of $\bar{\Gamma}^s$ and contracts in the fiber directions, thus ‘flattening out’ any reasonable initial piece Γ_\bullet^s to $\bar{\Gamma}^s$ as $t \rightarrow \infty$. To prove this rigorously, we shall describe Γ^s more indirectly in a way analogous to our construction of Γ^u .

Denote by $\bar{\Gamma}^s(\epsilon)$ an ϵ -neighborhood of Γ inside of \mathcal{X} ; with $T(\epsilon)$ as in (2.21), put then $\Gamma_0^s(\epsilon) = T(\epsilon) \times \bar{\Gamma}^s(\epsilon)$. Let also $U(\epsilon)$ denote the ball bundle of radius ϵ inside of $U \rightarrow \Gamma_0^s$. Write points on Γ_0^s as (t, x^S) , $x^S \in \bar{\Gamma}^s$. Consider then the space of sections

$$\Sigma(\epsilon) = \{ \text{sections } \sigma: \Gamma_0^s(\epsilon) \rightarrow U(\epsilon) :$$

$$\sigma|_{\Gamma_0^s(\epsilon) \cap \Gamma_0^s((t_0(\epsilon), t_0(\epsilon) + \frac{3}{2}))} = \sigma_\bullet, |\sigma| \leq C_\Sigma \rho(t), L_{(t, x^S)}(\check{\sigma}) \leq C_\Sigma \rho(t) \}$$

which initially agree with Γ_\bullet^s (we omit the ϵ -dependence of the latter from the notation); we write $\check{\sigma} = \pi_U \sigma$, where π_U projects from U onto the typical fiber of the trivial vector bundle U . Here, C_Σ will be fixed later; it is in particular chosen to be larger than a constant \tilde{C} , depending only on the perturbation of f_0 , so that σ_\bullet satisfies the L^∞ and Lipschitz bounds with constant \tilde{C} . When ϵ is sufficiently small, we can define a map $f^\sharp: \Sigma(\epsilon) \rightarrow \Sigma(\epsilon)$ by

$$f^\sharp \sigma(t, x^S) := \begin{cases} \sigma_\bullet(t, x^S), & t_0(\epsilon) < t < t_0(\epsilon) + \frac{3}{2}, \\ f^{-1} \sigma g^-(t, x^S), & t > t_0(\epsilon) + 1, \end{cases}$$

where g^- is a right inverse of $\pi_0^S f^{-1} \sigma$, with $\pi_0^S: U \rightarrow \Gamma_0^s$ denoting projection to the base. Note that by definition of σ_\bullet , the two expressions on the right hand side agree for $t_0(\epsilon) + 1 < t < t_0(\epsilon) + \frac{3}{2}$.

One can now proceed as in the case of Γ^u to infer that for small $\epsilon > 0$, f^\sharp is a well-defined contraction on $\Sigma(\epsilon)$, equipped with the complete C^0 metric. Indeed, up to additive $o(1)$ errors as $\epsilon \rightarrow 0$, the map g^- has Lipschitz constants $\max(\|\Gamma_p \bar{f}\|, 1)$, while f^{-1} contracts the fibers of U by no less than $m(\bar{N}_p^u f)^{-1}$. The product of these two quantities, which

arises in estimates on $f^\sharp\sigma$ similarly to before, is less than unity by 1-normal hyperbolicity. Moreover, the mapping properties of f^\sharp rely on the fact that \bar{f} preserves $\bar{\Gamma}^s$; see (2.35) for the analogous estimate for Γ^u .

Letting σ^s denote the unique fixed point of f^\sharp , the graph of σ^s is by definition equal to $\Gamma^s \cap U(\epsilon)$. We then proceed as before, improve the regularity of σ^s to pointwise differentiability, then to $\rho\mathcal{C}_b^1$, and then inductively, using Theorem 2.2, to $\sigma^s \in \rho\mathcal{C}_b^r$ under $\rho\mathcal{C}_b^r$ -assumptions on the perturbation of f_0 . This finishes the proof of part (4), and thus of the theorem. \square

This theorem remains true if we relax assumption (I.4) to eventual relative r -normal hyperbolicity as in [HPS77, Definition 3], that is: for some constants $0 < \mu < 1 < \lambda < \infty$ and $C > 1$, we have

$$m(\bar{N}_p^u \bar{f}^n) \geq C^{-1} \lambda^n \|\Gamma_p \bar{f}^n\|^k, \quad \|\bar{N}_p^s \bar{f}^n\| \leq C \mu^n m(\Gamma_p \bar{f}^n)^k, \quad \forall p \in \Gamma, \quad 0 \leq k \leq r, \quad n \geq 0. \quad (2.44)$$

Indeed, for all $n \geq N_0$, with N_0 sufficiently large, the map \bar{f}^n is then immediately relatively r -normally hyperbolic, and f^n satisfies the assumptions of Theorem 2.3 upon rescaling time by a factor of n . Then, denoting by Γ^u the corresponding unstable manifold, note that

$$f^n(f(\Gamma^u)) = f(f^n(\Gamma^u)) = f(\Gamma^u), \quad (2.45)$$

so $f(\Gamma^u)$ is another $\rho\mathcal{C}_b^1$ -perturbation of Γ_0^u which is invariant under f^n and approaches $f_0(\Gamma_0^u) = \Gamma_0^u$ as $t \rightarrow \infty$; by uniqueness, Γ^u is therefore f -invariant as well. Part (4) of Theorem 2.3 applies as well to f^n ; to ensure that Γ^s is f -invariant, we work with initial sections—which now need to have t -size at least n —which are f -invariant; in the notation of the above proof, we can take this to be $\bigcup_{j=0}^{n-1} f^{-j}\Gamma_\bullet^s$.

We end this section by noting that other notions of regularity carry over from the perturbation to the invariant manifolds. We only explicitly state one case of interest. Namely, working in $t \geq 1$, let $\mathcal{C}_{b,b}^0 \equiv \mathcal{C}_b^0$, and define the space $\mathcal{C}_{b,b}^r$ exactly like \mathcal{C}_b^r in (2.2), but in addition allowing any number of the V_i to be equal to $t\partial_t$.¹³ We strengthen the assumption (2.1) correspondingly by requiring

$$|(t\partial_t)^k \rho| \leq C_k \rho \quad \forall k \in \mathbb{N}_0.$$

This is satisfied for polynomial weights $\rho(t) = \langle t \rangle^{-\alpha}$, $\alpha > 0$.

Theorem 2.5. *Under the assumptions (I.1)–(I.4) and (II.1), and assuming in assumption (II.2) that $\tilde{V} \in \rho\mathcal{C}_{b,b}^\infty$, the unstable manifold $\Gamma^u \subset \mathcal{M}$ is, resp. the stable manifold Γ^s can be chosen to be, the graph over Γ_0^u , resp. Γ_0^s , of a function in $\rho\mathcal{C}_{b,b}^\infty$. This remains true upon replacing $\mathcal{C}_{b,b}^\infty$ by $\mathcal{C}_{b,b}^r$ for $r \in \mathbb{N}$ fixed.*

Proof. Introducing the new time coordinate $\tilde{t} := \log t$, we have $\partial_{\tilde{t}} = t\partial_t$, and we need to prove $\rho\mathcal{C}_b^\infty$ -regularity of the (un)stable manifold over $\mathbb{R}_{\tilde{t}} \times \mathcal{X}$. But this follows by a minor adaptation of the proof of Theorem 2.3. Indeed, the only fact about the t -behavior of f and f_0 used in the proof is that $\pi_T f(\tilde{t}, x) = \tilde{t} - \delta(\tilde{t}) = \pi_T f_0(\tilde{t}, x)$ change \tilde{t} by the same amount for every fixed \tilde{t} ; for the Lipschitz estimate (2.34), it suffices to assume $(\tilde{t} - \delta(\tilde{t})) - (\tilde{t}' - \delta(\tilde{t}')) \geq (1 - o(1))(\tilde{t} - \tilde{t}')$ as $\tilde{t}, \tilde{t}' \rightarrow \infty$. Note that $t \mapsto t - 1$ means $\tilde{t} \mapsto \log(e^{\tilde{t}} - 1) = \tilde{t} - \delta(\tilde{t})$ with $\delta(\tilde{t}) = \log(1 + \frac{1}{e^{\tilde{t}} - 1})$, which fits this assumption. \square

¹³The reason for the notation is that the vector fields on \mathcal{X} together with $t\partial_t$ span the space of b-vector fields on the compactification (1.4) of \mathcal{M} at future infinity.

2.4. Stable/unstable manifold theorem for flows. For the analogous theorem for flows on the closed manifold \mathcal{X} and the spacetime $\mathcal{M} = \mathbb{R}_t \times \mathcal{X}$, we make the following assumptions:

- (I_F.1) Γ is a closed C^∞ submanifold of \mathcal{X} ;
- (I_F.2) $\bar{V} \in \mathcal{V}(\mathcal{X})$ is a smooth vector field tangent to Γ ;
- (I_F.3) for all $r \in \mathbb{N}$, the time one flow $\bar{f} := e^{\bar{V}}$ is immediately relatively r -normally hyperbolic at Γ , see (I.3)–(I.4), or in fact just eventually relatively r -normally hyperbolic in the sense of (2.44).

Denote by $V_0 := -\partial_t + \bar{V} \in \mathcal{V}(\mathcal{M})$ the spacetime extension which moves at speed 1 in the t -direction. We consider the following class of perturbations $V \in \mathcal{V}(\mathcal{M})$:

- (II_F.1) $Vt = V_0t = -1$;
- (II_F.2) V is a $\rho\mathcal{C}_b^r$ -perturbation of V_0 for every r . That is,

$$\tilde{V} := V - V_0 \in \rho\mathcal{C}_b^\infty(\mathcal{M}; T\mathcal{X}).$$

Theorem 2.6. *Under these assumptions, and using the notation used in the statement of Theorem 2.3, in particular (2.16), there exists a submanifold $\Gamma^u \subset M$ such that:*

- (1) V is tangent to Γ^u ;
- (2) Γ^u approaches Γ_0^u as $t \rightarrow \infty$ in a $\rho\mathcal{C}_b^\infty$ sense;
- (3) Γ^u is the unique manifold satisfying (1) within the class of manifolds approaching Γ_0^u in a $\rho\mathcal{C}_b^1$ sense.

Furthermore:

- (4) there exists a (nonunique) manifold $\Gamma^s \subset M$ approaching Γ_0^s as $t \rightarrow \infty$ in a $\rho\mathcal{C}_b^\infty$ sense and such that V is tangent to Γ^s .

In these statements, the regularity of Γ^u and Γ^s is $\rho\mathcal{C}_b^r$ if we relax assumption (II_F.2) by only assuming $\rho\mathcal{C}_b^r$ -regularity of \tilde{V} . Moreover, if $\tilde{V} \in \rho\mathcal{C}_{b,b}^\infty$, then Γ^u is, resp. Γ^s can be chosen to be, a $\rho\mathcal{C}_{b,b}^\infty$ -graph over Γ_0^u , resp. Γ_0^s . This remains true upon replacing $\mathcal{C}_{b,b}^\infty$ by $\mathcal{C}_{b,b}^r$ for $r \in \mathbb{N}$ fixed.

Proof. The time one flow, denoted f , of V satisfies the assumptions of Theorem 2.3 (in the relaxed form using eventual hyperbolicity as around (2.44) under the assumption on eventual relative normal hyperbolicity) and, for the final part, those of Theorem 2.5. An argument analogous to (2.45) shows that the time t flow e^{tV} preserves Γ^u for all t , in particular as $t \rightarrow 0$, so V is tangent to Γ^u . For the construction of Γ^s , and using the notation of the proof of Theorem 2.3, we need to define Γ_\bullet^s slightly more carefully; for instance, we can take

$$\Gamma_\bullet^s = \bigcup_{t=0}^T e^{-tV} \Gamma_0^s(\{t_0(\epsilon)\})$$

for large and fixed T depending on the constants in the eventual hyperbolicity of f . \square

3. MICROLOCAL ESTIMATES AT NORMALLY HYPERBOLIC TRAPPING

Our microlocal estimates for PDEs with Hamilton flow having a normally hyperbolic trapped set require certain nondegeneracy assumptions which go beyond the purely dynamical results of §2. Thus, for clarity and simplicity, rather than indirectly controlling the (un)stable manifold via dynamical assumptions, we state the assumptions on them as well as the nondegeneracy requirements directly. This section can therefore be read independently of the previous one.

We first sketch the proof of microlocal estimates in the technically simpler setting of closed manifolds in §3.1. A detailed proof in the setting of asymptotically stationary space-times, with microlocal analysis taking place at future infinity, is given in §3.2.

3.1. Estimates on closed manifolds. We first consider microlocal estimates at normally hyperbolic trapping on closed manifolds without boundary and for the simplest class of operators, in order to present the main ideas without the technical complications caused by microlocal analysis on manifolds with boundary.

Thus, let X denote a closed manifold with a fixed volume density, and let $P \in \Psi^m(X)$ be a self-adjoint classical pseudodifferential operator of order $m \in \mathbb{R}$, with (real) principal symbol p . The Hamilton vector field $H_p \in \mathcal{V}(T^*X)$ is homogeneous of degree $m - 1$; if $\widehat{\rho} \in \mathcal{C}^\infty(T^*X \setminus o)$ denotes a positive function which is homogeneous of degree -1 , the rescaled vector field

$$H_p := \widehat{\rho}^{m-1} H_p \in \mathcal{V}(S^*X)$$

induces a vector field on the cosphere bundle $S^*X = (T^*X \setminus o)/\mathbb{R}_+$. We will occasionally identify subsets of S^*X with their conic extensions into $T^*X \setminus o$. Let

$$\Sigma := p^{-1}(0) \setminus o \subset T^*X \setminus o$$

denote the characteristic set of P . Suppose then that H_p is tangent to a closed \mathcal{C}^∞ submanifold $\Gamma \subset \Sigma$, and $dp \neq 0$ in a neighborhood of Γ , so Σ has codimension 1 there. Assume that $\widehat{\rho}$ is such that

$$H_p \widehat{\rho} \equiv 0 \quad \text{near } \Gamma; \tag{3.1}$$

this will allow us to shift the regularity in our estimate at will.¹⁴ Suppose there are local orientable manifolds $\Gamma^{u/s} \subset \Sigma$ near Γ to which H_p is tangent; we assume they are of class \mathcal{C}^∞ , orientable, and nontrivial, i.e. $\Gamma^{u/s} \supsetneq \Gamma$, and with $\text{codim}_\Sigma \Gamma^{u/s} = 1$. Let $\phi^{u/s} \in \mathcal{C}^\infty(S^*X)$ be such that $\phi^{u/s}$ are defining functions for $\Gamma^{u/s}$ within $\Sigma \cap S^*X$;¹⁵ we extend them to functions on $T^*X \setminus o$ by homogeneity of degree 0. We make the hyperbolicity and nondegeneracy assumptions

$$H_p \phi^u = -w^u \phi^u, \quad H_p \phi^s = w^s \phi^s, \quad w^{u/s} > 0, \tag{3.2}$$

$$\widehat{\rho}^{-1} H_p \phi^u \phi^s = \widehat{\rho}^{-1} \{\phi^u, \phi^s\} > 0, \tag{3.3}$$

¹⁴This can always be arranged *locally* along the H_p flow; here however we make this assumption in a full neighborhood of Γ .

¹⁵The case of higher codimension can easily be treated by using vector-valued $\phi^{u/s}$ here and in the proof below.

on Γ .¹⁶ Note that $H_{\phi^u}\phi^u = 0$ and $H_{\phi^u}p = -H_p\phi^u = 0$ on Γ^u ; arguing similarly for H_{ϕ^s} shows that $H_{\phi^u/s}|_{\Gamma^u/s} \in T\Gamma^u/s$. Assumption (3.3) on the other hand ensures that $H_{\phi^u/s}$ is transversal to $\Gamma^{s/u}$ within Σ .

