18.950/9501: Differential geometry of curves and surfaces.

02/04/2020

Course website: link on my homepage (math.mit.edu/~phintz)
Homework: due Thu in class; no late hw accepted, but 2 lowest scores dropped.
Grade: midterm (30%), final (40%), homework (30%).
Textbook: do Carmo, see website (available online for free).

Classical differential geometry: study of local properties of curves and surfaces. "Local" = only depending on the behavior of the curve/surface near a point of interest — e.g. curvature

Later: global diff. geo.: how local properties influence behavior of entire curve/surface.

I. Curves in $\mathbb{R}^3$.

I.1 Basic notions.

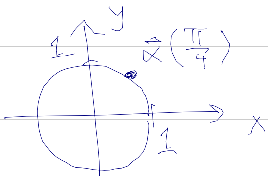
Def. A parameterized smooth curve is a differentiable map $\vec{\alpha}: I \to \mathbb{R}^3$ of an open interval $I = (a,b)$ into $\mathbb{R}^3$.
(That is: $\vec{\alpha}(t) = (x(t), y(t), z(t))$, $x, y, z: I \to \mathbb{R}$ smooth.)
Call $\vec{\alpha}'(t) = (x'(t), y'(t), z'(t))$ the tangent vector of $\vec{\alpha}$ at t, and $\vec{\alpha}(I) = \{\alpha(t) : t \in I\}$ the image (or trace).

Comments: (i) allow $a = -\infty$, $b = +\infty$

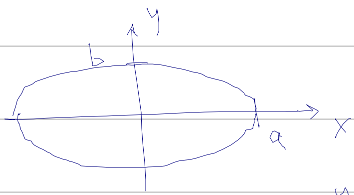
(ii) "smooth" means arbitrarily differentiable: x, y, z have derivatives of all orders, so we can differentiate $\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''', \dots$
(Do Carmo says "differentiable", but probably different from your calculus or real analysis class!)

Examples First in $\mathbb{R}^2$.

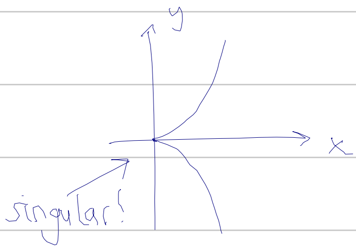
(1) $\vec{\alpha}(t) = (\cos(t), \sin(t))$



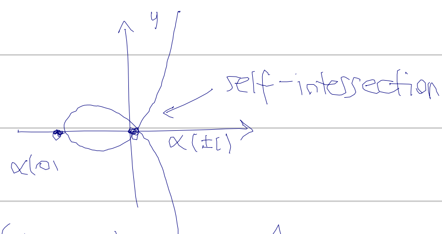
(2) $\vec{\alpha}(t) = (a\cos(t), b\sin(t))$, $a, b > 0$.



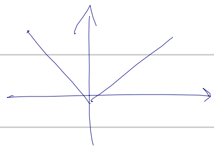
(3) $\vec{\alpha}(t) = (t^2, t^3)$



(4) $\vec{\alpha}(t) = (t^2-1, t(t^2-1))$

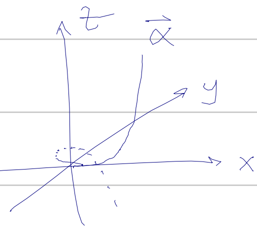


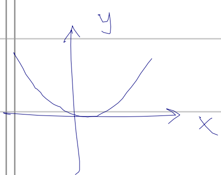
(5) $\vec{\alpha}(t) = (t, |t|)$
(Counterexample)

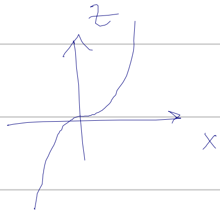


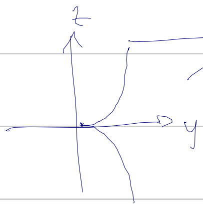
Not smooth!

$\mathbb{R}^3$. (6) Moment curve: $\vec{\alpha}(t) = (t, t^2, t^3)$

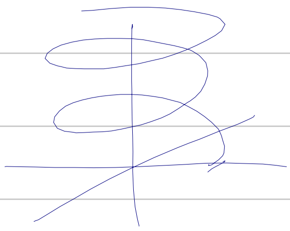








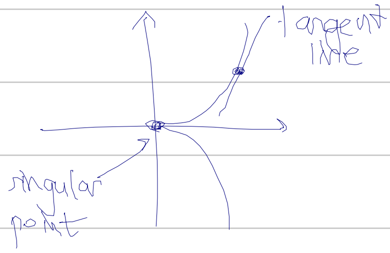
(7) Helix: $\vec{\alpha}(t) = (a\cos t, a\sin t, bt)$



Example (3) shows a typical issue in diff geo.: the occurrence of singularities.

Note: When $\vec{\alpha}'(t) \neq 0$, $\exists$ well-defined line in the direction $\vec{\alpha}'(t)$ passing through $\vec{\alpha}(t)$, called tangent line of $\vec{\alpha}$ at t.

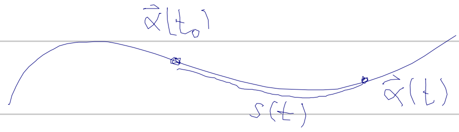
Call point t where
- $\vec{\alpha}'(t) \neq 0$: regular point
- $\vec{\alpha}'(t) = 0$: singular point



Def. A parameterized smooth curve $\vec{\alpha}: I \to \mathbb{R}^3$ is regular if $\vec{\alpha}'(t) \neq 0 \quad \forall t \in I$

- Arc length: $\vec{\alpha}: I \to \mathbb{R}^3$ regular, $t_0 \in I$. Arc length of $\vec{\alpha}$ from t_0 to $t \in I$:

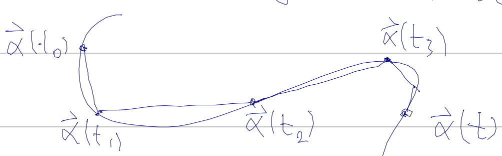
$$s(t) := \int_{t_0}^{t} \|\vec{\alpha}'(t')\| \, dt', \text{ where } \|\vec{\alpha}'(t)\| = \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right)^{1/2}.$$



We say that $\vec{\alpha}$ is parameterized by arc length if $\|\vec{\alpha}'(t)\| = 1 \quad \forall t \in I$.

In this case, $s(t) = t - t_0$, so $t - t_0 =$ arc length from $\vec{\alpha}(t_0)$ to $\vec{\alpha}(t)$.

Rmk. For a regular curve $\vec{\alpha}$, the arc length $s(t)$ from $\vec{\alpha}(t_0)$ to $\vec{\alpha}(t)$ is equal to the limit of lengths of polygonal approximations:

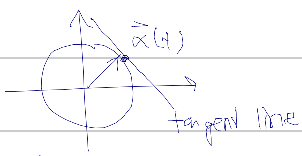


(as one takes more and more points closer and closer together).

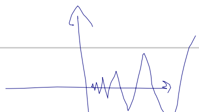
$\rightsquigarrow$ HW 1 for the precise statement.

- Examples: (1) If $\|\vec{\alpha}(s)\|$ is constant, then $\vec{\alpha}(s), \vec{\alpha}'(s)$ are orthogonal.

$\mathbb{R}^2$:



(2) $\vec{\alpha}(t) = (t, t\sin(\frac{\pi}{t}))$: $(0,1) \to \mathbb{R}^2$.

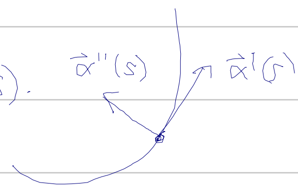


Then $\int_{1/N}^{1} \|\vec{\alpha}'(s)\| \, ds \geq C \sum_{n=1}^{N} \frac{1}{n}$

($\Rightarrow \vec{\alpha}$ has infinite length).

I.2 Local theory of curves parameterized by arc length.

$\vec{\alpha}: I \to \mathbb{R}^3$, $\|\vec{\alpha}'(s)\| = 1 \ \forall s \in I$.



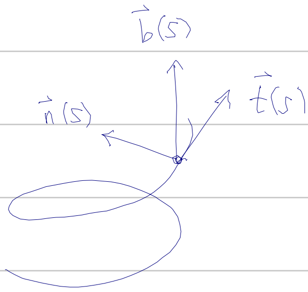
Note: $\vec{\alpha}'(s) \cdot \vec{\alpha}''(s) = 0$ (dot product), i.e. $\vec{\alpha}'(s) \perp \vec{\alpha}''(s)$.

· Definition $\vec{t}(s) = \vec{\alpha}'(s)$: tangent vector ($\|t(s)\| = 1$)

$\vec{n}(s) = \dfrac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|}$: normal vector, $k(s) = \|\vec{\alpha}''(s)\|$: curvature.

$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$: binormal vector.

($\vec{n}(s), \vec{b}(s)$ are only defined when $k(s) \neq 0$.)



$\Longrightarrow$ $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ forms a right-handed orthonormal frame.

How do $\vec{t}, \vec{n}, \vec{b}$ evolve along $\vec{\alpha}$?

· $\vec{t}'(s) = \vec{\alpha}''(s) = k(s)\vec{n}(s)$

· $\vec{b}'(s) \cdot \vec{b}(s) = 0$ (since $\|\vec{b}(s)\| = 1 \ \forall s$), and $\vec{b}'(s) = \overbrace{\vec{t}'(s) \times \vec{n}(s)}^{= k(s)\vec{n}(s) \times \vec{n}(s) = 0} + \vec{t}(s) \times \vec{n}'(s)$

$= \vec{t}(s) \times \vec{n}'(s)$,

so $\vec{b}'(s) \cdot \vec{t}(s) = 0 \Rightarrow \vec{b}'(s) \parallel \vec{n}(s)$.

Def. The number $\tau(s)$ s.t. $\vec{b}'(s) = \tau(s)\vec{n}(s)$ is the torsion of $\vec{\alpha}$ at s.

· Since $\vec{n}(s) = \vec{b}(s) \times \vec{t}(s)$, we have

$\vec{n}'(s) = \vec{b}'(s) \times \vec{t}(s) + \vec{b}(s) \times \vec{t}'(s) = -\tau(s)\vec{b}(s) - k(s)\vec{t}(s)$.

Summary : Frenet formulas : $\vec{t}' = k\vec{n}$, $\vec{n}' = -k\vec{t} - \tau\vec{b}$, $\vec{b}' = \tau\vec{n}$,

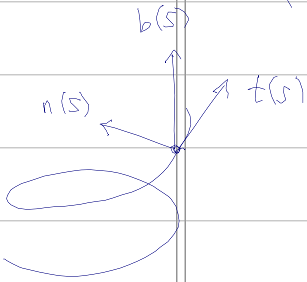
or $\underbrace{\left(\vec{t}' \,\middle|\, \vec{n}' \,\middle|\, \vec{b}'\right)}_{3 \times 3 \text{ matrix}} = \left(\vec{t} \,\middle|\, \vec{n} \,\middle|\, \vec{b}\right) \begin{pmatrix} 0 & -k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$.

02/06/2020

Examples (1) Helix $\vec{\beta}(t) = (a\cos t,\ a\sin t,\ bt)$, $a,b>0$.

(i) Reparameterize by arc length (i.e. find monotone function $t(s)$ s.t. $\vec{\alpha}(s) = \vec{\beta}(t(s))$ has $\|\alpha'(s)\|=1$)

$\rightsquigarrow \vec{\alpha}(s) = \left(a\cos\left(\frac{s}{c}\right),\ a\sin\left(\frac{s}{c}\right),\ b\frac{s}{c}\right)$, $c = \sqrt{a^2+b^2}$.



(ii) $\cdot\ \vec{t}(s) = \vec{\alpha}'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right),\ \frac{a}{c}\cos\left(\frac{s}{c}\right),\ \frac{b}{c}\right)$,

$\cdot\ \vec{t}'(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right),\ -\frac{a}{c^2}\sin\left(\frac{s}{c}\right),\ 0\right)$

$= k(s)\cdot\vec{n}(s)$,

$k(s) = \frac{a}{c^2}$, $\vec{n}(s) = \left(-\cos\left(\frac{s}{c}\right),\ -\sin\left(\frac{s}{c}\right),\ 0\right)$

$\cdot\ \vec{b}(s) = \vec{t}(s)\times\vec{n}(s) = \left(\frac{b}{c}\sin\left(\frac{s}{c}\right),\ -\frac{b}{c}\cos\left(\frac{s}{c}\right),\ \frac{a}{c}\right)$

$\cdot\ \vec{b}'(s) = \left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right),\ \frac{b}{c^2}\sin\left(\frac{s}{c}\right),\ 0\right)$

$= \tau(s)\cdot\vec{n}(s)$,

$\tau(s) = -\frac{b}{c^2}$.

(2) A smooth curve $\vec{\alpha}(s): I\to\mathbb{R}^3$ param. by arc length is a straight line iff $k(s)=0$.

Pf. ($\Rightarrow$) $\vec{\alpha}(s) = \vec{\alpha}_0 + s\vec{v}$, $\vec{v}\in\mathbb{R}^3$, $\|\vec{v}\|=1 \Rightarrow \vec{t}(s) = \vec{\alpha}'(s) = \vec{v}$

$\Rightarrow k(s) = \|\vec{t}'(s)\| = 0$.

($\Leftarrow$) If $k(s)=0\ \forall s$, then $\vec{t}'(s) = 0 \Rightarrow \vec{t}(s) = \text{constant} = \vec{v}$ ($\|\vec{v}\|=1$)

WLOG $0\in I \Rightarrow \vec{\alpha}(s) = \vec{\alpha}(0) + \int_0^s \vec{t}(s')\,ds' = \vec{\alpha}(0) + s\vec{v}$. $\square$

(3) A smooth curve $\vec{\alpha}(s): I\to\mathbb{R}^3$, param. by arc length and with $k(s)\neq 0$, is planar (i.e. its image is contained in a plane) iff its torsion vanishes.

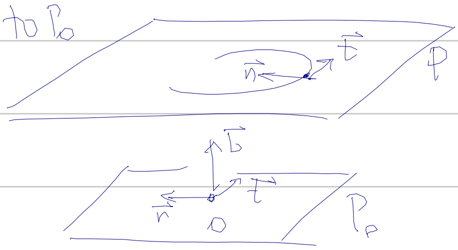
Pf ($\Rightarrow$) If $\vec{\alpha}(s)$ lies in a plane P for all s, then $\vec{t}(s), \vec{n}(s)$ span the plane P_0 parallel to P and passing through O.

$\Rightarrow \vec{b}(s) =$ normal vector to P_0

$=$ const

$\Rightarrow \vec{b}'(s) = 0$

$\Rightarrow \tau(s) = 0$.



($\Leftarrow$) If $\tau = 0$, then $\vec{b}(s) = \vec{b}_0$ is constant. Consider the function

$$f(s) = \vec{\alpha}(s)\cdot\vec{b}_0 \Rightarrow f'(s) = \vec{\alpha}'(s)\cdot\vec{b}_0 = \vec{t}(s)\cdot\vec{b}(s) = 0$$

$$\Rightarrow f(s) = \text{constant} = c$$

$$\Rightarrow \vec{\alpha}(s) \in \{\vec{x}\in\mathbb{R}^3 : \vec{x}\cdot\vec{b}_0 = c\}.$$ $\square$

- Local canonical form.

Prop. Let $\vec{\alpha}$ be a smooth arc length-parameterized curve. If $\vec{\alpha}(0)=0$, $\vec{\alpha}''(0)\neq 0$, then we have

$$\vec{\alpha}(s) = \left(s - k(0)^2\frac{s^3}{3!}\right)\vec{t}(0) + \left(k(0)\frac{s^2}{2} + k'(0)\frac{s^3}{3!}\right)\vec{n}(0) - k(0)\tau(0)\frac{s^3}{3!}\vec{b}(0) + \vec{R}(s)$$

where the remainder $\vec{R}(s)$ satisfies $\lim_{s\to 0}\frac{\|\vec{R}(s)\|}{s^3} = 0$.

Pf. $\vec{\alpha}'(0) = \vec{t}(0)$, $\vec{\alpha}''(0) = k(0)\vec{n}(0)$,

$$\vec{\alpha}'''(0) = (k\vec{n})'(0) = k'(0)\vec{n}(0) + k(0)\vec{n}'(0)$$

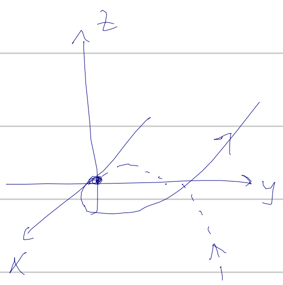
$$= k'(0)\vec{n}(0) - k(0)^2\vec{t}(0) - k(0)\tau(0)\vec{b}(0).$$

Taylor expansion. $\square$

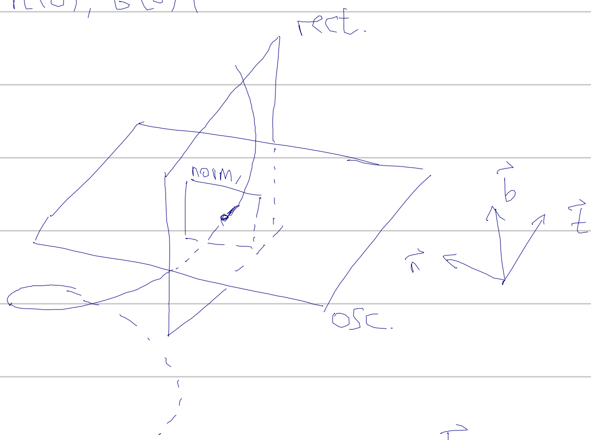
Can use coordinate system on $\mathbb{R}^3$ s.t. $\vec{t}(0) = (1,0,0)$, $\vec{n}(0) = (0,1,0)$, $\vec{b}(0) = (0,0,1)$

$$\Rightarrow \begin{cases} x(s) = s - \frac{k^2s^3}{6} + \cdots \\ y(s) = \frac{ks^2}{2} + \frac{k's^3}{6} + \cdots \\ z(s) = -\frac{k\tau s^3}{6} + \cdots \end{cases}$$ = loc. canonical form.

Sketch ($k>0$, $\tau<0$): (very similar to moment map!)

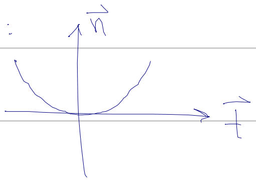


- Fundamental planes at $p = \vec{\alpha}(0)$ with $\vec{\alpha}''(0) \neq 0$:
  - (i) osculating plane $= \text{span}\{\vec{t}(0), \vec{n}(0)\}$
  - (ii) rectifying plane $= \text{span}\{\vec{t}(0), \vec{b}(0)\}$
  - (iii) normal plane $= \text{span}\{\vec{n}(0), \vec{b}(0)\}$

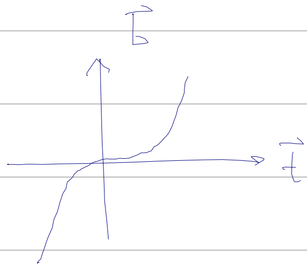


Local geometry of projections:

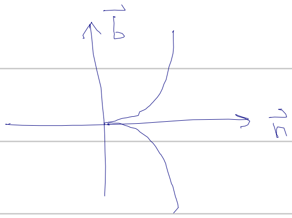
osculating: (almost contains the curve)



rectifying:



normal:



(Cf. local canonical form.)

Ex. All curves below have $k(s) \neq 0$.

(1) Claim. If all normal planes of a curve $\vec{\alpha}$ pass through one point, then $\vec{\alpha}$ lies on a sphere.

Pf. Let the point be $\vec{0}$. Then

$$\vec{0} = \vec{\alpha}(s) - \mu(s)\vec{n}(s) - \lambda(s)\vec{b}(s) \Rightarrow \vec{\alpha}(s) = \mu(s)\vec{n}(s) + \lambda(s)\vec{b}(s)$$

$$\Rightarrow \frac{1}{2}\frac{d}{ds}\|\vec{\alpha}(s)\|^2 = \vec{\alpha}(s)\cdot\vec{\alpha}'(s)$$

$$= (\mu\vec{n} + \lambda\vec{b})\cdot\vec{t}(s) = 0$$

$$\Rightarrow \|\vec{\alpha}(s)\| = \text{const.}$$ $\square$

(2) Claim. If all osculating planes of $\vec{\alpha}: I \to \mathbb{R}^3$ pass through $\vec{0}$, then $\vec{\alpha}$ is planar.

Pf. $\vec{\alpha}(s) = \mu(s)t(s) + \lambda(s)n(s)$

$$\Rightarrow t = (\mu' - k\lambda)t + (\lambda' + k\mu)n - \tau\lambda b \Rightarrow \tau\lambda = 0.$$

If $\lambda(s_0)\neq 0$, then $\tau(s_0)=0$. Thus, $\tau=0$ on the closure of $\{s\in I : \lambda(s)\neq 0\}$ in I.

If $\lambda(s)=0$ for s in some nonempty open subset $J\subset I$, then $\vec{\alpha}(s)=\mu(s)\vec{t}(s)$

$= \mu(s)\alpha'(s)$ for $s\in J$; but then $\vec{\alpha}' = \mu'\vec{\alpha}' + \underbrace{\mu\vec{\alpha}''}_{\perp\vec{\alpha}'}$ and

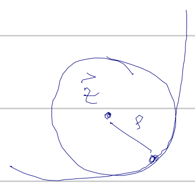
$k=|\vec{\alpha}''|\neq 0$ force $\mu(s)=0 \Rightarrow \vec{\alpha}(s)=\vec{0}$, $s\in J$,

contradiction (to the assumption that $\vec{\alpha}$ is regular). □

(3) Suppose $\vec{\alpha}$ is an arclength parameterized curve, $\vec{p}=\vec{\alpha}(0)$, $k(0)\neq 0$. The

<u>osculating circle</u> $\mathcal{C}$ at p is the limiting position of the circle through $\vec{p}$,

$\vec{\alpha}(s)$ and $\alpha(t)$ as $s,t\to 0$

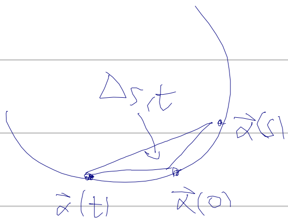


<u>Claim.</u> $\mathcal{C}$ has center $\vec{z}=\vec{p}+\frac{1}{k(0)}\vec{n}(0)$ and radius $\rho=\frac{1}{k(0)}$.

<u>Pf.</u> WLOG $\vec{p}=0$.

- Radius $\rho_{s,t}$ of circle with points $\vec{\alpha}(0),\vec{\alpha}(s),\vec{\alpha}(t)$ is

$$\rho_{s,t} = \frac{\|\vec{\alpha}(s)-\vec{\alpha}(0)\|\,\|\vec{\alpha}(t)-\vec{\alpha}(0)\|\,\|\vec{\alpha}(s)-\vec{\alpha}(t)\|}{4\,\text{area}(\Delta_{s,t})} = \frac{\|\vec{\alpha}(s)\|\,\|\vec{\alpha}(t)\|\,\|\vec{\alpha}(s)-\vec{\alpha}(t)\|}{2\|\vec{\alpha}(s)\times\vec{\alpha}(t)\|}.$$



Normal form: $\vec{\alpha}(s)=(s, \frac{s^2}{2}k(0), 0)+O(s^3)$.

$\Rightarrow \|\vec{\alpha}(s)\|=|s|+O(s^2)$, $\quad \vec{\alpha}(s)\times\vec{\alpha}(t)=(0,0,\frac{st}{2}k(0)(t-s))+O(|s^3t|+|st^3|)$

$\|\vec{\alpha}(s)-\vec{\alpha}(t)\|=|s-t|+O(s^2+t^2)$, $\quad \Rightarrow 2\|\vec{\alpha}(s)\times\vec{\alpha}(t)\|=|s|\,|t|\,|t-s|\,k(0)+O(|s^3t|+|st^3|)$

$\Rightarrow \rho_{s,t}=\frac{1}{k(0)}+O(|s|+|t|)\xrightarrow{s,t\to 0}\rho=\frac{1}{k(0)}$.

- Center $\vec{z}_{s,t}$ of circle lies in plane $P_{s,t}$ spanned by $\vec{\alpha}(s),\vec{\alpha}(t)$, so

$P_{s,t}=\{\vec{v}\in\mathbb{R}^3 : \vec{v}\cdot(\vec{\alpha}(s)\times\vec{\alpha}(t))=\vec{0}\}\xrightarrow{s,t\to 0}\{\vec{v} : \vec{v}\cdot(0,0,1)=0\}=:P$

$=$ osculating plane.

- Have $\vec{z}_{s,t}=\frac{\vec{\alpha}(s)}{2}+\vec{v}(s,t)$, $\quad\vec{v}(s,t)\cdot\vec{\alpha}(s)=0$, $\|\vec{v}(s,t)\|$ bounded as $s,t\to 0$.

$\Rightarrow \vec{z}_{s,t}\cdot\vec{t}(0)=\underbrace{\vec{t}(0)\cdot\frac{\vec{\alpha}(s)}{2}}_{\xrightarrow{s\to 0}0}+\underbrace{\vec{t}(0)\cdot\vec{v}(s,t)}_{\underbrace{\frac{\vec{\alpha}(s)}{s}+O(s)}_{=0+O(s)}}\xrightarrow{s,t\to 0}0$

$\Rightarrow$ Center $\vec{z}$ of $\mathcal{C}$ lies on

normal line through $\vec{\alpha}(0)$ $\Rightarrow \vec{z}=\rho\vec{n}(0)=\frac{1}{k(0)}\vec{n}(0)$. □

(Justify why we cannot have $\vec{z}=-\rho\vec{n}(0)$!)

02/11/2020

Fundamental theorem of curve theory. (i) Given smooth functions $k(s)>0$, $s\in I=(a,b)$, and $\tau(s)$, $s\in I$, there exists a regular parameterized curve $\vec{\alpha}: I\to\mathbb{R}^3$ such that k,τ are the curvature, torsion of $\vec{\alpha}$.

(ii) 2 curves $\vec{\alpha},\vec{\alpha}^*: I\to\mathbb{R}^3$, parameterized by arclength, have the same curvature $k(s)$ and torsion $\tau(s)$ iff $\vec{\alpha}$ and $\vec{\alpha}^*$ differ by a rigid motion, that is, $\exists$ orthogonal matrix $A\in\mathbb{R}^{3\times3}$, $\det(A)>0$, and $\vec{b}\in\mathbb{R}^3$ so that $\vec{\alpha}^*(s)=A\vec{\alpha}(s)+\vec{b}$.

Pf of (ii). $(\Leftarrow)$ $\frac{d}{ds}\vec{\alpha}^*(s)=A\vec{\alpha}'(s)\Rightarrow \vec{t}^*(s)=A\vec{t}(s)\Rightarrow k^*(s)\vec{n}^*(s)=k(s)A\vec{n}(s)$.

Since A is orthogonal, $\|A\vec{n}(s)\|=1$, so $\vec{n}^*(s)=A\vec{n}(s)$, $k^*(s)=k(s)$.

Moreover, $\vec{b}^*(s)=\vec{t}^*(s)\times\vec{n}^*(s)=A\vec{t}(s)\times A\vec{n}(s)=A(\vec{t}(s)\times\vec{n}(s))=A\vec{b}(s)$

(since $\det A>0$: A is orientation preserving).

Therefore, $\frac{d}{ds}\vec{b}^*(s)=A\frac{d}{ds}\vec{b}(s)=A\tau(s)\vec{n}(s)=\tau(s)\vec{n}^*(s)\Rightarrow\tau^*(s)=\tau(s)$.

$(\Rightarrow)$ Suppose $k=k^*$, $\tau=\tau^*$. Let A denote the (unique) orthogonal, orientation-preserving ($\det A>0$) matrix so that

$$\vec{t}^*(0)=A\vec{t}(0),\quad \vec{n}^*(0)=A\vec{n}(0),\quad \vec{b}^*(0)=A\vec{b}(0).$$

Let $\vec{b}=\vec{\alpha}^*(0)-\vec{\alpha}(0)$, put $\psi(\vec{x})=A\vec{x}+b$, set $\vec{\alpha}_*(s)=\psi(\vec{\alpha}(s))$.

Claim: $\vec{\alpha}_*(s)=\vec{\alpha}^*(s)$ $\forall s$.

Indeed, from "$(\Leftarrow)$", have $k_*=k^*=k$, $\tau_*=\tau^*=\tau$; and $\vec{\alpha}_*(0)=\vec{\alpha}^*(0)$, $\vec{t}_*(0)=\vec{t}^*(0)$, ...

Compute $\frac{d}{ds}\left[\frac{1}{2}\left(|\vec{t}_*(s)-\vec{t}^*(s)|^2+|\vec{n}_*(s)-\vec{n}^*(s)|^2+|\vec{b}_*(s)-\vec{b}^*(s)|^2\right]\right.$

$$=\langle\vec{t}_*-\vec{t}^*,k(\vec{n}_*-\vec{n}^*)\rangle-\langle\vec{n}_*-\vec{n}^*,k(\vec{t}_*-\vec{t}^*)\rangle-\langle\vec{n}_*-\vec{n}^*,\tau(\vec{b}_*-\vec{b}^*)\rangle$$
$$+\langle\vec{b}_*-\vec{b}^*,\tau(\vec{n}_*-\vec{n}^*)\rangle=0$$

$\Rightarrow \vec{t}_*(s)=\vec{t}^*(s)$ $\forall s$ $\Rightarrow$ $\vec{\alpha}_*(s)=\vec{\alpha}_*(0)+\int_0^s\vec{t}_*(s')\,ds'$

$$=\vec{\alpha}^*(0)+\int_0^s\vec{t}^*(s')\,ds'=\vec{\alpha}^*(s).\qquad\square$$

I.3 Isoperimetric inequality

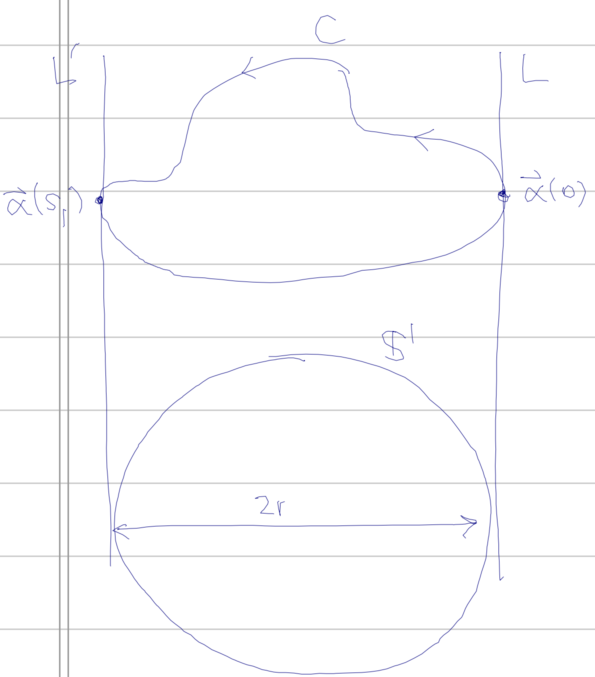
Thm. Let C be a simple closed smooth curve in $\mathbb{R}^2$ with length ℓ, and let A be the area enclosed by C. Then

$$\ell^2 \geq 4\pi A.$$

Equality holds iff C is a circle.

("Simple" means: no self-intersections.)

Pf.:



- Let $\vec{\alpha}(s) = (x(s), y(s))$ be a parameterization of C by arc length, $s \in [0, \ell]$.
- L, L': parallel lines, tangent to C, with C contained in strip between L', L.
- S^1: circle, parameterized by $\vec{\beta}(s) = (x(s), \tilde{y}(s))$.