Theorem 3.1. *There exist operators $B_0, B_1, G \in \Psi^0(X)$, with $\text{WF}'(B_0), \text{WF}'(B_1), \text{WF}'(G)$ contained in any fixed neighborhood of Γ , such that B_0 is elliptic at Γ , while $\text{WF}'(B_1) \cap \Gamma^u = \emptyset$, and G is elliptic near Γ , such that for $s, N \in \mathbb{R}$ and some $C > 0$,*

$$\|B_0v\|_{H^s} \leq C(\|B_1v\|_{H^{s+1}} + \|GPv\|_{H^{s-m+2}} + \|v\|_{H^{-N}}). \quad (3.4)$$

This holds in the strong sense that if all norms on the right are finite, then so is the norm on the left, and the inequality holds.

The geometric control condition for (3.4) to hold is that $\text{WF}'(B_0) \subset \text{Ell}(G)$, and all backward null-bicharacteristics from $\text{WF}'(B_0)$ either reach $\text{Ell}(B_1)$ in finite time or tend to Γ , all while remaining in $\text{Ell}(G)$. See Figure 1.3.

Remark 3.2. It suffices to assume that P has real scalar principal symbol, as long as the imaginary part $\widehat{\rho}^{m-1}\sigma(\frac{1}{2i}(P - P^*))$ of the subprincipal symbol is not too large relative to w^s and w^u , see Remark 3.6. One can also let P be a principally scalar operator acting on sections of a vector bundle. We shall consider these more general cases in §3.2 as they are crucial for applications in relativity.

Remark 3.3. Theorem 3.1 is different from (and in a certain sense more degenerate than) the saddle point radial estimates considered in [Vas13, §2], see also [DZ19, §E.5.2]. Indeed, in the simplest situation where $\Gamma \subset T^*X \setminus o$ is a single ray, assumption (3.1) above implies that the Hamilton vector field H_p vanishes at $\Gamma \subset T^*X \setminus o$, hence weights in the fiber such as $\widehat{\rho}^{-2s+m-1}$ cannot give positivity in positive commutator estimates. On the other hand, if in this situation H_p were nontrivially radial at Γ , i.e. $\widehat{\rho}^{-1}H_p\widehat{\rho} \neq 0$ at Γ , one would have the estimate (3.4) with the H^s , resp. H^{s-m+1} norm on B_1v , resp. GPv on the right for s below or above (depending on the sign of $\widehat{\rho}^{-1}H_p\widehat{\rho}$) a certain threshold, by considering commutants of the form $\chi(\phi^u)\chi(\phi^s)\widehat{\rho}^{-2s+m-1}$.

Remark 3.4. For $v \in \mathcal{C}^\infty(X)$, the estimate (3.4) can be proved by means of an argument by contradiction using defect measures, as done by Dyatlov [Dya16] in the semiclassical setting. A simple approximation argument proves (3.4) more generally for all $v \in H^s(X)$ for which the norms on the right are finite, namely, one applies the estimate to smoothed versions $J_\epsilon v$ of v , where $J_\epsilon = \text{Op}((1 + \epsilon\widehat{\rho}^{-1})^{-2})$ is uniformly bounded in $\Psi^0(X)$ and tends to I in $\Psi^\delta(X)$ for all $\delta \rightarrow 0$, and thus strongly on $H^\sigma(X)$ for all σ , and uses that $\|[GP, J_\epsilon]v\|_{H^{s-m+2}} \leq C\|v\|_{H^s}$ remains uniformly bounded in view of (3.1); one thus obtains the boundedness of $B_0J_\epsilon v$ in H^s , and thus by a weak compactness argument the membership $B_0v \in H^s$ with the estimate (3.4). What one *cannot* obtain in this fashion is the strong version of Theorem 3.1, which allows us to *conclude* H^s -membership of v at Γ ; this is proved by regularizing the (positive commutator) argument itself.

¹⁶Condition (3.3) is independent of the choice of defining functions, up to an overall sign. It holds in the following important situation: Γ can be extended off Σ to a conic symplectic submanifold of T^*X (of codimension 2), see also [Dya15b, §5.1]. In condition (3.2), $w^{u/s}$ can be controlled precisely for carefully chosen $\phi^{u/s}$ if the flow of H_p is absolutely 1-normally hyperbolic, see [Dya15b, Lemma 5.1] and also [HPS77, Theorem 2.2].

Example 3.5. We demonstrate that operators P as above exist: take X to be the 3-torus $X = \mathbb{S}_x^1 \times \mathbb{S}_y^1 \times \mathbb{S}_z^1$, and let

$$P = D_y + (\sin x)D_x.$$

Writing covectors as $\xi dx + \eta dy + \zeta dz$, the principal symbol of P is $p = \eta + (\sin x)\xi$, whose differential is nonvanishing, so $\Sigma = p^{-1}(0) \setminus o$ is smooth. The Hamilton vector field is $H_p = \partial_y + (\sin x)\partial_x - (\cos x)\xi\partial_\xi$, the conic submanifold

$$\Gamma = \{x = 0, \xi = 0\} \cap \Sigma = \{(x = 0, y, z; \xi = 0, \eta = 0, \zeta) : \zeta \neq 0\}$$

is invariant under the H_p -flow, and $\widehat{\rho} = |\zeta|^{-1}$, defined in a conic neighborhood of Γ , Poisson-commutes with p . Let $\phi^u = |\zeta|^{-1}\xi$ and $\phi^s = x$, then the manifolds

$$\Gamma^u = (\phi^u)^{-1}(0) \cap \Sigma = \{(x, y, z; 0, 0, \zeta) : \zeta \neq 0\},$$

$$\Gamma^s = (\phi^s)^{-1}(0) \cap \Sigma = \{(0, y, z; \xi, 0, \zeta) : (\xi, \zeta) \neq (0, 0)\}$$

are H_p -invariant; moreover, $\widehat{\rho}^{-1}\{\phi^u, \phi^s\} = 1$, and (3.2) holds with $w^{u/s} = 1$ at Γ . Thus, Theorem 3.1 applies. (Note that the trapping at Γ is not the only delicate structure of P : for example, over $x = 0$ there is also a radial set at $\{x = 0, \eta = 0, \zeta = 0\} \cap \Sigma = N^*\{x = 0\} \setminus o$, which however is disjoint from Γ .)

The key ingredient for the proof is a quantitative propagation estimate for classical operators $L \in \Psi^m$ with real principal symbol ℓ and subprincipal part $\ell_1 = \widehat{\rho}^{m-1}\sigma(\frac{1}{2i}(L - L^*))$. Working on S^*X , suppose $H_\ell = \widehat{\rho}^{m-1}H_\ell$ is transversal to a hypersurface Z , and let $K \subset Z$ be compact. For compact intervals $I \subset \mathbb{R}$, we write, schematically, $\|u\|_{H^s(I)}$ for the “ H^s norm of u on $\{\exp(xH_\ell)z : z \in K, x \in I\}$ ”, by which we mean, roughly, the H^s norm of the microlocalization of u to a small neighborhood of this set; this will be made rigorous later. Then for $T_1, T_2 > 0$ and $\delta_0 > 0$, and assuming $\ell_1 \leq 0$, we have (up to numerical constants independent of T_1, T_2, δ_0 , and using a slight enlargement of K in the norms on the right)

$$\begin{aligned} \|v\|_{H^s([0, T_2])} &\leq \sqrt{\frac{T_2}{T_1}} \|v\|_{H^s([-T_1, 0])} + (T_2 + \sqrt{T_1 T_2}) \|Lv\|_{H^{s-m+1}([-T_1, T_2 + \delta_0])} \\ &\quad + C \|v\|_{H^{s-1/2}([-T_1, T_2 + \delta_0])}. \end{aligned} \quad (3.5)$$

For general ℓ_1 , one can apply this to $L_\gamma = e^{-\gamma s} L e^{\gamma s}$ and $v_\gamma = e^{-\gamma s} v$ for $\gamma \geq \max\{0, \sup \ell_1\}$ —thus L_γ has nonpositive subprincipal part—and obtain the same estimate but with an overall factor of $e^{\gamma(T_1 + T_2)}$ on the right. Thus, if L is symmetric, we get a quantitative propagation estimate depending on the lengths of the control and conclusion regions. If L has a subprincipal part, the constants scale accordingly; the prototypical example is $L = D_x + i\ell_1$ on \mathbb{R}_x , $\ell_1 \in \mathbb{R}$, which in particular annihilates $v = e^{\ell_1 x}$. We sketch the proof of (3.5) at the end of this section.

We shall only sketch the main steps of the proof of Theorem 3.1, leaving the details to the reader; we give a detailed proof in the spacetime setting, which is of primary interest, in §3.2. We work in a fixed neighborhood $|\phi^u| < \epsilon_0$ of Γ^u ; by $\|\cdot\|_{H^s(I)}$, we shall mean the H^s norm on the subdomain where $\phi^s \in I$. We shall also drop localizers to the characteristic set, as well as irrelevant constants. Recall that in (3.4) we are effectively assuming a priori H^s -control on v in a punctured neighborhood of Γ^u .

- (1) Let $\Phi^u \in \Psi^0$ denote a quantization of ϕ^u . The weak localization $\Phi^u v$ off the unstable manifold satisfies a better equation due to (3.2),

$$(P - iW^u)\Phi^u v = \Phi^u P v + R v,$$

with $R \in \Psi^{m-2}$, and $W^u = (W^u)^* \in \Psi^{m-1}$ quantizing $\widehat{\rho}^{-m+1} w^u$. A simple commutator argument exploiting the positivity of w^u gives, for fixed small $\epsilon_0 > 0$,

$$\|\Phi^u v\|_{H^{s+1}([- \epsilon_0, \epsilon_0])} \leq \|P v\|_{H^{s-m+2}} + \|v\|_{H^s([-2\epsilon_0, 2\epsilon_0])} + \|v\|_{H^{s+1}(\{|\phi^u| > \epsilon_0\})}; \quad (3.6)$$

the second term comes from R , and the last one is an error term arising from the localization near Γ^u .

- (2) The quantitative estimate (3.5) applied to $L = \Phi^u$ allows us to control, for $\delta > 0$ to be chosen later,

$$\|v\|_{H^s([-2\delta, 2\delta])} \leq \delta^{1/2} (\|v\|_{H^s(\pm[2\delta, \epsilon_0])} + \|\Phi^u v\|_{H^{s+1}([- \epsilon_0, \epsilon_0])}) + \|v\|_{H^{s-1/2}}); \quad (3.7)$$

see Figure 3.1. This uses assumption (3.3), which implies that ϕ^s is comparable to the affine parameter along the integral curves of $\widehat{\rho}^{-1} H_{\phi^u}$.

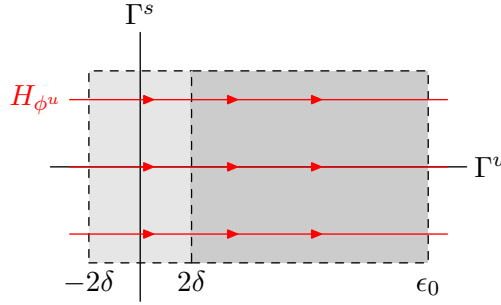


FIGURE 3.1. Quantitative propagation estimate for v in terms of $\Phi^u v$: the norm of v in the light gray box is small, $\mathcal{O}(\delta^{1/2})$, compared to that in the dark gray box due its small size in the H_{ϕ^u} direction (red). The labels indicate the ϕ^s -variable. Also shown is the H_{ϕ^u} flow along which we propagate.

For small $\delta > 0$, combined with (3.6) and absorbing the piece $\|v\|_{H^s([-2\delta, 2\delta])}$ of the second term on the right in (3.6) (which gets the small prefactor $\delta^{1/2}$ from (3.7)) into the left hand side of (3.7), this controls v close to Γ by a *small* constant times v away from Γ :

$$\begin{aligned} \|v\|_{H^s([-2\delta, 2\delta])} &\leq \delta^{1/2} (\|v\|_{H^s(-[2\delta, 2\epsilon_0])} + \|v\|_{H^s([2\delta, 2\epsilon_0])} + \|P v\|_{H^{s-m+2}}) \\ &\quad + \|v\|_{H^{s+1}(\{|\phi^u| > \epsilon_0\})} + \|v\|_{H^{s-1/2}}. \end{aligned} \quad (3.8)$$

This is analogous to the Lipschitz estimate [Dya16, Lemma 3.2] on the defect measure in the semiclassical setting.

- (3) Using a quantitative propagation estimate (3.5) for P , we can conversely estimate v in $\phi^s \in \pm[2\delta, 2\epsilon_0]$ by v in $\pm[\delta, 2\delta]$; this requires time $\sim \log \delta^{-1}$ propagation along

H_p by the unstable dynamics of H_p within Γ^u .¹⁷ Thus,

$$\begin{aligned} \|v\|_{H^s(\pm[2\delta, 2\epsilon_0])} &\leq (\log \delta^{-1})^{1/2} \|v\|_{H^s(\pm[\delta, 2\delta])} + (\log \delta^{-1}) \|Pv\|_{H^{s-m+1}} \\ &\quad + \|v\|_{H^s(\{|\phi^u| > \epsilon_0\})} + \|v\|_{H^{s-1/2}}; \end{aligned} \quad (3.9)$$

see Figure 3.2.

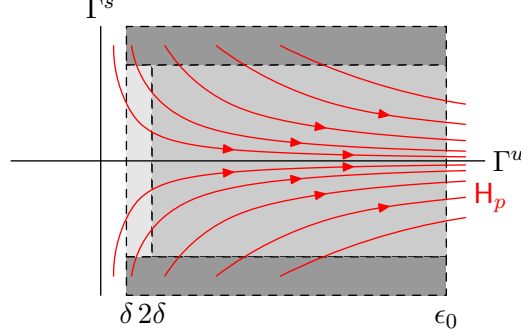


FIGURE 3.2. Control of v for $\phi^s \in [2\delta, \epsilon_0]$ (medium gray) by v on $\phi^s \in [\delta, 2\delta]$ (light gray) and v away from Γ^u (dark gray) by time $\sim \log \delta^{-1}$ propagation along H_p (red). There is an analogous, symmetric, picture for $\phi^s < 0$.

- (4) Plugging (3.9) into (3.8), we can absorb $\|v\|_{H^s(\pm[\delta, 2\delta])}$ into the left hand side of (3.8), giving unconditional control on $\|v\|_{H^s([-2\delta, 2\delta])}$. Plugged back into (3.9), this gives control on v near Γ^u , resulting in the estimate (3.4) with $N = s - \frac{1}{2}$. To obtain the better error term, one proceeds iteratively, applying this estimate to a microlocalized version $B'_0 v$ of v , where $B'_0 \in \Psi^0(X)$ is elliptic at Γ , and take B_0, B_1 to have

$$\text{WF}'(B_0), \text{WF}'(B_1) \subset \text{Ell}(B'_0), \quad \text{and} \quad \text{WF}'(I - B'_0) \cap \text{WF}'(G) = \emptyset;$$

one can then estimate the error term $\|B'_0 v\|_{H^{s-1/2}}$ by the analogous estimate with s reduced by $\frac{1}{2}$ and with slightly enlarged elliptic sets of B_1, G .

Remark 3.6. If $p_1 := \widehat{\rho}^{m-1} \sigma(\frac{1}{2i}(P - P^*))$ was nonzero, one would get an extra factor $\sim \delta^{-\gamma/w^s}$ on the right in (3.9), with $\gamma \geq \max\{0, \sup p_1\}$, which can still be absorbed in (3.8) for $\gamma < \frac{1}{2}w^s$. (On the other hand, step (1) requires $\gamma < w^u$.)

We finally indicate the proof of the model estimate (3.5); the key tool is Gårding's inequality, which allows one to translate symbolic bounds into operator bounds. Namely, dropping localizers to Σ and to Γ^u (the latter can be taken to be H_p -invariant, identically one in a small neighborhood of K) and choosing coordinates in which $H_p = \partial_x$, we consider commutants $a = \widehat{\rho}^{-2s+m-1} \tilde{a}$, where we design $\tilde{a} = \tilde{a}(x)$ with support in $[-T_1, T_2 + \delta_0]$ such that $0 \leq \tilde{a} \leq 1$, and $H_p \tilde{a} \leq \frac{2}{T_1}$ on $[-T_1, 0]$ and $H_p \tilde{a} \leq -\frac{1}{2T_2}$ on $[0, T_2]$. The latter implies $\tilde{a} \leq 2T_2(-H_p \tilde{a})$ on $[0, T_2]$, which we arrange on all of $[0, T_2 + \delta_0]$. See Figure 3.3.

We can arrange that $\tilde{a} = \tilde{a}_1^2 + \tilde{a}_2^2$, where $\tilde{a}_2 = \sqrt{\tilde{a}}$ on $[0, T_2 + \delta_0]$, and $0 \leq \tilde{a}_1 \leq 1$ is supported in $[-T_1, 0)$, and so that for $a_j = \widehat{\rho}^{-s+(m-1)/2} \tilde{a}_j$, $j = 1, 2$,

$$H_p a = -b^2 - \frac{1}{4T_2} \widehat{\rho}^{-m+1} a_2^2 + e,$$

¹⁷To control v in the fixed ϵ_0 -neighborhood of Γ^u , we again need to use the a priori control away from Γ^u , propagated along H_p , but the constants for this part of the estimates do not matter.

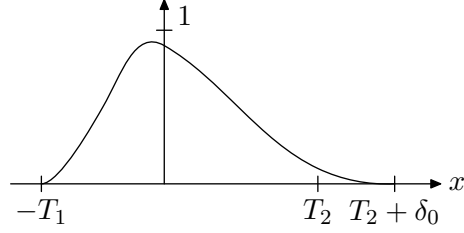


FIGURE 3.3. The commutant $\tilde{a} = \tilde{a}(x)$ used in the quantitative propagation estimate; the slope of \tilde{a} is controlled from above on $[-T_1, 0]$, and has definite decay on $[0, T_2]$.

where b is nonnegative, with lower bound $b \geq \frac{1}{2\sqrt{T_2}}\widehat{\rho}^{-s}$ on $[0, T_2]$, while e is supported in $[-T_1, 0]$ with $|e| \leq \frac{2}{T_1}\widehat{\rho}^{-2s}$. For $A = A_1^*A_1 + A_2^*A_2$, B , E denoting quantizations of a (and a_1, a_2), b , e , one gets by means of Gårding's inequality

$$\begin{aligned} 2\operatorname{Im}\langle Pu, Au \rangle &= \langle i[P, A]u, u \rangle \\ &\leq -\|Bu\|_{L^2}^2 - \frac{1}{4T_2}\|A_2u\|_{H^{-(m-1)/2}}^2 + |\langle Eu, u \rangle| + C\|u\|_{H^{s-1/2}}^2 \\ &\leq -\frac{1}{4T_2}\|u\|_{H^s([0, T_2])}^2 - \frac{1}{4T_2}\|A_2u\|_{H^{-(m-1)/2}}^2 + \frac{2}{T_1}\|u\|_{H^s([-T_1, 0])}^2 + C\|u\|_{H^{s-1/2}}^2. \end{aligned}$$

The left hand side can be estimated by

$$\begin{aligned} 2|\langle Pu, Au \rangle| &\geq -2\|A_1u\|_{H^{-(m-1)/2}}\|A_1Pu\|_{H^{(m-1)/2}} - 2\|A_2u\|_{H^{-(m-1)/2}}\|A_2Pu\|_{H^{(m-1)/2}} \\ &\geq -\frac{2}{T_1}\|u\|_{H^s([-T_1, 0])}^2 - \frac{1}{4T_2}\|A_2u\|_{H^{-(m-1)/2}}^2 \\ &\quad - \left(\frac{T_1}{2} + 4T_2\right)\|Pu\|_{H^{s-m+1}([-T_1, T_2+\delta_0])}^2. \end{aligned}$$

Combining the two gives the desired estimate

$$\|u\|_{H^s([0, T_2])}^2 \leq \frac{16T_2}{T_1}\|u\|_{H^s([-T_1, 0])}^2 + (2T_1T_2 + 16T_2^2)\|Pu\|_{H^{s-m+1}([-T_1, T_2+\delta_0])}^2 + C\|u\|_{H^{s-1/2}}^2.$$

When making this precise (in particular the H^s norms here) in the cusp setting in §3.2, we shall simply keep writing $4T_2\|Bu\|^2$ for the term on the left. The basic estimate then is:

Lemma 3.7. *Let $B, B' \in \Psi^s(X)$ with $\operatorname{WF}'(B') \subset \operatorname{Ell}(B)$, and suppose that their rescaled symbols $b = \widehat{\rho}^s\sigma(B)$, $b' = \widehat{\rho}^s\sigma(B')$ satisfy $|b'| \leq b$ on $\operatorname{WF}'(B')$. Then for any $\delta > 0$, there exists a constant C such that*

$$\|B'u\|_{H^s} \leq (1 + \delta)\|Bu\|_{H^s} + C\|u\|_{H^{s-1/2}}.$$

Proof. The principal symbol of $(1 + \delta)^2B^*B - (B')^*B'$ has a smooth real square root $e \in S^s(T^*X)$: near $\operatorname{WF}'(B')$, this follows from $(1 + \delta)b > |b'|$, while away from $\operatorname{WF}'(B')$, we simply have $e = (1 + \delta)\sigma(B)$. Thus, $(1 + \delta)^2B^*B - (B')^*B' - E^*E \in \Psi^{2s-1}(X)$. Applying this to u and then pairing with u gives the desired result.¹⁸ \square

¹⁸The sharp Gårding inequality allows one to take $\delta = 0$. On the other hand, for $\delta > 0$, one can replace the error term by $C\|u\|_{H^{-N}}$ for any fixed $N \in \mathbb{R}$ by improving the square root construction in the proof: one takes e to be the full symbol of $(1 + \delta)B$ in a fixed small neighborhood of $\operatorname{WF}'(B')$, hence one only needs to solve away the $\Psi^{2s-1}(X)$ -error term near $\operatorname{WF}'(B')$, where e is positive, so the usual square root

3.2. Estimates in the spacetime setting. Let X denote a closed $(n - 1)$ -dimensional manifold, and let

$$M^\circ = \mathbb{R}_t \times X.$$

We describe the (partial) compactification M of M° at future infinity and the relevant operator algebra and function spaces—(complete) cusp pseudodifferential operators and weighted cusp Sobolev spaces—in §3.2.1, before stating and proving the microlocal estimate at trapped sets lying over ∂M in §3.2.2. While our main applications concern wave equations on Lorentzian spacetimes, our arguments in the present section are entirely *microlocal*, and thus apply to any pseudodifferential operator with suitable null-bicharacteristic flow and subprincipal symbol conditions.

3.2.1. Compactification and the cusp algebra. The cusp ps.d.o. algebra was introduced by Mazzeo–Melrose [MM99] in more generality than needed here, hence we give a simplified account adapted to present interests. As a further reference, we refer the reader to the lecture notes [Vas18]. We partially compactify M° by introducing

$$M = (M^\circ \sqcup ([0, \infty)_\tau \times X)) / \sim, \quad (t, x) \sim (\tau = t^{-1}, x), \quad t > 0. \quad (3.10)$$

The manifold M has a natural smooth structure, with functions near $\tau = 0$ being smooth if and only if they are smooth functions of (τ, x) . Consider then the Lie algebra of *cusp vector fields*,

$$\mathcal{V}_{\text{cu}}(M) := \{V \in \mathcal{V}(M) : V\tau \in \tau^2 \mathcal{C}^\infty(M)\}.$$

If x^1, \dots, x^{n-1} denotes local coordinates on X , $\mathcal{V}_{\text{cu}}(M)$ is generated over $\mathcal{C}^\infty(M)$ by $\tau^2 \partial_\tau = -\partial_t$ and $\partial_{x^1}, \dots, \partial_{x^{n-1}}$; hence $\mathcal{V}_{\text{cu}}(M)$ is a uniform version of $\mathcal{V}(M^\circ)$ as $t \rightarrow \infty$, encoding the stationary nature (t -invariance) of M° .¹⁹ This also shows directly that $\mathcal{V}_{\text{cu}}(M)$ is the space of smooth sections of a natural vector bundle ${}^{\text{cu}}TM$, called *cusp tangent bundle*, with local frame

$$\tau^2 \partial_\tau, \partial_{x^1}, \dots, \partial_{x^{n-1}}; \quad (3.11)$$

we emphasize that $\tau^2 \partial_\tau$ is nonvanishing as a cusp vector field *down to* $\tau = 0$.²⁰ The dual bundle ${}^{\text{cu}}T^*M$, called *cusp cotangent bundle*, thus has local sections $\frac{d\tau}{\tau^2}, dx^1, \dots, dx^{n-1}$, with $\frac{d\tau}{\tau^2} = -dt$ nonsingular as a cusp 1-form down to $\tau = 0$. We note that there are natural smooth maps ${}^{\text{cu}}TM \rightarrow TM$ and $T^*M \rightarrow {}^{\text{cu}}T^*M$ which are isomorphisms over M° , but not over ∂M , where they have 1-dimensional kernel and cokernel.

Of particular importance in applications is the symmetric second tensor power $S^2 {}^{\text{cu}}T^*M$. Indeed, stationary metrics are smooth nondegenerate sections of $S^2 {}^{\text{cu}}T^*M$ *down to* $\tau = 0$; conversely, smooth nondegenerate sections $g \in \mathcal{C}^\infty(M; S^2 {}^{\text{cu}}T^*M)$ are ‘asymptotically stationary’, in that the components of $g - g|_{\partial M}$ in the frame (3.11) decay at a polynomial

construction applies. Neither improvement will be needed in our application, so we settle for the simplest version here.

¹⁹In previous works, see in particular [HV18b, §3.3], we encoded stationarity by compactifying using $\tau' = e^{-t}$ as the defining function of future infinity and the b-structure corresponding to the natural smooth structure generated by $\mathcal{C}^\infty(X)$ and smooth functions of τ' . The relationship between the two structures is $\tau' = e^{-1/\tau}$, $\tau' \partial_{\tau'} = \tau^2 \partial_\tau$. Since we are mainly interested in *polynomial* weights in t , as appropriate for perturbations of the Kerr spacetime, the cusp setting is more natural; but we could equally well work in the b-setting with logarithmic weights $|\log \tau'|^\alpha$.

²⁰For general manifolds M , not necessarily arising from our specific construction, ${}^{\text{cu}}TM$ is well-defined once one fixes the equivalence class of a defining function $\tau \in \mathcal{C}^\infty(M)$ of ∂M modulo $\tau^2 \mathcal{C}^\infty(M)$ (note that the latter space is independent of τ).

rate as $t \rightarrow \infty$. We shall also make use of the cusp density bundle whose sections are of the form $a|\frac{d\tau}{\tau^2}dx^1 \cdots dx^{n-1}| = a|dt dx^1 \cdots dx^{n-1}|$ with $a \in \mathcal{C}^\infty(M)$; by a *cusp volume density*, we mean a *positive* such density, i.e. $a > 0$.

Recall the filtered algebra of *cusp differential operators*

$$\text{Diff}_{\text{cu}}(M) = \bigcup_{m \geq 0} \text{Diff}_{\text{cu}}^m(M),$$

where $\text{Diff}_{\text{cu}}^m(M)$ is generated over $\mathcal{C}^\infty(M)$ by up to m -fold products of cusp vector fields; for $m = 0$, we set $\text{Diff}_{\text{cu}}^0(M) := \mathcal{C}^\infty(M)$. If $E, F \rightarrow M$ are smooth vector bundles, we define $\text{Diff}_{\text{cu}}(M; E, F)$ in local trivialisations of E, F as $\text{rank } F \times \text{rank } E$ matrices of elements of $\text{Diff}_{\text{cu}}(M)$. Thus, elements of $\text{Diff}_{\text{cu}}(M; E, F)$ define continuous linear maps $\mathcal{C}^\infty(M; E) \rightarrow \mathcal{C}^\infty(M; F)$ and $\dot{\mathcal{C}}^\infty(M; E) \rightarrow \dot{\mathcal{C}}^\infty(M; F)$; here $\dot{\mathcal{C}}^\infty$ consists of smooth functions or sections vanishing to infinite order at ∂M .

The (in our setup canonical) cusp vector field $\tau^2 \partial_\tau$ induces a natural fiber-linear function²¹

$$\sigma(\varpi) := -\varpi(\tau^2 \partial_\tau), \quad \varpi \in {}^{\text{cu}}T^*M. \quad (3.12)$$

This is the same as the coordinate σ in the set of local fiber coordinates on ${}^{\text{cu}}T^*M$ defined by writing cusp 1-forms as

$$-\sigma \frac{d\tau}{\tau^2} + \xi_i dx^i = \sigma dt + \xi_i dx^i. \quad (3.13)$$

Given an operator

$$P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(\tau, x) (-\tau^2 D_\tau)^j D_x^\alpha \in \text{Diff}_{\text{cu}}^m(M), \quad a_{j\alpha} \in \mathcal{C}^\infty(M),$$

we define its principal symbol in the coordinates (3.13) by

$$\sigma_{\text{cu}}^m(P) := \sum_{j+|\alpha|=m} a_{j\alpha} \sigma^j \xi^\alpha;$$

we often simply write $\sigma(P)$ for brevity. The symbol has the usual properties

$$\sigma_{\text{cu}}^{m_1+m_2}(P_1 \circ P_2) = p_1 p_2, \quad \sigma_{\text{cu}}^{m_1+m_2-1}(i[P_1, P_2]) = H_{p_1} p_2 \quad (3.14)$$

for $P_j \in \text{Diff}_{\text{cu}}^{m_j}(M)$, $p_j = \sigma_{\text{cu}}^{m_j}(P_j)$, $j = 1, 2$, where for a function $p \in \mathcal{C}^\infty({}^{\text{cu}}T^*M)$ its Hamilton vector field H_p is given by

$$H_p = -(\partial_\sigma p) \tau^2 \partial_\tau + (\partial_{\xi_i} p) \partial_{x^i} + (\tau^2 \partial_\tau p) \partial_\sigma - (\partial_{x^i} p) \partial_{\xi_i} \in \mathcal{V}_{\text{cu}}({}^{\text{cu}}T^*M).$$

In particular, $H_p t = \partial_\sigma p$ is a smooth function down to $\tau = 0$.

The algebra $\text{Diff}_{\text{cu}}(M)$ can be microlocalized, which amounts to allowing as symbols more general functions than polynomials in the fibers. Thus, denote by $S^m({}^{\text{cu}}T^*M)$ the space of symbols of order m , i.e. functions $a \in \mathcal{C}^\infty({}^{\text{cu}}T^*M)$ satisfying the bounds

$$|\partial_\tau^j \partial_x^\alpha \partial_\sigma^k \partial_\xi^\beta a(\tau, x, \sigma, \xi)| \leq C_{j\alpha k\beta} (1 + |\sigma| + |\xi|)^{m-(k+|\beta|)} \quad (3.15)$$

²¹For general M and with τ fixed modulo $\tau^2 \mathcal{C}^\infty(M)$, this function is well-defined at ${}^{\text{cu}}T_{\partial M}^*M$.

in local coordinates; the quantization $\text{Op}(a)$ of $a \in S^m$ acts on smooth functions $u \in \dot{\mathcal{C}}^\infty(M)$ supported in a coordinate chart near $\tau = 0$ via the usual formula²²

$$\text{Op}(a)u(t, x) = (2\pi)^{-n} \iiint e^{i((t-t')\sigma + (x-x')\xi)} a(\tau, x, \sigma, \xi) u(t', x') dt' dx' d\sigma d\xi,$$

For general u , define $\text{Op}(a)u$ using a partition of unity. The space $\Psi_{\text{cu}}^m(M)$ then consists of all operators of the form $\text{Op}(a)$. The principal symbol map extends to $\sigma_{\text{cu}}^m: \Psi_{\text{cu}}^m(M) \rightarrow S^m/S^{m-1}(\text{cu}T^*M)$ enjoying the properties (3.14).