Area bounded by C: $A = \int_0^\ell x(s) y'(s)\, ds$

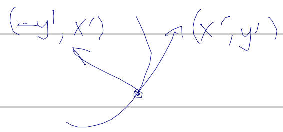
Area bounded by S^1: $\pi r^2 = -\int_0^\ell \tilde{y}(s) x'(s)\, ds$.

(Stokes' theorem:

$A = \iint \operatorname{div}(x\partial_x)\, dx\, dy$

$= \int_C \langle x\partial_x, \vec{n}(s)\rangle\, ds$,

$\vec{n}(s) = (-y'(s), x'(s))$.)



$$\Rightarrow A + \pi r^2 = \int_0^\ell x(s)y'(s) - \tilde{y}(s)x'(s)\, ds$$

$$= \int_0^\ell \langle (x(s), -\tilde{y}(s)), (y'(s), x'(s))\rangle\, ds$$

$$\leq \int_0^\ell \sqrt{x^2 + \tilde{y}^2}\sqrt{(x')^2 + (y')^2}\, ds$$

arclength param. $\rightarrow$

$$\overset{\downarrow}{=} \int_0^\ell \underbrace{\sqrt{x^2 + \tilde{y}^2}}_{=r}\, ds = r\ell.$$

Therefore, $\sqrt{A\pi r^2} \leq \frac{1}{2}(A + \pi r^2) = \frac{1}{2} r \ell$. $\square$

(Equality: left as an exercise.)

II Surfaces.

Def. A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$ and a map $\vec{x}: U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S$ such that:

(i) $\vec{x}$ is smooth; that is, if we write
$$\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$
then $x(u,v), y(u,v), z(u,v)$ have continuous partial derivatives of all orders.

(ii) $\vec{x}$ is a homeomorphism; that is, $\exists\, \vec{x}^{-1}: V \cap S \to U$, continuous.

(iii) (Regularity condition) For each $q \in U$, the differential $d\vec{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

Differential: $q \in U$. Let $\vec{\alpha}: (-1,1) \to \mathbb{R}^2$ be a smooth curve such that $\alpha(0) = q$.

Then $d\vec{x}_q : \vec{\alpha}'(0) \longmapsto \frac{d}{dt} \vec{x}(\vec{\alpha}(t))\big|_{t=0}$

Computation of $d\vec{x}_q$: Let $\vec{\alpha}: (-1,1) \to \mathbb{R}^2$, $\vec{\alpha}(0) = q$. Then

$$\frac{d}{dt}\vec{x}(\vec{\alpha}(t)) = \left(D_{u,v}\vec{x}\big|_{\vec{\alpha}(t)}\right) \cdot \vec{\alpha}'(t) \quad \text{(chain rule)}, \quad D_{u,v}\vec{x}\big|_q = \left.\begin{pmatrix} \frac{\partial}{\partial u}x & \frac{\partial}{\partial v}x \\ \frac{\partial}{\partial u}y & \frac{\partial}{\partial v}y \\ \frac{\partial}{\partial u}z & \frac{\partial}{\partial v}z \end{pmatrix}\right|_{(u,v)=q}$$

$$\Rightarrow \underbrace{\frac{d}{dt}\vec{x}(\vec{\alpha}(t))\Big|_{t=0}}_{= d\vec{x}_q(\vec{\alpha}'(0))} = \left(D_{u,v}\vec{x}\big|_q\right) \cdot \vec{\alpha}'(0)$$

Thus, $d\vec{x}_q = D_{u,v}\vec{x}\big|_{q=(u,v)}$.

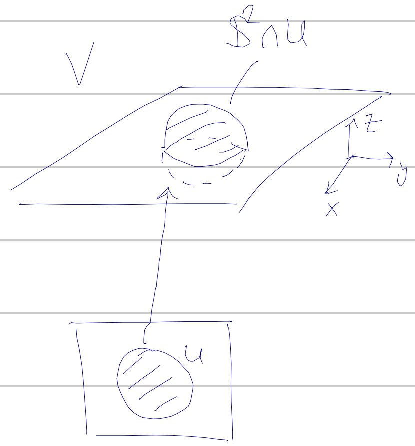
Remark. $d\vec{x}_q$ is injective $\iff \frac{\partial}{\partial u}\vec{x}(q) \times \frac{\partial}{\partial v}\vec{x}(q) \neq 0$

$\iff$ The determinant of one of the 2×2 matrices
$$\begin{pmatrix} \frac{\partial}{\partial u}x & \frac{\partial}{\partial v}x \\ \frac{\partial}{\partial u}y & \frac{\partial}{\partial v}y \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial u}x & \frac{\partial}{\partial v}x \\ \frac{\partial}{\partial u}z & \frac{\partial}{\partial v}z \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial u}y & \frac{\partial}{\partial v}y \\ \frac{\partial}{\partial u}z & \frac{\partial}{\partial v}z \end{pmatrix}$$
is nonzero.

- <u>Example 1</u> The unit sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is a regular surface.

<u>Proof</u> • Consider $V = \{(x,y,z) : z > 0\}$, and let

$$\vec{x}_1(x,y) = (x, y, \sqrt{1-x^2-y^2}) \text{ for}$$
$$(x,y) \in U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}.$$



Check conditions of definition:

(i) Since $x^2+y^2<1$ on U, $\sqrt{1-x^2-y^2}$ is smooth on U

$\Rightarrow \vec{x}_1$ is smooth on U.

(ii) The projection $(x,y,z) \mapsto (x,y)$ is the inverse of $\vec{x}_1$, and is continuous.

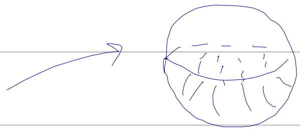
(iii) We have

$$\frac{\partial}{\partial x}\vec{x}_1(x,y) = \left(1, 0, -\frac{x}{\sqrt{1-x^2-y^2}}\right), \quad \frac{\partial}{\partial y}\vec{x}_1(x,y) = \left(0, 1, -\frac{y}{\sqrt{1-x^2-y^2}}\right),$$

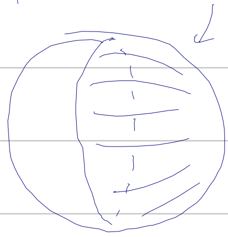
which are linearly independent. $\Rightarrow d_q\vec{x}_1$ is injective for all $q \in U$.

- We can similarly treat $\{(x,y,z) : z<0\}$, using

$$\vec{x}_2(x,y) = (x, y, -\sqrt{1-x^2-y^2}), \quad (x,y) \in U.$$



- What about points in $\mathbb{S}^2 \cap \{z=0\}$? $\rightsquigarrow$ Use $\vec{x}_3(x,z) = (x, \sqrt{1-x^2-z^2}, z)$, etc.
- <u>Summary</u>: can cover $\mathbb{S}^2$ with 6 such parametrizations.

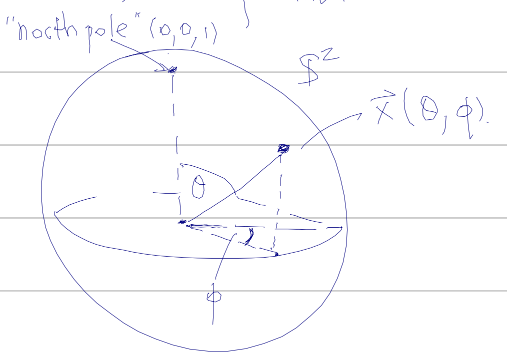


$\square$

<u>Another proof</u>: Polar coordinates:

$$\vec{x}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$
$$\theta \in (0,\pi), \ \phi \in (0,2\pi).$$

Can cover $\mathbb{S}^2$ with 2 parametrizations of this form. $\square$



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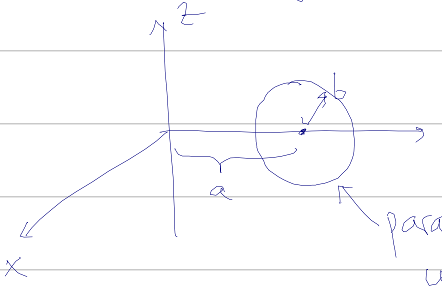
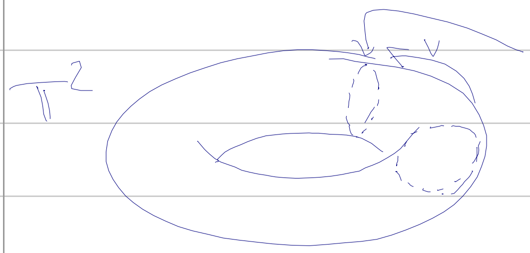
· Example 2. Is the 2-torus a regular surface?



$(0 < b < a.)$

parameterize circle by

$u \longmapsto (0,\ a + b\cos u,\ b\sin u).$



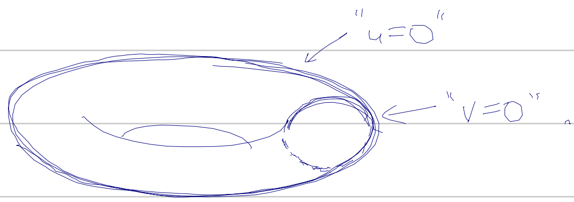
parameterize this rotated circle by

$u \longmapsto ((a+b\cos u)\sin v,\ (a+b\cos u)\cos v,\ b\sin u).$

So $\vec{x}(u,v) = ((a+b\cos u)\sin v,\ (a+b\cos u)\cos v,\ b\sin u),$

$(u,v) \in (0,2\pi)\times(0,2\pi),$

parameterizes $\mathbb{T}^2$ except for



Condition (i): ✓

(iii): $\frac{\partial}{\partial u}\vec{x}(u,v) = (-b\sin u\sin v,\ -b\sin u\cos v,\ b\cos u),$

$\frac{\partial}{\partial v}\vec{x}(u,v) = ((a+b\cos u)\cos v,\ -(a+b\cos u)\sin v,\ 0)$

$\Rightarrow \frac{\partial}{\partial u}\vec{x} \times \frac{\partial}{\partial v}\vec{x} = (a+b\cos u)b\cdot\left(\cos u\sin v,\ \cos u\cos v,\ \sin u\right) \neq \vec{0}.$ ✓

(ii)? Will follow from a general result below.

· Example 3. Let $U \subset \mathbb{R}^2$ be open and $f: U \to \mathbb{R}$ smooth. Then

$\mathrm{graph}(f) = \{(u,v,f(u,v)) : (u,v) \in U\}$ is a regular surface.

(Check: $\vec{x}(u,v) := (u,v,f(u,v))$.

(i): ✓. (ii): $\vec{x}^{-1}$ is the restriction of the (continuous) projection

$(u,v,w) \longmapsto (u,v)$ to graph(f). ✓

(iii) $\frac{\partial}{\partial u}\vec{x}$, $\frac{\partial}{\partial v}\vec{x}$ are linearly independent. ✓)

In the first example (S^2), we "solved for z" in order to find parameterizations. We describe a general procedure soon. First, recall:

<u>Thm.</u> (Inverse function theorem) Let $F: U \to \mathbb{R}^n$ be a smooth function, $U \subset \mathbb{R}^n$ open, $p \in U$. Suppose $d_pF: \mathbb{R}^n \to \mathbb{R}^n$ (the differential of F at p — matrix of partial derivatives) is invertible. Then there exists a neighborhood $V \subset U$ of p and a neighborhood $W \subset \mathbb{R}^n$ of $F(p)$ such that $F: V \to W$ is a diffeomorphism (smooth, invertible, with smooth inverse).

<u>Prop.</u> If $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $a \in f(U)$ is a <u>regular value</u> of f, then $f^{-1}(a) \subset \mathbb{R}^3$ is a regular surface.

(a is a regular value $:\Leftrightarrow$ For all $p \in U$ s.t. $f(p) = a$, the differential $d_pf: \mathbb{R}^3 \to \mathbb{R}$ is surjective $\Leftrightarrow$ $d_pf \neq 0$.)

<u>Ex</u> $f(x,y,z) = x^2+y^2+z^2$, $a=1$. If $f(p)=1$, $p=(x,y,z)$, then $d_pf = (2x, 2y, 2z) \neq 0$, so $a=1$ is a regular value of f. The Prop. thus implies that $S^2 = f^{-1}(1)$ is a regular surface.

<u>Proof of the proposition.</u> Let $p = (x_0, y_0, z_0) \in f^{-1}(a)$. By relabeling the coordinates, we may assume $f_z(p) := \frac{\partial}{\partial z} f(p) \neq 0$. Define

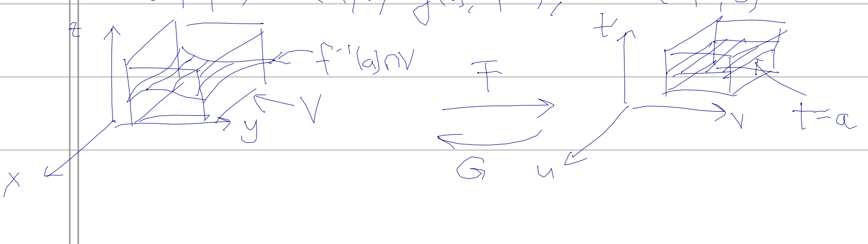
$F: U \to \mathbb{R}^3$, $F(x,y,z) = (x, y, f(x,y,z))$. (<u>Idea</u>: Want to use f as a replacement for the z-coordinate.)

Then:

$$d_pF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}$$ is invertible since $\det(d_pF) = f_z \neq 0$.

Apply inverse function theorem: $\exists\, V \subset U$ nbhd. of p, $W \subset \mathbb{R}^3$ nbhd. of $F(p)$, s.t. $F: V \to W$ is a diffeomorphism; so $G = F^{-1}: W \to V$ is smooth. Write

$G(u,v,t) = (u, v, g(u,v,t))$, $(u,v,t) \in W$.

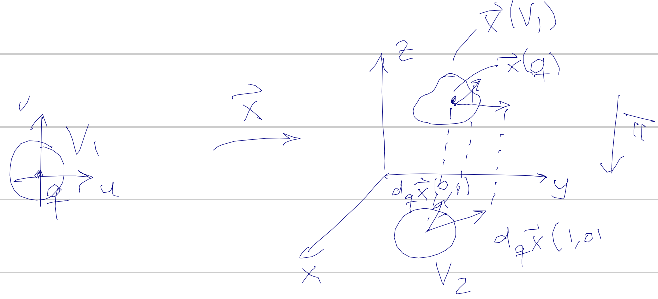


But $F(f^{-1}(a)\cap V) = W \cap \{(u,v,t) : t=a\}$,
so letting $U = \{(u,v) : (u,v,a)\in W\}$, $h(u,v) = g(u,v,a) : U \to \mathbb{R}$
$f^{-1}(a)\cap V$ is the graph of h. By example 3 above, this shows that $f^{-1}(a)\cap V$ is a coordinate neighborhood of p. Since $p\in f^{-1}(a)$ was arbitrary, $f^{-1}(a)$ is therefore a regular surface. $\square$

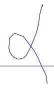
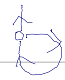
We next give a result which makes checking condition (ii) in the Definition of a regular surface straightforward.

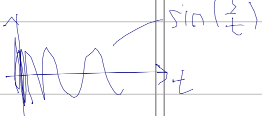
Prop. Let $S\subset\mathbb{R}^3$, $p\in S$, $V\subset\mathbb{R}^3$ open nbhd. of p, $U\subset\mathbb{R}^2$ open, $\vec{x} : U \to V\cap S$ smooth (condition (i) of Def.), surjective, and assume that $d_q\vec{x} : \mathbb{R}^2\to\mathbb{R}^3$ is injective for all $q\in U$ (condition (iii)).
If $\vec{x}$ is injective, then $\vec{x}^{-1} : V\cap S \to U$ is continuous. (I.e. condition (ii) holds.)

Proof.
- Write $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. Let $q\in U$. By relabeling coordinates, may assume that $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$.
- Let $\pi(x,y,z) = (x,y)$ denote the projection, then $\pi\circ\vec{x} : U \to \mathbb{R}^2$ has invertible differential at q.



- IFT $\Rightarrow$ $\exists$ nbhds. $V_1\subset U$ of q, $V_2\subset\mathbb{R}^2$ of $\pi(\vec{x}(q))$, s.t. $\pi\circ\vec{x} : V_1 \to V_2$ is a diffeomorphism. Since $\vec{x}$ is injective, we have, on $\vec{x}(V_1)$ (open neighborhood of $\vec{x}(q)$!)
$$\vec{x}^{-1} = (\pi\circ\vec{x})^{-1}\circ\pi,$$
which is continuous. Since $q\in U$ was arbitrary, $\vec{x}^{-1} : \vec{x}(U)\to U$ is continuous. $\square$

Remark. (Connection to theory of curves.) We define a <u>regular curve</u> (in analogy with regular surfaces) as a subset $S\subset\mathbb{R}^3$ s.t. $\forall p\in S$ $\exists$ open nbhd. $V\subset\mathbb{R}^3$ of p, and a smooth bijection $\vec{\alpha} : U\subset\mathbb{R} \to S\cap V$ such that $d_s\vec{\alpha} = \vec{\alpha}'(s) \neq 0$ $\forall s\in U$.
- This definition is more restrictive than our def. of regular parameterized curves, and forbids self-intersections  and other pathologies: , or

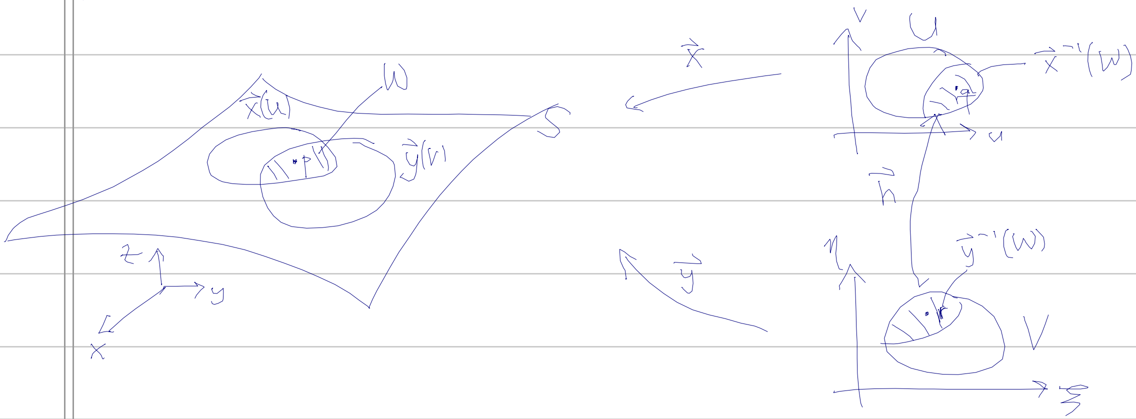


$(\{(0,y) : |y|<1\} \cup \{(t, \sin(\tfrac{1}{t})) : t \in (0,1)\}$ (topologist's sine curve).

- On the positive side, if $\vec{\alpha}: I \to \mathbb{R}^3$ is a regular parameterized curve and $s \in I$, then $\exists$ nbhd. $U \subset I$ of s so that $\vec{\alpha}(U)$ is a regular curve (in the above, more restrictive, sense). Prove this!

II.1 Change of parameters

Prop. Let p be a point on a regular surface $S \subset \mathbb{R}^3$. Let $\vec{x}: U \subset \mathbb{R}^2 \to S$, $\vec{y}: V \subset \mathbb{R}^2 \to S$ be two parameterizations of S such that $p \in \vec{x}(U) \cap \vec{y}(V) =: W$. Then the "change of coordinates" $\vec{h} = \vec{x}^{-1} \circ \vec{y} : \vec{y}^{-1}(W) \to \vec{x}^{-1}(W)$ is a diffeomorphism.



Proof. Take $q \in \vec{x}^{-1}(W)$, and set $r = \vec{y}^{-1}(\vec{x}(q)) \in \vec{y}^{-1}(W)$.

Write $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. WLOG $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ at q.

Let $\pi(x,y,z) = (x,y)$. Then, by the IFT, $\exists$ nbhds $U_1 \subset \vec{x}^{-1}(W)$ of q, $\tilde{W} \subset \mathbb{R}^2$ of $\pi(p)$, so that $\pi \circ \vec{x} : U_1 \to \tilde{W}$ is a diffeomorphism, and

$$\vec{x}^{-1} = (\pi \circ \vec{x})^{-1} \circ \pi \text{ on } \vec{x}(U_1).$$

Thus, on $\vec{y}^{-1}(\vec{x}(U_1))$, we have $\vec{h} = (\pi \circ \vec{x})^{-1} \circ \pi \circ \vec{y}$, which is smooth.
(neighborhood of r).

Since $r \in \vec{y}^{-1}(W)$ was arbitrary, $\vec{h}$ is smooth on $\vec{y}^{-1}(W)$.

Finally, reversing the roles of $\vec{x}, \vec{y}$ proves that $\vec{h}^{-1} = \vec{y}^{-1} \circ \vec{x}$ is smooth, too. $\square$

Definition A function $f: V \subset S \to \mathbb{R}$, $V \subset S$ open (in the relative topology), is smooth at $p \in V$ if, for some parameterization $\vec{x}: U \subset \mathbb{R}^2 \to S$ of S with $p \in \vec{x}(U) \subset V$, the composition $f \circ \vec{x}: U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth.
f is smooth if it is smooth at all $p \in V$.

$f \circ \vec{y} = f \circ \vec{x} \circ (\vec{x}^{-1} \circ \vec{y})$

Rmk. By the previous proposition, this is independent of the chosen parameterization $\vec{x}$!
This is the first of many instances we will see in this course where an object or property defined in terms of local coordinates on a surface is, in fact, independent of the parameterization.

Example Given a smooth $f: \mathbb{R}^3 \to \mathbb{R}$ and a regular surface S, the restriction $f|_S$ is a smooth function on S.

Definition A continuous map $\phi: V \subset S_1 \to S_2$ from an open subset V of a regular surface S_1 to a regular surface S_2 is smooth at $p \in V_1$ if, given parameterizations $\vec{x}_1: U_1 \subset \mathbb{R}^2 \to V_1$, $\vec{x}_2: U_2 \subset \mathbb{R}^2 \to S_2$,
$p \in U_1$, $\phi(p) \in U_2$,
the map $\vec{x}_2^{-1} \circ \phi \circ \vec{x}_1: U_1 \to U_2$ is smooth (has continuous partial derivatives of all orders) at $q = \vec{x}_1^{-1}(p)$.

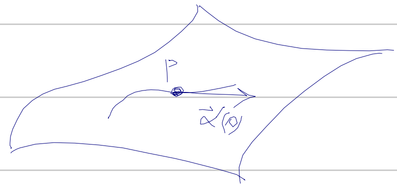
Definition A smooth map $\phi: S_1 \to S_2$ between two regular surfaces is a diffeomorphism if ϕ is bijective and $\phi^{-1}: S_2 \to S_1$ is smooth.
We say that S_1 and S_2 are diffeomorphic if $\exists$ diffeomorphism $S_1 \to S_2$.

02/20/2020

II.2 Tangent plane, differentials.

Let $S \subset \mathbb{R}^3$ be a regular surface.

Def. A tangent vector to S at $p \in S$ is the vector $\vec{\alpha}'(0)$ for a smooth parameterized curve $\vec{\alpha} : (-1,1) \to S$, $\vec{\alpha}(0) = p$.



Prop. Let $\vec{x} : U \subset \mathbb{R}^2 \to S$ be a parameterization of S, $q \in U$. Then $d\vec{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3$ is equal to the set of tangent vectors at $\vec{x}(q)$.

Pf. · If $\vec{\alpha} : (-1,1) \to \vec{x}(u)$, $\vec{\alpha}(0) = p = \vec{x}(q)$, $v = \vec{\alpha}'(0)$, then $\vec{\beta}(s) := \vec{x}^{-1} \circ \vec{\alpha}(s) : (-1,1) \to U$ is smooth, and $d\vec{x}_q(\vec{\beta}'(0)) = \frac{d}{ds} \vec{x}(\vec{\beta}(s))\big|_{s=0} = \frac{d}{ds}\vec{\alpha}(s)\big|_{s=0} = v$.

· Conversely, if $v = d\vec{x}_q(w)$, $w \in \mathbb{R}^2$, then v is the tangent vector of $(-\varepsilon, \varepsilon) \ni s \mapsto \vec{x}(q + sw)$. $\square$

· Significance: $T_pS = d\vec{x}_q(\mathbb{R}^2)$ $(p = \vec{x}(q))$ is independent of the parameterization. Called tangent plane to S at p. (Caution: DoCarmo bases T_pS at p. Our definition by contrast makes T_pS a linear subspace of $\mathbb{R}^3$.)

Choice of $\vec{x}$ gives coordinates on T_pS: $v = d\vec{x}_q(w)$, $w = (w_1, w_2)$, has coordinates (w_1, w_2). (Coordinates do depend on $\vec{x}$!)

· Can now capture first order behavior of smooth maps between surfaces.

Def. $\phi : S_1 \to S_2$ smooth, $p \in S_1$. Then $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$ (the differential) maps $d\phi_p(\vec{\alpha}'(0)) := \frac{d}{ds}\phi(\vec{\alpha}(s))\big|_{s=0}$ where $\vec{\alpha} : (-1,1) \to S_1$ is a smooth parameterized curve with $\vec{\alpha}(0) = p$.

Prop. $d\phi_p$ is well-defined and linear.

Pf. In local coordinates on S_1, S_2, $\phi(u,v) = (\phi_1(u,v), \phi_2(u,v))$.
(That is, $\vec{x} : U \subset \mathbb{R}^2 \to S_1$, $\vec{y} : V \subset \mathbb{R}^2 \to S_2$, $(\vec{y}^{-1} \circ \phi \circ \vec{x})(u,v) = (\phi_1(u,v), \phi_2(u,v))$.)

Consider $v = d\vec{x}_q(w)$, $w = (w_1, w_2)$ (where $p = \vec{x}(q)$). Then

$$d\phi_p(v) = d\vec{y}_r(w') \quad \text{where } w' = \begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{pmatrix} w.$$

(i.e. this is the matrix of $d\phi_p : T_pS_1 \to T_pS_2$ in local coordinates) $\square$

<u>Examples.</u> (1) Let $V \subset \mathbb{R}^3$ be a plane. Then $T_qV = V_0$ $\forall q \in V$, where $V_0 \subset \mathbb{R}^3$ is the plane parallel to V which passes through $\vec{0} \in \mathbb{R}^3$.

(2) Let $S \subset \mathbb{R}^3$ be the graph of a smooth function $f: \mathbb{R}^2 \to \mathbb{R}$. Find the tangent plane at $p = (x_0, y_0, f(x_0,y_0))$ based at p.

<u>Solution.</u> Parameterize S by $\vec{x}(x,y) = (x, y, f(x,y))$. Then

$$T_pS = \text{span}\{\vec{x}_x(x_0,y_0), \vec{x}_y(x_0,y_0)\} = \text{span}\{(1,0,f_x(x_0,y_0)), (0,1,f_y(x_0,y_0))\}.$$

Normal vector: $\begin{pmatrix}1\\0\\f_x\end{pmatrix} \times \begin{pmatrix}0\\1\\f_y\end{pmatrix} = \begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix}$

$\Rightarrow$ tangent plane based at p has equation

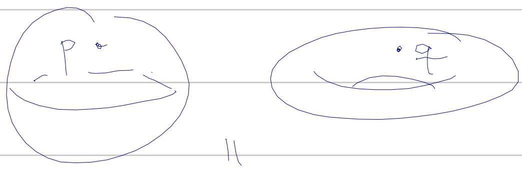
$$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

(3) Let $f: S \to \mathbb{R}$ be given by $f(p) = |p-p_0|^2$, where $p_0 \in \mathbb{R}^3$ is fixed. Show that $df_p(w) = 2w\cdot(p-p_0)$, $w \in T_pS$.

<u>Pf.</u> Consider $\vec{\alpha}: (-1,1) \to S$, $\vec{\alpha}(0) = p$, $\vec{\alpha}'(0) = w$. Then

$$df_p(w) = \frac{d}{ds} f(\vec{\alpha}(s))\Big|_{s=0} = \frac{d}{ds}|\vec{\alpha}(s) - p_0|^2\Big|_{s=0} = 2\vec{\alpha}'(s)\cdot(\vec{\alpha}(s)-p_0)\Big|_{s=0}$$
$$= 2w\cdot(p-p_0).$$ $\square$

<u>Definition</u> A regular surface S is <u>connected</u> if for any two points $p, q \in S$ there exists a continuous curve $\vec{\alpha}: [t_0, t_1] \to S$ with $\vec{\alpha}(t_0) = p$, $\vec{\alpha}(t_1) = q$.



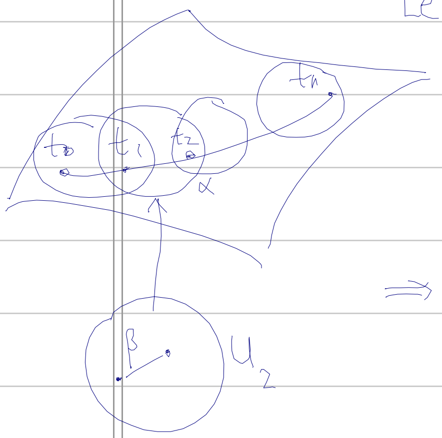
$\parallel$

S : not connected

<u>Example (4)</u> Let S be a connected regular surface. If $f: S \to \mathbb{R}$ is smooth, and $df_p = 0$ $\forall p \in S$, then f is constant.

<u>Pf.</u> Fix $p \in S$, and let $q \in S$, $\vec{\alpha}: [0,1] \to S$ continuous. There exists a partition $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$ s.t. $\vec{\alpha}(t_{i-1}), \vec{\alpha}(t_i)$ lie in a coordinate

neighborhood $\vec{x}_i : U_i \to V_i \cap S$, $\quad U_i = \{(u,v)\in\mathbb{R}^2 : u^2+v^2<1\}$.



Let $\vec{\beta}_i(s) = (1-s)\,\vec{x}_i^{-1}(\vec{\alpha}(t_{i-1})) + s\,\vec{x}_i^{-1}(\vec{\alpha}(t_i))$,

then $f(\vec{\alpha}(t_i)) - f(\vec{\alpha}(t_{i-1})) = f(\vec{x}_i(\vec{\beta}_i(1))) - f(\vec{x}_i(\vec{\beta}_i(0)))$

$$= \int_0^1 \underbrace{df_{\vec{x}_i(\vec{\beta}_i(s))}}_{=0}\left((d\vec{x}_i)_{\vec{\beta}_i(s)}(\vec{\beta}_i'(s))\right) ds = 0$$

$\Rightarrow f(q) = f(p)$. $\square$

II.3 The first fundamental form

Inner product on $\mathbb{R}^3$: $\langle \vec{v}, \vec{w}\rangle_{\mathbb{R}^3}$ $(\vec{v},\vec{w}\in\mathbb{R}^3)$.

Let $S\subset\mathbb{R}^3$ be a regular surface, $p\in S$. Then $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^3$ induces an inner product $\langle\cdot,\cdot\rangle_p$ on T_pS as follows:

$$\vec{v},\vec{w}\in T_pS\subset\mathbb{R}^3 \Rightarrow \langle\vec{v},\vec{w}\rangle_p := \langle\vec{v},\vec{w}\rangle_{\mathbb{R}^3}.$$

Thus, $\langle\cdot,\cdot\rangle_p : T_pS\times T_pS\to\mathbb{R}$ is a symmetric bilinear form on T_pS.

$$\langle\vec{v},\vec{w}\rangle_p = \langle\vec{w},\vec{v}\rangle_p$$

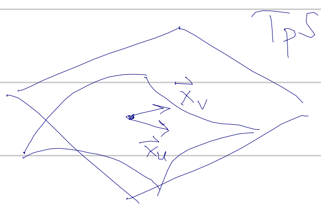
Def. The quadratic form $I_p(\vec{w}) := \langle\vec{w},\vec{w}\rangle_p = \|\vec{w}\|^2 \geq 0$, $\vec{w}\in T_pS$, on T_pS is called the first fundamental form of S at p.