We further define the operator wave front set $\text{WF}'_{\text{cu}}(A) \subset \text{cu}T^*M \setminus o$ of $A = \text{Op}(a)$ as the essential support $\text{esssupp } a$ of the full symbol a , which is the complement of all $\varpi \in \text{cu}T^*M \setminus o$ such that a is of order $-\infty$ in a conic neighborhood of ϖ . This is a conic set by definition, and hence can be identified with its image in the quotient space

$$\text{cu}S^*M := (\text{cu}T^*M \setminus o)/\mathbb{R}_+,$$

the cusp cosphere bundle of M . It is often convenient to radially compactify $\text{cu}T^*M$, namely, in a local trivialization $\mathcal{U} \times \mathbb{R}^n$ of $\text{cu}T^*M$, we set

$$\overline{\text{cu}T^*M} := \mathcal{U} \times \overline{\mathbb{R}^n},$$

where $\overline{\mathbb{R}^n}$ is the radial compactification

$$\overline{\mathbb{R}^n} := \left(\mathbb{R}^n \sqcup ([0, \infty)_\rho \times \mathbb{S}^{n-1}) \right) / \sim, \quad \mathbb{R}^n \setminus \{0\} = r\omega \sim (\rho = r^{-1}, \omega).$$

We can then identify $\text{cu}S^*M$ with the new boundary ‘at fiber infinity’ of $\overline{\text{cu}T^*M}$. Thus, functions, resp. vector fields, which are homogeneous of degree 0 are identified with functions, resp. induce vector fields, on S^*M . (In the case of vector fields, the restriction ${}^bT(\overline{\text{cu}T^*M})|_{\text{cu}S^*M} \rightarrow T(\text{cu}S^*M)$ to S^*M has a nontrivial kernel generated by the fiber-radial vector field.) We shall identify conic subsets of $\text{cu}T^*M \setminus o$ with their closures in $\overline{\text{cu}T^*M} \setminus o$ and also with their boundary at fiber at infinity, or equivalently their quotient in $\text{cu}S^*M$.

Working on M° , we can define another closely related class of operators, $\Psi_\infty^m(M)$: given a *uniform symbol* $a \in S_\infty^m(T^*M^\circ)$ satisfying

$$|\partial_t^j \partial_x^\alpha \partial_\sigma^k \partial_\xi^\beta a(t, x, \sigma, \xi)| \leq C_{j\alpha k\beta} (1 + |\sigma| + |\xi|)^{m-(k+|\beta|)} \quad (3.16)$$

(the difference to S^m being that we use ∂_t , not ∂_τ), we define the quantization $\text{Op}(a)$ of a by the same formula as above; its Schwartz kernel is again super-polynomially decaying off the diagonal. This defines the *uniform algebra* $\Psi_\infty(M)$ which generalizes $\Psi_{\text{cu}}(M)$ in that the symbols are no longer required to be smooth down to $\tau = 0$, so $\Psi_{\text{cu}}(M) \subset \Psi_\infty(M)$; on the other hand, uniform symbols do not have well-defined limits at $\tau = 0$. Since the regularity of $a \in S_\infty^m$ in the base is precisely boundedness with respect to iterative application of cusp vector fields, we shall think of elements of $\Psi_\infty(M)$ as cusp ps.d.o.s with coefficients having infinite cusp regularity. They form a filtered algebra, $\Psi_\infty^{m_1}(M) \circ \Psi_\infty^{m_2}(M) \subset \Psi_\infty^{m_1+m_2}(M)$, and the symbolic properties (3.14) continue to hold in $\tau > 0$.

Both classes of operators are invariant under conjugation by polynomial weights in t or equivalently $\tau = t^{-1}$, which allows us to define bi-filtered algebras $\tau^\alpha \Psi_{\text{cu}}^m(M)$ and $\tau^\alpha \Psi_\infty^m(M)$, $\alpha \in \mathbb{R}$, with both orders additive upon composition of operators. We point

²²The Schwartz kernel of $\text{Op}(a)$ in (t, x) -variables is super-polynomially decreasing away from the diagonal, which is sufficient for $\text{Op}(a)$ to act on the polynomially weighted function spaces below. Thus, unlike in the b-setting [HV15, §2], we do not need to insert a cutoff for t near t' .

out that if for $P \in \tau^\alpha \Psi_{\text{cu}}^m(M)$ one has $\sigma(P) = 0 \in \tau^\alpha S^m / \tau^\alpha S^{m-1}(\text{cu}T^*M)$, then one can only conclude that $P \in \tau^\alpha \Psi_{\text{cu}}^{m-1}(M)$, likewise for the uniform algebra; thus, the principal symbol captures only the leading order behavior of P in the sense of differential order, but not in the sense of decay at $\tau = 0$.

Closely related to uniform symbols (3.16) are cusp-conormal functions

$$\mathcal{A}_{\text{cu}}(M) := \{u \in L^\infty(M) : Au \in L^\infty(M) \forall A \in \text{Diff}_{\text{cu}}(M)\}, \quad \mathcal{A}_{\text{cu}}^\alpha(M) := \tau^\alpha \mathcal{A}_{\text{cu}}(M). \quad (3.17)$$

Indeed, $\mathcal{A}_{\text{cu}}(M) = \mathcal{A}_{\text{cu}}^0(M) \subset S_\infty^0(T^*M^\circ)$. In the notation of §2, we have (restricting supports to $t \geq 1$)

$$\mathcal{A}_{\text{cu}}^\alpha(M) = t^{-\alpha} \mathcal{C}_b^\infty(M^\circ).$$

Standard conormal functions are defined using b- instead of cusp operators, to wit

$$\mathcal{A}_b(M) := \{u \in L^\infty(M) : Au \in L^\infty(M) \forall A \in \text{Diff}_b(M)\}, \quad \mathcal{A}_b^\alpha(M) := \tau^\alpha \mathcal{A}_b(M). \quad (3.18)$$

Thus, $\mathcal{A}_b^\alpha(M) = t^{-\alpha} \mathcal{C}_{b,b}^\infty(M^\circ) \subset \mathcal{A}_{\text{cu}}^\alpha(M)$. Conormal functions valued in vector bundles are defined using local trivializations, or equivalently using connections to define derivatives of sections.

Finally, we define the natural L^2 -based function spaces for cusp analysis. Being interested in uniform analysis near $\tau = 0$, we shall implicitly assume that all functions have support in, say, $\tau \leq 1$ (i.e. $t \geq 1$). Fixing any cusp volume density²³ ν , we define the L^2 -space

$$H_{\text{cu}}^0(M) \equiv L_{\text{cu}}^2(M) := L^2(M; \nu),$$

as well as its weighted analogue $\tau^\alpha L_{\text{cu}}^2(M) \equiv H_{\text{cu}}^{0,\alpha}(M)$, $\alpha \in \mathbb{R}$, consisting of all $u \in L_{\text{loc}}^2(M)$ for which $\tau^{-\alpha}u \in L_{\text{cu}}^2(M)$. For integers $k \geq \mathbb{N}_0$, we define the *weighted cusp Sobolev space*

$$H_{\text{cu}}^{k,\alpha}(M) := \{u \in \tau^\alpha L_{\text{cu}}^2(M) : Pu \in \tau^\alpha L_{\text{cu}}^2(M) \forall P \in \text{Diff}_{\text{cu}}^k(M)\}.$$

This becomes a Hilbert space with the norm $\|u\|_{H_{\text{cu}}^{k,\alpha}(M)}^2 = \sum_j \|P_j u\|_{\tau^\alpha L_{\text{cu}}^2(M)}^2$, where $\{P_j\} \subset \text{Diff}_{\text{cu}}^k(M)$ is a finite collection of operators spanning $\text{Diff}_{\text{cu}}^k(M)$ over $\mathcal{C}^\infty(M)$. For general $k \in \mathbb{R}$, $H_{\text{cu}}^{k,\alpha}(M)$ can be defined by duality and interpolation. Any operator $P \in \tau^{-\beta} \Psi_{\text{cu}}^m(M)$ or $P \in \tau^{-\beta} \Psi_\infty^m(M)$ then defines a continuous linear map

$$P: H_{\text{cu}}^{s,\alpha}(M) \rightarrow H_{\text{cu}}^{s-m,\alpha-\beta}(M), \quad s, \alpha \in \mathbb{R}.$$

Using local trivializations and partitions of unity, one can similarly define $\Psi_{\text{cu}}^m(M; E, F)$, $\Psi_\infty^m(M; E, F)$ for vector bundles $E, F \rightarrow M$, as well as function spaces $H_{\text{cu}}^{s,\alpha}(M; E)$.

Lastly, we remark that the basic lemma 3.7 is proved using a purely symbolic argument, and hence applies to cusp ps.d.o.s and the uniform algebra as well.

3.2.2. Setup and proof of the microlocal estimate. Define M by (3.10), and let $E \rightarrow M$ denote a smooth vector bundle such as $E = S^2 \text{cu}T^*M$. Let

$$P \in \Psi_{\text{cu}}^m(M; E)$$

denote a principally scalar operator with principal symbol $p = \sigma_{\text{cu}}^m(P) \in \mathcal{C}^\infty(\text{cu}T^*M)$. Its characteristic set is denoted

$$\Sigma := p^{-1}(0) \subset \text{cu}T^*M \setminus o.$$

²³All choices lead to the same function spaces with equivalent norms since we are restricting supports to the fixed compact subset $\tau \leq 1$ of M .

We define the stationary model at ∂M as follows: denote the restriction of the Hamilton vector field $H_p \in \mathcal{V}_{\text{cu}}(\text{cu}T^*M)$ to $\text{cu}T^*_{\partial M}M$ as a *cuspidal vector field* by H_{p_0} ; in local coordinates, this simply means

$$H_{p_0} = -(\partial_\sigma p_0)\tau^2\partial_\tau + (\partial_{\xi_i} p_0)\partial_{x^i} - (\partial_{x^i} p_0)\partial_{\xi_i}, \quad p_0 := p|_{\tau=0},$$

the point being that we keep the $\tau^2\partial_\tau$ -term. We can identify this with the stationary vector field (i.e. it commutes with \mathcal{L}_{∂_t})

$$(\partial_\sigma p_0)\partial_t + (\partial_{\xi_i} p_0)\partial_{x^i} - (\partial_{x^i} p_0)\partial_{\xi_i} \in \mathcal{V}(T^*M^\circ),$$

which is indeed the same as the Hamilton vector field H_{p_0} (and thus justifies the notation) if we define $p_0 \in \mathcal{C}^\infty(T^*M^\circ)$ by stationary extension, to wit, $p_0(t, x, \sigma, \xi) \equiv p_0(x, \sigma, \xi)$.

We make the following assumptions near the compact \mathcal{C}^∞ submanifold $\Gamma \subset \Sigma \cap \text{cu}S^*_{\partial M}M$:

- (P.1) $dp_0 \neq 0$ on Σ near Γ ;
- (P.2) there exists a function $\hat{\rho} \in \mathcal{C}^\infty(\text{cu}T^*M \setminus o)$, $\hat{\rho} > 0$, which is homogeneous of degree -1 such that $H_{p_0}\hat{\rho} = 0$ near Γ ;
- (P.3) the rescaled Hamilton vector field for the stationary model, $\mathbf{H}_{p_0} := \hat{\rho}^{m-1}H_{p_0} \in \mathcal{V}_{\text{cu}}(\text{cu}S^*M)$, is tangent to Γ , and satisfies

$$\inf_{\Gamma} \mathbf{H}_{p_0} t > 0.$$

- (P.4) there are \mathcal{C}^∞ orientable submanifolds $\bar{\Gamma}^{u/s} \subset \Sigma \cap \text{cu}S^*_{\partial M}M$ near Γ intersecting transversally at Γ which have codimension²⁴ 1 inside of $\Sigma \cap \text{cu}S^*_{\partial M}M$ and to which \mathbf{H}_{p_0} is tangent;
- (P.5) there exist defining functions $\bar{\phi}^{u/s} \in \mathcal{C}^\infty(\text{cu}S^*_{\partial M}M)$ of $\bar{\Gamma}^{u/s}$ inside of $\Sigma \cap \text{cu}S^*_{\partial M}M$, defined locally near Γ , such that in a neighborhood of Γ inside Σ ,

$$\mathbf{H}_{p_0}\bar{\phi}^u = -\bar{w}^u\bar{\phi}^u, \quad \mathbf{H}_{p_0}\bar{\phi}^s = \bar{w}^s\bar{\phi}^s, \quad (3.19)$$

$$\hat{\rho}^{-1}H_{\bar{\phi}^u}\bar{\phi}^s = \hat{\rho}^{-1}\{\bar{\phi}^u, \bar{\phi}^s\} > 0, \quad (3.20)$$

where we denote the homogeneous degree 0 and τ -independent extensions of $\bar{\phi}^{u/s}$ to $\text{cu}T^*M \setminus o$ by the same letters; we assume that

$$\nu_{\min} := \min\left\{\inf_{\Gamma} \bar{w}^u, \inf_{\Gamma} \bar{w}^s\right\} > 0. \quad (3.21)$$

- (P.6) there exist \mathcal{C}^∞ submanifolds $\Gamma^{u/s} \subset \Sigma$ such that $\Gamma^{u/s} \cap \text{cu}S^*_{\partial M}M = \bar{\Gamma}^{u/s}$, and so that $\mathbf{H}_p := \hat{\rho}^{m-1}H_p$ is tangent to $\Gamma^{u/s}$. There exist defining functions $\phi^{u/s} \in \mathcal{C}^\infty(\text{cu}S^*M)$ of $\Gamma^{u/s}$ inside of Σ so that $\phi^{u/s} - \bar{\phi}^{u/s} \in \tau\mathcal{C}^\infty(\text{cu}S^*M)$.

Remark 3.8. The conditions in assumptions (P.5)–(P.6) only depend on the restrictions of $\bar{\phi}^{u/s}$, resp. $\phi^{u/s}$, to $\Sigma \cap \text{cu}S^*_{\partial M}M$, resp. $\Sigma \cap \text{cu}S^*M$.

It is important for our application to relax the regularity of P and $\Gamma^{u/s}$. Thus, we shall allow

$$P \in \Psi_{\text{cu}}^m(M; E) + \tau^\alpha \Psi_\infty^m(M; E), \quad \alpha > 0. \quad (3.22)$$

(The characteristic set of such P is still smooth, and will be described precisely in Lemma 4.6 below.) We then relax assumption (P.6) to:

²⁴As in §3.2, the case of higher codimension can be treated similarly, but is not needed for our application.

(P.6') there exist subsets $\Gamma^{u/s} \subset \Sigma$, which are C^∞ submanifolds of S^*M° in $\tau > 0$, such that $\Gamma^{u/s} \cap {}^{\text{cu}}S^*_\partial M M = \bar{\Gamma}^{u/s}$, and so that $H_p := \widehat{\rho}^{m-1} H_p$ is tangent to $\Gamma^{u/s}$ in $\tau > 0$. There exist functions $\phi^{u/s} \in C^\infty(S^*M^\circ)$ such that $\Gamma^{u/s} = \Sigma \cap (\phi^{u/s})^{-1}(0)$ near Γ , and so that

$$\phi^{u/s} - \bar{\phi}^{u/s} \in \mathcal{A}_{\text{cu}}^\alpha({}^{\text{cu}}S^*M);$$

see also Figures 2.1 and 2.2.