Calculation in local coordinates.

Let $\vec{x}: U\subset\mathbb{R}^2\to S$ be a parameterization of S near $p=\vec{x}(q)$.

Let $\vec{w}\in T_pS$, then there exists $\vec{\beta}:(-1,1)\to U$ such that $\vec{\beta}(0)=q$, $d\vec{x}_q(\vec{\beta}'(0))=\vec{w}$.

Write $\vec{\beta}'(0)=(\beta_1,\beta_2)$. Then



$$I_p(\vec{w}) = \langle\vec{w},\vec{w}\rangle_p = \langle\vec{w},\vec{w}\rangle_{\mathbb{R}^3} = \langle d\vec{x}_q\cdot\vec{\beta}'(0), d\vec{x}_q\cdot\vec{\beta}'(0)\rangle_{\mathbb{R}^3}$$

$$= \langle\vec{\beta}'(0), (d\vec{x}_q^T d\vec{x}_q)\vec{\beta}'(0)\rangle_{\mathbb{R}^2}$$

$$= \left\langle \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right\rangle_{\mathbb{R}^2} = E\beta_1^2 + 2F\beta_1\beta_2 + G\beta_2^2,$$

where $\begin{pmatrix}E & F\\ F & G\end{pmatrix} = \begin{pmatrix} - \vec{x}_u(q) - \\ - \vec{x}_v(q) - \end{pmatrix}\begin{pmatrix} | & | \\ \vec{x}_u(q) & \vec{x}_v(q) \\ | & | \end{pmatrix}$, so $\begin{cases} E = \langle\vec{x}_u(q),\vec{x}_u(q)\rangle_{\mathbb{R}^3}, \\ F = \langle\vec{x}_u(q),\vec{x}_v(q)\rangle_{\mathbb{R}^3}, \\ G = \langle\vec{x}_v(q),\vec{x}_v(q)\rangle_{\mathbb{R}^3}. \end{cases}$

Allowing $p = \vec{x}(q)$ to vary, E, F, G become smooth functions on U.

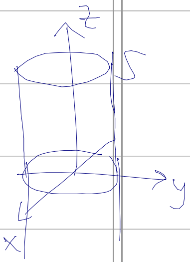
Examples (1) P = plane through $\vec{p}_0, \vec{p}_0 + \vec{w}_1, \vec{p}_0 + \vec{w}_2$, where $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^3$ have $\|\vec{w}_1\| = \|\vec{w}_2\| = 1$ and $\langle \vec{w}_1, \vec{w}_2 \rangle = 0$. Parameterize P by $\vec{x}(u,v) = \vec{p}_0 + u\vec{w}_1 + v\vec{w}_2$, $(u,v) \in \mathbb{R}^2$.

Then $E = \langle \vec{w}_1, \vec{w}_1 \rangle = 1$, $F = 0$, $G = 1$.

($\Rightarrow$ Length^2 of tangent vector to P with coordinates a, b in basis $\vec{w}_1, \vec{w}_2$ is $Ea^2 + 2Fab + Gb^2 = a^2 + b^2$. Pythagoras!)

(2) Let S be the right cylinder over the unit circle in the xy plane,

$$\vec{x}(u,v) = (\cos u, \sin u, v) \Rightarrow \vec{x}_u = (-\sin u, \cos u, 0)$$
$$\vec{x}_v = (0, 0, 1)$$



$$\Rightarrow E = \|\vec{x}_u\|^2 = 1, \quad F = \langle \vec{x}_u, \vec{x}_v \rangle = 0, \quad G = \|\vec{x}_v\|^2 = 1.$$

(Same as (1)! Discuss significance later. Note that a sheet of paper can be rolled up to a cylinder.)

(3) Use I to compute length of curves intrinsically: let $\vec{\alpha} : (-1,1) \to S$ be a smooth parameterized curve. Then for $[a,b] \subset (-1,1)$,

$$\text{Length}(\vec{\alpha}|_{[a,b]}) = \int_a^b \|\vec{\alpha}'(s)\|_{\mathbb{R}^3}\, ds = \int_a^b \sqrt{\mathrm{I}_{\vec{\alpha}(s)}(\vec{\alpha}'(s))}\, ds.$$

In local coordinates, $\vec{\alpha}(s) = \vec{x}(u(s), v(s))$, and

$$\int_a^b \sqrt{\mathrm{I}_{\vec{\alpha}(s)}(\vec{\alpha}'(s))}\, ds = \int_a^b \sqrt{E\, u'(s)^2 + 2F\, u'(s) v'(s) + G\, v'(s)^2}\, ds.$$

$\uparrow$ $E(u(s), v(s))$ $\qquad \uparrow \ldots \qquad \uparrow \ldots$

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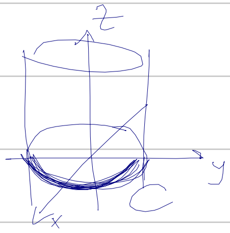
(4) The 1st fundamental form $\mathrm{I}_p : T_pS \to \mathbb{R}$ is defined independently of local coordinates. The coefficients E, F, G on the other hand do depend on coordinates.

E.g.: S = right cylinder, $\vec{y}(u,v) = (\sqrt{1-u^2}, u, v)$

$$\Rightarrow \vec{y}_u = \left(-\frac{u}{\sqrt{1-u^2}}, 1, 0\right), \quad \vec{y}_v = (0, 0, 1) \Rightarrow \begin{cases} E = \frac{1}{1-u^2} \\ F = 0 \\ G = 1. \end{cases}$$

($\Leftrightarrow$ example (2)).

Consider the curve $C=\{z=0,\ x>0\}\cap S$.



- If $\vec{\alpha}(t)=(t,0)$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$(\vec{x}\circ\vec{\alpha})(-\tfrac{\pi}{2},\tfrac{\pi}{2})=C,$$

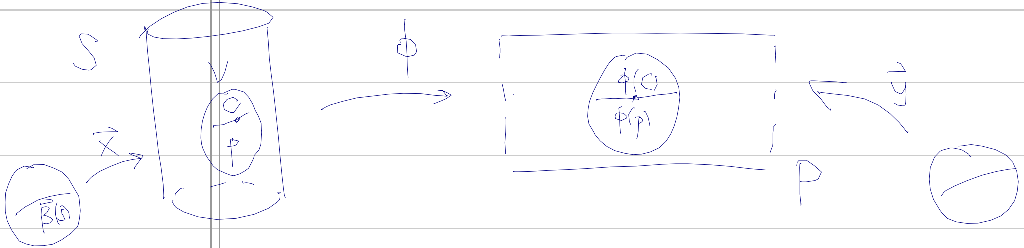
and $$\text{length}(\vec{x}\circ\vec{\alpha})=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{I_{\vec{x}(\vec{\alpha}(s))}\left((\vec{x}\circ\vec{\alpha})'(s)\right)}\,ds=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\left\langle\vec{\alpha}'(s),\begin{pmatrix}1&0\\0&1\end{pmatrix}\vec{\alpha}'(s)\right\rangle}\,ds$$
$$=\int_{-\pi/2}^{\pi/2}1\,ds=\pi.$$

- But also $C=(\vec{y}\circ\vec{\beta})(-1,1)$ for $\vec{\beta}(t)=(t,0)$, $\quad t=\sin\theta,\ \theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

and $$\text{length}(\vec{y}\circ\vec{\beta})=\int_{-1}^{1}\sqrt{\left\langle\vec{\beta}'(t),\begin{pmatrix}\frac{1}{1-t^2}&0\\0&1\end{pmatrix}\vec{\beta}'(t)\right\rangle}\,dt=\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt=\int_{-\pi/2}^{\pi/2}d\theta=\pi.$$

$\Rightarrow$ Same result! ✓

(5) As a first glimpse of the relationship between the plane (1) and the right cylinder (2), we show: for each $p\in S$, there exists an open neighborhood $V\subset S$ and a smooth map $\phi: V\to P$ with the following property: if $C\subset V$ is a regular curve, then $\phi(C)$ is a regular curve, and $\text{length}(C)=\text{length}(\phi(C))$.



Pf. Parameterize S by $\vec{x}:(0,2\pi)\times\mathbb{R}\ni(u,v)\mapsto(\cos u,\sin u, v)$,

and P by $\vec{y}:\mathbb{R}^2\ni(u,v)\mapsto\vec{p}_0+u\vec{w}_1+v\vec{w}_2$ ($\vec{w}_1,\vec{w}_2$ orthonormal).

For $p=\vec{x}(u_0,v_0)$, let $V=\vec{x}((u_0-\pi,u_0+\pi)\times\mathbb{R})$ and
$$\phi:V\to P,\quad \phi(\vec{x}(u,v))=\vec{y}(u,v).$$

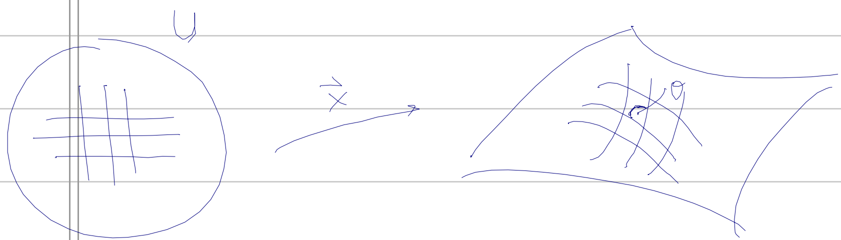
If $C=(\vec{x}\circ\vec{\beta})((-1,1))$, then $\phi(C)=(\vec{y}\circ\vec{\beta})((-1,1))$, and therefore
$$\text{length}(\phi(C))=\int_{-1}^{1}\sqrt{I((\vec{y}\circ\vec{\beta})'(s))}\,ds=\int_{-1}^{1}\|\vec{\beta}'(s)\|\,ds$$
$$=\int_{-1}^{1}\sqrt{I((\vec{x}\circ\vec{\beta})'(s))}\,ds=\text{length}(C)\qquad\square$$

S and P are "locally isometric".

Def. The angle under which two parameterized smooth curves $\vec{\alpha}: I \to S$, $\vec{\beta}: I \to S$ intersect at $p = \vec{\alpha}(t_0) = \vec{\beta}(t_0)$ is

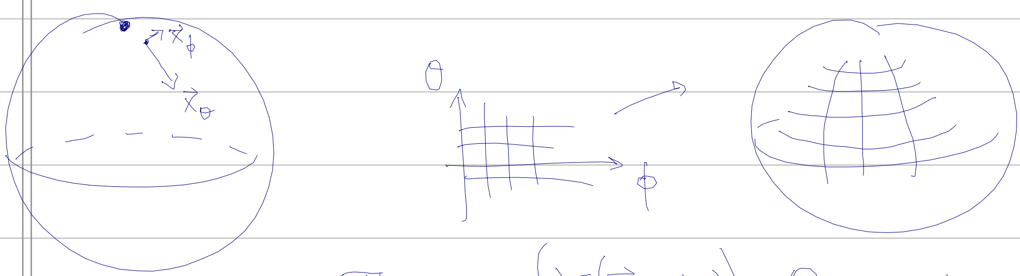
$$\cos\theta = \frac{\langle \vec{\alpha}'(t_0), \vec{\beta}'(t_0)\rangle_p}{\|\vec{\alpha}'(t_0)\| \, \|\vec{\beta}'(t_0)\|}$$ (computable only using the 1st fund. form!)

Example (1) $\vec{x}: U \subset \mathbb{R}^2 \to S$ parameterization, $\vec{\alpha}(t) = \vec{x}(t, 0)$, $\vec{\beta}(t) = \vec{x}(0, t)$



$$\cos\theta = \frac{\vec{x}_u \cdot \vec{x}_v}{\|\vec{x}_u\| \, \|\vec{x}_v\|} = \frac{F}{\sqrt{EG}}$$

(2) Consider the parameterization $\vec{x}: (\theta, \phi) \in (0, \pi) \times (0, 2\pi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in S^2 \subset \mathbb{R}^3$. Then $E = \|\vec{x}_\theta\|^2 = 1$, $F = \langle \vec{x}_\theta, \vec{x}_\phi \rangle = 0$, $G = \|\vec{x}_\phi\|^2 = \sin^2\theta$

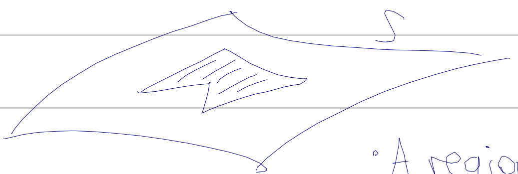


Thus, $\cos(\measuredangle(\vec{x}_\theta, \vec{x}_\phi)) = 0$ everywhere (because $F = 0$).

Parameterizations for which $F = 0$ are called orthogonal parameterizations.
(They always exist locally — later!)

II. 4 Area

Definition · A regular domain of a regular surface S is an open, connected subset of S whose boundary is the image of a circle by a smooth homeomorphism which is regular except at a finite number of points.



· A region of S is the union of a regular domain with its boundary.

· A region is bounded if it is contained in a ball $\subset \mathbb{R}^3$ of finite radius.

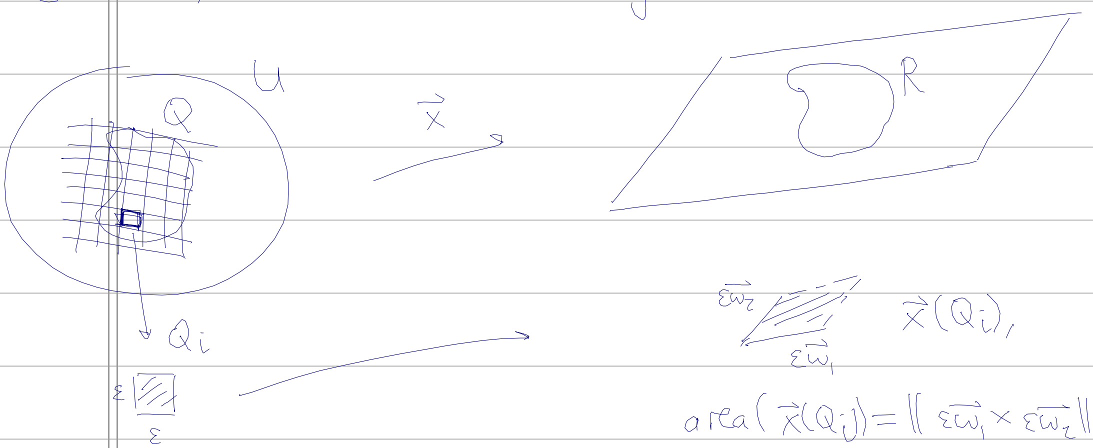
We will only consider regions R such that $R \subset \vec{x}(U)$ for some parameterization $\vec{x}: U \to S$.

<u>Def.</u> Let $R \subset S$ be a bounded region of a regular surface, and assume $R \subset \vec{x}(U)$ as above. Let $Q = \vec{x}^{-1}(R)$. Then

$$A(R) := \iint_Q \|\vec{x}_u \times \vec{x}_v\| \, du\, dv$$

is called the <u>area</u> of R.

· <u>Why?</u> Suppose S is a plane, parameterized by $\vec{x}(u,v) = \vec{p}_0 + u\vec{w}_1 + v\vec{w}_2$. Let $R = \vec{x}(Q) \subset S$ be a bounded region.



$$\text{area}(\vec{x}(Q_i)) = \|\varepsilon\vec{w}_1 \times \varepsilon\vec{w}_2\| = \varepsilon^2 \|\vec{w}_1 \times \vec{w}_2\|,$$

hence
$$A(R) = A\left(\vec{x}\left(\bigcup_i Q_i\right)\right) + \text{small error (when } \varepsilon \text{ is small)}$$
$$= \sum_i A(\vec{x}(Q_i)) + \text{small}$$
$$= \left(\sum_i \varepsilon^2\right) \|\vec{w}_1 \times \vec{w}_2\| = \int_Q \|\vec{x}_u \times \vec{x}_v\| \, du\, dv + \text{small}$$
$$\xrightarrow{\varepsilon \to 0} \int_Q \|\vec{x}_u \times \vec{x}_v\|.$$

A general regular surface is locally close to planes. (See textbook for details).

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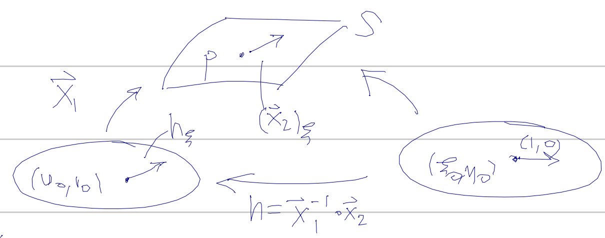
<u>Prop.</u> The area $A(R)$ is independent of the choice of parameterization.

<u>Pf.</u> Let $\vec{x}_i : V_i \subset \mathbb{R}^2 \to S$, $i = 1, 2$, be two parameterizations with $R \subset \vec{x}_i(U_i)$; set $Q_i = \vec{x}_i^{-1}(R)$. Let $h = \vec{x}_1^{-1} \circ \vec{x}_2 : U_2 \to U_1$ denote the change of variables, write $(u,v) = h(\xi,\eta) = (h_1(\xi,\eta), h_2(\xi,\eta))$, with <u>Jacobian</u> $\frac{\partial(u,v)}{\partial(\xi,\eta)} := \begin{pmatrix} \partial_\xi h_1 & \partial_\eta h_1 \\ \partial_\xi h_2 & \partial_\eta h_2 \end{pmatrix}$.

By definition, if $\vec{x}_2(\xi_0,\eta_0) = p = \vec{x}_1(u_0,v_0) \in S$, then

$$(\vec{x}_2)_\xi(\xi_0,\eta_0) = (\vec{x}_1 \circ h)_\xi(\xi_0,\eta_0) = (d\vec{x}_1)_{(u_0,v_0)} \cdot h_\xi(\xi_0,\eta_0),$$

and $(\vec{x}_2)_\eta(\xi_0,\eta_0) = (d\vec{x}_1)_{(u_0,v_0)} \cdot h_\eta(\xi_0,\eta_0)$.



Therefore,

$$\iint_{Q_2} \|(\vec{x}_2)_\xi \times (\vec{x}_2)_\eta\| \, d\xi\, d\eta = \iint_{Q_2} \|(d\vec{x}_1 \cdot h_\xi) \times (d\vec{x}_1 \cdot h_\eta)\| \, d\xi\, d\eta$$

$$= \iint_{Q_2} \left\| \det\left(\frac{\partial(u,v)}{\partial(\xi,\eta)}\right) \cdot (\vec{x}_1)_u \times (\vec{x}_1)_v \right\| d\xi\, d\eta$$

$$= \iint_{Q_2} \|(\vec{x}_1)_u \times (\vec{x}_1)_v\| \left| \det\left(\frac{\partial(u,v)}{\partial(\xi,\eta)}\right)\right| d\xi\, d\eta$$

$$= \iint_{Q_1} \|(\vec{x}_1)_u \times (\vec{x}_1)_v\| \, du\, dv$$ by the change of variables formula. $\square$

Relationship with 1^{st} fund. form.

Let $\vec{x}: U \subset \mathbb{R}^2 \to S$ be a parameterization of a regular surface.

Then $\|\vec{x}_u \times \vec{x}_v\|^2 = \|\vec{x}_u\|^2 \|\vec{x}_v\|^2 - |\vec{x}_u \cdot \vec{x}_v|^2 = EG - F^2$.

Therefore, for a bounded region $R = \vec{x}(Q)$,

$$A(R) = \iint_Q \sqrt{EG-F^2}\, du\, dv = \iint_Q \sqrt{\det\begin{pmatrix} E & F \\ F & G\end{pmatrix}}\, du\, dv.$$

Example (1) Consider the sphere $S \subset \mathbb{R}^3$ parameterized by

$\vec{x}(\theta,\phi) = a(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $a>0$.

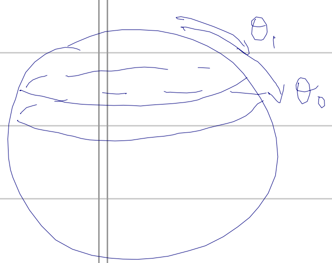
For $-a < z_0 < z_1 < a$, prove that the area $A(S \cap \{z_0 < z < z_1\})$ only depends on $z_1 - z_0$.

Pf. Write $z_i = a\cos\theta_i$, $i=0,1$. We have $E=a^2$, $F=0$, $G=a^2\sin^2\theta$.

Therefore, $\sqrt{EG-F^2} = a^2\sin\theta$.

$$\Rightarrow A(S\cap\{z_0<z<z_1\}) = \int_{\theta_1}^{\theta_0}\int_0^{2\pi} a^2\sin\theta\, d\phi\, d\theta = 2\pi a^2 \cdot (-\cos\theta)\Big|_{\theta_1}^{\theta_0}$$

$$= 2\pi a(z_1 - z_0).$$ $\square$



(2) Let $\vec{\alpha}:[0,\ell]\to\mathbb{R}^3$ be a regular curve, parameterized by arc length, with $k\neq 0$ (curvature).
For small $r>0$, let T denote the tube of radius r around $\vec{\alpha}$.
Show: $A(T)=2\pi r\cdot\ell$.

Solution. Parameterize T by

$$\vec{x}:[0,\ell]\times(0,2\pi)\ni(s,\theta)$$
$$\longmapsto \vec{\alpha}(s)+r(\vec{n}(s)\cos\theta+\vec{b}(s)\sin\theta)\quad(\vec{n},\vec{b}=\text{normal, binormal vector of }\vec{\alpha}).$$

Compute $\vec{x}_s=\vec{t}(s)+r(\vec{n}'(s)\cos\theta+\vec{b}'(s)\sin\theta)$

$$=(1-rk(s)\cos\theta)\vec{t}(s)+r\tau(s)\sin\theta\,\vec{n}(s)-r\tau(s)\cos\theta\,\vec{b}(s)$$

(Frenet formulas!),

$$\vec{x}_\theta=r(-\vec{n}(s)\sin\theta+\vec{b}(s)\cos\theta)$$

$$\Longrightarrow E=\|\vec{x}_s\|^2=r^2\tau(s)^2+(1-rk(s)\cos\theta)^2$$
$$F=\langle\vec{x}_s,\vec{x}_\theta\rangle=-r^2\tau(s)$$
$$G=\|\vec{x}_\theta\|^2=r^2.$$

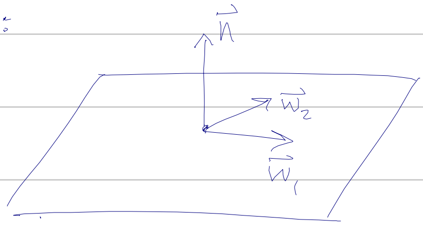
$$\Rightarrow \text{area}(T)=\int_0^\ell\int_0^{2\pi}\sqrt{EG-F^2}\,d\theta\,ds=\int_0^\ell\int_0^{2\pi}\sqrt{r^4\tau(s)^2+r^2(1-rk(s)\cos\theta)^2-r^4\tau(s)^2}\,d\theta\,ds$$
$$=\int_0^\ell\int_0^{2\pi} r-r^2k(s)\cos\theta\,d\theta\,ds$$
$$=2\pi\ell r.\qquad\square$$

(3) This remains true also without the condition $k\neq 0$. (Exercise.)

II.5 Orientation.

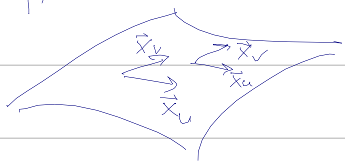
Let $P\subset\mathbb{R}^3$ be a plane. Then an orientation of P is fixed be choosing the direction of its unit normal vector:

two vectors $\vec{w}_1,\vec{w}_2\in T_pP$ are positively oriented if $\vec{w}_1\times\vec{w}_2=\lambda\vec{n}$, $\lambda>0$.



Let S be a regular surface, $p \in S$. The choice of an orientation of T_pS induces an orientation of T_qS for q near p.

· More precisely: Let $\vec{x} : U \subset \mathbb{R}^2 \to S$ be a local parameterization of S near p.
(We might e.g. declare $\vec{x}_u(q), \vec{x}_v(q) \in T_{\vec{x}(q)}S$ to give us an orientation on $\vec{x}(U)$.)
If $\vec{y} : V \subset \mathbb{R}^2 \to S$ is another parameterization of S near p,
we say that the bases



$\{\vec{x}_u, \vec{x}_v\}$ and $\{\vec{y}_u, \vec{y}_v\}$ determine
the same orientation of T_pS if the Jacobian determinant of the change of variables is positive.

Concretely, let $\vec{h} = \vec{x}^{-1} \circ \vec{y}$, Jacobian $= \begin{pmatrix} \partial_\xi h_1 & \partial_\eta h_1 \\ \partial_\xi h_2 & \partial_\eta h_2 \end{pmatrix}$.

Recall $\underbrace{\vec{y}_\xi \times \vec{y}_\eta}_{\neq 0 \text{ normal vector to } T_pS} = \left(\det \begin{pmatrix} \partial_\xi h_1 & \partial_\eta h_1 \\ \partial_\xi h_2 & \partial_\eta h_2 \end{pmatrix}\right) \underbrace{\vec{x}_u \times \vec{x}_v}_{\neq 0, \text{ normal vector to } T_pS}$.

Therefore, $\{\vec{x}_u, \vec{x}_v\}$ and $\{\vec{y}_u, \vec{y}_v\}$ determine the same orientation iff $\vec{x}_u \times \vec{x}_v$ and $\vec{y}_u \times \vec{y}_v$ point in the same direction.

Def. · A regular surface S is orientable if one can cover it with a family of coordinate charts in such a way that if a point $p \in S$ belongs to two such charts, the change of coordinates has positive Jacobian determinant.
· The choice of such a family is called an orientation, and S is then called oriented.
· If no orientation exists, S is called nonorientable.

Prop. A regular surface $S \subset \mathbb{R}^3$ is orientable iff there exists a smooth field of unit normal vectors on S, i.e. a smooth map $N : S \to \mathbb{R}^3$ s.t. $N(p) \perp T_pS$ for all $p \in S$.

Proof ($\Rightarrow$) If S is orientable, cover it with local coordinate charts as in the above definition. For one such chart $\vec{x}: U \subset \mathbb{R}^2 \to S$, define

$$N(p) = \frac{\vec{x}_u(q) \times \vec{x}_v(q)}{\|\vec{x}_u(q) \times \vec{x}_v(q)\|} \qquad (p = \vec{x}(q) \in \vec{x}(U)).$$

If $\vec{y}: V \subset \mathbb{R}^2 \to S$ is another such chart, $h = \vec{x}^{-1} \circ \vec{y}$, and $J = \det \begin{pmatrix} \partial_\xi h_1 & \partial_\eta h_1 \\ \partial_\xi h_2 & \partial_\eta h_2 \end{pmatrix}$, then

$$\frac{\vec{y}_\xi \times \vec{y}_\eta}{\|\vec{y}_\xi \times \vec{y}_\eta\|} = \frac{J \cdot (\vec{x}_u \times \vec{x}_v)}{\|J (\vec{x}_u \times \vec{x}_v)\|} = \frac{J}{|J|} N(p) = N(p) \text{ since } J > 0.$$

Thus, the definition of $N(p)$ does not depend on the choice of chart of our orientation. Moreover, $N(p)$ depends smoothly on $p \in S$ (since $\vec{x}$ is regular).

($\Leftarrow$) Let $N: S \to \mathbb{R}^3$ be a smooth field of unit normal vectors.

For each local parameterization $\vec{x}: U \subset \mathbb{R}^2 \to S$, with U connected, consider the function

$$f(p) = \left\langle N(p), \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} \right\rangle = \pm 1, \quad p \in \vec{x}(U).$$

Since f is continuous on U, $f \equiv 1$ or $f \equiv -1$. Switching the roles of u, v if necessary, we can ensure that $f \equiv 1$. This proves that we can cover S with charts $\vec{x}: U \subset \mathbb{R}^2 \to S$ so that $N(p) = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$.

For any two such charts $\vec{x}: U \subset \mathbb{R}^2 \to S$ and $\vec{y}: V \subset \mathbb{R}^2 \to S$, we have $J > 0$, as otherwise $N(p) = \frac{\vec{y}_\xi \times \vec{y}_\eta}{\|\vec{y}_\xi \times \vec{y}_\eta\|} = \operatorname{sign}(J) \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} = -N(p)$, contradiction. $\square$

Examples (1) Claim. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be smooth, and let $a \in \mathbb{R}$ be a regular value of f. Then $S = f^{-1}(a)$ is orientable.

Proof Let $p = (x_0, y_0, z_0) \in S$. If $\vec{\alpha}(t) = (x(t), y(t), z(t))$ is a smooth parameterized curve on S, $\vec{\alpha}(t_0) = p$, then

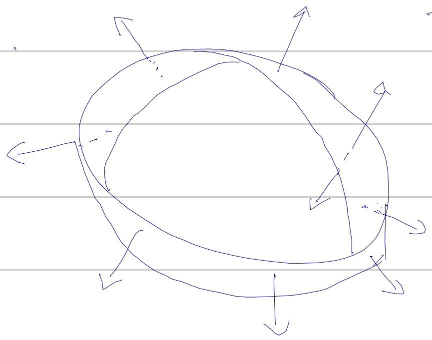
$$f(\vec{\alpha}(t)) = a \Rightarrow df_p(\vec{\alpha}'(t_0)) = \frac{d}{dt} f(\vec{\alpha}(t))\Big|_{t=t_0} = 0,$$

that is,

$$f_x(p)\, x'(t_0) + f_y(p)\, y'(t_0) + f_z(p)\, z'(t_0) = 0.$$

Since $\vec{\alpha}'(t_0)$ can be an arbitrary vector in T_pS, this shows that $N(p) := (f_x(p), f_y(p), f_z(p)) / \sqrt{f_x(p)^2 + f_y(p)^2 + f_z(p)^2}$ is a smooth field of normal vectors on S. (Note: $\sqrt{\cdots} > 0$ since $df_p \neq 0$!) $\square$

(2) Möbius strip.



There exists no differentiable field of unit vectors.

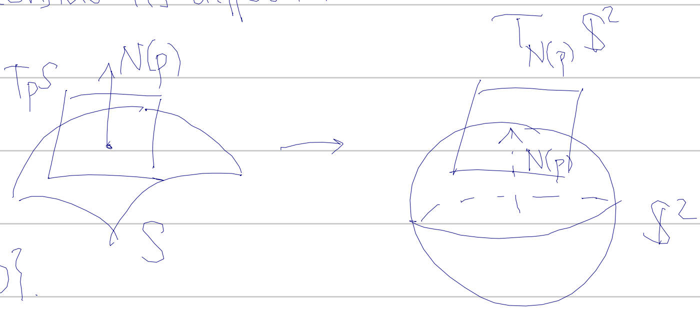
03/02/2020

III The Gauss map

Measure how a regular surface S "pulls away" from T_pS near a point $p\in S$. Idea: Look at how a normal vector to S changes near $p\in S$.

Def. Let S be a surface with an orientation N (= choice of smooth field of unit normal vectors). Let $\mathbb{S}^2=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2=1\}$. Then the map $N: S\to\mathbb{S}^2$ is the Gauss map of S.

The map N is smooth; thus, we can consider its differential
$$dN_p: T_pS\longrightarrow T_{N(p)}\mathbb{S}^2.$$



Lemma $T_{N(p)}\mathbb{S}^2=T_pS\subset\mathbb{R}^3$.

Proof $T_pS=\{\vec{v}\in\mathbb{R}^3: \langle\vec{v},N(p)\rangle_{\mathbb{R}^3}=0\}$.

On the other hand, if $\vec{\alpha}:(-1,1)\to\mathbb{S}^2$, $\vec{\alpha}(0)=N(p)$, then
$$0=\frac{d}{ds}\|\vec{\alpha}(s)\|^2\Big|_0=2\vec{\alpha}(0)\cdot\vec{\alpha}'(0)=2N(p)\cdot\vec{\alpha}'(0),$$
so $T_{N(p)}\mathbb{S}^2=\{\vec{v}\in\mathbb{R}^3: N(p)\cdot\vec{v}=0\}$. $\square$

Therefore, $dN_p: T_pS\to T_pS$.

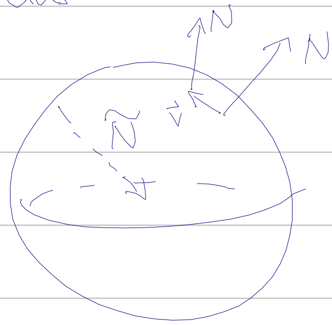
Even if S is non-orientable: every $p_0 \in S$ has an open neighborhood $V \subset S$ which is orientable, thus $\exists$ unit normal field $N: V \to \mathbb{S}^2$ on V, and $dN_p: T_pS \to T_pS$, $p \in V$.

<u>Examples</u>. (1) $P = \{\vec{p}_0 + u\vec{w}_1 + v\vec{w}_2\}$, $\vec{w}_1, \vec{w}_2$ orthonormal

$\Rightarrow N(p) = \vec{w}_1 \times \vec{w}_2$, thus $dN_p = 0$. (That is, $dN_p(\vec{v}) = \vec{0} \in T_pS$ for all $\vec{v} \in T_pS$.)

(2) $\mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2=1\}$ has unit normal vector fields

$N(x,y,z) = (x,y,z)$, $\tilde{N}(x,y,z) = (-x,-y,-z)$.



Consider a curve $\vec{\alpha}: (-1,1) \to \mathbb{S}^2$, $\vec{\alpha}(t) = (x(t), y(t), z(t))$.

· Then $dN_{\vec{\alpha}(0)}(\vec{\alpha}'(0)) = \frac{d}{dt} N(\vec{\alpha}(t))\Big|_{t=0}$

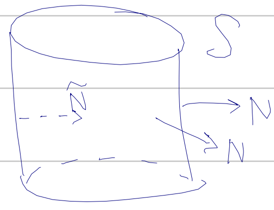
$= \frac{d}{dt}\vec{\alpha}(t)\Big|_{t=0} = \vec{\alpha}'(0)$.

$\Rightarrow dN_p(\vec{v}) = \vec{v} \ \forall \vec{v} \in T_p\mathbb{S}^2$

· For $\tilde{N}$, compute $d\tilde{N}_{\vec{\alpha}(0)}(\vec{\alpha}'(0)) = \frac{d}{dt}(-\vec{\alpha}(t))\Big|_{t=0} = -\vec{\alpha}'(0)$, so

$d\tilde{N}_p(\vec{v}) = -\vec{v} \ \forall \vec{v} \in T_p\mathbb{S}^2$: differs from dN_p by an overall sign.

(3) Let $S = \{(x,y,z) : x^2+y^2 = c^2\}$, $c > 0$.



Then $N = (x,y,0)/c$ and $\tilde{N} = -N$ are two fields of unit normal vectors.

Let $\vec{\alpha}(t) = (x(t), y(t), z(t))$ be a smooth curve, $p = \vec{\alpha}(0)$. Then

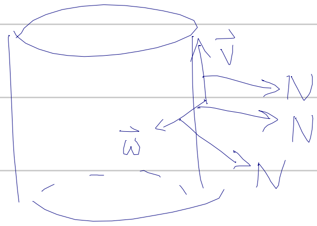
$dN_p(\vec{\alpha}'(0)) = \frac{d}{dt} N(\vec{\alpha}(t))\Big|_{t=0} = \frac{d}{dt}(x(t), y(t), 0)/c\Big|_{t=0} = \frac{1}{c}(x'(0), y'(0), 0)$.

Thus, if $\vec{v} = \vec{\alpha}'(0) \in T_pS$ is parallel to the z-axis (i.e. $x'(0) = y'(0) = 0$),

then $dN_p(\vec{v}) = 0$;

if $\vec{w} = \vec{\alpha}'(0) \in T_pS$ is parallel to the xy-plane (i.e. $z'(0) = 0$),

then $dN_p(\vec{w}) = \frac{1}{c}\vec{w}$.

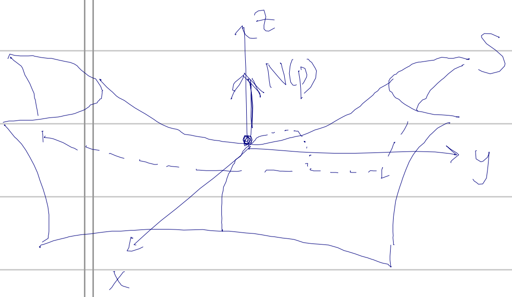


$\Rightarrow \vec{v}, \vec{w}$ are eigenvectors of dN_p with eigenvalues $0, \frac{1}{w}$, respectively.

Moreover, $d\tilde{N}_p = -dN_p$, so $\vec{v}, \vec{w}$ are eigenvectors of

$d\tilde{N}_p$ with eigenvalues $0, -\frac{1}{c}$.

(4) $S = \{z = y^2 - x^2\} = \vec{x}(\mathbb{R}^2)$, $\vec{x}(u,v) = (u, v, v^2 - u^2)$: hyperbolic paraboloid,



$p = (0,0,0) \in S$.

Normal vector: compute $\vec{x}_u = (1, 0, -2u)$,

$\vec{x}_v = (0, 1, 2v)$

$$\Rightarrow N(u,v) = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} = \left(\frac{2u}{\sqrt{1+4u^2+4v^2}}, -\frac{2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}} \right).$$

- Consider the curve $\vec{\alpha}(t) = \vec{x}(t,0)$, then $\vec{\alpha}'(0) = (1,0,0)$,

$$dN_p(\vec{\alpha}'(0)) = \frac{d}{dt} N(t,0)\Big|_{t=0} = N_u\Big|_{(0,0)} = (2,0,0) = 2\vec{\alpha}'(0).$$

- For $\vec{\beta}(t) = \vec{x}(0,t)$, $\vec{\beta}'(0) = (0,1,0)$,

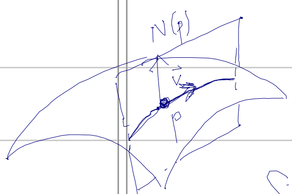
$$dN_p(\vec{\beta}'(0)) = N_v\Big|_{(0,0)} = (0,-2,0) = -2\vec{\beta}'(0).$$

Therefore, dN_p has eigenvalues $\begin{cases} +2 \\ -2 \end{cases}$ with eigenvectors $\begin{cases} (1,0,0) \in T_pS \\ (0,1,0) \in T_pS. \end{cases}$

(5) Relationship with curvature of curves:

Consider $p \in S$, S = regular surface. Let $\vec{v} \in T_pS$, $\|\vec{v}\| = 1$,

and let P = plane through p spanned by $\vec{v}$, $N(p)$.



Let $\vec{\alpha} : I \to S \cap P$ be an arc length parameterization near p, with $\vec{\alpha}(0) = p$, $\vec{\alpha}'(0) = \vec{v}$.

Since $\vec{\alpha}(s) \in P$ $\forall s$, the normal vector of $\vec{\alpha}$ satisfies $\vec{n}(0) = \pm N(p)$.

$$\Rightarrow \pm k(0) = \langle k(0)\vec{n}(0), N(p) \rangle = \langle \vec{T}'(0), N(\vec{\alpha}(0)) \rangle_{\mathbb{R}^3}$$

$$= \frac{d}{ds} \underbrace{\langle \vec{T}(s), N(\vec{\alpha}(s)) \rangle_{\mathbb{R}^3}}_{=0 \text{ always}} \Big|_{s=0} - \langle \vec{T}(0), dN_p(\vec{T}(0)) \rangle_{\mathbb{R}^3}$$

$$= -\langle dN_p(\vec{T}(0)), \vec{T}(0) \rangle_{\mathbb{R}^3}.$$

Prop. The differential $dN_p : T_pS \to T_pS$ is self-adjoint. That is,

$$(*) \quad \langle dN_p(\vec{v}), \vec{w} \rangle = \langle \vec{v}, dN_p(\vec{w}) \rangle, \quad \vec{v}, \vec{w} \in T_pS.$$

Corollary There exists an orthonormal basis $\vec{e}_1, \vec{e}_2$ of T_pS and real numbers $k_1 \geq k_2 \in \mathbb{R}$ s.t. $dN_p(\vec{e}_1) = -k_1 \vec{e}_1$, $dN_p(\vec{e}_2) = -k_2 \vec{e}_2$.

Definition k_1, k_2 are called the principal curvatures of S at p.

A vector $\vec{e} \in T_pS$, $\|\vec{e}\|=1$, s.t. $dN_p(\vec{e}) = -k_1 \vec{e}$ or $dN_p(\vec{e}) = -k_2 \vec{e}$ is called a principal direction at p. (If $k_1 \neq k_2$, the principal directions are $\pm\vec{e}_1, \pm\vec{e}_2$.)

Proof of Proposition It suffices to check $(*)$ for a basis $\vec{v}, \vec{w}$ of T_pS.

Let $\vec{x} : U \subset \mathbb{R}^2 \to S$ be a parameterization of S near $p = \vec{x}(u_0, v_0)$ and let $N : U \to \mathbb{R}^3$ denote the Gauss map in local coordinates (i.e. $N(u,v)$ = unit normal vector at $\vec{x}(u,v)$), then $dN_p(\vec{x}_u(u_0,v_0)) = \frac{d}{ds} N(u_0+s, v_0)\big|_{s=0} = N_u(u_0,v_0)$.

Taking $\vec{v} = \vec{x}_u$, $\vec{w} = \vec{x}_v$, we need to show:

$$\langle N_u, \vec{x}_v \rangle = \langle \vec{x}_u, N_v \rangle \qquad (\#)$$

Now, we have

$$\langle N, \vec{x}_v \rangle \equiv 0 \overset{\partial_u}{\Longrightarrow} \langle N_u, \vec{x}_v \rangle + \langle N, \vec{x}_{vu} \rangle = 0, \qquad (1)$$

$$\langle \vec{x}_u, N \rangle \equiv 0 \overset{\partial_v}{\Longrightarrow} \langle \vec{x}_u, N_v \rangle + \langle \vec{x}_{uv}, N \rangle = 0. \qquad (2)$$

Since $\vec{x}_{vu} = \vec{x}_{uv}$, subtracting the 2^{nd} from the 1^{st} eqn. proves $(\#)$. $\square$

Def. The quadratic form II_p on T_pS, defined by

$$\mathrm{II}_p(\vec{v}) = -\langle dN_p(\vec{v}), \vec{v} \rangle, \quad \vec{v} \in T_pS,$$

is called the second fundamental form of S at p.

Also write $\mathrm{II}(\vec{v}, \vec{w}) = -\langle dN_p(\vec{v}), \vec{w} \rangle$, $\vec{v}, \vec{w} \in T_pS$ (symmetric bilinear form).

In particular, $k_1 = \max\limits_{\substack{\vec{v} \in T_pS \\ \|\vec{v}\|=1}} \mathrm{II}_p(\vec{v})$, $k_2 = \min\limits_{\substack{\vec{v} \in T_pS \\ \|\vec{v}\|=1}} \mathrm{II}_p(\vec{v})$. (Indeed, if $\vec{e}_1, \vec{e}_2$ are

orthonormal principal directions, then $II_p(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2) = k_1\cos^2\theta + k_2\sin^2\theta$.
Maximal for $\theta = 0, \pi$, minimal for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

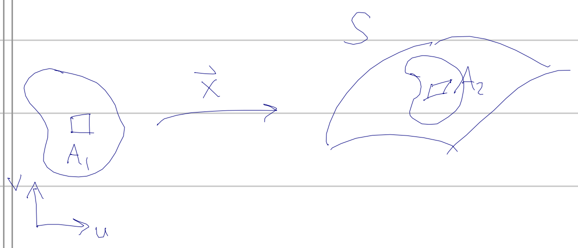
<u>Definition</u> Let $p \in S$. Then the <u>Gauss curvature</u> of S at p is

$$K(p) := \det(dN_p) = k_1 k_2,$$

and the <u>mean curvature</u> is

$$H(p) = -\tfrac{1}{2}\mathrm{tr}(dN_p) = \frac{k_1+k_2}{2}.$$

· <u>Recall</u>:



$$\mathrm{area}(A_2) \approx \|\vec{x}_u \times \vec{x}_v\|\,\mathrm{area}(A_1) = \sqrt{EG-F^2}\,\mathrm{area}(A_1).$$

<u>Now</u>:

$$\mathrm{area}(A_3) \approx \|dN_p(\vec{e}_1) \times dN_p(\vec{e}_2)\|\,\mathrm{area}(A_2) = |k_1k_2|\,\|\vec{e}_1 \times \vec{e}_2\|\,\mathrm{area}(A_2) = |K|\,\mathrm{area}(A_2).$$

(Gauss' original interpretation.)

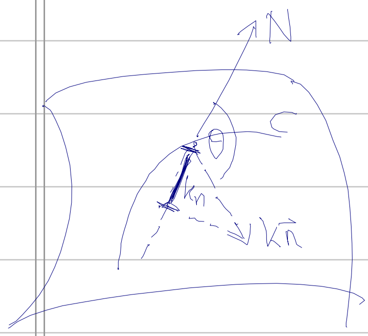
· <u>Examples</u>
- plane: $k_1 = 0$, $k_2 = 0 \Rightarrow K = 0$, $H = 0$
- cylinder: $k_1 = 0$, $k_2 = -1 \Rightarrow K = 0$, $H = -\frac{1}{2}$.
- unit sphere: $k_1 = k_2 = -1 \Rightarrow K = 1$, $H = -1$.
- hyperbolic paraboloid, at $p = (0,0,0)$: $k_1 = 2$, $k_2 = -2 \Rightarrow K = -4$, $H = 0$.

<u>Definition</u> A point $p \in S$ is called
<u>elliptic</u> if $K > 0$, <u>hyperbolic</u> if $K < 0$, <u>parabolic</u> if $K = 0$ but $dN_p \neq 0$,
<u>planar</u> if $K = 0$ and $dN_p = 0$.

03/04/2020

More on the connection of II_p to the theory of curves.

- Def. Let $C \subset S$ be a regular curve passing through $p \in S$, $k, \vec{n}$ the curvature and normal vector of C at p, and $\cos\theta = \langle \vec{n}, N(p)\rangle$. Then $k_n := k\cos\theta$ is the normal curvature of C at p.



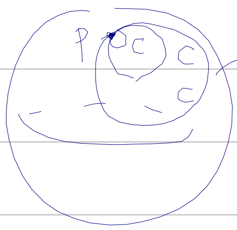
Calculation: $\vec{\alpha}(s)$ parameterization of C by arc length, $\vec{\alpha}(0) = p$, $N(s) := N(\vec{\alpha}(s))$.

Then $k_n(p) = \langle N(p), k\vec{n}\rangle = \langle N(0), \vec{\alpha}''(0)\rangle$

$$= \frac{d}{ds}\langle N(s), \vec{\alpha}'(s)\rangle\Big|_{s=0} - \langle N'(0), \vec{\alpha}'(0)\rangle$$

$$= -\langle dN_{\vec{\alpha}(0)}(\vec{\alpha}'(0)), \vec{\alpha}'(0)\rangle = \mathrm{II}_p(\vec{\alpha}'(0)).$$

Cor. k_n only depends on p and the unit tangent vector of C at p!



all these C have same normal curvature at p.

- Def. Let $p \in S$. An asymptotic direction of S at p is a direction of T_pS (i.e. a line through $0 \in T_pS$) for which the normal curvature is zero. An asymptotic curve is a regular curve $C \subset S$ such that for all $p \in C$, the tangent line to C at p is an asymptotic direction.

Thus, if $\vec{\alpha}: I \to S$, $\vec{\alpha}(0) = p$, $\vec{0} \neq \vec{\alpha}'(0)$ is an asymptotic direction of S at p, then $\mathrm{II}(\vec{\alpha}'(0)) = 0$. If $\vec{\alpha}'(0) = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2$, this means $\alpha_1^2 k_1 + \alpha_2^2 k_2 = 0$.

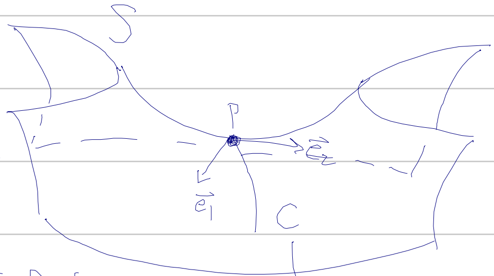
principal directions

Thus, asymptotic directions only exist if $K \leq 0$.

Example (Beltrami–Enneper theorem.) Let $C \subset S$ be an asymptotic curve with nonzero curvature. Then the torsion τ of C at a point $p \in C$ satisfies $|\tau| = \sqrt{-K(p)}$.

Pf.: Since C is asymptotic curve, we have

$\mathrm{II}_p(\vec{v}) = 0$, $\vec{v}$ = unit tangent vector of C at p.

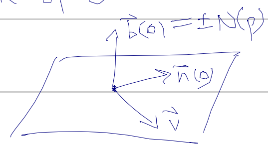


- Write $\vec{v} = \cos\phi\, \vec{e}_1 + \sin\phi\, \vec{e}_2$,

  where $\vec{e}_1, \vec{e}_2$ are the principal directions of S at p.

  Principal curvatures $k_1 \geq 0 \geq k_2$

  $\Rightarrow 0 = \mathrm{II}_p(\vec{v}) = k_1 \cos^2\phi + k_2 \sin^2\phi$. (#)



- Let $\vec{n}(0)$ = normal vector to C at p. We have

  $k_n = \mathrm{II}_p(\vec{v}) = 0$, and $k_n = \langle k(0)\vec{n}(0), N(p)\rangle = k(0)\langle \vec{n}(0), N(p)\rangle$.

  Gauss map

  normal curvature of C at p.

  By assumption, $k(0) \neq 0 \Rightarrow \vec{n}(0) \perp N(p) \Rightarrow \vec{n}(0) \in T_pS$.

  Moreover, $\vec{n}(0)\cdot\vec{v} = 0 \Rightarrow \vec{n}(0) = \pm(\sin\phi\, \vec{e}_1 - \cos\phi\, \vec{e}_2)$.

- Binormal vector of C at p: $\vec{b}(0) = \vec{v} \times \vec{n}(0) = \pm N(p)$

- Definition of torsion:

  $\langle N(\vec{\alpha}(s)), \vec{n}(s)\rangle \equiv 0$ for $\vec{\alpha}$ = arclength parameterization of C.

$$|\tau(s)| = |\langle \vec{b}(0), \vec{n}'(0)\rangle| = |\langle N(p), \vec{n}'(0)\rangle| = |\langle dN_p(\vec{v}), \vec{n}(0)\rangle|$$

$$= |\langle k_1 \cos\phi\, \vec{e}_1 + k_2 \sin\phi\, \vec{e}_2,\ \sin\phi\, \vec{e}_1 - \cos\phi\, \vec{e}_2\rangle|$$

$$= |k_1 \cos\phi \sin\phi - k_2 \cos\phi \sin\phi| = (|k_1| + |k_2|)|\cos\phi||\sin\phi|$$

From (#), $\sqrt{k_1}|\cos\phi| = \sqrt{-k_2}|\sin\phi|$

$$\Rightarrow |\tau(s)| = |k_1| \cdot \frac{\sqrt{-k_2}}{\sqrt{k_1}} |\sin\phi|^2 + |k_2| \frac{\sqrt{k_1}}{\sqrt{-k_2}} |\cos\phi|^2 = \sqrt{-k_1 k_2} = \sqrt{-K}. \qquad \square$$

Def. A point $p \in S$ is called an umbilical point if $k_1 = k_2$.

Prop. Let S be a connected regular surface. If all points of S are umbilical, then S is either contained in a plane or a sphere.

Proof $\vec{x}: U \subset \mathbb{R}^2 \to S$ parameterization, U connected. Let $N: U \to \mathbb{R}^3$ be a smooth unit normal field in local coordinates ($N(\vec{x}(q)) \perp T_{\vec{x}(q)}S$).

We have $dN_p(\vec{w}) = \lambda(p)\vec{w}$ $\forall\, \vec{w} \in T_pS$, $p \in S$, with $\lambda: S \to \mathbb{R}$ smooth.

Take $\vec{w} = \vec{x}_u, \vec{x}_v$, then $\begin{cases} N_u = \lambda \vec{x}_u, \\ N_v = \lambda \vec{x}_v, \end{cases}$ where $\lambda : U \to \mathbb{R}$ is smooth.

(1) Claim: λ is constant.

Indeed, $\begin{array}{l} N_{uv} = \lambda_v \vec{x}_u + \lambda \vec{x}_{uv} \\ \quad \| \\ N_{vu} = \lambda_u \vec{x}_v + \lambda \vec{x}_{vu} \end{array} \Rightarrow \lambda_v \vec{x}_u = \lambda_u \vec{x}_v.$

But $\vec{x}_u, \vec{x}_v$ are linearly independent $\Rightarrow \lambda_u = \lambda_v = 0$ in U. Since U is connected, $\lambda =$ const.

(2) If $\lambda \equiv 0$, then $N_u = 0$, $N_v = 0$ $\Rightarrow$ $N = N_0$ is constant, and therefore the function $f(u,v) = \langle \vec{x}(u,v), N(u,v) \rangle$ satisfies

$$f_u = \langle \vec{x}_u, N \rangle + \langle \vec{x}, N_u \rangle = 0,$$

$$f_v = 0$$

$\Rightarrow$ $f =$ constant $\Rightarrow$ $\vec{x}(u,v) \cdot N_0 =$ const, so $\vec{x}(U) \subset$ plane.

(3) If $\lambda \neq 0$, the point $\vec{z}(u,v) := \vec{x}(u,v) - \frac{1}{\lambda} N(u,v)$ is constant:

we have $\vec{z}_u = \vec{x}_u - \frac{1}{\lambda} N_u = \vec{x}_u - \frac{1}{\lambda} \lambda \vec{x}_u = 0$,

$$\vec{z}_v = 0,$$

Therefore, $\vec{z}(u,v) = \vec{z}_0$; and $\| \vec{x}(u,v) - \vec{z}_0 \| = \frac{1}{|\lambda|}$.

$\Rightarrow$ $\vec{x}(U) \subset$ sphere centered at $\vec{z}_0$ with radius $\frac{1}{|\lambda|}$.

(4) This proves the proposition locally on S. Use connectedness of S to get global result. $\square$

III.1 Gauss map in local coordinates.

S = oriented regular surface, $N : S \to \mathbb{R}^3$ Gauss map, parameterization $\vec{x} : U \subset \mathbb{R}^2 \to S$ with

$$N = \frac{\vec{x}_u \times \vec{x}_v}{\| \vec{x}_u \times \vec{x}_v \|}.$$

· Let $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$ be a parameterized curve, $\vec{\alpha}(0) = p \in S$.

$\Rightarrow \vec{\alpha}'(0) = \vec{x}_u u' + \vec{x}_v v'$,

$dN_p(\vec{\alpha}'(0)) = N_u u' + N_v v'$.

Since $N_u \in T_{N(p)} S^2 = T_p S$, can write $\begin{cases} N_u = a_{11}\vec{x}_u + a_{21}\vec{x}_v, \\ N_v = a_{12}\vec{x}_u + a_{22}\vec{x}_v. \end{cases}$

$\Rightarrow dN_p(\vec{\alpha}'(0))$

$= (a_{11}u' + a_{12}v')\vec{x}_u + (a_{21}u' + a_{22}v')\vec{x}_v$.

Therefore, in the basis $\vec{x}_u, \vec{x}_v$ of T_pS, $dN_p = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. (*) (Not necessarily a symmetric, unless $\vec{x}_u, \vec{x}_v$ = orthonormal!)

· 2^{nd} fund. form:

$II_p(\vec{\alpha}'(0)) = -\langle dN_p(\vec{\alpha}'(0)), \vec{\alpha}'(0)\rangle = -\langle N_u u' + N_v v', \vec{x}_u u' + \vec{x}_v v'\rangle$

$= e(u')^2 + 2f u'v' + g(v')^2$,

where $\begin{cases} e = -\langle N_u, \vec{x}_u\rangle = \langle N, \vec{x}_{uu}\rangle, & (\leftarrow \text{use } \langle N, \vec{x}_u\rangle = 0) \\ f = -\langle N_u, \vec{x}_v\rangle = \langle N, \vec{x}_{uv}\rangle \quad (= -\langle N_v, \vec{x}_u\rangle), & (\leftarrow \text{use } \langle N, \vec{x}_v\rangle = 0) \\ g = -\langle N_v, \vec{x}_v\rangle = \langle N, \vec{x}_{vv}\rangle \end{cases}$

· Relate e, f, g to a_{ij} in (*):

$$\left.\begin{aligned} -f &= \langle N_u, \vec{x}_v\rangle = a_{11}F + a_{21}G, \\ -f &= \langle N_v, \vec{x}_u\rangle = a_{12}E + a_{22}F, \\ -e &= \langle N_u, \vec{x}_u\rangle = a_{11}E + a_{21}F, \\ -g &= \langle N_v, \vec{x}_v\rangle = a_{12}F + a_{22}G. \end{aligned}\right\} \Leftrightarrow -\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Therefore, dN_p in the basis $\vec{x}_u, \vec{x}_v$ is given by the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} e & f \\ f & g \end{pmatrix}; \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

· Gauss curvature: $K = \det(dN_p) = \det\begin{pmatrix} e & f \\ f & g \end{pmatrix} \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \dfrac{eg - f^2}{EG - F^2}$.

· Mean curvature: $H = -\frac{1}{2}\operatorname{tr}(dN_p) = -\dfrac{a_{11} + a_{22}}{2}$.

· Principal curvatures k_1, k_2 are roots of characteristic polynomial

$\det(dN_p + k\,\mathbb{1}_{2\times 2}) = \det\begin{pmatrix} a_{11}+k & a_{12} \\ a_{21} & a_{22}+k \end{pmatrix} = k^2 + k(a_{11} + a_{22}) + K = k^2 - 2Hk + K$, so $k_{1/2} = H \pm \sqrt{H^2 - K}$.

In particular, putting $k_1 = H + \sqrt{H^2-K}$, $k_2 = H - \sqrt{H^2-K}$, the functions k_1, k_2 are continuous, and smooth except possibly at the points where $H^2 = K$.

(Ex: These are precisely the umbilical points.)

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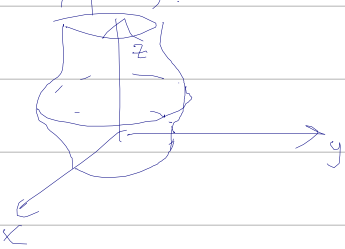
Example Surfaces of revolution S:

$$\vec{x}(u,v) = (\varphi(v)\cos u,\ \varphi(v)\sin u,\ \psi(v)), \quad 0 < u < 2\pi,\ a < v < b,\ \varphi(v) > 0.$$

- First fund. form: $\vec{x}_u = (-\varphi(v)\sin u, \varphi(v)\cos u, 0)$, $\vec{x}_v = (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v))$.

$$E = \varphi^2,\quad F = 0,\quad G = (\varphi')^2 + (\psi')^2.$$



For convenience: parameterize generating curve by arc length, so $G = 1$.

- $N(u,v) = \dfrac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} = (\varphi \cdot (\psi'\cos u, \psi'\sin u, -\varphi'))/\|\cdots\|$

$$= (\psi'\cos u, \psi'\sin u, -\varphi').$$

$$\vec{x}_{uu} = (-\varphi\cos u, -\varphi\sin u, 0), \qquad e = \langle N, \vec{x}_{uu}\rangle = -\varphi\psi'$$

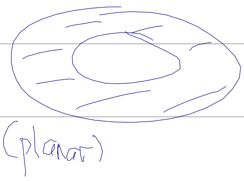
$$\vec{x}_{uv} = (-\varphi'\sin u, \varphi'\cos u, 0), \quad \Rightarrow \quad f = \langle N, \vec{x}_{uv}\rangle = 0$$

$$\vec{x}_{vv} = (\varphi''\cos u, \varphi''\sin u, \psi''), \qquad g = \langle N, \vec{x}_{vv}\rangle = \varphi''\psi' - \varphi'\psi''.$$

$$\Rightarrow K = \frac{eg - f^2}{EG - F^2} = -\frac{\varphi\psi'(\varphi''\psi' - \varphi'\psi'')}{\varphi^2 \cdot 1} \underset{\varphi'\varphi'' = -\psi'\psi''}{=} -\frac{(\psi')^2\varphi'' + (\varphi')^2\varphi''}{\varphi} = -\frac{\varphi''}{\varphi}.$$

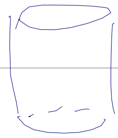
Rmk $K = 0 \Leftrightarrow \varphi'' = 0 \Leftrightarrow \varphi$ linear, thus $\varphi' = \text{const} \Rightarrow \psi'$ const. $\Rightarrow \psi$ linear.

(i) $\psi = \text{const.}$: (ii) $\varphi = \text{const.}$: (iii) $\varphi, \psi \neq \text{const}$:

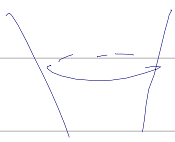


(planar)

(cylinder)



(cone)

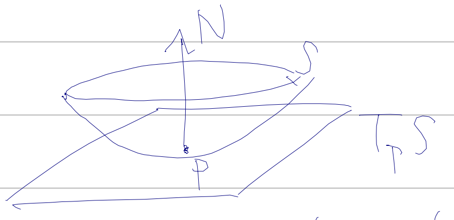


- Mean curvature: $a_{11} = -\dfrac{e}{E} = \dfrac{\psi'}{\varphi}$, $a_{22} = -\dfrac{g}{G} = \varphi'\psi'' - \varphi''\psi'$,

$$H = \frac{a_{11} + a_{22}}{2} = \ldots$$

- Principal curvatures: $k_1, k_2 = H \pm \sqrt{H^2 - K} = -a_{11}, -a_{22}$

Example Let S be a regular surface, $p \in S$. After rotation and translation, we may assume $p = \vec{0} \in \mathbb{R}^3$, $T_pS =$ xy-plane $\{z=0\}$.



Near p, S is a graph, with local parameterization

$\vec{x}(u,v) = (u, v, h(u,v))$, where $h(0,0)=0$, $h_u(0,0)=0$, $h_v(0,0)=0$.

$\Rightarrow N(0,0) = (0,0,1)$. Compute at p:

1st fund. form: $E=1$, $F=0$, $G=1$.

2nd fund. form: $e = \langle N, \vec{x}_{uu} \rangle = h_{uu}$, $f = h_{uv}$, $g = h_{vv}$.

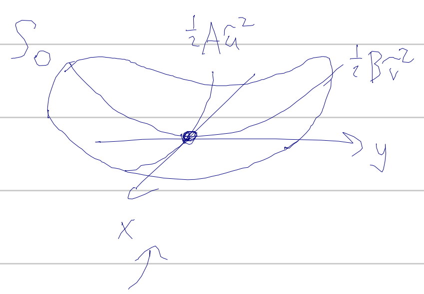
$\Rightarrow$ Gauss curvature $K = \frac{eg-f^2}{EG-F^2} = h_{uu}h_{vv} - h_{uv}^2$,

mean curvature $H = -\left(\frac{e}{E} + \frac{g}{G}\right)/2 = -\frac{h_{uu}+h_{vv}}{2}$.