The weight in (P.2) can be taken to be τ -independent by replacing it with $(\tau, x) \mapsto \widehat{\rho}(0, x)$. Assumption (P.3) implies a uniform lower bound for $H_p t$ near Γ , and thus ensures that null-bicharacteristics of P on Γ^u tend to ∂M . A crucial consequence of assumption (P.6') is that

$$H_p \phi^u = -w^u \phi^u, \quad H_p \phi^s = w^s \phi^s, \quad (3.23)$$

in Σ , where $w^{u/s} - \bar{w}^{u/s} \in \mathcal{A}_{\text{cu}}^\alpha({}^{\text{cu}}S^*M)$ (or $C^\infty({}^{\text{cu}}S^*M)$ in the smooth case). We can now state the main microlocal theorem of this paper:

Theorem 3.9. *Suppose E is equipped with a positive definite fiber inner product so that*

$$\sup_\Gamma \mathfrak{p}_1 < \frac{1}{2} \nu_{\min}, \quad \mathfrak{p}_1 := \widehat{\rho}^{m-1} \sigma_{\text{cu}}^{m-1} \left(\frac{1}{2i} (P - P^*) \right). \quad (3.24)$$

Then there exist operators $B_0, B_1, G \in \Psi_{\text{cu}}^0(M)$, with $\text{WF}'_{\text{cu}}(B_0), \text{WF}'_{\text{cu}}(B_1), \text{WF}'_{\text{cu}}(G)$ contained in any fixed neighborhood of Γ , such that B_0 is elliptic at Γ , while $\text{WF}'_{\text{cu}}(B_1) \cap \bar{\Gamma}^u = \emptyset$, and G is elliptic near Γ , such that for all $s, r, N \in \mathbb{R}$, we have

$$\|B_0 v\|_{H_{\text{cu}}^{s,r}} \leq C (\|B_1 v\|_{H_{\text{cu}}^{s+1,r}} + \|G P v\|_{H_{\text{cu}}^{s-m+2,r}} + \|v\|_{H_{\text{cu}}^{-N,r}}) \quad (3.25)$$

for some constant $C > 0$. This holds in the strong sense that if $v \in H_{\text{cu}}^{-\infty,r}$ and the quantities on the right are finite, then so is the quantity on the left, and the inequality holds.

This estimate also holds for P^ in place of P and for suitable choices of B_0, B_1, G , where the assumption on B_1 is now $\text{WF}'_{\text{cu}}(B_1) \cap \Gamma^s = \emptyset$.*

Thus, we can propagate $H_{\text{cu}}^{s,r}$ -control from $\Sigma \setminus \bar{\Gamma}^u$ into a neighborhood of Γ , and from there by standard real principal type propagation, out along $\bar{\Gamma}^u$. The last statement concerns propagation in the opposite direction for the adjoint problem: control from $\Sigma \setminus \Gamma^s$ can be propagated into Γ and out along Γ^s (in particular, into $\tau > 0$). See Figure 3.4.

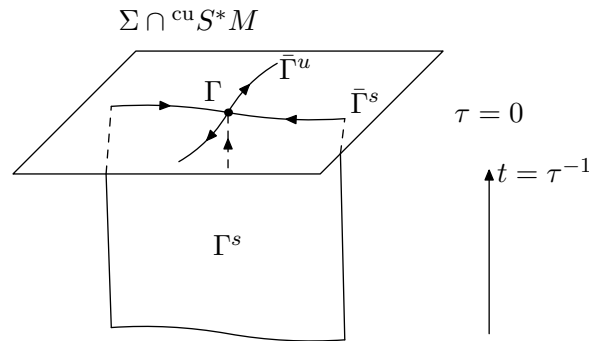


FIGURE 3.4. Illustration of the setup of Theorem 3.9.

Remark 3.10. In the statement of Theorem 3.9 and in its proof below, inequalities between self-adjoint bundle endomorphisms of E and real numbers are meant in the sense of quadratic forms, so for example the condition (3.24) means $\sup_{p \in \Gamma, e \in E_{\pi(p)}, \|e\|=1} \langle \mathfrak{p}_1 e, e \rangle < \frac{1}{2} \nu_{\min}$, where $\pi: {}^{\text{cu}}S^*M \rightarrow M$ is the projection to the base, and $\langle -, - \rangle$ is the fiber inner product on E , with corresponding norm denoted $\| \cdot \|$. Any smooth positive self-adjoint section of $\text{End}(E)$ has a unique pointwise positive self-adjoint square root, which is again a smooth section of $\text{End}(E)$; in positive commutator constructions below, this will be applied to sections of the pullback bundle $\pi^* \text{End}(E) \rightarrow {}^{\text{cu}}S^*M$.

Remark 3.11. One can replace the weighted H_{cu} -norms in the theorem by weighted H_{b} -norms, with the same indices, on the exponential compactification

$$M_e = [0, \infty)_{\tau_e} \times X, \quad \tau_e = e^{-t}, \quad (3.26)$$

of M , provided $P \in \Psi_{\text{b}}^m(M_e)$. Note that then r is the order of an *exponential* weight. One can reduce to the case $r = 0$ (for which $H_{\text{cu}}^{s,0}(M) = H_{\text{b}}^{s,0}(M_e)$) by conjugating P by the exponential e^{-rt} ; this changes \mathfrak{p}_1 , and we therefore only obtain an estimate at Γ on $H_{\text{b}}^{s,r}$ when $r < r_0$ for some threshold $r_0 \in \mathbb{R}$; in fact r_0 is positive when P is symmetric to leading and subleading order, i.e. modulo Ψ_{b}^{m-2} . In this way, Theorem 3.9 extends [HV14, Theorem 3.2], which applies for $P - P^* \in \Psi_{\text{b}}^{m-2}$ and $r < 0$, to a wider range of weights, in particular providing direct estimates on exponentially decaying function spaces which however lose an additional derivative once one crosses the weight $r = 0$.

Remark 3.12. The analogues of the observations in Remark 3.4 apply also here. Furthermore, with nonlinear applications in mind, we recall that positive commutator arguments, as used in the proof below, can be generalized easily to the case that P has coefficients with high, but only finite regularity, with quantitative control of C in (3.25) on their H^s -norm (roughly speaking); in fact, one can prove more precise *tame estimates* as in [Hin16, HV16]. Details will be discussed elsewhere.

Proof of Theorem 3.9. We drop the bundle E from the notation. We shall first consider the case that $P \in \Psi_{\text{cu}}^m$ and work under the stronger assumption (P.6). Denote by $\Phi^{u/s} = (\Phi^{u/s})^* \in \Psi_{\text{cu}}^0(M)$ quantizations of $\phi^{u/s}$. In order to not overburden the notation, we will re-use symbols for microlocal cutoffs and adjust their size to our needs as we proceed.

Steps 1–4 of the proof follow the outline (1)–(4) in the closed manifold setting. We first prove the estimate (3.25) for $v \in \mathcal{C}^\infty(M)$, leaving the regularization argument giving the stronger statement to step 5. In step 6, we consider the general case (P.6’); in step 7, we prove the adjoint estimate.

Step 1: equation and estimate for $\Phi^u v$. Let $W^{u/s} \in \Psi_{\text{cu}}^{m-1}(M)$ denote quantizations of $\widehat{\rho}^{-m+1} w^{u/s}$; then by (3.23), we have

$$i[P, \Phi^u] = -W^u \Phi^u + iR_1 P + iR_2, \quad R_1 \in \Psi_{\text{cu}}^{-1}, \quad R_2 \in \Psi_{\text{cu}}^{m-2};$$

the term $R_1 P$ arises because (3.23) only controls the principal symbol of the commutator on the characteristic set, see also equation (3.31) below. Therefore,

$$P' v' = f', \quad P' := P - iW^u, \quad v' := \Phi^u v, \quad f' := (\Phi^u + R_1) P v + R_2 v. \quad (3.27)$$

We record that f' satisfies the estimate

$$\|G f'\|_{H_{\text{cu}}^{s-m+2,r}} \leq C(\|\widetilde{G} P v\|_{H_{\text{cu}}^{s-m+2,r}} + \|\widetilde{G} v\|_{H_{\text{cu}}^{s,r}} + \|v\|_{H_{\text{cu}}^{s-1,r}}) \quad (3.28)$$

for $G, \tilde{G} \in \Psi_{\text{cu}}^0(M)$ with $\text{WF}'_{\text{cu}}(G) \subset \text{Ell}_{\text{cu}}(\tilde{G})$.

We point out that this is the step which requires Γ^u —even though in the dynamical setting of Theorem 2.3 and 2.6 (where we consider backwards propagation, hence the roles of Γ^s and Γ^u are reversed relative to the present discussion) it is *not* canonically defined—to be preserved by the \mathbf{H}_p flow. Indeed, without arranging $\mathbf{H}_p \phi^u = 0$ on Γ^u , one would get an error term from the commutator $[P, \Phi^u]$ which on Σ does not factor through Φ^u and which thus, while decaying, has too high order in the differential sense to be absorbed later on: f' would get an extra term $\tau \Psi_{\text{cu}}^{m-1}(M)v$, but since we need to estimate the $H_{\text{cu}}^{s-m+2,r}$ norm of f' , this would necessitate control of the $H_{\text{cu}}^{s+1,r-1}$ norm of v near Γ , which is of higher order in the sense of differential orders than the $H_{\text{cu}}^{s,r}$ control we will be able to get (having a weaker weight $r-1$ does not affect this issue).

Noting that $\mathbf{p}'_1 := \widehat{\rho}^{m-1} \sigma(\frac{1}{2i}(P' - (P')^*)) = \mathbf{p}_1 - w^u$ on Σ , the assumption (3.24) implies that $\mathbf{p}'_1 < -\frac{1}{2}\nu_{\min}$ has a favorable sign at Γ . Quantitatively, letting $\mathbf{p} = \widehat{\rho}^m p$, define the commutant

$$a = \check{a}^2, \quad \check{a} = \tau^{-r} \widehat{\rho}^{-s-1+(m-1)/2} \chi^u((\phi^u)^2) \chi^s((\phi^s)^2) \chi_T(\tau) \chi_\Sigma(\mathbf{p}), \quad (3.29)$$

where $\chi^{u/s}, \chi_T, \chi_\Sigma \in \mathcal{C}_c^\infty(\mathbb{R})$ have nonpositive derivatives on $[0, \infty)$, and $\sqrt{-\chi^{u/s}(\chi^{u/s})'}$, $\sqrt{-\chi_T \chi_T'} \in \mathcal{C}^\infty([0, \infty))$. We can choose the supports of the cutoffs so small that $\text{supp } a$ is localized in any fixed neighborhood of Γ inside ${}^{\text{cu}}S^*M$.

With $\check{A} \in \tau^{-r} \Psi_{\text{cu}}^{s+1-(m-1)/2}(M)$ and $A = \check{A}^* \check{A}$ denoting quantizations of \check{a} and a , we then consider the L_{cu}^2 pairing

$$\text{Im} \langle f', Av' \rangle = \frac{1}{2} i (\langle Av', P'v' \rangle - \langle P'v', Av' \rangle) = \left\langle \left(\frac{1}{2} i [P', A] + \frac{P' - (P')^*}{2i} A \right) v', v' \right\rangle. \quad (3.30)$$

Writing

$$\mathbf{H}_p \phi^u = -w^u \phi^u + r^u \mathbf{p}, \quad \mathbf{H}_p \phi^s = w^s \phi^s + r^s \mathbf{p}, \quad (3.31)$$

and fixing $0 < c < -\inf_\Gamma \mathbf{p}'_1$, we compute the principal symbol of the operator on the right as

$$\begin{aligned} \check{a} \mathbf{H}_p \check{a} + \widehat{\rho}^{-m+1} \mathbf{p}'_1 \check{a}^2 &= -c \widehat{\rho}^{-m+1} \check{a}^2 - (\widehat{\rho}^{-s-1} b_0)^2 - (\widehat{\rho}^{-s-1} b^s)^2 \\ &\quad + (\widehat{\rho}^{-s-1} b^u)^2 + (\widehat{\rho}^{-s-1} b_T)^2 + hp, \end{aligned} \quad (3.32)$$

where, writing $\chi = \chi^u \chi^s \chi_T \chi_\Sigma$, and $\widehat{\chi}^u = \chi^s \chi_T \chi_\Sigma$ etc,

$$\begin{aligned} b_0 &= \tau^{-r} \chi(-\mathbf{p}'_1 - c + r\tau(\tau^{-2} \mathbf{H}_p \tau) + (s+1 - (m-1)/2)(\widehat{\rho}^{-1} \mathbf{H}_p \widehat{\rho}))^{1/2}, \\ b^{u/s} &= \tau^{-r} \widehat{\chi}^{u/s} \phi^{u/s} \sqrt{-2w^{u/s} \chi^{u/s} (\chi^{u/s})'}, \\ b_T &= \tau^{-r} \widehat{\chi}_T \sqrt{\tau \chi_T \chi_T' (\tau^{-2} \mathbf{H}_p \tau)}, \\ h &= 2\tau^{-2r} \widehat{\rho}^{-2s-2+m} \chi(\widehat{\chi}^u (\chi^u)' \phi^u r^u + \widehat{\chi}^s (\chi^s)' \phi^s r^s + m \widehat{\chi}_\Sigma \chi'_\Sigma (\widehat{\rho}^{-1} \mathbf{H}_p \widehat{\rho})); \end{aligned}$$

note here that $\tau^{-2} \mathbf{H}_p \tau = -\mathbf{H}_p t < 0$ on Γ by assumption (P.3), and $\widehat{\rho}^{-1} \mathbf{H}_p \widehat{\rho} = o(1)$ as $\tau \rightarrow 0$ by assumption (P.2), hence all square roots are well-defined and smooth provided $\text{supp } a$ is sufficiently close to Γ . Writing $B_0 = \text{Op}(b_0)$ etc, we thus find that

$$\begin{aligned} c \|\check{A} v'\|_{H_{\text{cu}}^{(m-1)/2}}^2 + \|B_0 v'\|_{H_{\text{cu}}^{s+1,r}}^2 + \|B^s v'\|_{H_{\text{cu}}^{s+1,r}}^2 \\ \leq \|B^u v'\|_{H_{\text{cu}}^{s+1,r}}^2 + \|B_T v'\|_{H_{\text{cu}}^{s+1,r}}^2 + |\langle P'v', H v' \rangle| \end{aligned}$$

$$+ \left(\|Gv'\|_{H_{\text{cu}}^{s+1/2,r}}^2 + C\|v'\|_{H_{\text{cu}}^{s-1/2,r}}^2 \right) + c\|\check{A}v'\|_{H_{\text{cu}}^{(m-1)/2}}^2 + \frac{1}{2c}\|\check{A}P'v'\|_{H_{\text{cu}}^{-(m-1)/2}}$$

for some constant $C \in \mathbb{R}$, where the terms in parentheses on the last line are due to the equality (3.32) only concerning the *principal* symbols; we keep a microlocalization $G \in \Psi_{\text{cu}}^0(M)$ near $\text{supp } a$, which is estimated in a norm which is $\frac{1}{2}$ weaker than the control afforded by B_0 , to set up an iterative argument, given momentarily, for improving the error term to $\|v'\|_{H_{\text{cu}}^{s-1/2,r}}^2$. Absorbing the second to last term into the left hand side, and simplifying by dropping the last term on the left, we obtain

$$\|B_0\Phi^u v\|_{H_{\text{cu}}^{s+1,r}} \leq \|B_1\Phi^u v\|_{H_{\text{cu}}^{s+1,r}} + \|Gf'\|_{H_{\text{cu}}^{s-m+2,r}} + \|G\Phi^u v\|_{H_{\text{cu}}^{s+1/2,r}} + C\|\Phi^u v\|_{H_{\text{cu}}^{s-1/2,r}}; \quad (3.33)$$

here, $B_1 \in \Psi_{\text{cu}}^0(M)$ is a suitable operator, microlocalized near Γ , which is elliptic on $\text{Ell}_{\text{cu}}(B^u) \cup \text{Ell}_{\text{cu}}(B_T)$ and satisfies $\text{WF}'_{\text{cu}}(B_1) \cap \bar{\Gamma}^u = \emptyset$. This estimate can be iterated for the term $G\Phi^u v$, i.e. replacing B_0 by G , and replacing the control terms on the right by ones with slightly larger microsupport. (Recall that our arguments can be localized arbitrarily closely to Γ .) Iterating twice, we can combine the resulting term $\|G''\Phi^u v\|_{H_{\text{cu}}^{s+1/2-2\cdot 1/2,r}}$, with $\text{WF}'_{\text{cu}}(G'')$ near Γ , into the final term; estimating Gf' using (3.28) gives

$$\|B_0\Phi^u v\|_{H_{\text{cu}}^{s+1,r}} \leq C(\|B_1\Phi^u v\|_{H_{\text{cu}}^{s+1,r}} + \|\tilde{G}Pv\|_{H_{\text{cu}}^{s-m+2,r}} + \|\tilde{G}v\|_{H_{\text{cu}}^{s,r}} + \|v\|_{H_{\text{cu}}^{s-1/2,r}}) \quad (3.34)$$

upon slightly re-defining B_0, B_1, \tilde{G} ; their properties are: for some small fixed $\epsilon_0 > 0$,

$$\mathcal{U}_{3\epsilon_0} := \{\tau, |\phi^u|, |\phi^s|, |\mathbf{p}| < 3\epsilon_0\} \subset \text{Ell}_{\text{cu}}(B_0),$$

while $\text{WF}'_{\text{cu}}(B_1)$ is disjoint from an ϵ_0 -neighborhood of $\bar{\Gamma}^u$, and \tilde{G}, B_0, B_1 are all microsupported in $\mathcal{U}_{4\epsilon_0}$.

Step 2: quantitative propagation for Φ^u . We now estimate v close to $\bar{\Gamma}^s$ as the solution of the equation $\Phi^u v = v'$ via a propagation estimate. We choose a new commutant

$$a = \check{a}^2, \quad \check{a} = \tau^{-r} \widehat{\rho}^{-s-1/2} \psi(\phi^s) \chi^u((\phi^u)^2) \chi_T(\tau) \chi_\Sigma(\mathbf{p}), \quad (3.35)$$

where $\chi^u, \chi_T, \chi_\Sigma$ as before, identically 1 in $[-\epsilon_0, \epsilon_0]$, and with their supports and $\text{supp } \psi$ chosen small enough so that $\text{supp } a \subset \mathcal{U}_{3\epsilon_0}$. Concretely, for $0 < \delta < \frac{\epsilon_0}{3}$ to be chosen later, we arrange for the smooth function ψ to satisfy $\text{supp } \psi \subset [-4\delta, 2\epsilon_0]$, $0 \leq \psi \leq 1$, with $\psi' \geq 0$ in $\phi^s \leq 0$, $\psi(-3\delta) \geq \frac{1}{4}$, and $\psi' \geq \frac{1}{12\delta}$ on $[-3\delta, 3\delta]$, while $\psi' \geq -\frac{1}{\epsilon_0}$ in $\phi^s \geq 0$. In $[-3\delta, 3\delta]$, this implies $\psi \leq 12\delta\psi'$; we arrange for this to hold in all of $[-4\delta, 3\delta]$.

Split $\psi^2 = \psi_-^2 + \psi_+^2$, where $\psi_- = \psi$ in $\phi^s \leq 3\delta$ and $\text{supp } \psi_- \subset [-4\delta, 4\delta]$, and $0 \leq \psi_+ \leq 1$ has $\text{supp } \psi_+ \subset (3\delta, 2\epsilon_0]$. We can then write

$$\psi\psi' = \frac{1}{24\delta}\psi_-^2 + \frac{1}{24\delta}\tilde{b}_\delta^2 - \tilde{e},$$

where $\text{supp } \tilde{e} \subset (3\delta, 2\epsilon_0]$, $|\tilde{e}| \leq \frac{1}{\epsilon_0}$, and $\tilde{b}_\delta \geq 0$, $\text{supp } \tilde{b}_\delta \subset [-4\delta, 4\delta]$, and, crucially, $\tilde{b}_\delta \geq \frac{1}{4}$ in $[-3\delta, 3\delta]$.²⁵ We define \check{a}_\pm by replacing ψ by ψ_\pm in the definition (3.35) of \check{a} , so $\check{a}^2 = \check{a}_-^2 + \check{a}_+^2$.

By assumption (3.20), and shrinking ϵ_0 if necessary (also in the construction of the commutant), we can choose $C_\phi > 1$ so that

$$C_\phi^{-1} \leq \mathbf{H}_{\phi^u} \phi^s \leq C_\phi \quad \text{on } \mathcal{U}_{3\epsilon_0}, \quad \mathbf{H}_{\phi^u} := \widehat{\rho}^{-1} H_{\phi^u}.$$

²⁵Smoothness of \tilde{b}_δ is only an issue near the left boundary of its support (while near its right boundary we may simply cut \tilde{b}_δ off smoothly in $[3\delta, 4\delta]$); but this can be ensured by taking ψ to be of the form $e^{-1/(\phi^s+4\delta)}$ in $\phi^s > -4\delta$ near $\phi^s = -4\delta$. This relies on the fact that for $f(x) = e^{-1/x}$ ($x > 0$), one has $f' = x^{-2}f$, so e.g. $ff' - Cf^2 = (x^{-1}f\sqrt{1-Cx^2})^2$ indeed has a smooth square root near $x = 0$ since $x^{-1}f$ is smooth.

Therefore,

$$\check{a}H_{\phi^u}\check{a} = \frac{1}{48C_\phi\delta}\widehat{\rho}\check{a}_-^2 + (\tau^{-r}\widehat{\rho}^{-s}b_-)^2 + \frac{1}{24C_\phi\delta}(\tau^{-r}\widehat{\rho}^{-s}b_\delta)^2 + \tau^{-2r}\widehat{\rho}^{-2s}e_1 - \tau^{-2r}\widehat{\rho}^{-2s}e_2, \quad (3.36)$$

where

$$\begin{aligned} b_- &= \psi_- \chi^u \chi_T \chi_\Sigma \left(\frac{1}{24\delta} (\mathbf{H}_{\phi^u} \phi^s - \frac{1}{2} C_\phi^{-1}) - r\tau(\tau^{-2} \mathbf{H}_{\phi^u} \tau) - (s + \frac{1}{2})(\widehat{\rho}^{-1} \mathbf{H}_{\phi^u} \widehat{\rho}) \right)^{1/2}, \\ b_\delta &= \chi^u \chi_T \chi_\Sigma \widetilde{b}_\delta \sqrt{C_\phi(\mathbf{H}_{\phi^u} \phi^s)}, \\ e_1 &= \psi^2 (\chi^u)^2 (\tau^2 \chi_T \chi'_T \chi_\Sigma^2 (\tau^{-2} \mathbf{H}_{\phi^u} \tau) + \chi_T^2 \chi_\Sigma \chi'_\Sigma (\mathbf{H}_{\phi^u} \mathbf{p})), \\ e_2 &= (\chi^u)^2 \chi_T^2 \chi_\Sigma^2 \left((\mathbf{H}_{\phi^u} \phi^s) \widetilde{e} + \psi_+^2 (r\tau(\tau^{-2} \mathbf{H}_{\phi^u} \tau) + (s + \frac{1}{2})(\widehat{\rho}^{-1} \mathbf{H}_{\phi^u} \widehat{\rho})) \right). \end{aligned}$$

The supports of χ^u , χ_T , and χ_Σ can be taken to be independent of δ . Taking $\delta > 0$ small ensures that the square roots in b_- and b_δ are well-defined and smooth. The key point is then that

$$b_\delta \geq \frac{1}{4} \quad \text{on } \mathcal{U}_{\epsilon_0} \cap \{-3\delta \leq \phi^s \leq 3\delta\},$$

while the a priori control term e_2 has support in $\mathcal{U}_{3\epsilon_0}$ and satisfies

$$\phi^s > 3\delta \quad \text{on } \text{supp } e_2, \quad |e_2| \leq C_\phi \epsilon_0^{-1} + C'$$

for some δ -independent constant C' . The error term e_1 is the sum of a piece with support in $\tau > 0$ and a piece supported away from the characteristic set of P ; both places are away from $\bar{\Gamma}^u$ and hence we have a priori control there.

Quantizing these symbols, with the corresponding operators denoted by upper case letters, evaluation of the pairing $\text{Im} \langle v', Av \rangle = \langle \frac{1}{2} i [\Phi^u, A] v, v \rangle$ gives

$$\begin{aligned} & \frac{1}{48C_\phi\delta} \|\check{A}_- v\|_{H_{\text{cu}}^{-1/2,0}}^2 + \|B_- v\|_{H_{\text{cu}}^{s,r}}^2 + \frac{1}{24C_\phi\delta} \|B_\delta v\|_{H_{\text{cu}}^{s,r}}^2 \\ & \leq |\langle E_1 v, v \rangle| + |\langle E_2 v, v \rangle| + |\langle \check{A}_- v', \check{A}_- v \rangle| + |\langle \check{A}_+ v', \check{A}_+ v \rangle| + \widetilde{C} \|v\|_{H_{\text{cu}}^{s-1/2,r}}^2 \\ & \leq |\langle E_2 v, v \rangle| + \|B_1 v\|_{H_{\text{cu}}^{s,r}}^2 + \frac{1}{48C_\phi\delta} \|\check{A}_- v\|_{H_{\text{cu}}^{-1/2,0}}^2 + 24C_\phi\delta \|\check{A}_- \Phi^u v\|_{H_{\text{cu}}^{1/2,0}}^2 \\ & \quad + \frac{1}{2\epsilon_0} \|\check{A}_+ v\|_{H_{\text{cu}}^{-1/2,0}}^2 + \epsilon_0 \|\check{A}_+ \Phi^u v\|_{H_{\text{cu}}^{1/2,0}}^2 + \widetilde{C} \|v\|_{H_{\text{cu}}^{s-1/2,r}}^2, \end{aligned}$$

where $B_1 \in \Psi_{\text{cu}}^0(M)$ (with implicit dependence on δ) is elliptic on $\text{supp } e_1$ and has wave front set disjoint from an ϵ_0 -neighborhood of $\bar{\Gamma}^u$; the constant \widetilde{C} depends on δ and arises from the fact that (3.36) only captures *principal* symbols. The third term on the right can be absorbed into the left hand side, and the second term on the left can be dropped. In view of the bounds on the symbols, and using Gårding's inequality in the form of Lemma 3.7, we obtain, after overall multiplication by $24C_\phi\delta$, the estimate (cf. the estimate (3.5))

$$\begin{aligned} \|B_\delta v\|_{H_{\text{cu}}^{s,r}} & \leq C(\sqrt{\delta/\epsilon_0} \|\widetilde{E}_\delta v\|_{H_{\text{cu}}^{s,r}} + \|B_1 v\|_{H_{\text{cu}}^{s,r}} \\ & \quad + (\delta + \sqrt{\delta\epsilon_0}) \|G\Phi^u v\|_{H_{\text{cu}}^{s+1,r}} + \widetilde{C} \|v\|_{H_{\text{cu}}^{s-1/2,r}}), \end{aligned} \quad (3.37)$$

where C is a δ -independent constant; here, $\widetilde{E}_\delta \in \Psi_{\text{cu}}^0(M)$ controls the terms involving $E_2 v$ and $\check{A}_+ v$, has principal symbol bounded from above by 1, and has $\phi^s > 3\delta$ on $\text{WF}'_{\text{cu}}(\widetilde{E}_\delta)$; and

$$B_\delta \in \Psi_{\text{cu}}^0(M), \quad \sigma(B_\delta) \geq \frac{1}{4} \quad \text{on } \mathcal{U}_{\epsilon_0} \cap \{-3\delta \leq \phi^s \leq 3\delta\}. \quad (3.38)$$

The operator $G \in \Psi_{\text{cu}}^0(M)$ is microsupported in $\mathcal{U}_{3\epsilon_0}$. A similar estimate holds in $\phi^s < 0$.

Let us now appeal to the estimate (3.34) (which is of course independent of δ), with the present G taking the place of B_0 there. Its third term gives a contribution $C(\delta + \sqrt{\delta\epsilon_0})\|\tilde{G}v\|_{H_{\text{cu}}^{s,r}}$ in (3.37), where we may assume that $|\sigma(\tilde{G})| \leq 1$ and that C is independent of δ . But then we can bound

$$\|\tilde{G}v\|_{H_{\text{cu}}^{s,r}} \leq 2\|B_\delta v\|_{H_{\text{cu}}^{s,r}} + 2\|E_\delta v\|_{H_{\text{cu}}^{s,r}} + \tilde{C}\|B_1 v\|_{H_{\text{cu}}^{s,r}} + \tilde{C}\|v\|_{H_{\text{cu}}^{s-1/2,r}}, \quad (3.39)$$

where $E_\delta \in \Psi_{\text{cu}}^0(M)$ satisfies $|\sigma(E_\delta)| \leq 1$,

$$\begin{aligned} \sigma(E_\delta) &= 1 \text{ on } \{3\delta \leq \phi_s \leq 4\epsilon_0, |\phi^u| \leq \epsilon_0, |\mathbf{p}| \leq \epsilon_0\}, \\ \text{WF}'_{\text{cu}}(E_\delta) &\subset \{\tfrac{5}{2}\delta \leq \phi_s \leq 5\epsilon_0, |\phi^u| \leq 2\epsilon_0, |\mathbf{p}| \leq 2\epsilon_0\}. \end{aligned}$$

The third term $\|B_1 v\|$, takes care of $\tilde{G}v$ more than ϵ_0 away from $\bar{\Gamma}^u$. Choosing $\delta > 0$ small so that $2C(\delta + \sqrt{\delta\epsilon_0}) \leq \frac{1}{2}$, we can absorb the first term in (3.39) into the left hand side of (3.37). Dropping irrelevant δ -dependencies and fixing ϵ_0 , we conclude that

$$\|B_\delta v\|_{H_{\text{cu}}^{s,r}} \leq C(\delta^{1/2}\|E_\delta v\|_{H_{\text{cu}}^{s,r}} + \|B_1 v\|_{H_{\text{cu}}^{s+1,r}} + \|\tilde{G}Pv\|_{H_{\text{cu}}^{-m+2,r}} + \tilde{C}\|v\|_{H_{\text{cu}}^{s-1/2,r}}). \quad (3.40)$$

We summarize the key features: C, \tilde{C} are constants, with C (but not \tilde{C}) independent of δ ; B_δ is as in (3.38), while $\text{WF}'_{\text{cu}}(B_1)$ is disjoint from an ϵ_0 -neighborhood of $\bar{\Gamma}^u$; and $\text{WF}'_{\text{cu}}(\tilde{G}) \subset \mathcal{U}_{4\epsilon_0}$. (We give ourselves some extra room compared to the sketch of the argument in §3.2.)

Step 3: quantitative propagation for P . We now use quantitative real principal type propagation for P to estimate $\|E_\delta v\|_{H_{\text{cu}}^{s,r}}$ by $\|B_\delta v\|_{H_{\text{cu}}^{s,r}}$ in (3.40). The two qualitative differences to the previous step are (1) the repelling nature of H_p at Γ^s , necessitating a different time scale $\log \delta^{-1}$ for propagation from $\phi^s = \delta$ to $\phi^s = \epsilon_0$, and (2) the presence of the subprincipal part \mathbf{p}_1 of P , leading to exponential (in the propagation time) growth of the constants in the propagation estimate. Thus, fixing $\beta \in \mathbb{R}$ subject to

$$\max\left\{\frac{\sup_{\Gamma} \mathbf{p}_1}{\nu_{\min}}, 0\right\} < \beta < \frac{1}{2}, \quad (3.41)$$

we consider a commutant of the form

$$a = \check{a}^2, \quad \check{a} = \tau^{-r} \check{\rho}^{-s+(m-1)/2} |\phi^s|^{-\beta} \psi(\log |\phi^s/\delta|) \chi^u((\phi^u)^2) \chi_T(\tau) \chi_\Sigma(\mathbf{p}), \quad (3.42)$$

where $\chi^u, \chi_T, \chi_\Sigma$ are cutoffs as above and chosen to be identically 1 on $[-3\epsilon_0, 3\epsilon_0]$, while $\psi \in \mathcal{C}_c^\infty((0, \log(6\epsilon_0\delta^{-1})))$, which in particular cuts out the singularity of $|\phi^s|^{-\beta}$ at $\phi^s = 0$. Since the weight $|\phi^s|^{-\beta}$ will give positivity, as we explain below, we may choose ψ less carefully than above (and with various sign switches, as the direction of propagation is now *away* from Γ^s).²⁶ Namely, we arrange $0 \leq \psi \leq 1$, further $\psi' \leq 1$ on $[0, \log \frac{9}{4}]$, and $\psi' < 0$ and $\psi \geq \frac{1}{4}$ on $[\log \frac{9}{4}, \log(5\epsilon_0\delta^{-1})]$. We may then further arrange

$$\psi\psi' = -\tilde{b}_\ell^2 + \tilde{e},$$

where $\text{supp } \tilde{e} \subset [0, \log \frac{9}{4}]$, $|\tilde{e}| \leq 1$, while $\tilde{b}_\ell \geq 0$, with $\text{supp } \tilde{b}_\ell \subset [\log \frac{3}{2}, \log(6\epsilon_0\delta^{-1})]$, is smooth. We again write $\psi^2 = \psi_+^2 + \psi_-^2$, where now $\psi_+ = \psi$ on $[\log \frac{9}{4}, \log(6\epsilon_0\delta^{-1})]$, and $0 \leq \psi_- \leq 1$ has $\text{supp } \psi_- \subset [0, \log \frac{9}{4}]$.