- Note: Only 2nd order Taylor series of h matters; the second fund. form of S at p is the same as that of the paraboloid

$S_0 = \{(u, v, \tfrac{1}{2}au^2 + buv + \tfrac{1}{2}cv^2)\}$, $a = h_{uu}$, $b = h_{uv}$, $c = h_{vv}$!

Recall: every quadratic form can be diagonalized. Thus, by a rotation in the xy-plane, $S_0 = \{(\tilde{u}, \tilde{v}, \tfrac{1}{2}(A\tilde{u}^2 + B\tilde{v}^2))\}$, $A, B \in \mathbb{R}$.



Principal curvatures: $-A, -B$

Gauss curvature: AB,

mean curvature: $-\frac{A+B}{2}$,

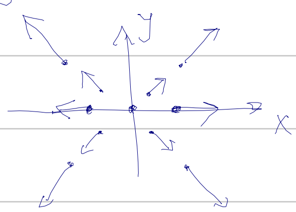
principal directions: $(1,0,0)$, $(0,1,0)$.

- Rmk. The reduction to S_0 demonstrates that dN_p has an orthonormal basis of eigenvectors (using the diagonalizability of quadratic forms (spectral theorem!)) with real eigenvalues. This implies that dN_p is self-adjoint!

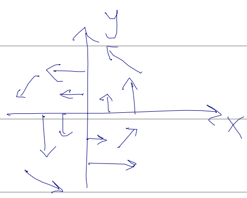
III.2 Vector fields

(i) In the plane. A vector field is a smooth map $w : U \subset \mathbb{R}^2 \to \mathbb{R}^2$.

Examples (1) $w(x,y) = (x,y)$ (i.e. $w(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}$)



(2) $w(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$



A trajectory of w is a curve $\vec{\alpha}(t)$, $t \in I$, satisfying $\vec{\alpha}'(t) = w(\vec{\alpha}(t))$.

(Writing $\vec{\alpha}(t) = (x(t), y(t))$, $w(x,y) = (a(x,y), b(x,y))$, this means

$$\begin{cases} x'(t) = a(x(t), y(t)) \\ y'(t) = b(x(t), y(t)). \end{cases}$$
)

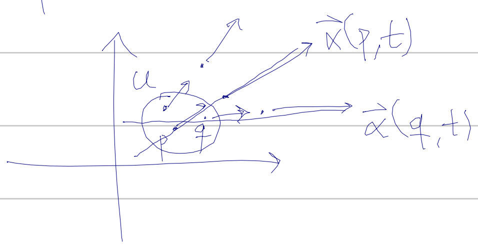
· Thm. Let $w : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field. For each $p \in U$, there exist a neighborhood $V \subset U$ of p and an open interval $I \subset \mathbb{R}$, and a map $\vec{\alpha} : V \times I \to U$ s.t.

(i) for $q \in V$, the curve $\vec{\alpha}(q, \cdot) : I \to U$ is the (unique) trajectory of w passing through q; that is,

$$\vec{\alpha}(q, 0) = q, \qquad \frac{\partial \vec{\alpha}}{\partial t}(q,t) = w(\vec{\alpha}(q,t));$$

(ii) $\vec{\alpha}$ is smooth.

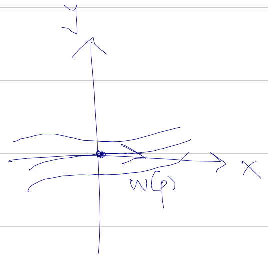
(The map $\vec{\alpha} : I \times V \to U$ is called the (local) flow of w (near p).)



Lemma Let $w: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field, and let $p \in U$ be such that $w(p) \neq 0$. Then $\exists$ neighborhood $W \subset U$ of p and a smooth function $f: W \to \mathbb{R}$ such that (i) f is constant along each trajectory of w,

(ii) $df_q \neq 0$ for all $q \in W$.

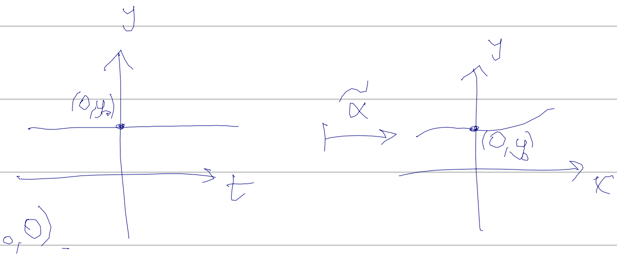
Proof WLOG $p = 0$, $w(p) = (a_0, 0)$, $a_0 \neq 0$.



- Let $\vec{\alpha}: V \times I \to U$ be the local flow of w at p, $V \subset U$. Consider

$$\tilde{\alpha}(y,t) := \vec{\alpha}(0,y,t) \quad \text{for } (0,y) \in V,\ t \in I.$$

(Idea: parameterize space of trajectories near p by the y-intercept.)



- Then $d\tilde{\alpha}_{(0,0)}((y',0)) = (0,y')$,

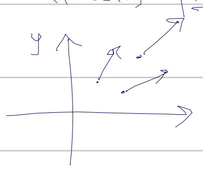
$d\tilde{\alpha}_{(0,0)}((0,t')) = \frac{\partial \tilde{\alpha}}{\partial t}(0,0,0) = w(p) = (a_0, 0)$.

$\Rightarrow$ $d\tilde{\alpha}_{(0,0)}$ is invertible.

- Inverse function theorem $\Rightarrow$ $\tilde{\alpha}$ is a local diffeomorphism near $(y,t) = (0,0)$. Write $\tilde{\alpha}^{-1}(x,y) = (f(x,y), \tau(x,y))$, then $f(x,y)$ is constant along the trajectories of w, and $df_{(0,0)}((0,y')) = y'$, so $df_p \neq 0$ for $p = (0,0)$, hence $df_q \neq 0$ for $q \in U$ near p. $\square$

Terminology: f is a (local) first integral of w near p.

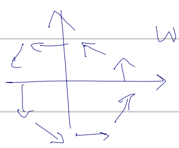
Examples (1)



$w(x,y) = (x,y)$, $p = (x_0, y_0)$, $x_0, y_0 > 0$.

$\rightsquigarrow$ E.g. $f(x,y) = y/x$.

(2)



$w(x,y) = (-y, x)$, $p \neq (0,0)$.

$\rightsquigarrow$ E.g. $f(x,y) = x^2 + y^2$.

(ii) Vector fields on surfaces.

Def A vector field w in an open subset $U \subset S$ of a regular surface S is a map $w: S \to \mathbb{R}^3$ assigning to each $p \in U$ a vector $w(p) \in T_pS$.

The vector field w is smooth at $p \in \mathcal{U}$ if, for some parameterization $\vec{x}(u,v)$ of S at p, the functions $a(u,v), b(u,v)$ defined by

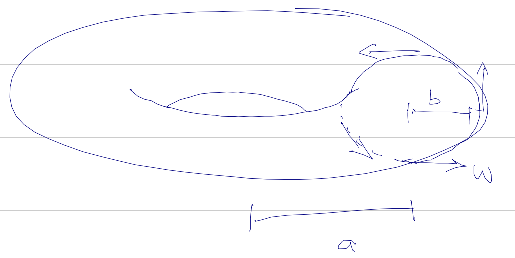
$$w(\vec{x}(u,v)) = a(u,v)\,\vec{x}_u + b(u,v)\,\vec{x}_v$$

are smooth at $(u,v) = \vec{x}^{-1}(p)$.

· Ex. Consider the torus S parameterized by

$$\vec{x}(u,v) = \big((a+b\cos u)\cos v,\ (a+b\cos u)\sin v,\ b\sin u\big) \qquad (0<b<a).$$

Then $w(\vec{x}(u,v)) := \vec{x}_u(u,v) = (-b\sin u\cos v, -b\sin u\sin v, b\cos u)$ is a smooth vector field on S.



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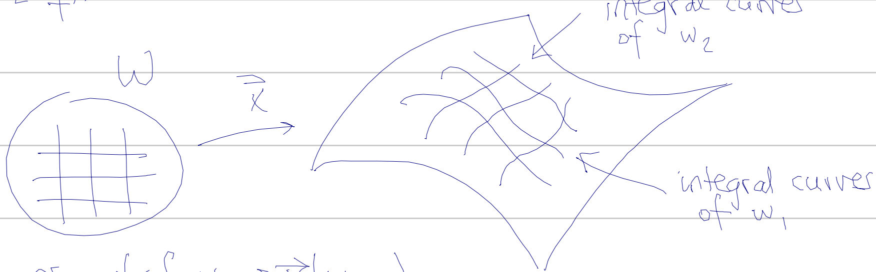
· Thm. Let w_1, w_2 be two smooth vector fields on an open set $\mathcal{U} \subset S$. Suppose $p \in \mathcal{U}$ is s.t. $w_1(p), w_2(p)$ are linearly independent. Then it is possible to parameterize a neighborhood $V \subset \mathcal{U}$ of p using $\vec{x}: W \subset \mathbb{R}^2 \to V$ such that at each $q = \vec{x}(r)$, the "coordinate vector fields" $\vec{x}_u(r), \vec{x}_v(r)$ are parallel to $w_1(q), w_2(q)$.



(Thus, the images of $\begin{cases} u \mapsto \vec{x}(u,v_0) \\ v \mapsto \vec{x}(u_0,v) \end{cases}$ are the images of the trajectories of w_1, w_2 passing through $\vec{x}(u_0,v_0)$.)

Proof. Let $\mathcal{U}_0 \subset \mathcal{U}$ denote a neighborhood of p in which there exist first integrals f_1, f_2 of w_1, w_2. Define $\varphi: \mathcal{U}_0 \to \mathbb{R}^2$ by

$\varphi(q) = (f_2(q), f_1(q))$, $q \in U_0$. $\quad$ ($(df_1)_p \neq 0$, $(df_1)_p(w_1) = 0$, and w_2 is not parallel to w_1!)

Then $\quad d\varphi_p \circ w_2 = ((df_2)_p(w_2), (df_1)_p(w_2)) = (0, b)$, $\quad 0 \neq b \in \mathbb{R}$,

likewise $d\varphi_p \circ w_1 = (a, 0)$, $\quad 0 \neq a \in \mathbb{R}$.

$\Rightarrow d\varphi_p$ is nonsingular $\Rightarrow$ φ is locally invertible: $\varphi : V \subset U_0 \to W \subset \mathbb{R}^2$ is a diffeomorphism.

Then $\vec{x} = \varphi^{-1} : W \to V$ does the job. $\square$

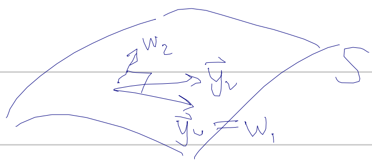
Corollary. Let S be a regular surface, $p \in S$. Then there exists an orthogonal parameterization $\vec{x}(u,v)$ of a neighborhood $V \subset S$ of p. (That is, the coordinate curves $u = \text{const.}$, $v = \text{const.}$ are orthogonal; i.e. $\vec{x}_u \cdot \vec{x}_v \equiv 0$.)

Proof Let $\vec{y} : U \subset \mathbb{R}^2 \to S$ be an arbitrary parameterization near p;

let $w_1 = \vec{y}_u$, $\quad w_2 = \vec{y}_v - a\vec{y}_u$, then $w_1 \cdot w_2 = \vec{y}_u \cdot \vec{y}_v - a|\vec{y}_u|^2 = 0$

provided we choose $\quad a = \dfrac{\vec{y}_u \cdot \vec{y}_v}{|\vec{y}_u|^2}$.

Since w_1, w_2 are lin. indep., the Theorem applies. $\square$



· Another typical use of vector fields is to differentiate functions on regular surfaces.

Let X be a vector field on $U \subset S$, and let $f : U \to \mathbb{R}$ be smooth.

Then $\quad X(f)(q) := \frac{d}{ds} f(\vec{\alpha}(s))\big|_{s=0}$, where $\vec{\alpha} : I \to S$ is the integral curve $\vec{\alpha}(0) = q$, $\vec{\alpha}'(s) = X(\vec{\alpha}(s))$.

Claim: (i) $X(f)$ is smooth in U for all smooth f iff X is smooth.

(ii) $X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)$, $\quad X(fg) = f\,Xg + g\,Xf$

Proof (i) "$\Leftarrow$" Write $X(u,v) = a(u,v)\vec{x}_u + b(u,v)\vec{x}_v$, then

$$X(f)(u_0, v_0) = df_{(u_0,v_0)}(a(u_0,v_0), b(u_0,v_0)),$$

which does depend smoothly on (u_0, v_0).

"$\Rightarrow$" Let f_1, f_2 be first integrals of $\vec{x}_u, \vec{x}_v$. Then

$$X(f_1)(q) = (df_1)_q(a\vec{x}_u + b\vec{x}_v) = b\,\underbrace{(df_1)_q(\vec{x}_v)}_{\text{smooth, nonzero}}$$

$\Rightarrow b$ is smooth.

Similarly, $X(f_2)(q) = a\,\underbrace{(df_2)_q(\vec{x}_u)}_{\text{smooth}, \neq 0} \Rightarrow a$ is smooth. $\square$

III.3 Minimal surfaces.

Def. A regular parameterized surface $\vec{x}$ ($\vec{x}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$, smooth, $d\vec{x}_q$ injective $\forall q \in U$) is minimal if the mean curvature vanishes: $H \equiv 0$.

In what sense are they minimal?

Let $\vec{x}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parameterized surface.

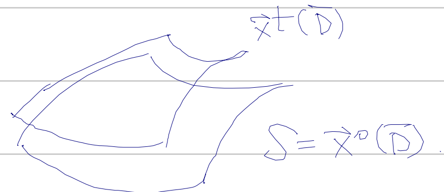
Let $D \subset U$ denote a bounded domain, and let

$h: \overline{D} = D \cup \partial D \to \mathbb{R}$ be a smooth function. Fix the normal vector $N(u,v) = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$.

The normal variation of $\vec{x}(\overline{D})$ determined by h is the map

$\varphi: \overline{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$,

$\varphi(u,v,t) = \vec{x}(u,v) + t\,h(u,v)\,N(u,v)$.



For $t \in (-\varepsilon, \varepsilon)$, let $\vec{x}^t(u,v) = \varphi(u,v,t)$.

Then $\vec{x}^t: U \to \mathbb{R}^3$ is a parameterized regular (for small ε) surface with

$$\vec{x}^t_u = \vec{x}_u + t h N_u + t h_u N$$
$$\vec{x}^t_v = \vec{x}_v + t h N_v + t h_v N.$$

First fund. form:

$$E^t = E + th \cdot 2\langle \vec{x}_u, N_u \rangle + t^2 h^2 \|N_u\|^2 + t^2 h_u^2,$$
$$F^t = F + th \cdot (\langle \vec{x}_u, N_v \rangle + \langle \vec{x}_v, N_u \rangle) + t^2 h^2 \langle N_u, N_v \rangle + t^2 h_u h_v,$$
$$G^t = G + th \cdot 2\langle \vec{x}_v, N_v \rangle + t^2 h^2 \|N_v\|^2 + t^2 h_v^2.$$

Recall: $e = -\langle \vec{x}_u, N_u \rangle$, $f = -\langle \vec{x}_u, N_v \rangle = \langle \vec{x}_v, N_u \rangle$, $g = -\langle \vec{x}_v, N_v \rangle$,

$$H = \frac{1}{2} \frac{Eg - 2fF + Ge}{EG - F^2}$$

$$\Rightarrow E^t G^t - (F^t)^2 = EG - F - 2th(Eg + Ge - 2Ff) + R$$
$$= (EG - F)(1 - 4thH) + R,$$

where $\lim_{t \to 0} |R(t)/t| = 0$.

$\Rightarrow A(t) := \text{area}(\vec{x}^t(\bar{D})) = \iint_D \sqrt{E^t G^t - (F^t)^2}\, du\, dv$

$= \iint_D \sqrt{EG - F^2}\, (1 - 4thH + R)^{1/2}\, du\, dv.$

Rate of change:

$$A'(0) = -\iint_D \sqrt{EG-F^2} \cdot \tfrac{1}{2} \cdot 4hH\, du\,dv = -\iint_D H \cdot 2h \sqrt{EG - F^2}\, du\, dv.$$

This shows:

Prop. Let $\vec{x}: U \to \mathbb{R}^3$ be a regular parameterized surface. Then $\vec{x}$ is minimal if and only if $A'(0) = 0$ for all bounded domains $D \subset U$ and all normal variations of $\vec{x}(\bar{D})$.

· Easy checks for minimality for certain parameterizations:

Def. A parameterization $\vec{x}(u,v)$ is isothermal if $\langle \vec{x}_u, \vec{x}_u \rangle = \langle \vec{x}_v, \vec{x}_v \rangle$ and $\langle \vec{x}_u, \vec{x}_v \rangle = 0$.

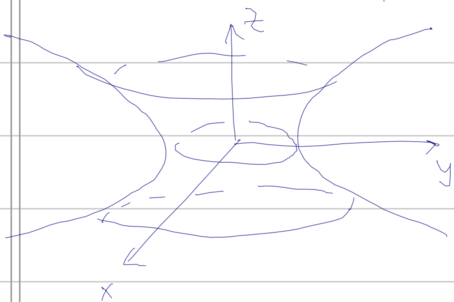
Fact: If $\vec{x}$ is isothermal, then $\vec{x}_{uu} + \vec{x}_{vv} = 2\lambda^2 \vec{H}$, where $\lambda^2 = \langle \vec{x}_u, \vec{x}_u \rangle = \langle \vec{x}_v, \vec{x}_v \rangle$, and $\vec{H} = HN$.

This implies: For $h = H$, $A'(0) = -\iint 2 \|\vec{H}\|^2 \lambda^2\, du\, dv = 0$ iff $H \equiv 0$ iff $\vec{x}_{uu} + \vec{x}_{vv} = 0$.

Example. Catenoid: $\vec{x}(u,v) = (a \cosh v \cos u,\ a \cosh v \sin u,\ av)$, $0 < u < 2\pi$, $v \in \mathbb{R}$.

$\Rightarrow E = G = a^2 \cosh^2 v$, $F = 0 \Rightarrow \vec{x}$ is isothermal.

Since $\vec{x}_{uu} + \vec{x}_{vv} = 0$, $\vec{x}$ is minimal.



$y = a\cosh(z/a)$

03/31/2020

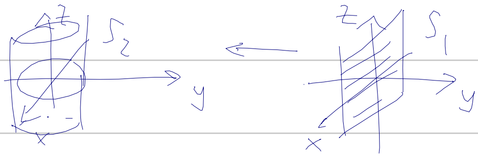
IV Intrinsic geometry of surfaces ("local geometry using the first fundamental form")

IV.1. Isometries, conformal maps.

Def. • A diffeomorphism $\phi: S_1 \to S_2$ between regular surfaces is an isometry if for all $p \in S_1$ and all $\vec{w} \in T_pS_1$, we have $I_p(\vec{w}) = I_{\phi(p)}(d\phi_p(\vec{w}))$.

($\Leftrightarrow \langle \vec{w}_1, \vec{w}_2 \rangle_p = \langle d\phi_p(\vec{w}_1), d\phi_p(\vec{w}_2)\rangle_{\phi(p)}$.) S_1 and S_2 are then said to be isometric.

• A map $\phi: V \subset S_1 \to S_2$ is a local isometry at $p \in V$ if there exist neighborhoods $V_1 \subset V$ and $V_2 \subset S_2$ so that $\phi: V_1 \to V_2$ is an isometry.



Example $S_1 = \{y=0\}$ (xz-plane),

$S_2 = \{x^2+y^2=1\}$ (right cylinder over the unit circle).

Consider the map $\phi: (x,0,z) \in S_1 \longmapsto (\cos x, \sin x, z) \in S_2$.

Take $\vec{w} = (a,0,b) \in T_pS_1$, $p = (x_0, 0, z_0) \in S_1$. Then

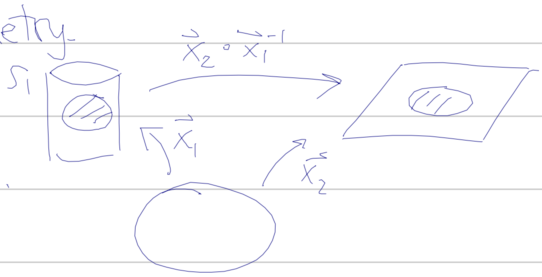
$I_p(\vec{w}) = \|\vec{w}\|^2 = a^2+b^2$, and $d\phi_p(\vec{w}) = a \cdot \begin{pmatrix} -\sin x_0 \\ \cos x_0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so

$$I_{\phi(p)}(d\phi_p(\vec{w})) = \| d\phi_p(\vec{w})\|^2 = a^2+b^2$$

$\Rightarrow \phi$ is a local isometry.

More generally: Assume there are parameterizations $\vec{x}_1: U \to S_1$, $\vec{x}_2: U \to S_2$ such that $E_1 = \|(\vec{x}_1)_u\|^2 = E_2$, $F_1 = \langle (\vec{x}_1)_u, (\vec{x}_1)_v \rangle = F_2$, $G_1 = G_2$ (as functions on U).

Then $\phi: \vec{x}_2 \circ \vec{x}_1^{-1}: \vec{x}_1(U) \to S_2$ is a local isometry.



Proof Let $p \in \vec{x}_1(U)$, $\vec{w} \in T_pS_1$; then

$\vec{w} = (\vec{x}_1 \circ \vec{\alpha})'(0)$ for some curve $\vec{\alpha}(t) = (u(t), v(t))$.

Thus, $\vec{w} = u'(0)(\vec{x}_1)_u(\vec{\alpha}(0)) + v'(0)(\vec{x}_1)_v(\vec{\alpha}(0))$.

By definition, $d\phi_p(\vec{w}) = (\phi \circ (\vec{x}_1 \circ \vec{\alpha}))'(0) = (\vec{x}_2 \circ \vec{\alpha})'(0)$

$= u'(0)(\vec{x}_2)_u(\vec{\alpha}(0)) + v'(0)(\vec{x}_2)_v(\vec{\alpha}(0))$.

Since $E_1(\vec{\alpha}(0)) = \|(\vec{x}_1)_u(\vec{\alpha}(0))\|^2 = \|(\vec{x}_2)_u(\vec{\alpha}(0))\|^2 = E_2(\vec{\alpha}(0))$ etc., we get

$I_p(w) = E_1(u')^2 + 2F_1u'v' + G_1(v')^2 = E_2(u')^2 + 2F_2u'v' + G_2(v')^2 = I_{\phi(p)}(d\phi_p(\vec{w}))$. $\square$

Using I, we can also define:

<u>Def</u> Let S be a connected regular surface. Then the <u>(intrinsic) distance</u> between p and $q \in S$ is

$$d(p,q) := \inf \ell(\vec{\alpha}),$$

where the inf is taken over all piecewise smooth curves $\vec{\alpha}$ starting at p and ending at q.

<u>Prop.</u> Let $\phi: S_1 \to S_2$ be an isometry between connected regular surfaces. Then $d(p,q) = d(\phi(p), \phi(q))$ for all $p, q \in S_1$.

<u>Proof.</u> · Given $p, q \in S_1$, $\varepsilon > 0$, and a piecewise smooth curve $\vec{\alpha}: [0,1] \to S_1$, $\vec{\alpha}(0) = p$, $\vec{\alpha}(1) = q$ with $d(p,q) \geq \ell(\vec{\alpha}) - \varepsilon$, we have

$$\ell(\phi \circ \vec{\alpha}) = \int_0^1 \sqrt{I_{\phi(\vec{\alpha}(t))}((\phi \circ \vec{\alpha})'(t))}\, dt = \int_0^1 \sqrt{I_{\vec{\alpha}(t)}(\vec{\alpha}'(t))}\, dt = \ell(\vec{\alpha}),$$

therefore $d(\phi(p), \phi(q)) \leq \ell(\vec{\alpha}) \leq d(p,q) + \varepsilon$ (since $\phi \circ \vec{\alpha}$ connects $\phi(p)$ and $\phi(q)$). Since $\varepsilon > 0$ was arbitrary, this shows that

$$d(\phi(p), \phi(q)) \leq d(p,q).$$

· The converse inequality follows by applying the same reasoning to ϕ^{-1}. $\square$

<u>Def.</u> · A diffeomorphism $\phi: S_1 \to S_2$ is called a <u>conformal map</u> if

$$\langle d\phi_p(v_1), d\phi_p(v_2) \rangle_{\phi(p)} = \lambda(p)^2 \langle v_1, v_2 \rangle_p \quad \forall\, p \in S_1,\ v_1, v_2 \in T_p S_1,$$

where $\lambda: S_1 \to \mathbb{R}$ is a smooth nonzero function. In this case, S_1 and S_2 are <u>conformal</u>.

· A smooth map $\phi: V \subset S_1 \to S_2$ is a <u>local conformal map</u> at $p \in V$ if there exist neighborhoods $V_1 \subset S_1$ of p and $V_2 \subset S_2$ of $\phi(p)$ s.t. $\phi: V_1 \to V_2$ is a conformal map.

<u>Geometric meaning</u>: ϕ preserves angles between tangent vectors (but not necessarily lengths).

Indeed, if $\vec{\alpha}: (-1,1) \to S_1$ and $\vec{\beta}: (-1,1) \to S_1$ are smooth curves, $\vec{\alpha}(0) = \vec{\beta}(0) = p$, then $\cos\left(\measuredangle\left(d\phi_p(\vec{\alpha}'(0)), d\phi_p(\vec{\beta}'(0))\right)\right) = \dfrac{\langle d\phi(\vec{\alpha}'), d\phi(\vec{\beta}')\rangle}{\|d\phi(\vec{\alpha}')\|\,\|d\phi(\vec{\beta}')\|} = \dfrac{\lambda^2\langle\vec{\alpha}',\vec{\beta}'\rangle}{\lambda\|\vec{\alpha}'\|\cdot\lambda\|\vec{\beta}'\|}$

$= \dfrac{\langle\vec{\alpha}',\vec{\beta}'\rangle}{\|\vec{\alpha}'\|\,\|\vec{\beta}'\|} = \cos(\measuredangle(\vec{\alpha}'(0), \vec{\beta}'(0)))$.

Rmk. This property characterizes local conformal maps.

Theorem. Any two regular surfaces are locally conformal.

Pf. This boils down to finding local coordinates on any given regular surface in which $E(u,v) = \lambda^2(u,v) > 0$, $F(u,v) = 0$, $G(u,v) = \lambda^2(u,v)$.

$\leadsto$ PDE class! $\square$

Example(s) (Mercator projection) Parameterize the unit sphere S^2 by

$$\vec{x}(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \quad (\theta,\varphi)\in U = (0,\pi)\times(0,2\pi).$$

Reparameterize this using

$$u = \log\tan(\tfrac{1}{2}\theta) \in (-\infty,\infty), \quad v = \varphi \in (0,2\pi).$$

Then $\vec{y}(u,v) = (\operatorname{sech} u\cos v, \operatorname{sech} u \sin v, \tanh u)$ $\quad (\operatorname{sech} = \frac{1}{\cosh})$

(Indeed, $\sin\theta = \sin(2\arctan e^u) = 2\sin(\arctan(e^u))\cos(\arctan(e^u))$

$= 2\dfrac{e^u}{\sqrt{1+e^{2u}}}\cdot\dfrac{1}{\sqrt{1+e^{2u}}} = \dfrac{2}{e^u+e^{-u}} = \operatorname{sech} u.$

$\cos\theta = \cos(2\arctan e^u) = 1 - 2\sin^2(2\arctan e^u) = 1 - 2\dfrac{e^{2u}}{1+e^{2u}} = \tanh u.$)

Compute: $E = \|\vec{y}_u\|^2 = \operatorname{sech}^2 u$,

$F = \langle\vec{y}_u, \vec{y}_v\rangle = 0$,

$G = \|\vec{y}_v\|^2 = \operatorname{sech}^2 u$.

$\Rightarrow \vec{y}^{-1}: \vec{x}(U) \subset S^2 \to \mathbb{R}^2$ is a conformal map taking parallels (θ = const.) and meridians (φ = const.) into horizontal and vertical lines on the plane.

Medieval use for navigation: Sail in fixed compass direction = travel along curve $\vec{\alpha}(t) = (\theta(t), \varphi(t))$ which has constant angle β with the meridians φ = const.

$\Leftrightarrow \cos\beta = \dfrac{\langle\vec{\alpha}'(t), \vec{x}_\theta\rangle}{\|\vec{\alpha}'(t)\|\,\|\vec{x}_\theta\|} = \dfrac{\theta'}{\sqrt{(\theta')^2 + \sin^2\theta\,(\varphi')^2}}$

$\Leftrightarrow -(\theta')^2\sin^2\beta + \sin^2\theta\,(\varphi')^2\cos^2\beta = 0$

$$\Longleftrightarrow \quad \frac{\theta'}{\sin\theta} = \pm \frac{\varphi'}{\tan\beta} \quad \Longleftrightarrow \quad \log\tan\left(\frac{\theta}{2}\right) = \pm(\varphi + c)\cot\beta.$$

$\Rightarrow \vec{y}^{-1}(\vec{x}(\vec{\alpha}(t))) \in \{(u,v) : u = \pm(v+c)\cot\beta\}$ is a line!

(These curves are called loxodromes ($\lambda o\xi o s$ = oblique, $\delta\rho \acute{o}\mu o s$ = running)

(2) Stereographic projection $\varphi : S^2 \setminus \{N\} \to \mathbb{R}^2$.

This is a conformal map. (Homework 2, ex. 1)

(3) A diffeomorphism $\varphi : S_1 \to S_2$ is area-preserving if the area of any region $R \subset S_1$ is equal to the area of $\varphi(R)$. Prove that if φ is area-preserving and conformal, then φ is an isometry.

Proof: $\vec{x} : U \subset \mathbb{R}^2 \to S_1$ regular parameterization $\Rightarrow \varphi \circ \vec{x} : U \to S_2$ is a regular parameterization. Consider a region $R = \vec{x}(Q) \subset S_1$, then

$$\text{Area}(\varphi(R)) = \int_Q \|(\varphi\circ\vec{x})_u \times (\varphi\circ\vec{x})_v\| \, du\, dv$$

$$\| \qquad (*)$$

$$\text{Area}(R) = \int_Q \|\vec{x}_u \times \vec{x}_v\| \, du\, dv.$$

- φ is conformal, so $\langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle = \lambda(p)^2 \langle w_1, w_2\rangle$

$$\Rightarrow E_2G_2 - F_2^2 = (\lambda(p)^2)^2(E_1G_1 - F_1^2).$$

Plug into $(*)$: $\displaystyle\int_Q \sqrt{E_2G_2 - F_2^2}\, du\, dv = \int_Q \lambda(u,v)^2 \sqrt{E_1G_1 - F_1^2}\, du\, dv$

$$\| \, (*)$$

$$\int_Q \sqrt{E_1G_1 - F_1^2}\, du\, dv.$$

Since Q is arbitrary, this forces $\lambda(u,v)^2 \equiv 1$. $\square$

(4) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(u,v) = (u(x,y), v(x,y))$, where $x, y : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions satisfying the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \qquad (\Leftrightarrow z = x + iy \mapsto u + iv \text{ is complex differentiable.})$$

Then $\varphi: \mathbb{R}^2 \setminus \mathcal{C} \to \mathbb{R}^2$ is a local conformal map, where $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : u_x = u_y = 0\}$.
(E.g. $\varphi(x,y) = (x^2-y^2, 2xy)$ $\quad (z = x+iy \mapsto x^2-y^2+2ixy = z^2)$)

Pf. Compute $\|d\varphi_p(1,0)\|^2 = \left\|\begin{pmatrix} u_x \\ v_x \end{pmatrix}\right\|^2 = u_x^2 + v_x^2$,

$$\langle d\varphi_p(1,0), d\varphi_p(0,1)\rangle = \left\langle \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \begin{pmatrix} u_y \\ v_y \end{pmatrix}\right\rangle = u_x \cdot (-v_x) + v_x \cdot u_x = 0,$$

$$\|d\varphi_p(0,1)\|^2 = \left\|\begin{pmatrix} u_y \\ v_y \end{pmatrix}\right\|^2 = v_x^2 + u_x^2 \quad \checkmark. \qquad \square$$

04/02/2020

IV.2 Christoffel symbols.

- In study of curves, had Frenet frame $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ and the accompanying Frenet formulae $\vec{t}' = \kappa\vec{n}$, $\vec{n}' = -k\vec{t} - \tau\vec{b}$, $\vec{b}' = \tau\vec{n}$.
- Analogously for surfaces with parameterization $\vec{x}(u,v)$: work with basis $\{\vec{x}_u, \vec{x}_v, N\}$, $N = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$ (not orthonormal!), and consider

$$(1) \quad \vec{x}_{uu} = \Gamma_{11}^1 \vec{x}_u + \Gamma_{11}^2 \vec{x}_v + \langle \vec{x}_{uu}, N\rangle N$$

$$(2) \quad \vec{x}_{uv} = \Gamma_{12}^1 \vec{x}_u + \Gamma_{12}^2 \vec{x}_v + \langle \vec{x}_{uv}, N\rangle N$$

$$(2') \quad \vec{x}_{vu} = \Gamma_{21}^1 \vec{x}_u + \Gamma_{21}^2 \vec{x}_v + \langle \vec{x}_{vu}, N\rangle N$$

$$\left.\begin{matrix} (2) \\ (2') \end{matrix}\right\} \Rightarrow \Gamma_{12}^1 = \Gamma_{21}^1, \quad \Gamma_{12}^2 = \Gamma_{21}^2.$$

$$(3) \quad \vec{x}_{vv} = \Gamma_{22}^1 \vec{x}_u + \Gamma_{22}^2 \vec{x}_v + \langle \vec{x}_{vv}, N\rangle N$$

$$N_u = a_{11}\vec{x}_u + a_{21}\vec{x}_v \qquad (\langle N_u, N\rangle N = 0 \cdot N = \vec{0})$$

$$N_v = a_{12}\vec{x}_u + a_{22}\vec{x}_v. \qquad ((a_{ij}) = dN \text{ in local coordinates})$$

Recall: $\langle \vec{x}_{uu}, N\rangle = -\langle \vec{x}_u, N_u\rangle = -\langle \vec{x}_u, dN_p(\vec{x}_u)\rangle = \mathrm{II}_p(\vec{x}_u) = e$.

The $\Gamma_{ij}^k = \Gamma_{ij}^k(u,v)$ are called <u>Christoffel symbols</u>.

- How to <u>compute</u>? Take inner product of (1) with $\vec{x}_u$

$$\Rightarrow \langle \vec{x}_{uu}, \vec{x}_u\rangle = E\,\Gamma_{11}^1 + F\,\Gamma_{11}^2, \text{ and } \langle \vec{x}_{uu}, \vec{x}_u\rangle = \frac{1}{2}\frac{\partial}{\partial u}\|\vec{x}_u\|^2 = \frac{1}{2}E_u.$$

Inner product of (1) with $\vec{x}_v$: $\langle \vec{x}_{uu}, \vec{x}_v\rangle = F\,\Gamma_{11}^1 + G\,\Gamma_{11}^2$, and

$$\langle \vec{x}_{uu}, \vec{x}_v\rangle = \partial_u\langle \vec{x}_u, \vec{x}_v\rangle - \langle \vec{x}_u, \vec{x}_{uv}\rangle = F_u - \frac{1}{2}\partial_v\|\vec{x}_u\|^2 = F_u - \frac{1}{2}E_v;$$

in matrix form: $\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix} \Rightarrow \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$.

· Similarly, $\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix}$, $\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$.

Note: The Γ_{ij}^k only depend on the 1st fundamental form!

Example Sphere $\vec{x}(u,v) = (\sin u\cos v, \sin u \sin v, \cos u)$, $u\in(0,\pi)$, $v\in(0,2\pi)$.

$E = \|\vec{x}_u\|^2 = 1$, $\quad E_u = 0$, $\quad E_v = 0$,

$F = 0$, $\quad F_u = 0$, $\quad F_v = 0$, $\quad \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \begin{pmatrix} E^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix}$

$G = \|\vec{x}_v\|^2 = \sin^2 u$, $\quad G_u = 2\sin u\cos u$, $\quad G_v = 0$.

$$\begin{cases} \Gamma_{11}^1 = E^{-1}\left(\frac{1}{2}E_u\right) = 0, \\ \Gamma_{11}^2 = G^{-1}\left(F_u - \frac{1}{2}E_v\right) = 0, \end{cases} \qquad \begin{cases} \Gamma_{12}^1 = E^{-1}\frac{1}{2}E_v = 0, \\ \Gamma_{12}^2 = G^{-1}\frac{1}{2}G_u = \cot u, \end{cases}$$

$$\begin{cases} \Gamma_{22}^1 = E^{-1}\left(F_v - \frac{1}{2}G_u\right) = -\sin u\cos u \\ \Gamma_{22}^2 = G^{-1}\frac{1}{2}G_v = 0. \end{cases}$$

· Compatibility equations. Q: Does there exist $f: \mathbb{R}^2 \to \mathbb{R}$, $f_u = u+v$, $f_v = v^2$?

A: No! Necessary condition: $\underset{1}{\underset{\parallel}{(f_u)_v}} = f_{uv} \underset{\neq}{=} f_{vu} = \underset{0}{\underset{\parallel}{(f_v)_u}}$.

For our 3 functions $\vec{x}_u, \vec{x}_v, N$,

we correspondingly get 3 such compatibility conditions.

Namely, $(\vec{x}_{uu})_v - (\vec{x}_{uv})_u = 0$, (1)

$(\vec{x}_{vv})_u - (\vec{x}_{uv})_v = 0$, (2)

$N_{uv} - N_{vu} = 0$. (3)

Relation (1) is $\Gamma_{11}^1\vec{x}_{uv} + \Gamma_{11}^2\vec{x}_{vv} + (\Gamma_{11}^1)_v\vec{x}_u + (\Gamma_{11}^2)_v\vec{x}_v + e_vN + eN_v$

$= \Gamma_{12}^1\vec{x}_{uu} + \Gamma_{12}^2\vec{x}_{uv} + (\Gamma_{12}^1)_u\vec{x}_u + (\Gamma_{12}^2)_v\vec{x}_v + f_uN + fN_u$

$\Leftrightarrow \vec{x}_u\left(\Gamma_{11}^1\Gamma_{12}^1 + \Gamma_{11}^2\Gamma_{22}^1 + (\Gamma_{11}^1)_v + e\,a_{12}\right)$

$+\vec{x}_v\left(\Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + (\Gamma_{11}^2)_v + e a_{22}\right)$

$+N\left(f\Gamma_{11}^1 + g\Gamma_{11}^2 + e_v\right)$

$= \vec{x}_u\left(\Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{12}^1 + (\Gamma_{12}^1)_u + f a_{11}\right)$

$+\vec{x}_v\left(\Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 + (\Gamma_{12}^2)_v + f a_{21}\right) + N\left(e\Gamma_{12}^1 + f\Gamma_{12}^2 + f_u\right)$

The $\vec{x}_u$-component gives

$$(\Gamma_{11}^1)_v + \Gamma_{11}^2\Gamma_{22}^1 + e\,a_{12} = (\Gamma_{12}^1)_u + \Gamma_{12}^2\Gamma_{12}^1 + f a_{11}$$

$$\Longleftrightarrow \quad a_{12}e - a_{11}f = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1.$$

Recall: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} e & f \\ f & g \end{pmatrix}$

so $a_{12}e - a_{11}f = -(EG-F^2)^{-1}\left(e\cdot(Gf-Fg) - f\cdot(Ge-Ff)\right)$

$\quad = F\,(EG-F^2)^{-1}(eg-f^2) = FK.$

$\Rightarrow$ Prop.

$$\begin{cases} EK = (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)^2, & (*) \\ FK = (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1, \\ GK = (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - (\Gamma_{12}^1)^2 - \Gamma_{12}^2\Gamma_{22}^1. \end{cases}$$

These are called Gauss equations.

Theorem: The Gauss curvature K can be computed only using the 1^{st} fundamental form! Thus, it is invariant under local isometries. ("Theorema Egregium.")

Proof: Expression $(*)$. $\square$

$\Rightarrow$ K can be computed without reference to the space containing the surface (as soon as the 1^{st} fund. form is given).

- Codazzi equations (or Mainardi-Codazzi equations): N-component of $(\vec{x}_{uu})_v = (\vec{x}_{uv})_u$:

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2.$$

Likewise: $f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2$ (from N-component of $(\vec{x}_{uv})_v = (\vec{x}_{vv})_u$).

The Gauss and Codazzi equations are the only compatibility equations:

Theorem (Bonnet). Given smooth $E, F, G, e, f, g: V \subset \mathbb{R}^2 \to \mathbb{R}$ such that $E > 0$, $G > 0$, $EG - F^2 > 0$, and such that the Gauss and Codazzi equations are satisfied formally (i.e. replacing Γ_{ij}^k by their expressions in terms of E, F, G). Then for all $q \in V$, there exists a neighborhood $U \subset V$ of q and a diffeomorphism $\vec{x}: U \to \vec{x}(U) \subset \mathbb{R}^3$ such that the regular surface $\vec{x}(U)$ has E, F, G and e, f, g as the coefficients of its 1^{st} & 2^{nd} fundamental form, respectively.

- If U is connected, and if $\vec{y}: U \to \vec{y}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exists a rigid motion ρ of $\mathbb{R}^3$ (translation & rotation) s.t. $\vec{y} = \rho \circ \vec{x}$.

Pf. Existence: PDE theorem

Uniqueness: similar to uniqueness proof in fundamental theorem of curves (how does $\vec{x}_u, \vec{x}_v, N$ evolve along some curve?). $\square$

04/07/2020

- **Theorem** (Liebmann) Let S be a compact, connected, regular surface with constant Gauss curvature K. Then S is a sphere.

The proof proceeds in several steps.

Step 1 Suppose $S \subset \mathbb{R}^3$ is a compact regular surface. Then $\exists\, p \in S$ s.t. $K(p) > 0$.

Proof Let $p \in S$ be the point where the function $f(q) = \|q\|^2_{\mathbb{R}^3}$, $q \in S$, attains its maximum. Consider $\vec{\alpha}: (-\varepsilon, \varepsilon) \to S$, parameterized by arc-length, with $\vec{\alpha}(0) = p$. The normal curvature of $\vec{\alpha}$ at p is $\geq \frac{1}{\|f(p)\|}$. (Homework 5, ex 4.) $\Rightarrow$ All normal curvatures at p are $\geq \frac{1}{\|f(p)\|} \Rightarrow K(p) \geq \frac{1}{\|f(p)\|^2} > 0$. $\square$

Step 2: Suppose every point of S is umbilic, i.e. $dN_p(w) = \lambda(p) w \;\; \forall p \in S, w \in T_pS$. Then the conclusion holds.

Proof We already proved that S must be contained in a plane or sphere.

Since $K>0$ on S, S must be contained in a sphere Σ. Since S is a regular surface, S is open in Σ; since S is compact, S is also closed in Σ. Since Σ is connected, we must have $S=\Sigma$.

(the larger one of the 2 principal curvatures)

Step 3 Suppose $\exists\, p\in S$ which is not umbilic. Then $k_1(p)\neq\sqrt{K}$, therefore $k_2(p)<\sqrt{K}<k_1(p)$. The continuous function k_1 attains a maximum at some point $q\in S$. Since $k_2=\frac{K}{k_1}$, k_2 attains a minimum at q, and $k_2(q)<k_1(q)$. The following lemma shows that this contradicts $K>0$, hence we are done. $\square$

Lemma (Hilbert) Suppose $k_1(p)>k_2(p)$, and p is a local $\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}$ of $\begin{matrix}k_1\\ k_2\end{matrix}$.
Then $K(p)\le 0$.

Proof. There exists a neighborhood V of p s.t. $k_1>k_2$ on U; therefore, k_1, k_2 are smooth on V, and so are the principal directions X_1, X_2.
$\Rightarrow$ The vector fields X_1, X_2 on V are smooth, and $X_1(p), X_2(p)$ are linearly independent.
$\Rightarrow \exists$ parameterization $\vec{x}: U\to S$ of a neighborhood of p s.t.

$$\vec{x}_u = aX_1,\quad \vec{x}_v = bX_2 \quad \text{for some smooth } a,b>0.$$

But then $F=\langle \vec{x}_u, \vec{x}_v\rangle = ab\langle X_1, X_2\rangle = 0$

and $f=\langle \vec{x}_u, N_v\rangle = \langle \vec{x}_u, dN(\vec{x}_v)\rangle = \langle \vec{x}_u, -k_1\vec{x}_v\rangle = 0$;

hence $k_1=\frac{e}{E}$, $k_2=\frac{g}{G}$ (might need to exchange u,v to make this so).

- Codazzi equations become

$$e_v = \frac{E_v}{2}(k_1+k_2)$$

$$g_u = \frac{G_u}{2}(k_1+k_2).$$

(use $k_1>k_2$!)

- Differentiate $e=Ek_1$ in v: $\frac{E_v}{2}(k_1+k_2) = E_v k_1 + E(k_1)_v$

$$\Leftrightarrow E(k_1)_v = \frac{E_v}{2}(-k_1+k_2) \Leftrightarrow E_v = -\frac{2E(k_1)_v}{k_1-k_2}.$$

Differentiate $g=Gk_2$ in u: $G(k_2)_u = \frac{G_u}{2}(k_1-k_2) \Leftrightarrow G_u = \frac{2G(k_2)_u}{k_1-k_2}.$

- At p, this implies $E_v=0$, $G_u=0$.

Since $F=0$, the intrinsic formula for K reads (exercise!)

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

$\Rightarrow$ $-2EG\cdot K = E_{vv} + G_{uu} + (\cdots)E_v + (\cdots)G_u = E_{vv} + G_{uu}$ at p.

· But $E_{vv}(p) = -\frac{2E}{k_1-k_2}(k_1)_{vv}$, $G_{uu}(p) = \frac{2G}{k_1-k_2}(k_2)_{uu}$.

Therefore,

$$-2EG\cdot K = \underbrace{\frac{2}{k_1-k_2}}_{\geq 0}\Big(-\overset{\geq 0}{\widetilde{E}}\underbrace{(k_1)_{vv}}_{\leq 0} + \overset{\geq 0}{\widetilde{G}}\underbrace{(k_2)_{uu}}_{\geq 0}\Big) \geq 0 \quad \text{at } p,$$

$\Rightarrow K(p) \leq 0$. $\square$

IV.3 Parallel transport, geodesics

How to $\begin{cases}\text{compare tangent vectors at different points on a regular surface } S? \\ \text{differentiate vector fields?}\end{cases}$

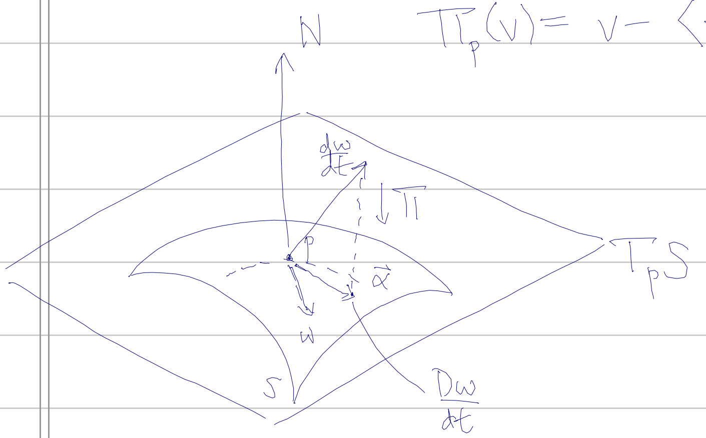
Def. Let w be a smooth vector field on $U \subset S \subset \mathbb{R}^3$. Let $p \in U$ and $\vec{y} \in T_pS$.
Let $\vec{\alpha}: (-1,1) \to S$ be a regular parameterized curve with $\vec{\alpha}(0) = p$, $\vec{\alpha}'(0) = \vec{y}$.
Let $w(t) := w(\vec{\alpha}(t))$, so $w: (-1,1) \to \mathbb{R}^3$ is the restriction of w to $\vec{\alpha}$.
Then the covariant derivative of w at p in the direction $\vec{y}$ is defined by

$$\frac{Dw}{dt}(0) = D_y w(p) := \Pi_p\left(\frac{dw}{dt}(0)\right) \in T_pS$$

where $\Pi_p: \mathbb{R}^3 \to T_pS$ is the orthogonal projection, defined by

$$\Pi_p(v) = v - \langle v, N(p)\rangle N(p).$$

normal vector to S at p



Claim: $D_y w$ only depends on w, y, and 1^{st} fund. form on S.

Proof Write $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$ and $w(t) = a(u(t), v(t))\,\vec{x}_u + b(u(t), v(t))\,\vec{x}_v$

$$= a(t)\,\vec{x}_u(u(t), v(t)) + b(t)\,\vec{x}_v(u(t), v(t)).$$

Then $\frac{dw}{dt} = a\cdot(\vec{x}_{uu} u' + \vec{x}_{uv} v') + b(\vec{x}_{uv} u' + \vec{x}_{vv} v') + a'\vec{x}_u + b'\vec{x}_v$

$$\Rightarrow \frac{Dw}{dt} = \left(a' + a(\Gamma^1_{11} u' + \Gamma^1_{12} v') + b(\Gamma^1_{12} u' + \Gamma^1_{22} v')\right)\vec{x}_u \qquad (*)$$
$$+ \left(b' + a(\Gamma^2_{11} u' + \Gamma^2_{12} v') + b(\Gamma^2_{12} u' + \Gamma^2_{22} v')\right)\vec{x}_v. \qquad \square$$

(using $\vec{x}(u,v) = \vec{x}_0 + u\vec{w}_1 + v\vec{w}_2$, w/ $\vec{w}_1, \vec{w}_2$ orthonormal)

Rmk. If S is a plane, parameterized so that $E=1$, $F=0$, $G=0$, then $\Gamma^k_{ij} = 0$ $\forall i,j,k = 1,2$, and thus $(*)$ implies $\frac{Dw}{dt} = a'\vec{x}_u + b'\vec{x}_v = \frac{dw}{dt}$.

$\Rightarrow \frac{D}{dt}$ generalizes the usual derivative of vector fields on $\mathbb{R}^2$ (or planes in $\mathbb{R}^3$) to curved surfaces.

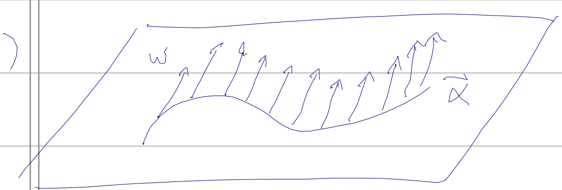
- For a vector field w along a smooth parameterized curve $\vec{\alpha}: I \to S$ (i.e. $w(t) \in T_{\vec{\alpha}(t)} S$), we can define $\frac{Dw}{dt}$ in the same way.

Ex. $w(t) = \vec{\alpha}'(t) \Rightarrow \frac{Dw}{dt} =$ tangential component of $\vec{\alpha}''$ = "intrinsic acceleration" of $\vec{\alpha}$.

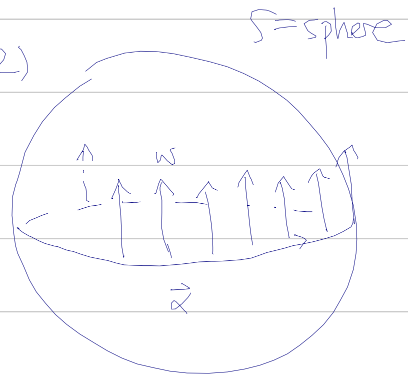
Def. A vector field w along a parameterized curve $\vec{\alpha}: I \to S$ is parallel if $\frac{Dw}{dt} = 0$ $\forall t \in I$.

Ex.

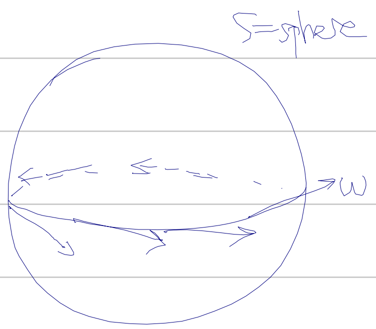
(1)



(2)



(3)



Examples (2) & (3) motivate: (i.e. v is parallel, and w is parallel!)

(*) Prop. Let w and v be parallel vector fields along $\alpha: I \to S$. Then $\langle v(t), w(t)\rangle$ is constant. Thus, $\|v(t)\|$, $\|w(t)\|$, and $\measuredangle(v(t), w(t))$ are constant.

Proof $\frac{Dw}{dt}=0$ means (by definition) that $\frac{dw}{dt}$ is normal to S. Therefore, $\langle \frac{dw}{dt}(t), v(t)\rangle = 0$. Likewise $\langle w, v'\rangle = 0$.

$\Rightarrow \frac{d}{dt}\langle v(t), w(t)\rangle = \langle v', w\rangle + \langle v, w'\rangle = 0.$ $\square$

Prop. Let $\vec{\alpha}: I \to S$ be a smooth parameterized curve in S and let $w_0 \in T_{\alpha(t_0)} S$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\vec{\alpha}(t)$ with $w(t_0) = w_0$.

Proof In local coordinates: $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$,

want to find $w(t) = a(t)\vec{x}_u(u(t), v(t)) + b(t)\vec{x}_v(u(t), v(t))$, $w(t_0) = w_0$,

s.t. $0 = \frac{Dw}{dt} = \left(a' + a(\Gamma_{11}^1 u' + \Gamma_{12}^1 v') + b(\Gamma_{12}^1 u' + \Gamma_{22}^1 v')\right)\vec{x}_u$
$+ \left(b' + a(\Gamma_{11}^2 u' + \Gamma_{12}^2 v') + b(\Gamma_{12}^2 u' + \Gamma_{22}^2 v')\right)\vec{x}_v$

$\Leftrightarrow$ both parentheses are $=0$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t)\end{pmatrix} = \begin{pmatrix} A & C \\ B & D\end{pmatrix}\begin{pmatrix} a(t) \\ b(t)\end{pmatrix}, \\ \begin{pmatrix} a(t_0) \\ b(t_0)\end{pmatrix} = (\text{components of } w_0)\end{cases}$$ where A, B, C, D are functions of E, F, G, u, v.

Linear ODE! Solution exists (and is unique) for all $t \in I$. $\square$

04/09/2020

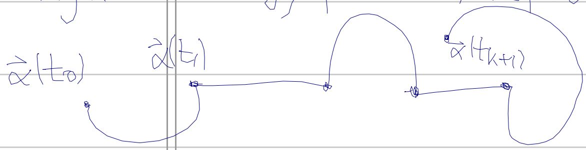
Def. Let $\vec{\alpha}: I \to S$ be a parameterized curve and $w_0 \in T_{\vec{\alpha}(t_0)} S$, $t_0 \in I$. Let w be the parallel vector field along $\vec{\alpha}$ w/ $w(t_0) = w_0$. Then $w(t_1)$ (for $t_1 \in I$) is called the <u>parallel transport</u> of w_0 along $\vec{\alpha}$ to $\vec{\alpha}(t_1)$.

Rmk. (1) By Proposition (*) above, parallel transport as in the Def. is a <u>linear isometry</u>
$T_{\vec{\alpha}(t_0)} S \to T_{\vec{\alpha}(t_1)} S$.

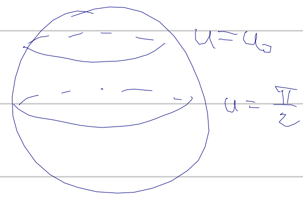
(2) Parallel transport only depends on the curve joining the start & end points:
let $\vec{\alpha}, \vec{\beta}$ be two regular parameterizations, $\vec{\beta}(s) = \vec{\alpha}(t(s))$.
Then $\frac{d}{dt} w(\vec{\alpha}(t)) = dw \cdot \vec{\alpha}'(t)$ and $\frac{d}{ds} w(\vec{\beta}(s)) = dw \cdot \vec{\beta}'(s) = \left(dw \cdot \vec{\alpha}'(t(s))\right) \cdot t'(s)$;

therefore, $\frac{d}{ds} w(\vec\beta(s))$ is $\perp T_{\vec\beta(s)} S$ iff $\frac{d}{dt} w(\vec\alpha(t))\Big|_{t=t(s)}$ is $\perp T_p S$.

One can define parallel transport along parameterized <u>piecewise</u> <u>regular curves</u> $\alpha : [0, \ell] \to S$: that is, $\exists\, 0 = t_0 < t_1 < \cdots < t_{k+1} = \ell$ s.t. $\vec\alpha|_{[t_i, t_{i+1}]}$ is regular. Namely, parallel transport $w_0 \in T_{\vec\alpha(t_0)} S$ along $\vec\alpha|_{[t_0, t_1]}$ to $T_{\vec\alpha(t_1)} S$ etc. until you reach $T_{\vec\alpha(t_{k+1})} S$.



<u>Example</u>: (1) S^2, parameterization $\vec x(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$

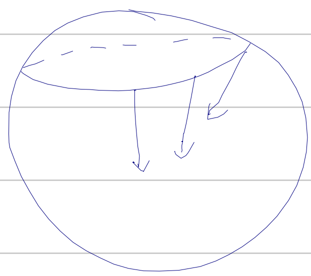


Fix a parallel $u = u_0 \in (0, \pi)$, parameterized by
$$\vec\alpha(t) = \vec x(u(t), v(t)), \quad \begin{cases} u(t) = u_0 \\ v(t) = t. \end{cases}$$
Compute parallel transport along $\vec\alpha$!

<u>Well</u>, recall Christoffel symbols: $\Gamma^1_{11} = 0$, $\Gamma^1_{12} = 0$, $\Gamma^1_{22} = -\cos u \sin u$,
$\Gamma^2_{11} = 0$, $\Gamma^2_{12} = \cot u$, $\Gamma^2_{22} = 0$.

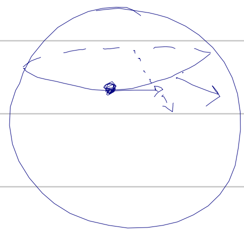
If $w(t) = a(t)\vec x_u + b(t)\vec x_v$, the equation $\frac{Dw}{dt} = 0$ reads
$$\begin{cases} a' - (\cos u_0 \sin u_0)\, b = 0, \\ b' + (\cot u_0) a \quad = 0. \end{cases}$$

Take another derivative $\Rightarrow$ $a'' = (\cos u_0 \sin u_0)\, b' = -(\cos^2 u_0)\, a$
$b'' = -(\cot u_0)\, a' = -(\cos^2 u_0)\, b$.



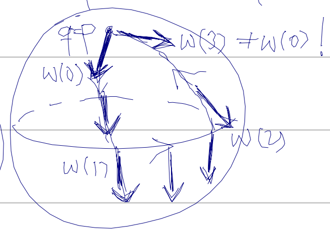
For $\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, get $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \cos(t \cos(u_0)) \\ -\frac{1}{\sin u_0} \sin(t \cos(u_0)) \end{pmatrix}$

For $\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, get $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \sin(u_0) \sin(t \cos(u_0)) \\ \cos(t \cos(u_0)) \end{pmatrix}$

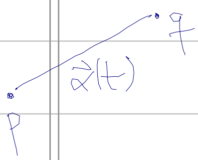


(2) Parallel transport from $p \in S^2$ to $q \in S^2$ depends on the chosen path:

(<u>Later</u>: this is indicative of <u>curvature</u> of S.)



• Geodesics. In the plane, for any 2 points $p, q \in \mathbb{R}^2$, the curve connecting p and q of minimal length l is the straight line $\vec{\alpha}(t)$, $t \in [0, l]$ (chosen to be parameterized by arc-length). This $\vec{\alpha}$ is characterized by $\vec{\alpha}'' = 0$ (and $\|\vec{\alpha}'(0)\| = 1$).



(That is, $\vec{\alpha}$ has vanishing curvature, i.e. does not make any "detours".)

Def. • A nonconstant, smooth parameterized curve $\vec{\gamma}: I \to S$ is called a geodesic if $\vec{\gamma}'(t)$ is parallel along $\vec{\gamma}(t)$, i.e. $\frac{D\vec{\gamma}'(t)}{dt} = 0$.

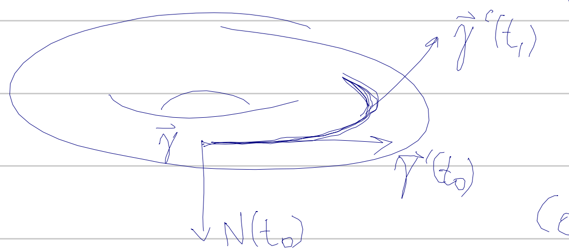
• A regular, connected curve $C \subset S$ is a geodesic if for every $p \in C$, the arc length parameterization $\vec{\gamma}(t)$ of a neighborhood of p is a geodesic (in the above sense).

Rmk. • $\vec{\gamma}$ geodesic $\Rightarrow \|\vec{\gamma}'\| = \text{const} \neq 0$.

• $\frac{D\vec{\gamma}'}{dt} = 0$ does not imply $\frac{d\vec{\gamma}'}{dt} = 0$ unlike in the planar case.

Rather, $\frac{d\vec{\gamma}'}{dt} = k(t) N(t)$. (#)

(= normal curvature of $\vec{\gamma}'$)



Conversely, (#) is an extrinsic characterization of geodesics.

• Parameterized geodesics may have self-intersections.

Prop. Given a point $p \in S$ and a vector $w \in T_pS$, $w \neq 0$, there exists $\varepsilon > 0$ and a unique parameterized geodesic $\vec{\gamma}: (-\varepsilon, \varepsilon) \to S$ with $\gamma(0) = p$, $\gamma'(0) = w$.

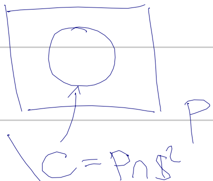
Proof Write $\vec{\gamma}(t) = \vec{x}(u(t), v(t))$, then $\vec{\gamma}'(t) = u'\vec{x}_u(u,v) + v'\vec{x}_v(u,v)$ and

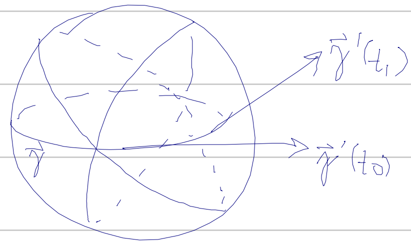
$$\frac{D\vec{\gamma}'}{dt} = \vec{x}_u\left(u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1\right)$$
$$+ \vec{x}_v\left(v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2\right) = 0$$

is equivalent to $\frac{d^2}{dt^2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \text{expressions involving} \\ u', v', \Gamma_{ij}^k(u,v) \end{pmatrix}$ — a 2^{nd} order (nonlinear!) ODE, w/ unique solution given initial point $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$ and initial velocity $\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$. □

Examples (1) Sphere S^2: every great circle (intersection of S^2 with a 2-dim. subspace P of $\mathbb{R}^3$) is a geodesic.

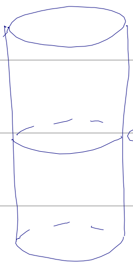
Indeed, $P \cap S^2 =$





The normals of C $(= \vec{\gamma}'')$ coincide with the normal vector to S^2.

(2) Right cylinder: S



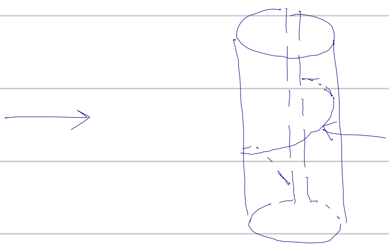
From the above argument, all curves $\vec{\alpha}(t) = (\cos t, \sin t, v_0)$ are geodesics $\forall v_0$.