²⁶In part (3) of the proof sketch in §3.2, we did not exploit the freedom of inserting a weight $|\phi^s|^{-\beta}$, and instead relied on quantitative decay of ψ on $[\log \frac{9}{4}, \log(2\epsilon_0\delta^{-1})]$ at a rate $\psi' \sim 1/\log \delta^{-1}$, giving in (3.43) a contribution of the same sign as $b_{\ell,+}$ but by a factor $1/\log \delta^{-1}$ smaller than the fixed size control afforded by $\beta > 0$.

Denote by \check{A} and $A = \check{A}^* \check{A}$ quantizations of \check{a} and a , respectively. Much as for (3.30), we then consider the commutator

$$\text{Im}\langle Pv, Av \rangle = \left\langle \left(\frac{1}{2}i[P, A] + \frac{P-P^*}{2i}A \right) v, v \right\rangle.$$

Fixing $c \in (0, \beta\nu_{\min} - \sup_{\Gamma} \mathbf{p}_1)$, we then calculate, using (3.31),

$$\check{a}H_p\check{a} + \widehat{\rho}^{-m+1}\mathbf{p}_1\check{a}^2 = -c\widehat{\rho}^{-m+1}\check{a}^2 - (\tau^{-r}\widehat{\rho}^{-s}b_{\ell,-})^2 - (\tau^{-r}\widehat{\rho}^{-s}b_{\ell,+})^2 + e_1 + e_2 + hp, \quad (3.43)$$

where now

$$\begin{aligned} b_{\ell,\pm} &= |\phi^s|^{-\beta}\psi_{\pm}\chi^u\chi_T\chi_{\Sigma}(\beta w^s - \mathbf{p}_1 - c \\ &\quad + r\tau(\tau^{-2}\mathbf{H}_p\tau) + (s - (m-1)/2)(\widehat{\rho}^{-1}\mathbf{H}_p\widehat{\rho}) + w^s\widetilde{b}_{\ell}^2)^{1/2}, \\ e_1 &= \tau^{-2r}\widehat{\rho}^{-2s}|\phi^s|^{-2\beta}\psi^2\chi_{\Sigma}^2(2(\phi^u\mathbf{H}_p\phi^u)\chi^u(\chi^u)'\chi_T^2 + (\tau^{-2}\mathbf{H}_p\tau)\tau^2(\chi^u)^2\chi_T\chi_T'), \\ e_2 &= \tau^{-2r}\widehat{\rho}^{-2s}|\phi^s|^{-2\beta}w^s\widetilde{e}(\chi^u)^2\chi_T^2\chi_{\Sigma}^2, \\ h &= \tau^{-2r}\widehat{\rho}^{-2s+m}|\phi^s|^{-2\beta}\psi^2(\chi^u)^2\chi_T^2((-\beta r^s + \psi\psi'\frac{r^s}{\phi^s})\chi_{\Sigma}^2 + m(\widehat{\rho}^{-1}\mathbf{H}_p\widehat{\rho})\chi_{\Sigma}\chi_{\Sigma}'). \end{aligned}$$

Here, the weight $|\phi^s|^{-\beta} = e^{-\beta \log|\phi^s|}$ in \check{a} , which is exponentially decreasing along the \mathbf{H}_p flow near $\bar{\Gamma}^u$, means that we are giving up control as we get farther away from Γ^s ; this counteracts the allowed growth of amplitudes caused by the subprincipal symbol \mathbf{p}_1 of P . Furthermore, e_2 , which has a quantitative bound coming from that of \widetilde{e} , captures errors near $\bar{\Gamma}^s$, which is where we propagate from; the term e_1 captures terms away from $\bar{\Gamma}^u$, where we have a priori control. The main term in this calculation in $b_{\ell,+} \geq 0$, which gives control away from Γ^s . Concretely, we have uniform lower and upper bounds

$$\begin{aligned} b_{\ell,+} &\geq c' && \text{on } \mathcal{U}_{3\epsilon_0} \cap \{|\phi^s| \in [\frac{9}{4}\delta, 5\epsilon_0]\}, \\ |e_2| &\leq C'\tau^{-2r}\widehat{\rho}^{-2s}\delta^{-2\beta} && \text{on } \text{supp } e_2 \subset \mathcal{U}_{4\epsilon_0} \cap \{|\phi^s| \in [\delta, \frac{9}{4}\delta]\} \end{aligned} \quad (3.44)$$

for some δ -independent constants $c', C' > 0$.

We now proceed as usual by quantizing the relation (3.43), obtaining

$$\begin{aligned} c\|\check{A}v\|_{H_{\text{cu}}^{(m-1)/2,0}}^2 + \|B_{\ell,+}v\|_{H_{\text{cu}}^{s,r}}^2 + \|B_{\ell,-}v\|_{H_{\text{cu}}^{s,r}}^2 \\ \leq |\langle E_1v, v \rangle| + |\langle E_2v, v \rangle| + |\langle Pv, Hv \rangle| + |\langle \check{A}Pv, \check{A}v \rangle| + \widetilde{C}\|v\|_{H_{\text{cu}}^{s-1/2,r}}^2 \\ \leq |\langle E_1v, v \rangle| + |\langle E_2v, v \rangle| + \|\widetilde{G}'Pv\|_{H_{\text{cu}}^{s-m+1,r}}^2 + \widetilde{C}\|\widetilde{G}'v\|_{H_{\text{cu}}^{s-1,r}}^2 \\ + c\|\check{A}v\|_{H_{\text{cu}}^{(m-1)/2,0}}^2 + \frac{1}{2c}\|\check{A}Pv\|_{H_{\text{cu}}^{-(m-1)/2,0}}^2 + \widetilde{C}\|v\|_{H_{\text{cu}}^{s-1/2,r}}^2, \end{aligned}$$

where $\widetilde{G}' \in \Psi_{\text{cu}}^0(M)$ is elliptic near $\text{supp } a$ and has wave front set contained in a small neighborhood thereof, and \widetilde{C} is a δ -dependent constant. We can absorb the fifth term on the right into the left hand side, drop the third term on the left, and combine the $\widetilde{G}'v$ term with the last, error, term. We estimate $|\langle E_2v, v \rangle|$ by means of Gårding's inequality and (3.44). Using the operators in (3.40), this gives the estimate

$$\|E_{\delta}v\|_{H_{\text{cu}}^{s,r}} \leq C(\delta^{-\beta}\|B_{\delta}v\|_{H_{\text{cu}}^{s,r}} + \|\widetilde{B}_1v\|_{H_{\text{cu}}^{s,r}} + \widetilde{C}\|\widetilde{G}'Pv\|_{H_{\text{cu}}^{s-m+1,r}} + \widetilde{C}\|v\|_{H_{\text{cu}}^{s-1/2,r}}) \quad (3.45)$$

for some δ -independent constant C , while \widetilde{C} may depend on δ ; here, \widetilde{B}_1 is a quantization of a symbol \widetilde{b}_1 which is elliptic in the complement of an $\frac{1}{2}\epsilon_0$ -neighborhood of $\bar{\Gamma}^u$ within $\mathcal{U}_{6\epsilon_0}$, and with $\text{WF}'_{\text{cu}}(\widetilde{B}_1)$ disjoint from an $\frac{1}{4}\epsilon_0$ -neighborhood of $\bar{\Gamma}^u$; this takes care of those parts of $|\langle E_1v, v \rangle|$ which lie outside the elliptic set of B_{δ} and are thus away from $\bar{\Gamma}^u$.

Step 4: combining the estimates. Plugging the estimate (3.45) into (3.40), we can absorb the resulting term $\delta^{1/2-\beta}\|B_\delta u\|$ (with $\beta < \frac{1}{2}$ by (3.41)) into the left hand side of (3.40) upon fixing $\delta > 0$ sufficiently small, giving the estimate

$$\|B_\delta v\|_{H_{\text{cu}}^{s,r}} \leq \tilde{C}(\|\tilde{B}_1 v\|_{H_{\text{cu}}^{s+1,r}} + \|\tilde{G}' P v\|_{H_{\text{cu}}^{s-m+2,r}} + \|v\|_{H_{\text{cu}}^{s-1/2,r}}), \quad (3.46)$$

which controls v in a δ -neighborhood of Γ . This finishes the proof of the estimate (3.25) for $N = s - \frac{1}{2}$; the case of general N follows as in §3.2 by applying (3.25) to a microlocalized version of v and iterating the estimate for the resulting microlocalized error term.

Step 5: regularization. The pairings and integrations by parts in the above argument are not justified if we only assume the terms on the right in (3.25) to be finite. This is easily remedied by a standard regularization argument: in the first step, we replace the commutant \check{a} in (3.29) by $\check{a}_\eta = \check{a}\varphi_\eta$, where $\varphi_\eta(\hat{\rho}) := (1 + \eta\hat{\rho}^{-1})^{-1}$, $\eta > 0$. In the commutator calculation (3.32), the H_p -derivative falling on φ_η can be absorbed by the main term b_0 in view of

$$\varphi_\eta H_p \varphi_\eta = \frac{\eta}{\eta + \hat{\rho}} \varphi_\eta^2 (\hat{\rho}^{-1} H_p \hat{\rho}), \quad \left| \frac{\eta}{\eta + \hat{\rho}} \right| \leq 1,$$

and the vanishing of $\hat{\rho}^{-1} H_p \hat{\rho}$ at $\tau = 0$; that is, this term can be made arbitrarily small by localizing close enough to $\tau = 0$. For $\eta > 0$, one can then quantize $a_\eta = \check{a}_\eta^2$, which for $\eta > 0$ is a symbol of 2 orders less, and proceed as written (starting with $s = -N + \frac{1}{2}$); this gives an estimate of the form (3.33) with B_0 etc. replaced by operators $B_{0,\eta}$ (quantizing $\varphi_\eta \hat{\rho}^{s+1} b_0$). Taking $\eta \rightarrow 0$ and using a weak*-compactness argument (see [Vas18] or [Mel94, Proof of Proposition 7 and §9]) then implies that $B_0 \Phi^u v \in H_{\text{cu}}^{s+1,r}$ and the estimate (3.33) holds. One argues similarly for steps 2–4, where one combines the regularized estimates from steps 2 and 3 to obtain a regularized version of (3.46) at which point one takes the regularization parameter to 0 to conclude. One obtains (3.25) for general N by the usual inductive argument.

Step 6: relaxing the regularity requirements. Under the weaker assumption (P.6'), the proof goes through with only minor changes. Indeed, the positivity of the Poisson brackets in the above positive (or negative) commutator arguments near $\tau = 0$ is preserved upon adding $\mathcal{O}(\tau^\alpha)$ errors, provided one localizes in a sufficiently small neighborhood of $\tau = 0$. Quantizations of unweighted zeroth order symbols now lie in $\Psi_{\text{cu}}^0 + \tau^\alpha \Psi_\infty^0$, with the orders of both summands shifted by the same amounts for weighted symbols of general order. If one defines the elliptic set of an operator $A = A_0 + \tilde{A}$, $A_0 \in \Psi_{\text{cu}}^0$, $\tilde{A} \in \tau^\alpha \Psi_\infty^0$ at $\tau = 0$ to be equal to $\text{Ell}_{\text{cu}}(A_0)$, microlocal elliptic regularity holds by the usual proof (inverting the principal symbol $\sigma(A) = \sigma(A_0) + \sigma(\tilde{A})$), hence the positivity of symbols gives microlocal control of v just as in the smooth setting.

Step 7: proof of the final statement (backward propagation). The estimate (3.25) for P^* in place of P , and with B_1 microlocalized away from Γ^s , is proved by a completely analogous argument. The key differences are: (1) the roles of ϕ^u and ϕ^s (and correspondingly w^u and w^s) are switched; (2) passing to P^* switches the sign of the imaginary part of its subprincipal symbol, but we must now propagate *backwards* along the H_p flow, as we assume a priori control on $\bar{\Gamma}^u \setminus \Gamma$ and want to propagate this into Γ —hence the threshold condition for step 3 above is still given by (3.24); (3) the $-H_p$ -derivative of the time cutoff $\chi_T(\tau)$ now has a *favorable* sign, as we are now propagating estimates in the direction of increasing τ —that is, we do not need to place a priori assumptions on $\text{supp } \chi'_T$. \square

4. APPLICATIONS TO WAVE EQUATIONS ON ASYMPTOTICALLY KERR SPACETIMES

For black hole masses $\mathbf{m} > 0$ and subextremal angular momenta $\mathbf{a} \in \mathbb{R}$, $|\mathbf{a}| < \mathbf{m}$, we consider the Kerr metric in Boyer–Lindquist coordinates,

$$g_{\mathbf{m},\mathbf{a}} = \frac{\Delta}{\rho^2} (dt - \mathbf{a} \sin^2 \theta d\varphi)^2 - \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) - \frac{\sin^2 \theta}{\rho^2} (\mathbf{a} dt - (r^2 + \mathbf{a}^2) d\varphi)^2, \quad (4.1)$$

$$\rho^2 = r^2 + \mathbf{a}^2 \cos^2 \theta, \quad \Delta = r^2 - 2\mathbf{m}r + \mathbf{a}^2,$$

as a stationary Lorentzian metric of signature $(+, -, -, -)$ on

$$M^\circ := \mathbb{R}_t \times (r_+, \infty)_r \times \mathbb{S}^2,$$

using spherical coordinates on \mathbb{S}^2 , and where $r_+ := \mathbf{m} + \sqrt{\mathbf{m}^2 - \mathbf{a}^2}$ is the radius of the event horizon. Denote by $G_{\mathbf{m},\mathbf{a}} = g_{\mathbf{m},\mathbf{a}}^{-1}$ the dual metric function. Partially compactifying M° as

$$M := \left(M^\circ \sqcup ([0, \infty)_\tau \times (r_+, \infty) \times \mathbb{S}^2) \right) / \sim, \quad (t, r, \omega) \sim (\tau = t^{-1}, r, \omega) \text{ for } t > 0, \quad (4.2)$$

the discussion in §3.2.1 implies that $g_{\mathbf{m},\mathbf{a}} \in \mathcal{C}^\infty(M; S^2 \text{cu}T^*M)$ is a smooth cusp metric. Denote the characteristic set by

$$\Sigma := G_{\mathbf{m},\mathbf{a}}^{-1}(0) \subset \text{cu}T^*M \setminus o;$$

it has two components, $\Sigma = \Sigma^+ \sqcup \Sigma^-$, where $\Sigma^\pm = \{\varpi \in \Sigma : \pm G_{\mathbf{m},\mathbf{a}}(\varpi, dt) > 0\}$. Identifying Σ with its closure inside $\overline{\text{cu}T^*M} \setminus o$, let

$$\mathcal{X}_\vee := \Sigma \cap \text{cu}T_{\partial M}^*M, \quad \mathcal{X} := \partial \mathcal{X}_\vee = \Sigma \cap \text{cu}S_{\partial M}^*M. \quad (4.3)$$

denote the characteristic set at future infinity (with ‘ \vee ’ indicating its conic version, rather than its boundary at fiber infinity \mathcal{X}).