Also, $\vec{\beta}(t) = (\cos u_0, \sin u_0, t)$ is a geodesic $\forall u_0$.

What else? Note: definition of geodesics only uses the 1st fundamental form!

(Therefore: If $C \subset S_1$ is a geodesic, $\phi: S_1 \to S_2$ local isometry $\Rightarrow \phi(C) \subset S_2$ is a geodesic, too.)

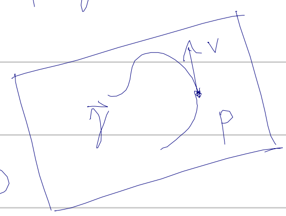
Consider



$\Rightarrow$ All geodesics on S are helices.

(3) Claim. If all geodesics of a connected surface $S \subset \mathbb{R}^3$ are planar curves (i.e. contained in $P \cap S$ for some plane $P \subset \mathbb{R}^3$), then S is contained in a plane or a sphere.

Proof. Let $p \in S$ and $v \in T_pS$, $\|v\| = 1$. The geodesic $\vec{\gamma}: (-\varepsilon, \varepsilon) \to S$ with $\vec{\gamma}(0) = p$, $\vec{\gamma}'(0) = v$ lies in a plane $P \subset \mathbb{R}^3$. Since $\vec{\gamma}$ is a geodesic,



$\frac{D\vec{\gamma}'}{dt} = 0$, so $\vec{\gamma}''(t) = \underbrace{k_n(t)}_{\text{normal curvature}} N(\vec{\gamma}(t))$

$= \mathrm{II}(\vec{\gamma}'(t))$

If $k_n(0)\neq 0$, then, viewing $\vec{\gamma}$ as a planar curve in $\mathbb{R}^3$, we have
$\vec{\gamma}'(t)=\vec{t}(t)$, $\vec{t}'=k\vec{n}$, so $\vec{n}(t)=N(\vec{\gamma}(t))$, $k(t)=k_n(t)$,
and $\vec{n}' \underset{\uparrow}{=} -k\vec{t}$, so $dN_p(v)=dN_p(\vec{\gamma}'(0))=\vec{n}'(0)=-k(0)\vec{t}(0)$
(torsion of $\vec{\gamma}$ is $=0$)

$$= -k_n(0)\vec{\gamma}'(0)$$
$$= -k_n(0)v.$$

We have shown: $\|v\|=1$, $\mathrm{II}(v)\neq 0 \Rightarrow dN_p(v) = -\mathrm{II}(v)\,v$. $(\ast)$

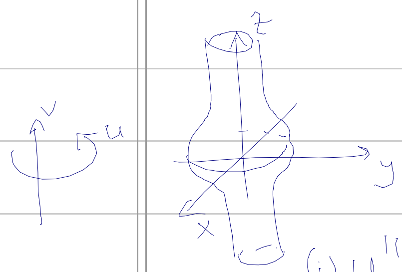
- If $\mathrm{II}(v)=0\ \forall v$, then $dN_p(v)=0\ \forall v$.
- If $\mathrm{II}(v)\neq 0$ for some v, then $\mathrm{II}(v)\neq 0$ for "most" v, and $(\ast)$ follows by continuity.

So: $dN_p(v) = -\mathrm{II}(v)v\ \ \forall v\in T_pS$, $\|v\|=1$. Every v is an eigenvector!

- Check: This implies that $dN_p(v)=\lambda(p)v$, $\lambda(p)\in\mathbb{R}$ independent of v.
  (Use that dN_p is a linear map.)
- Therefore, S is umbilic $\Rightarrow$ $S=$ plane or sphere. $\square$

04/14/2020

(4) Geodesics on surfaces of revolution. Parameterization $\vec{x}(u,v)=(f(v)\cos u, f(v)\sin u, g(v))$



Christoffel symbols: $\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{22}^1=0$,

$$\Gamma_{11}^2 = -\frac{ff'}{(f')^2+(g')^2},\quad \Gamma_{12}^1=\frac{f'}{f},\quad \Gamma_{22}^2=\frac{f'f''+g'g''}{(f')^2+(g')^2}.$$

Geodesic equation for $\vec{x}(u(t),v(t))=\vec{\gamma}(t)$:

$$\text{(i)}\ u''+\Gamma_{11}^1(u')^2+2\Gamma_{12}^1u'v'+\Gamma_{22}^1(v')^2$$
$$=u''+\frac{2f'}{f}u'v'=0,$$

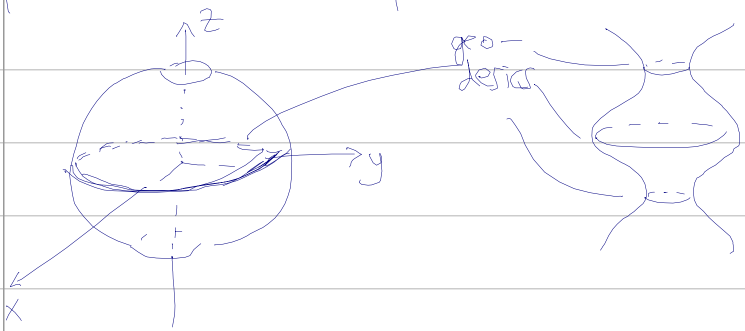
$$\text{(ii)}\ v''+\Gamma_{11}^2(u')^2+2\Gamma_{12}^2u'v'+\Gamma_{22}^2(v')^2$$
$$=v''-\frac{ff'}{(f')^2+(g')^2}(u')^2+\frac{f'f''+g'g''}{(f')^2+(g')^2}(v')^2$$

– Meridians $u=\text{const.}$, $v=v(t)$, parameterized by arclength, are geodesics:
$u'=u''=0 \Rightarrow$ (i) ✓, and to check (ii), we differentiate the relation
$((f')^2+(g')^2)(v')^2=1$ (arc length parameterization!) in t,
giving $2v'(f'f''+g'g'')(v')^2+2((f')^2+(g')^2)v'v''=0$ and thus (ii)
after division by v' ($\neq 0$!).

– Which parallels are geodesics? $v = v_0 = \text{const}$, $u = u(s)$ (arclength parameterization) satisfies (i) if $u' = \text{const} \neq 0$, and (ii) then gives

$$\frac{ff'}{(f')^2+(g')^2}(u')^2 = 0 \quad \Rightarrow \quad f'(v_0) = 0.$$

Thus, the generating curve $v \mapsto (f(v), 0, g(v))$ must have tangent at v_0 parallel to the axis of rotation.



– Clairaut's relation. Rewrite (i) as $(f^2u')' = f^2u'' + 2ff'u'v' = 0$, hence $f^2u' = c = \text{const}$. (The closer you are to axis of rotation, the faster you go around! Unless $u' \equiv 0$ of course.)

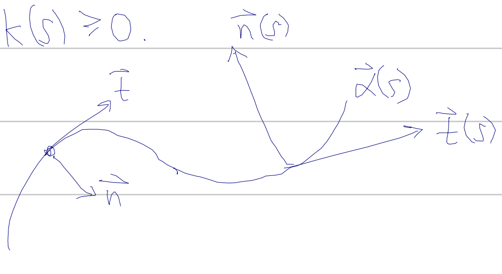
On the other hand, the angle $\theta \in [0, \frac{\pi}{2}]$ between $\vec{\gamma}$ (arc length parameterized) and a parallel is $\cos\theta = \frac{|\langle \vec{x}_u, \vec{x}_u u' + \vec{x}_v v' \rangle|}{|\vec{x}_u|} = |fu'|$.

Therefore, writing $r(t) = f(v(t))$ we have

$$\boxed{r(s)\cos\theta(s) = |c| = \text{const.}} \quad \text{(Clairaut)}.$$

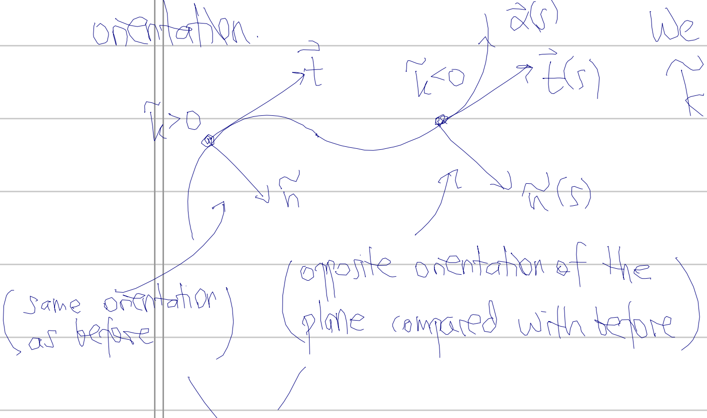
• Geodesic curvature. On oriented surfaces $S \subset \mathbb{R}^3$, one can define the intrinsic curvature of curves on S as a number, rather than merely a vector. (Thus, one can say whether a curve is positively or negatively curved.)

– First, consider a planar curve $\vec{\alpha}: I \to \mathbb{R}^2$, parameterized by arc length, then we defined $k(s) := \|\vec{t}'(s)\|$ $(\vec{t}(s) = \vec{\alpha}'(s))$, so $\vec{t}'(s) = k(s)\vec{n}(s)$, and $k(s) \geq 0$.



Note: The basis $\vec{t}(s), \vec{n}(s)$ defines an orientation of the plane; typically depends on value of s (as on the left).

If instead we fix from the outset an orientation of the plane, we can always choose a normal vector $\tilde{n}$ to $\vec{\alpha}$ uniquely s.t. $\vec{t}, \tilde{n}$ matches the chosen orientation. We can then define the numerical curvature $\tilde{k}$ so that $\vec{t}' = \tilde{k}\tilde{n}$.

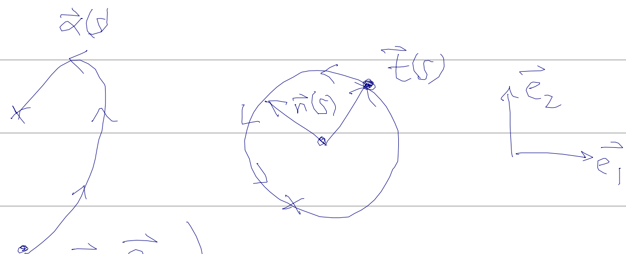


$\vec{t}(s)\ \tilde{n}(s)$ define a fixed orientation of the plane!
(Namely, same orientation as $\vec{e}_2, \vec{e}_1$.)

Example Let $\vec{\alpha} : [0, \ell] \to \mathbb{R}^2$ be an arclength parameterized curve, $\theta(s) =$ (angle between x-axis and $\vec{t}(s) = \vec{\alpha}'(s)$) $\in \mathbb{R}$ (chosen s.t. $\theta(s)$ is smooth in s),

so $\vec{t}(s) = (\cos\theta(s), \sin\theta(s))$.

Then $\vec{t}'(s) = \theta'(s)\underbrace{(-\sin\theta(s), \cos\theta(s))}_{=:\tilde{n}(s)}$



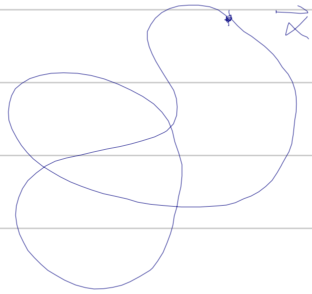
(So $\vec{t}, \tilde{n}$ are oriented with respect to $\vec{e}_1, \vec{e}_2$.)

The signed curvature is then $k(s) = \theta'(s)$.

If $\vec{\alpha}$ is a closed curve, then $\vec{t}(\ell) = \vec{t}(0)$,

so $\theta(\ell) - \theta(0) = 2\pi I$, $I \in \mathbb{Z}$ rotation/winding number of $\vec{\alpha}$.

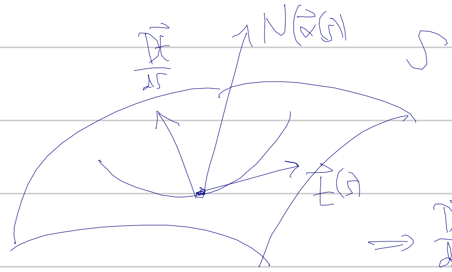
$$\theta(\ell) - \theta(0) = \int_0^\ell k(s)\,ds$$



– Let us generalize these ideas to surfaces. If $\vec{\alpha} : I \to S$ is parameterized by arc length, $\vec{t}(s) = \vec{\alpha}'(s)$, then

$$0 = \frac{d}{ds}\langle \vec{t}(s), \vec{t}(s)\rangle = 2\left\langle \frac{D\vec{t}}{ds}, \vec{t}\right\rangle \Rightarrow \frac{D\vec{t}}{ds} \in T_{\vec{\alpha}(s)}S \text{ is orthogonal}$$

to $\vec{t}(s)$. Now, an orientation of S is the same as the choice of a smooth unit normal field N.



The Darboux frame is $\{\vec{t}, N\times\vec{t}, N\}$;
$\vec{t}, N\times\vec{t}$ is an orthonormal basis of $T_{\vec{\alpha}(s)}S$.

$$\Rightarrow \frac{D\vec{t}}{ds}(s) = \underbrace{\left\langle \frac{D\vec{t}}{ds}(s), N\times\vec{t}(s)\right\rangle}_{=:\text{ signed curvature!}} (N\times\vec{t}(s))$$

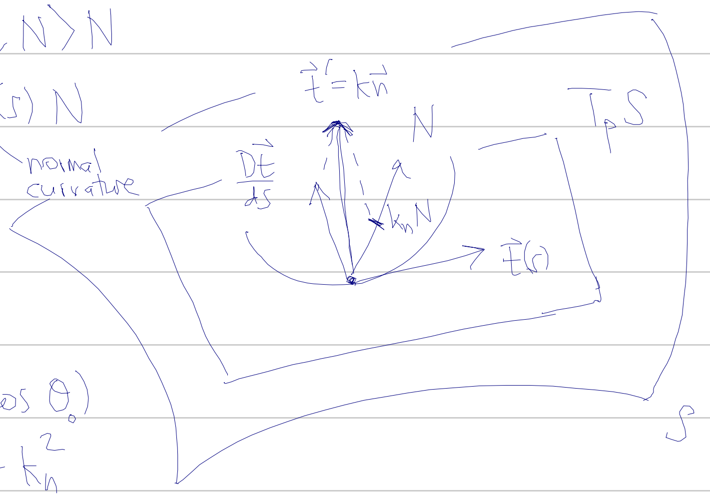
⇝ **Def.** Let C be a regular curve contained in an oriented surface with smooth unit normal vector field N. Let $\vec{\alpha}:(-\varepsilon,\varepsilon)\to S$ be an arclength parameterization of a neighborhood of $p\in C$. Then

$$k_g = \left\langle \frac{D\vec{\alpha}'(s)}{ds}, N(\vec{\alpha}(s))\times\vec{\alpha}'(s)\right\rangle$$

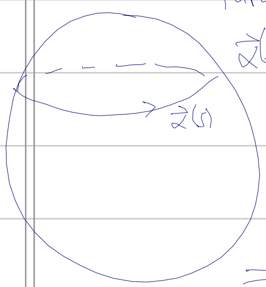
is called the geodesic curvature of C at p.

Rmk.
- $\vec{\alpha}$ is a parameterized geodesic, i.e. $\frac{D\vec{\alpha}'}{ds}=0$, if and only if $k_g=0$.
- Considering $\vec{\alpha}: I\to S\subset\mathbb{R}^3$, we have

$$\vec{t}'(s) = k(s)\vec{n}(s) = \frac{D\vec{\alpha}'(s)}{ds} + \langle \vec{t}'(s), N\rangle N$$
$$= k_g(s)(N\times\vec{t}(s)) + \underbrace{k_n(s)}_{\text{normal curvature}} N$$
$$\Rightarrow \quad k(s)^2 = k_g(s)^2 + k_n(s)^2.$$



Ex. (1) $\mathbb{S}^2$



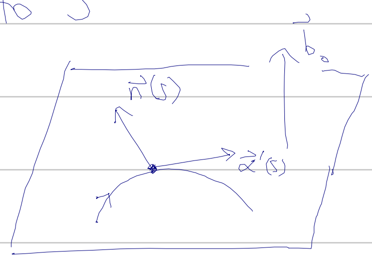
Parallel at $\theta=\theta_0$:
$$\vec{\alpha}(s) = \left(\cos\left(\frac{s}{\sin\theta_0}\right)\sin\theta_0, \ \sin\left(\frac{s}{\sin\theta_0}\right)\sin\theta_0, \ \cos\theta_0\right)$$
$$\Rightarrow k^2 = \frac{1}{\sin^2\theta_0} = k_g^2 + \underset{1}{\underset{\parallel}{k_n^2}}$$
$$\Rightarrow k_g^2 = \cot^2\theta_0,$$
so $k_g = \pm\cot\theta_0$ (sign depending on orientation of $\mathbb{S}^2$).

(2) S = plane, $\vec{\alpha}$ = arclength-p. curve in S, $\vec{b}_0$ = fixed unit normal to S

$\Rightarrow \vec{n}(s) := \vec{b}_0\times\vec{\alpha}'(s)\in T_{\vec{\alpha}(s)}S$ gives an oriented normal,

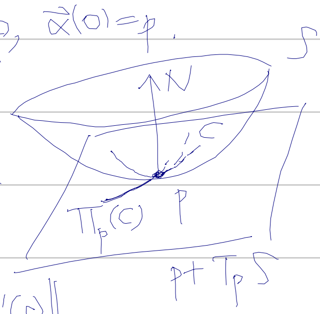
and $k(s) := \langle\vec{t}'(s), \vec{n}(s)\rangle = \langle\vec{t}'(s), \vec{b}_0\times\vec{t}(s)\rangle$

= signed curvature.



(3) Claim The geodesic curvature at p of a curve C in S is equal to the curvature at p of the projection of C to $p+T_pS$.

Proof $\vec{\alpha}(s)$ = arc length parameterization of C near p, $\vec{\alpha}(0)=p$.



Have $p+T_pS = p+\text{span}\{\vec{t}(0), N\times\vec{t}(0)\}$. WLOG $p=0$.

$\Rightarrow \Pi_p(\vec{\alpha}(s)) = \langle\vec{\alpha}(s),\vec{t}(0)\rangle\vec{t}(0) + \langle\vec{\alpha}(s), N\times\vec{t}(0)\rangle(N\times\vec{t}(0))$.

Let $k^*(s)$ = curvature of $\Pi_p(\vec{\alpha})$ at s; recall:

$$k^*(s) = \|(\Pi_p\circ\vec{\alpha})'(s)\|^{-3}\,\|(\Pi_p\circ\vec{\alpha})'(s)\times(\Pi_p\circ\vec{\alpha})''(s)\|$$

$$= \|\langle\vec{t}(s),\vec{t}(0)\rangle\vec{t}(0) + \langle\vec{t}(s), N\times\vec{t}(0)\rangle(N\times\vec{t}(0))\|^{-3}$$

$$\times\|[\langle\vec{t}(s),\vec{t}(0)\rangle\vec{t}(0)+\langle\vec{t}(s),N\times\vec{t}(0)\rangle(N\times\vec{t}(0))]$$

$$\times[\langle\vec{\alpha}''(s),\vec{t}(0)\rangle\vec{t}(0)+\langle\vec{\alpha}''(s),N\times\vec{t}(0)\rangle(N\times\vec{t}(0))]\|.$$

At $s=0$: $\langle\vec{\alpha}''(0),\vec{t}(0)\rangle=0$, $\langle\vec{t}(0), N\times\vec{t}(0)\rangle=0$

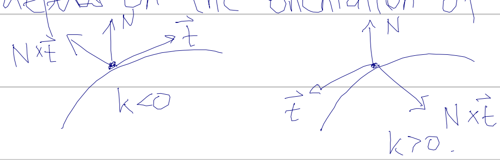
$$\Rightarrow k^*(0) = \|\vec{t}(0)\times(\langle\vec{\alpha}''(0), N\times\vec{t}(0)\rangle(N\times\vec{t}(0)))\|$$

$$= |\langle\vec{\alpha}''(0), N\times\vec{t}(0)\rangle| = \left|\left\langle \frac{D\vec{\alpha}'}{ds}(0), N\times\vec{t}(0)\right\rangle\right| = |k_g(0)|.$$

This proves the claim up to a sign; get same sign from orientation considerations (using N to give an orientation on S and T_pS). $\square$

04/16/2020

Rmk. The sign of the geodesic curvature depends on the orientation of S, and on that of the curve



In a similar manner, we can assign numerical values to covariant derivatives:

Def. Let w be a smooth field of unit vectors along $\vec{\alpha}: I\to S$, S = oriented surface. Then $\frac{dw}{dt}$ is normal to w, hence

$$\frac{Dw}{dt} = \lambda\cdot(N\times w),\quad\text{where } \lambda(t) = \left\langle\frac{Dw}{dt}, N\times w\right\rangle =: \left[\frac{Dw}{dt}\right] \text{ is}$$

the algebraic value of the covariant derivative of w at t.

Ex. $\vec{\alpha}$ = arc length parameterized curve $\Rightarrow k_g(t) = \left[\frac{D\vec{\alpha}'}{dt}(t)\right]$.

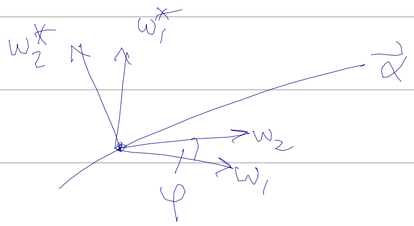
We shall obtain a neat formula for $\left[\frac{Dw}{dt}\right]$.

<u>Lemma</u> Let w_1 and w_2 be two smooth vector fields along $\vec{\alpha}: I \to S$ with $\|w_1(t)\| = \|w_2(t)\| = 1$ $\forall t \in I$. Then

$$\left[\frac{Dw_2}{dt}\right] - \left[\frac{Dw_1}{dt}\right] = \varphi'(t),$$

where $\varphi(t) = \measuredangle(w_1(t), w_2(t))$.

Proof Let $w_1^* = N \times w_1$, $w_2^* = N \times w_2$.



Then $w_2(t) = (\cos\varphi(t))\, w_1(t) + (\sin\varphi(t))\, w_1^*(t)$,

so $w_2^*(t) = -(\sin\varphi(t))\, w_1(t) + (\cos\varphi(t))\, w_1^*(t)$.

Compute $w_2'(t) = -\varphi' \sin\varphi\ w_1 + \varphi' \cos\varphi\ w_1^*$
$\qquad + \cos\varphi\ w_1' + \sin\varphi\ (w_1^*)'$,

therefore $\left[\frac{Dw_2}{dt}\right] = \langle w_2', w_2^* \rangle = \varphi'(\sin^2\varphi + \cos^2\varphi) - \sin^2\varphi \overbrace{\langle w_1, (w_1^*)' \rangle}^{= -\langle w_1', w_1^* \rangle}$
$\qquad + \cos^2\varphi \langle w_1', w_1^* \rangle$

$\qquad = \varphi' + \langle w_1', w_1^* \rangle = \varphi' + \left[\frac{Dw_1}{dt}\right]$ $\square$

In particular, if w_2 is parallel along $\vec{\alpha}$, then $\left[\frac{Dw_1}{dt}\right] = \varphi'(t)$ = rate of change of angle of $w_1(t)$ with a parallel vector field.

<u>Prop.</u> Let $\vec{x}(u,v)$ be an orthogonal parameterization of an oriented surface S (so that $N = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$), and let $w(t)$ be a smooth unit vector field on S along $\vec{x}(u(t), v(t))$. Then

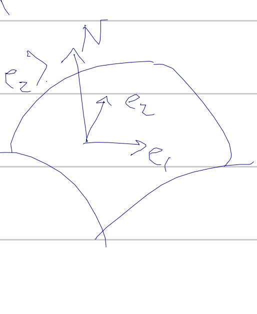
$$\left[\frac{Dw}{dt}\right] = \underbrace{\frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right)}_{\substack{\text{algebraic value of} \\ \text{cov. derivative of unit} \\ \text{vector in } \vec{x}_u\text{-direction!}}} + \varphi',$$

where φ = angle from $\vec{x}_u$ to $w(t)$.

Proof $e_1 = \frac{\vec{x}_u}{\sqrt{E}}$, $e_2 = \frac{\vec{x}_v}{\sqrt{G}}$ are orthonormal tangent vectors, and $e_1 \times e_2 = N$.

$\Rightarrow$ $\left[\frac{Dw}{dt}\right] = \left[\frac{De_1}{dt}\right] + \varphi'$, where $e_1(t) = e_1(u(t), v(t))$; and

$\left[\frac{De_1}{dt}\right] = \langle \frac{de_1}{dt}, N \times e_1 \rangle = \langle e_1', e_2 \rangle = u' \langle (e_1)_u, e_2 \rangle + v' \langle (e_1)_v, e_2 \rangle$.



Since $F = 0$, have $\langle \vec{x}_{uu}, \vec{x}_v \rangle = \partial_u F - \langle \vec{x}_u, \vec{x}_{vu} \rangle = -\frac{1}{2} E_v$, so

$\langle (e_1)_u, e_2 \rangle = \langle \left(\frac{\vec{x}_u}{\sqrt{E}}\right)_u, \frac{\vec{x}_v}{\sqrt{G}} \rangle = \langle \frac{\vec{x}_{uu}}{\sqrt{E}}, \frac{\vec{x}_v}{\sqrt{G}} \rangle = -\frac{1}{2} \frac{E_v}{\sqrt{EG}}$;

likewise $\langle (e_1)_v, e_2 \rangle = \frac{1}{2}\frac{G_u}{\sqrt{EG}}$. Plug in $\Rightarrow$ done. $\square$

<u>Example</u>. Revisit existence of parallel transport. If $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$, $\vec{x}$ = orthogonal parameterization of S, then w = unit vector field along $\vec{\alpha}$ is parallel if

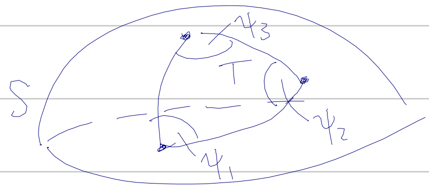
$$\varphi' = -\frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) =: B(t),$$

where $\varphi(t)$ = angle from $\vec{x}_u$ to $w(t)$; so $w(t)$ is determined by

$$\varphi(t) = \underbrace{\varphi(t_0)}_{\text{determined from initial value } w(t_0)} + \int_{t_0}^{t} B(z)\,dz.$$

<u>IV.4 Gauss-Bonnet Theorem</u> S = regular, oriented surface.

<u>Goal</u>:



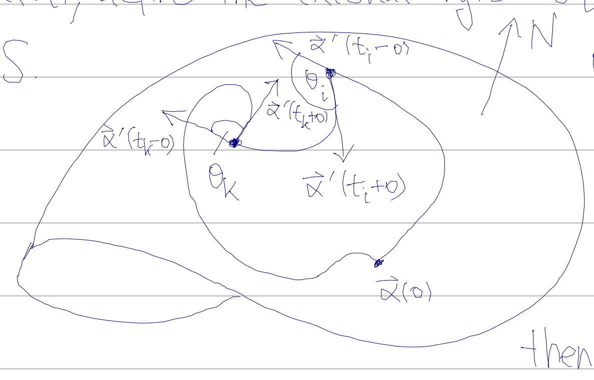
edges of T = geodesics.

$$\Rightarrow \quad \psi_1 + \psi_2 + \psi_3 = \pi + \underbrace{\int_T K\,d\sigma}_{\text{integral of Gauss curvature over the triangle } T}.$$

Will consider a more general situation: let $\vec{\alpha}: [0, l] \to S$ be a parameterized curve which is

- simple (no self-intersections: $\vec{\alpha}(t) \neq \vec{\alpha}(s)$ for $0 \le t, s < l$, $t \neq s$),
- closed ($\vec{\alpha}(0) = \vec{\alpha}(l)$),
- piecewise regular: $\vec{\alpha}|_{[t_i, t_{i+1}]}$ is regular for some partition $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$.

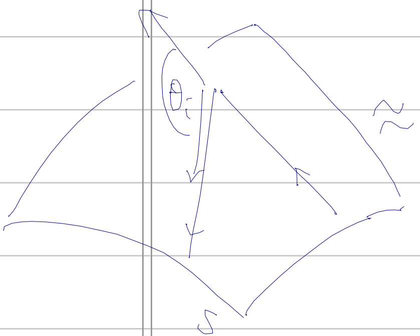
At the vertices $\vec{\alpha}(t_i)$, define the external angle $\theta_i \in [-\pi, \pi]$ conforming with the orientation of S.

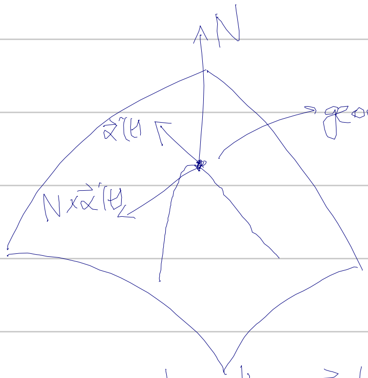


$\theta_i > 0$

$\theta_k < 0$

- Intuition for the sign of the θ_i: smooth out the corner at $\vec{\alpha}(t_i)$ by a sharp curve; the sign of θ_i is then the sign of the geodesic curvature. $\to$





geodesic curvature is positive: $\vec{\alpha}$ curves towards $N\times\vec{\alpha}'(t)$.

Suppose $\vec{\alpha}([0,\ell])\subset\vec{x}(U)$ for some parameterization $\vec{x}:U\subset\mathbb{R}^2\to S$.
Let $\varphi_i:[t_i,t_{i+1}]\to\mathbb{R}$ denote the angle from $\vec{x}_u$ to $\vec{\alpha}'(t)$ (chosen to be smooth in $t\in[t_i,t_{i+1}]$).

· Theorem of turning tangents. Under these assumptions,

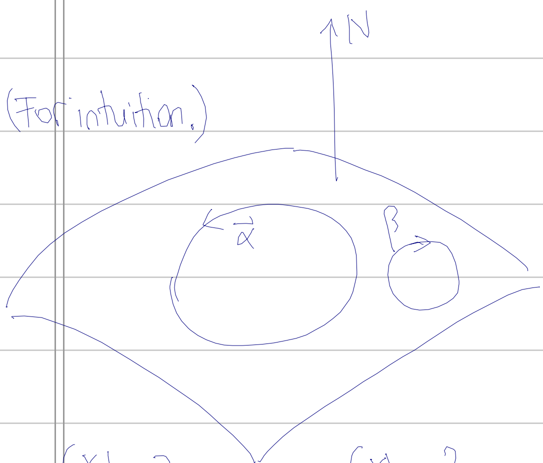
$$(*)\qquad \sum_{i=0}^{k}\left(\varphi_i(t_{i+1})-\varphi_i(t_i)\right)+\sum_{i=0}^{k}\theta_i=\pm 2\pi,$$

where the sign depends on the orientation of $\vec{\alpha}$.

Note. The first sum is
$$\sum_{i=0}^{k-1}\left(\varphi_i(t_{i+1})-\varphi_{i+1}(t_{i+1})\right)+\left(\varphi_k(t_{k+1})-\varphi_0(t_0)\right),$$
demonstrating that the role of $\vec{x}_u$ is merely to have a consistent definition of angles along $\vec{\alpha}$.

Rmk. Compare with our earlier example of a regular! planar curve $\vec{\alpha}(s)$ with $\vec{\alpha}'(s)=(\cos\theta(s),\sin\theta(s))$, and $\int_0^\ell\theta'(s)\,ds=2\pi I$, $I\in\mathbb{Z}$.
If in the theorem above, $\vec{\alpha}$ has no corners, $\pm 2\pi=\sum_{i=0}^{k}\varphi_i(t_{i+1})-\varphi_i(t_i)$
$=\sum\int_{t_i}^{t_{i+1}}\varphi_i'(s)\,ds=\int_0^\ell\varphi'(s)\,ds$ is of the same form, with $I=\pm1$. This restriction on I is a result of the assumption that $\vec{\alpha}$ is a simple curve. The angles θ_i at corners arise as limits of approximations as in the illustration above.