The Hamilton flow $H_{G_{\mathbf{m},\mathbf{a}}}$ restricted to Σ has a trapped set in $r > r_+$ at which the flow is r -normally hyperbolic for every r (the latter r being an integer, not the radius function!). This was first proved by Wunsch and Zworski [WZ11] for $|\mathbf{a}| \ll \mathbf{m}$, and extended to the full subextremal range by Dyatlov [Dya15a], see also [DZ13]. (The Kerr–de Sitter case was discussed by Vasy [Vas13].) This is usually phrased as a condition on the t -independent spacetime trapped set (though all geodesics in the spacetime trapped set escape to $t = \infty$, hence trapping really only occurs there). Concretely, define

$$\Gamma_\vee := \{\varpi \in \mathcal{X}_\vee : H_{G_{\mathbf{m},\mathbf{a}}} r = (H_{G_{\mathbf{m},\mathbf{a}}})^2 r = 0 \text{ at } \varpi\}, \quad \Gamma := \partial \Gamma_\vee \subset \mathcal{X}. \quad (4.4)$$

As shown in the references, $\Gamma_{(\vee)}$ is a smooth codimension 2 submanifold of $\mathcal{X}_{(\vee)}$, and $H_{G_{\mathbf{m},\mathbf{a}}}$ is tangent to Γ_\vee . Write

$$\Gamma^\pm := \Gamma \cap \Sigma^\pm.$$

We also recall that the fiber-linear function $\sigma = \partial_t(\cdot) \in \mathcal{C}^\infty(\text{cu}T^*M)$ (which was already defined in (3.12)) is nonzero on Γ_\vee ; indeed, $\pm\sigma > 0$ on $\Gamma_\vee^\pm = \Gamma_\vee \cap \Sigma^\pm$. Thus, $|\sigma|^{-1}$ is a smooth function, homogeneous of degree -1 , near Γ_\vee , and

$$H_{G_{\mathbf{m},\mathbf{a}}} \hat{\rho} = 0, \quad \hat{\rho} := |\sigma|^{-1}, \quad (4.5)$$

since the Kerr metric is stationary. Furthermore, $H_{G_{\mathbf{m},\mathbf{a}}} t \neq 0$ on Γ_\vee , hence the rescaled vector field

$$\mathbf{H} := \frac{1}{H_{G_{\mathbf{m},\mathbf{a}}} t} H_{G_{\mathbf{m},\mathbf{a}}}, \quad \mathbf{H} t = 1,$$

induces a vector field tangent to Γ ; and \mathbf{H} , or rather its projection to a cusp vector field on M over ∂M , is *future* null. The following is a reformulation of the results of [Dya15a, §3.2]:

Proposition 4.1. *The H-flow within \mathcal{X} is eventually absolutely r -normally hyperbolic at Γ for every r . The stable and unstable manifolds $\bar{\Gamma}^{u/s}$ at Γ are smooth and orientable, of codimension 1 within \mathcal{X} , intersect transversally at Γ , and admit defining functions $\bar{\phi}^{u/s} \in \mathcal{C}^\infty(\mathcal{X})$ which satisfy the nondegeneracy conditions (3.19)–(3.21).*

Let $\Xi \in \mathcal{C}^\infty(M)$, $\Xi > 0$, denote a smooth conformal factor. We denote by

$$\nu_{\min} > 0$$

the quantity defined in (3.21) with $H_{p_0} := \pm \Xi \widehat{\rho} H_{G_{\mathbf{m}, \mathbf{a}}}$ near Γ^\pm ;²⁷ $H_{p_0} = \Xi \sigma^{-1}(H_{G_{\mathbf{m}, \mathbf{a}}} t) \cdot \mathbf{H}$ is a smooth positive multiple of \mathbf{H} .

This verifies assumptions (P.1)–(P.5) of Theorem 3.9, and, by stationarity, also assumption (P.6) for the Kerr spacetime $(M, g_{\mathbf{m}, \mathbf{a}})$, thereby providing us with microlocal estimates at the trapped set for bundle-valued wave equations on exact Kerr. The main advance in the present paper is that we can consider significant perturbations of the Kerr spacetime:

Definition 4.2. Let $\alpha > 0$. A metric $g \in \mathcal{C}^\infty(M^\circ; S^2 T^* M^\circ)$ is an *asymptotically (subextremal) Kerr metric* if there exist parameters \mathbf{m} and $\mathbf{a} \in (-\mathbf{m}, \mathbf{m})$ of a subextremal Kerr black hole and a symmetric 2-tensor \tilde{g} such that

$$g = g_{\mathbf{m}, \mathbf{a}} + \tilde{g}, \quad \tilde{g} \in \mathcal{A}_{\text{cu}}^\alpha(M; S^2 \text{cu} T^* M).$$

In particular, smooth metrics $g \in \mathcal{C}^\infty(M; S^2 \text{cu} T^* M)$ with $g - g_{\mathbf{m}, \mathbf{a}} \in \tau \mathcal{C}^\infty$ are examples of asymptotically Kerr metrics, though the smoothness requirement is unnatural from the point of view of applications.

Combining the results of §2.4 (describing the null-geodesic flow near the trapped set of the asymptotically Kerr spacetime) with those of §3.2 (microlocal estimates), we obtain:

Theorem 4.3. *Let $g = g_{\mathbf{m}, \mathbf{a}} + \tilde{g}$ denote an asymptotically subextremal Kerr metric, and let $G = g^{-1}$. Let $\Sigma := G^{-1}(0) \subset \text{cu} T^* M \setminus o$ denote the characteristic set, and define \mathcal{X} as in (4.3). Define the trapped set Γ as in (4.3) and (4.4); further, let $\bar{\Gamma}^{u/s} \subset \mathcal{X}$ denote the unstable/stable manifold of Γ inside of \mathcal{X} , and denote by $\Gamma_0^s = [0, \infty)_\tau \times \bar{\Gamma}^s \subset \text{cu} S^* M$ the stationary extension of $\bar{\Gamma}^s$. Denote the speed 1 rescaling of the Hamilton vector field by*

$$\mathbf{H} := \frac{1}{H_G t} H_G \in (\mathcal{C}^\infty + \mathcal{A}_{\text{cu}}^\alpha) \mathcal{V}_{\text{cu}}(\overline{\text{cu} T^* M}),$$

defined in a neighborhood of Γ . Then:

- (1) *(Existence and smoothness of the stable manifold.) There exists a subset $\Gamma^s \subset \Sigma$, with $\Gamma^s \cap \mathcal{X} = \bar{\Gamma}^s$ and with $\Gamma^s \cap S^* M^\circ$ a smooth manifold, such that Γ^s approaches Γ_0^s in an $\mathcal{A}_{\text{cu}}^\alpha$ sense (see the statement of part (2) of Theorem 2.3), and such that \mathbf{H} is tangent to Γ^s .*
- (2) *(Microlocal estimates.) Let $E \rightarrow M$ be a vector bundle, let $P_0 \in \Psi_{\text{cu}}^2(M; E)$ denote a classical cusp ps.d.o. with $\sigma_{\text{cu}}^2(P_0) = \Xi G_{\mathbf{m}, \mathbf{a}}$, and let $\tilde{P} \in \tau^\alpha \Psi_\infty^2(M; E)$ denote an operator with $\sigma^2(\tilde{P}) = \Xi(G - G_{\mathbf{m}, \mathbf{a}})$; let*

$$P = P_0 + \tilde{P}.$$

²⁷In the Schwarzschild case $\mathbf{a} = 0$ and for $\Xi = \rho^2$, we have $\nu_{\min} = 6\sqrt{3}\mathbf{m}$; this can be read off from [Dyal15a, Proposition 3.8] and $\Xi = 9\mathbf{m}^2$, $\sigma^{-1} H_{G_{\mathbf{m}, 0}} t = 6$ at Γ .

Suppose that

$$\sup_{\Gamma} \rho_1 < \frac{1}{2} \nu_{\min}, \quad \rho_1 := \pm \widehat{\rho} \sigma_{\text{cu}}^1 \left(\frac{1}{2i} (P_0 - P_0^*) \right) \text{ on } \Gamma^{\pm}. \quad (4.6)$$

Then the conclusions of Theorem 3.9 hold for P .

If $\tilde{g} \in \mathcal{A}_b^\alpha(M; S^2 \text{cu} T^* M)$, then part (1) holds with \mathcal{A}_b in place of \mathcal{A}_{cu} .

Remark 4.4. Due to the structural stability of r -normally hyperbolic trapping [HPS77, Theorem (4.1)], and due to the fact that the proof of Theorem 3.9 for any fixed level of Sobolev regularity only requires a finite degree of smoothness of the coefficients of P , one can consider a much more general situation: suppose B is an open subset of a Banach space of parameters (which in Theorem 4.3 was taken to be $B = \{(\mathbf{m}, \mathbf{a}) : |\mathbf{a}| < \mathbf{m}\}$) smoothly parametrizing C^∞ stationary metrics by assigning $B \in b \mapsto g_b$, and let $b_0 \in B$ be such that g_{b_0} is a subextremal Kerr metric. Then, for regularity and weights confined to compact subsets of \mathbb{R} , the estimates of Theorem 4.3 hold at the trapped set of g_b when b is close to b_0 .

Remark 4.5. The main calculation required to apply the microlocal estimates is the verification of the subprincipal symbol condition (4.6). This was verified for the linearization of the gauge-fixed Einstein equation in [HV18b, §9.1] at the Schwarzschild–de Sitter metric, but the calculations there work directly for Schwarzschild metrics as well. By continuity, the condition (4.6) is verified also for slowly rotating Kerr black holes. More generally, it holds for wave equations on tensors on slowly rotating Kerr, as follows in the same manner from [Hin17], and can in fact be explicitly verified in the *full* subextremal range by using the relationship, explained in [Hin17], between (4.6) and parallel transport along trapped null-geodesics, with the latter being described by Marck [Mar83]; the details will be presented elsewhere.

The results of §2.4 take place on the stationary manifold

$$\mathcal{M} := \mathbb{R}_t \times \mathcal{X}. \quad (4.7)$$

One would like to take \mathcal{M} to be the characteristic set $\Sigma \cap S^* M^\circ$; however, since g is not stationary when $\tilde{g} \neq 0$, we need to relate $\Sigma \cap M^\circ$ and (4.7). To this effect, we have the following general result:

Lemma 4.6. *With $M = [0, \infty)_\tau \times X$, X compact, let $E \rightarrow M$ denote a vector bundle with zero section o , and let $SE \rightarrow M$ be the fiber bundle with fibers $SE_p := (E_p \setminus o_p)/\mathbb{R}_+$. Let $p_0 \in C^\infty(SE)$, and suppose that $dp_0 \neq 0$ on $\mathfrak{X} := p_0^{-1}(0) \cap SE|_{\partial M}$, which we assume to be nonempty. Put $\mathfrak{M}_0 := [0, \infty)_\tau \times \mathfrak{X} \subset SE$, which is thus smooth and of codimension 1. Let moreover $\alpha > 0$ and $\tilde{p} \in \mathcal{A}_{\text{cu}}^\alpha(SE)$, and let*

$$p = p_0 + \tilde{p}, \quad \mathfrak{M} := p^{-1}(0) \subset SE.$$

Let $\varpi \in \mathfrak{X}$, and let $\mathcal{U} \times (-1, 1)$ be a tubular neighborhood of an open neighborhood $\mathcal{U} \subset \mathfrak{X}$ of ϖ within $SE|_{\partial M}$; extend this to a tubular neighborhood $([0, \infty) \times \mathcal{U}) \times (-1, 1)$ of $[0, \infty)_\tau \times \mathcal{U} \subset \mathfrak{M}_0$ inside SE .²⁸ Then there exists $\tau_0 > 0$ such that in this neighborhood, $\mathfrak{M} \cap \{\tau < \tau_0\}$ is the graph over $\mathfrak{M}_0 \cap \{\tau < \tau_0\}$ of a function $f \in \mathcal{A}_{\text{cu}}^\alpha(\mathfrak{M}_0)$.

²⁸If the normal bundle of \mathfrak{X} is orientable, one can take $\mathcal{U} = \mathfrak{X}$ simply.

More generally, if we merely assume that \tilde{p} has finite regularity, namely there exists $k \in \mathbb{N}$ such that $\text{Diff}_{\text{cu}}^k(SE)\tilde{p} \subset \tau^\alpha L^\infty$, then the conclusion holds for a function f of matching regularity: $\text{Diff}_{\text{cu}}^k(\mathfrak{M}_0)f \subset \tau^\alpha L^\infty$.

The lemma also holds true if one replaces \mathcal{A}_{cu} by \mathcal{A}_{b} throughout.

Proof of Lemma 4.6. We may assume that in local coordinates $(x^0, x') \in \mathbb{R} \times \mathbb{R}^{n-2+\text{rank } E}$ on $SE|_{\partial M}$, we have $p_0 = x^0$, so $\mathfrak{X} = \{x^0 = 0\}$ and $\mathfrak{M}_0 = \{x^0 = 0\}$ in the product coordinate system; the defining equation for \mathfrak{M} reads $p(\tau, x^0, x') = x^0 + \tilde{p}(\tau, x^0, x') = 0$. Write $x^0 = \tau^\alpha y$ and $\tilde{q} = \tau^{-\alpha} \tilde{p}$, which satisfies $\text{Diff}_{\text{cu}}^k \cdot \tilde{q} \subset L^\infty$. We then need to solve $y + \tilde{q}(\tau, \tau^\alpha y, x') = 0$, which can be solved using the contraction mapping principle on the y -ball of radius $\|\tilde{q}\|_{L^\infty} + 1$. Higher regularity follows by differentiating this equation along $\tau^2 \partial_\tau$ and $\partial_{x'}$. \square

Proof of Theorem 4.3. By the previous lemma, $\Sigma \cap S^*M^\circ$ is an $\mathcal{A}_{\text{cu}}^\alpha$ graph over $\mathcal{M} \subset S^*M^\circ$ near $\tau = 0$. Let $\Phi: \mathcal{M} \rightarrow \Sigma \cap S^*M^\circ$ be the diffeomorphism $\mathcal{M} \ni \varpi \mapsto (\varpi, f(\varpi))$ (using the notation and the collar neighborhood of Lemma 4.6). Then

$$\Phi^*H =: V = V_0 + \tilde{V},$$

where $V_0 = \frac{1}{H_{G_{\text{m,a}}}} H_{G_{\text{m,a}}}$ is the stationary model, and $\tilde{V} \in \rho \mathcal{C}_b^\infty(\mathcal{M}; T\mathcal{X})$, $\rho(t) = t^{-\alpha}$, is the perturbation. The vector field $-V$ thus satisfies the assumptions of Theorem 2.6, proving part (1).

Theorem 2.6 also produces a (nonunique) unstable manifold $\Gamma^u \subset \Sigma$, with $\Gamma^u \cap \mathcal{X} = \bar{\Gamma}^u$, which is an $\mathcal{A}_{\text{cu}}^\alpha$ graph over $\Gamma_0^u = [0, \infty)_\tau \times \bar{\Gamma}^u$ near $\tau = 0$, and to which H is tangent; thus, we get defining functions $\phi^{u/s} \in (\mathcal{C}^\infty + \mathcal{A}_{\text{cu}}^\alpha)^{(\text{cu}S^*M)}$ of $\Gamma^{u/s}$ within Σ . Thus, assumption (P.6') of Theorem 3.9 is verified as well. In Σ^+ , we can thus apply this theorem to the operator P , while in Σ^- , we apply it to $-P$ since the Hamilton vector field of $-P$ is future null in Σ^- . This proves part (2). \square

Remark 4.7. For those parameters of Kerr–de Sitter black holes for which normal hyperbolicity was verified in [Dya15a, Vas13], we can similarly analyze asymptotically Kerr–de Sitter metrics g : in this case, natural metric perturbations \tilde{g} of such a Kerr–de Sitter metric lie in the class $e^{-\alpha t} \mathcal{A}_{\text{cu}}$ on the compactification (4.2) (or equivalently $\tau_e^\alpha \mathcal{A}_{\text{b}}$ on the exponential compactification (3.26)). Theorem 4.3 then provides a description of the spacetime stable/unstable manifolds and gives microlocal trapping estimates even on function spaces with (sufficiently mild) exponential decay in t ; see Remark 3.11.

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