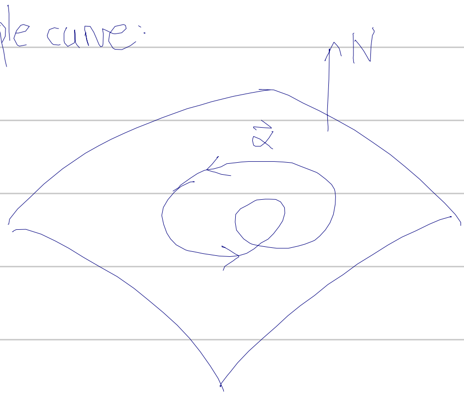
Ex. (For intuition.)



$(*)=2\pi$ for $\vec{\alpha}$.
Rotation number $I=1$

$(*)=-2\pi$ for $\vec{\beta}$.
$I=-1$

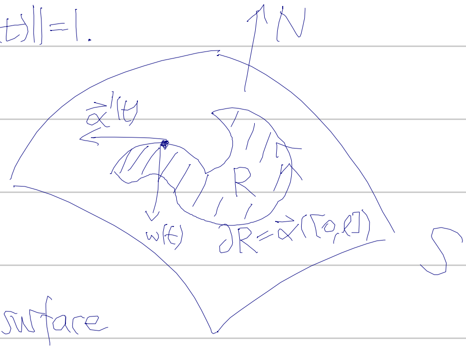
Non simple curve:



$I=2$.

Need more concepts:

- <u>Orientation</u>. Let $R \subset S$ be the union of a connected open subset $\subset S$ with its boundary. R is called <u>simple</u> if R is homeomorphic to a disk, and the boundary ∂R of R is the image of a simple, closed, piecewise regular, parameterized curve $\vec{\alpha}: I \to S$, $\|\vec{\alpha}'(t)\| = 1$.



We say that $\vec{\alpha}$ is <u>positively oriented</u> if for each $t \in [0, \ell]$, there exists an orthogonal basis $\{\vec{\alpha}'(t), w(t)\}$ of $T_{\vec{\alpha}(t)}S$ so that

(i) $\vec{\alpha}'(t) \times w(t) = N(\vec{\alpha}(t))$ (the fixed surface normal)

(ii) $w(t)$ points towards the region R.

(That is, $\vec{\alpha}$ moves counterclockwise around R.)

- <u>Surface integrals</u>. Let $R \subset \vec{x}(U) \subset S$ be entirely contained in a coordinate neighborhood $\vec{x}: U \to S$. If $f: S \to \mathbb{R}$ is a smooth function on S, then

$$\iint_R f \, d\sigma := \iint_{\vec{x}^{-1}(R)} f(\vec{x}(u,v)) \sqrt{EG - F^2} \, du \, dv.$$

(This is independent of the chosen parameterization, as in the definition of the area. Also, $\iint_R 1 \, d\sigma = \text{area}(R)$.)

Finally: <u>Theorem</u> (Local Gauss–Bonnet theorem.) Let $\vec{x}: U \to S$ be an orthogonal parameterization of an oriented surface S. Let $R \subset \vec{x}(U)$ be a simple region; let $\vec{\alpha}: [0, \ell] \to S$ be a simple, closed, piecewise regular, arc-length parameterized curve, $\vec{\alpha}([0,\ell]) = \partial R$, positively oriented, and let $\vec{\alpha}(s_0), \dots, \vec{\alpha}(s_k)$ and $\theta_0, \dots, \theta_k$ be the vertices and external angles of $\vec{\alpha}$.

Then
$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \underbrace{k_g(s)}_{\text{geodesic curvature of } \alpha} ds + \iint_R K \, d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$$

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<u>Note:</u> The quantities in this formula are all intrinsic (independent of the parameterization).
Will prove global theorem soon.

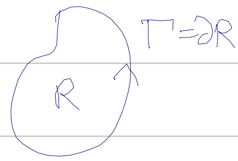
<u>Proof.</u> Write $\vec{\alpha}(s) = \vec{x}(u(s), v(s))$. Recall that

$$k_g(s) = \left[\frac{D\vec{\alpha}'}{ds}(s)\right] = \frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) + \varphi_i' \qquad (s \in [s_i, s_{i+1}]),$$

where φ_i = angle between $\vec{\alpha}'(s)$ and $\vec{x}_u(u(s), v(s))$ for $s \in [t_i, t_{i+1}]$.

$$\Rightarrow \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\, ds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) ds + \sum_{i=0}^{k}\left(\varphi_i(s_{i+1}) - \varphi_i(s_i)\right).$$

Recall Green's theorem $\oint_\Gamma F\,dx + G\,dy = \iint_R (G_x - F_y)\,dx\,dy$,



so

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}}\left(G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right) ds$$

$$= \int_{\vec{x}^{-1}(\vec{\alpha}([0,\ell]))} \frac{1}{2\sqrt{EG}}\left(-E_v\, du + G_u\, dv\right)$$

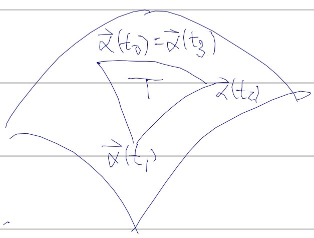
$$= \iint_{\vec{x}^{-1}(R)} \left[\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right] du\,dv$$

$$= \iint_{\vec{x}^{-1}(R)} -K\sqrt{EG}\, du\,dv = -\iint_{\vec{x}^{-1}(R)} K\sqrt{EG - F^2}\, du\,dv = -\iint_R K\, d\sigma.$$

Altogether, $\displaystyle\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\, ds + \iint_R K\, d\sigma = \sum_{i=0}^{k}\left(\varphi_i(s_{i+1}) - \varphi_i(s_i)\right)$

$$\overset{\substack{\text{Theorem of} \\ \text{turning tangents}}}{=} 2\pi - \sum_{i=0}^{k} \theta_i,$$ which gives the desired result. $\square$

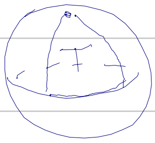
<u>Ex.</u> (1) Geodesic triangle T:
Interior angle at $\vec{\alpha}(t_i)$
$= \pi - (\text{exterior angle } \theta_i)$.



$\vec{\alpha}\big|_{[t_i, t_{i+1}]}$ = geodesic, so $k_g = 0$.

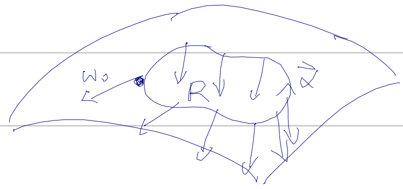
Gauss-Bonnet $\Rightarrow$ $\displaystyle 0 + \iint_T K\, d\sigma + \underbrace{\sum_{i=0}^{2}(\pi - \theta_i)}_{3\pi - \sum_{i=0}^{2}\theta_i} = 2\pi$

$$\Rightarrow \sum_{i=0}^{2} \theta_i = \pi + \iint_T K\, d\sigma.$$

Verify this for  S^2!

(2) Let $\vec{\alpha}:[0,\ell]\to S$ parametrize ∂R, $R=$ simple region, as in the statement of Gauss-Bonnet. Let $p=\vec{\alpha}(0)=\vec{\alpha}(\ell)$, $w_0\in T_pS$.

Let $w(s)=$ parallel transport of $w(0)=w_0$ along $\vec{\alpha}(s)$. Then



$$0=\int_0^\ell \left\langle \frac{Dw}{ds}(s),\, N\times\vec{\alpha}'(s)\right\rangle ds$$
$$=\int_0^\ell \left[\frac{Dw}{ds}(s)\right] ds=\int_0^\ell \frac{1}{2\sqrt{EG}}(G_u v'-E_v u')\,ds+\int_0^\ell \varphi'(s)\,ds$$
$$=-\iint_R K\,d\sigma+\varphi(\ell)-\varphi(0) \qquad (\varphi(s)=\measuredangle(\vec{x}_u,w(s)))$$

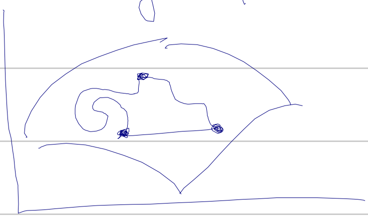
$\Rightarrow$ $\varphi(\ell)-\varphi(0)=\iint_R K\,d\sigma$. (Why is the LHS independent of w_0?)

<u>Corollary</u>. $\lim\limits_{R\to\{p\}}\dfrac{\varphi(\ell)-\varphi(0)}{\text{area}(R)}=K(p)$. (E.g. take $R=\vec{x}(\varepsilon$-ball around $(u_0,v_0)=\vec{x}^{-1}(p))$ and let $\varepsilon\to 0$.)

To state the global version of Gauss-Bonnet, we need further concepts from topology.

<u>Def</u>. A <u>triangle</u> is a simple region with three vertices and external angles $\theta_i\neq 0$.

<u>Def</u>. A connected region $R\subset S$ is <u>regular</u> if R is compact and its boundary ∂R is a finite union of simple piecewise curves which do not intersect.

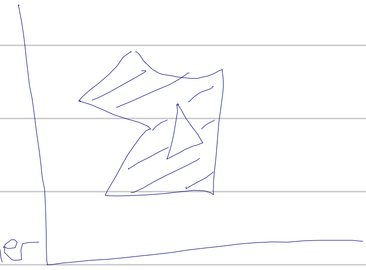


(A compact regular surface will be considered a regular region, with empty boundary.)

<u>Def</u>. A <u>triangulation</u> of a regular region $R\subset S$ is a finite family $\mathcal{T}=\{T_1,\dots,T_n\}$ of triangles $T_i\subset S$ s.t.

(i) $\bigcup\limits_{i=1}^n T_i=R$;

(ii) If $T_i\cap T_j\neq\emptyset$ and $i\neq j$, then $T_i\cap T_j$ is either a common edge or a common vertex of T_i and T_j.



- Given a triangulation $\mathcal{T}$ of R, let F,E,V denote the number of triangles (F, "faces"), edges (counted once), vertices (counted once) of the triangulation.

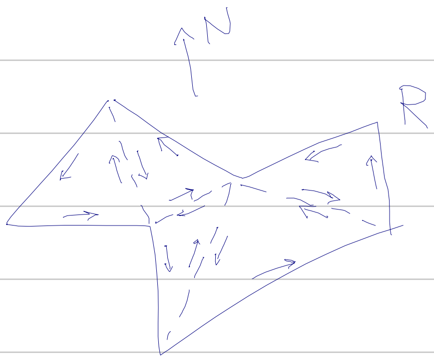
Then $\chi(R)=F-E+V$ is the Euler characteristic of R.

<u>Facts</u>: (i) Every regular domain admits a triangulation.

(ii) $\chi(R)$ is independent of the triangulation.

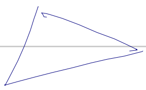
(iii) Let S be an oriented regular surface covered by coordinate charts $\vec{x}_i : U_i \subset \mathbb{R} \to S$ compatible with the orientation of S. Let $R \subset S$ be a regular region. Then $\exists$ triangulation $\mathcal{T}$ such that every $T \in \mathcal{T}$ is contained in a single chart $\vec{x}_i(U_i)$. Moreover if the boundary of each $T \in \mathcal{T}$ is positively oriented, then adjacent triangles determine opposite orientations on the common edge.

<u>Ex</u> (1)



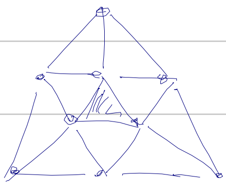
$F=5,\ E=11,\ V=7$
$\Rightarrow \chi(R) = 1.$

(2)



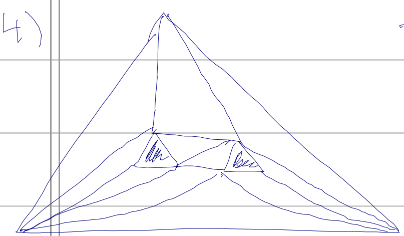
$F=1,\ E=3,\ V=3 \Rightarrow \chi(\triangle) = 1.$

(3)



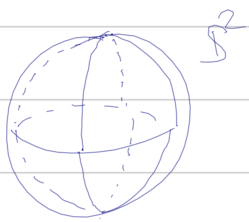
$F=9,\ E=18,\ V=9 \Rightarrow \chi(R) = 0$

(4)



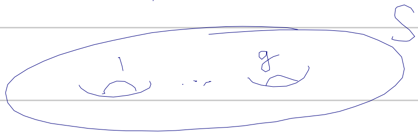
$F=11,\ E=21,\ V=9 \Rightarrow \chi(R) = -1$

(5)



$F=8,\ E=12,\ V=6 \Rightarrow \chi(S^2) = 2.$

(6)



Torus with g holes $\Rightarrow \chi(S) = 2-2g$ "genus"

Examples (5) & (6) comprise all compact connected orientable regular surfaces in $\mathbb{R}^3$ up to homeomorphism. Thus, if S is such a surface and $\chi(S) > 0$, then S is homeomorphic to S^2.

<u>Global Gauss-Bonnet Theorem</u>. Let $R \subset S$ be a regular region inside an oriented surface S; denote by $C_1, \dots, C_p$ the closed, simple, piecewise regular, positively oriented curves whose union is $= \partial R$. Let $\theta_1, \dots, \theta_q$ be the collection of external angles of all the C_i. Then

$$\underbrace{\sum_{i=1}^{p} \int_{C_i} k_g(s)\, ds}_{\substack{\text{sum of integrals over all}\\ \text{regular arcs of } C_i,\\ \text{parameterized by arc length}}} + \iint_R K\, d\sigma + \sum_{j=1}^{q} \theta_j = 2\pi \chi(R).$$

<u>Proof</u> From local Gauss-Bonnet + counting. $\square$

<u>Cor.</u> (Improved "local" Gauss-Bonnet). If $R \subset S$ is a simple region $\chi(R)=1.$, then

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{i=1}^{k} \theta_i = 2\pi.$$

<u>Cor.</u> Let S = orientable compact surface. Then

$$\iint_S K\, d\sigma = 2\pi \chi(S).$$

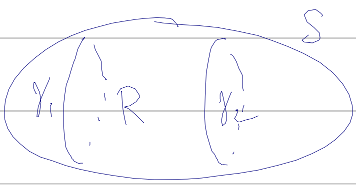
($\Rightarrow$ Total curvature is <u>independent</u> of the shape of S inside $\mathbb{R}^3$!)

<u>Applications</u> (1) S = compact, connected, orientable, $K > 0 \Rightarrow S$ is homeomorphic to a sphere.

<u>Pf.</u> $\chi(S) = \frac{1}{2\pi} \iint_S K\, d\sigma > 0$. $\square$

(2) S = compact, connected, orientable, $K > 0$. γ_1, γ_2 = two distinct simple closed geodesics $\Rightarrow \gamma_1 \cap \gamma_2 \neq \emptyset$.

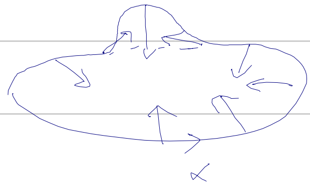
<u>Pf.</u> S is homeomorphic to a sphere. If $\gamma_1 \cap \gamma_2 = \emptyset$, then $\gamma_1 \cup \gamma_2 = \partial R$ for a region R with $\chi(R) = 0$. By Gauss-Bonnet, $\iint_R K\, d\sigma = 0$. Contradiction since $K > 0$. $\square$



(3) (Jacobi) Let $\vec{\alpha}: I \to \mathbb{R}^3$ be a closed, regular, parameterized curve with everywhere nonzero curvature. Assume the curve $\vec{n}(s)$ of normal vectors is simple. Then $\vec{n}(I) \subset S^2$ subdivides S^2 into two regions of equal area (2π).

Pf. WLOG $\vec{\alpha}(s)$ = parameterized by arc length

Let $\bar{s}$ = arc length of $\vec{n}(s)$; write $\dot{}$ for $\frac{d}{d\bar{s}}$.

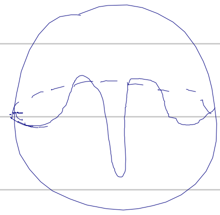


$\Rightarrow$ Geodesic curvature of $\vec{n}$ is $\bar{k}_g = \langle \ddot{\vec{n}}, \vec{n} \times \dot{\vec{n}} \rangle$.

Have $\dot{\vec{n}} = \frac{d\vec{n}}{d\bar{s}} = \frac{d\vec{n}}{ds}\frac{ds}{d\bar{s}} = (-k\vec{t} - \tau\vec{b})\frac{ds}{d\bar{s}}$,

$$\ddot{\vec{n}} = (-k\vec{t} - \tau\vec{b})\frac{d^2 s}{d\bar{s}^2} + (-k'\vec{t} - \tau'\vec{b})\left(\frac{ds}{d\bar{s}}\right)^2 - (k^2+\tau^2)\vec{n}\left(\frac{ds}{d\bar{s}}\right)^2,$$

$\to \vec{n}(I) =$

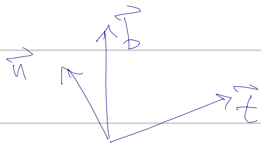


and $1 = \|\dot{\vec{n}}\|^2 = \left(\frac{ds}{d\bar{s}}\right)^2 \|\vec{n}'\|^2 = \left(\frac{ds}{d\bar{s}}\right)^2 (k^2+\tau^2)$

$$\Rightarrow \bar{k}_g = \left(\frac{ds}{d\bar{s}}\right)\langle \ddot{\vec{n}}, k\vec{b} - \tau\vec{t} \rangle$$

$$= \left(\frac{ds}{d\bar{s}}\right)^3 (-k\tau' + k'\tau) = -\frac{\tau' k - \tau k'}{k^2+\tau^2}\frac{ds}{d\bar{s}}$$

$$= -\frac{d}{ds}\arctan\left(\frac{\tau}{k}\right)\cdot\frac{ds}{d\bar{s}} = -\frac{d}{d\bar{s}}\arctan\left(\frac{\tau}{k}\right).$$



Apply Gauss-Bonnet to a region $R \subset S^2$ bounded by $\vec{n}(I)$:

$$2\pi = \iint_R \underbrace{K}_{=1}\, d\sigma + \underbrace{\int_{\partial R} \bar{k}_g(\bar{s})\, d\bar{s}}_{=0} = \text{area}(R). \qquad \square$$

IV.5. Exponential map. 04/23/2020

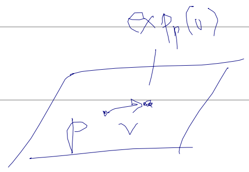
- We already encountered the idea: given $p \in S$, we can "shoot out" geodesics from p; this gives natural ways to parameterize a neighborhood of p, as we will see.
- Given $p \in S$, $v \in T_pS$, $\exists!$ parameterized geodesic $\gamma: (-\varepsilon, \varepsilon) \to S$, $\gamma(0) = p$, $\gamma'(0) = v$.

 Write $\gamma(v, t) := \gamma(t)$ to stress dependence on v.

Lemma. If the geodesic $\gamma(t, v)$ is defined for $t \in (-\varepsilon, \varepsilon)$, then $\gamma(t, \lambda v)$ with $\lambda > 0$ is defined for $t \in (-\varepsilon/\lambda, \varepsilon/\lambda)$, and $\gamma(t, \lambda v) = \gamma(\lambda t, v)$.

Proof. Let $\gamma(t) = \gamma(t, v)$, $\alpha(t) = \gamma(\lambda t)$, then $\alpha(0) = \gamma(0)$, $\alpha'(0) = \lambda\gamma'(0) = \lambda v$, and
$D_{\alpha'(t)}\alpha'(t) = \lambda^2 (D_{\gamma'(s)}\gamma'(s))|_{s=\lambda t} = 0 \Rightarrow \alpha$ is a geodesic. By uniqueness, $\alpha(t) = \gamma(t, \lambda v)$. $\square$

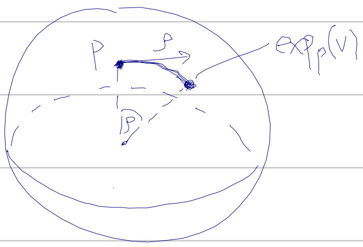
Def. If $v \in T_pS$, $\|v\| \neq 0$, is s.t. $\gamma(\|v\|, \frac{v}{\|v\|}) = \gamma(1, v)$ is defined, we set
$\exp_p(v) := \gamma(1, v)$.



Ex. (1) S = plane, $p \in S$, $v \in T_pS \Rightarrow \exp_p(v) = p + v$.

(2) $S = S^2$, $p = (0,0,1)$, $v = (\rho\cos\theta, \rho\sin\theta, 0)$

$\Rightarrow \exp_p(v) = (\cos\rho\cos\theta, \cos\rho\sin\theta, \sin\rho)$.



(So: ρ, θ are polar coordinates (usually denoted θ, φ in this order).)

Prop. Given $p \in S$, $\exists\, \varepsilon > 0$ s.t. $\exp_p$ is defined and smooth in $\{v \in T_pS : \|v\| < \varepsilon\} =: B_p(\varepsilon)$.

Pf. ODE theory for the geodesic equation. $\square$

Generalizing example (1), we can always use $\exp_p$ to produce parameterizations of S:

Prop. $\exp_p : B_p(\varepsilon) \subset T_pS \to S$ is a diffeomorphism $\exp_p : U \subset B_p(\varepsilon) \to \exp_p(U) \subset S$ in a neighborhood $U \subset B_p(\varepsilon)$ of $0 \in T_pS$.

Pf. We wish to apply the inverse function theorem. Compute

$$d(\exp_p)_0(v) = \frac{d}{dt}\exp_p(0 + tv)\Big|_{t=0} = \frac{d}{dt}\gamma(1, tv)\Big|_{t=0}$$
$$= \frac{d}{dt}\gamma(t, v)\Big|_{t=0} = v.$$

$\Rightarrow d(\exp_p)_0 : T_pS \to T_pS$ (really: $T_0(T_pS) \to T_pS$) is invertible. $\square$

Def. · If $U \subset T_pS$ is a neighborhood of 0, and $\exp_p : U \to \exp_p(U)$ is a diffeomorphism, then $V = \exp_p(U)$ is called a <u>normal neighborhood</u> of p.

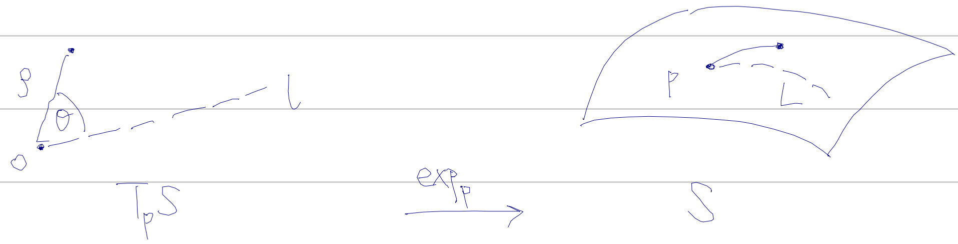
· Coordinate systems on V:

– <u>normal coordinates</u> corresponding to a system of rectangular coordinates in T_pS. (That is: $e_1, e_2 \in T_pS$ orthonormal $\Rightarrow \vec{x}(u,v) := \exp_p(ue_1 + ve_2)$, with (u,v) the normal coordinates.)

– <u>geodesic polar coordinates</u> corresponding to polar coordinates in T_pS.

Rmk. In normal coordinates (u,v), the geodesics through p are precisely the images under $\exp_p$ of straight lines $(u(t),v(t)) = (at, bt)$.
We have $E(p)=1$, $F(p)=0$, $G(p)=1$. Exercise: $E_u = E_v = F_u = F_v = G_u = G_v = 0$ at p.

We shall study geodesic polar coordinates in some detail.
Choose in T_pS, $p \in S$, a system of polar coordinates (ρ, θ) where ρ is the radius ($\rho = \|\cdot\|_p$) and $\theta \in (0, 2\pi)$ is the polar angle relative to a closed half line $l \subset T_pS$.



Set $L = \exp_p(l)$. With $V = \exp_p(U)$ a normal neighborhood of p,
$\exp_p : V \setminus l \to U \setminus L$ is still a diffeomorphism.

Def. The images under $\exp_p$ of $\begin{cases} \text{circles in } U \text{ centered at } 0 \\ \text{lines in } U \text{ through } 0 \end{cases}$ are $\begin{cases} \underline{\text{geodesic circles}} \text{ of } V. \\ \underline{\text{radial geodesics}}. \end{cases}$

(In $V \setminus L$, these are curves $\begin{cases} \rho = \text{const} \\ \theta = \text{const}. \end{cases}$)

Proposition. Let $\vec{x} : U \setminus l \to V \setminus L$ be a system of geodesic polar coordinates (ρ, θ).
Then $E = E(\rho,\theta)$, F, G satisfy $E = 1$, $F = 0$, $\lim_{\rho\to 0} G = 0$, $\lim_{\rho \to 0} (\sqrt{G})_\rho = 1$.

(Example $S = \{z = 0\} \subset \mathbb{R}^3$, $p = (0,0)$, $l = \{(x,0,0)\}$,
$\vec{x}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 0)$, $E = 1$, $F = 0$, $G = \rho^2$.)

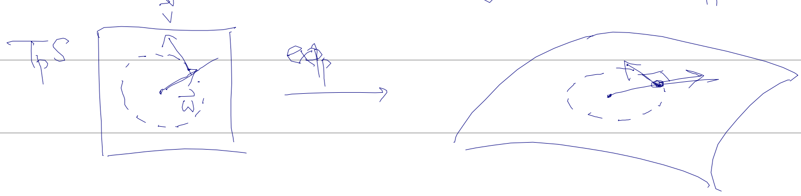
Proof We have $\vec{x}(\rho,\theta) = \exp_p(\rho\,\vec{w}_\theta)$, $\vec{w}_\theta$ = unit vector in T_pS, $\measuredangle(l, \vec{w}_\theta) = \theta$.

- $E = \langle \vec{x}_\rho, \vec{x}_\rho \rangle$, and $\rho \mapsto \vec{x}(\rho, \theta_0)$ is an arc length parameterized geodesic for all θ_0 $\Rightarrow$ $E = 1$.
- F: this is the content of Gauss' lemma, which we state separately.

Lemma (Gauss). Suppose $\exp_p(\vec{w})$ is defined. Then

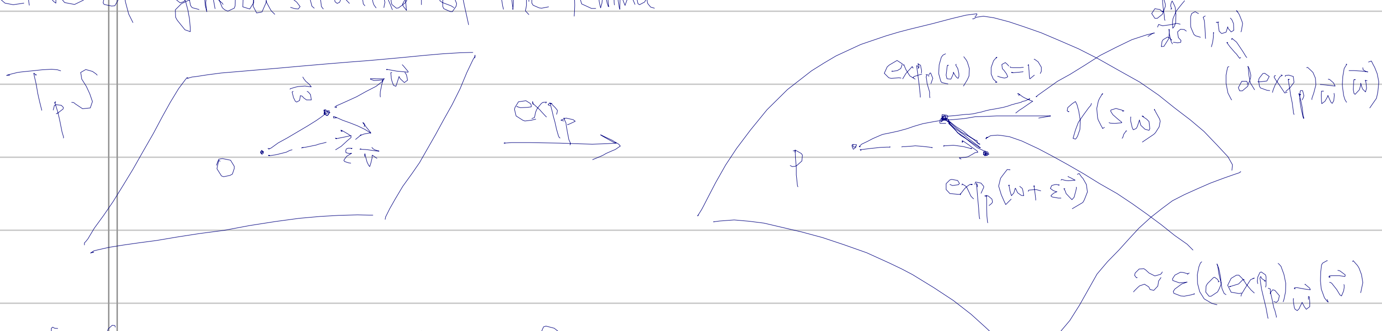
$$\langle (d\exp_p)_{\vec{w}}(\vec{w}),\ (d\exp_p)_{\vec{w}}(\vec{v})\rangle_{\exp_p(w)} = \langle \vec{w},\vec{v}\rangle_p \quad \forall\, \vec{v},\vec{w}\in T_pS.$$

Cor. For $\vec{v}\perp\vec{w}$, this implies that geodesic circles are orthogonal to radial geodesics.
($(d\exp_p)_{\vec{w}}(\vec{w})$ = tangent vector of radial geodesic; $(d\exp_p)_{\vec{w}}(\vec{v})$ = tangent vector of geodesic circle



($\Rightarrow F=0$ in the Prop.)

Picture of general situation of the lemma:



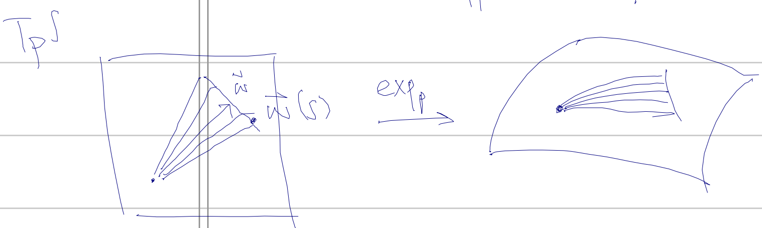
Proof of Lemma. Nothing to do if $\vec{w}=0$.

For $\vec{w}\neq 0$, write $\vec{v}=\vec{v}_T+\vec{v}_N$, $\vec{v}_T=c\vec{w}$ $(c\in\mathbb{R})$, $\langle\vec{v}_N,\vec{w}\rangle=0$.

Already know $\langle (d\exp_p)_{\vec{w}}(\vec{w}), (d\exp_p)_{\vec{w}}(\vec{v}_T)\rangle = c\|\vec{w}\|^2 = \langle \vec{w},\vec{v}_T\rangle$, so need to consider only the case $\vec{v}=\vec{v}_N\perp\vec{w}$.

Since $\exp_p$ is defined on open set (ODE theory), $\exists\,\varepsilon>0$ s.t. $\exp_p$ is defined for $t\vec{w}(s)$, $t\in[0,1]$, $\vec{w}(s)=\vec{w}+s\vec{v}$, $s\in(-\varepsilon,\varepsilon)$.

Let $F(t,s)=\exp_p(t\vec{w}(s))$; for fixed s, $t\mapsto F(t,s)$ is a geodesic.



Compute:

(i) $\frac{\partial}{\partial t}F(1,s)=\frac{d}{dt}\exp_p(t\vec{w}(s))\big|_{t=1}$

$= (d\exp_p)_{\vec{w}(s)}(\vec{w}(s))$

$\Rightarrow \frac{\partial}{\partial t}F(1,0)=(d\exp_p)_{\vec{w}}(\vec{w})$.

(ii) $\frac{\partial}{\partial s}F(1,s)=(d\exp_p)_{\vec{w}(s)}(\vec{w}'(s)) \Rightarrow \frac{\partial}{\partial s}F(1,0)=(d\exp_p)_{\vec{w}}(\vec{v})$.

Furthermore, $\frac{\partial}{\partial t}\langle \frac{\partial}{\partial t}F(t,s), \frac{\partial}{\partial s}F(t,s)\rangle = \langle \underbrace{\frac{D}{dt}\frac{\partial}{\partial t}F(t,s)}_{=0}, \frac{\partial}{\partial s}F(t,s)\rangle + \langle\frac{\partial}{\partial t}F(t,s), \frac{D}{dt}\frac{\partial}{\partial s}F(t,s)\rangle$

vector fields along the curve $t\mapsto F(t,s)$

$=0$ since $t\mapsto F(t,s)$ is a geodesic

$$= \left\langle \frac{\partial}{\partial t} F(t,s),\ \underbrace{\Pi_{F(t,s)}}_{\text{orth. projection to } T_{F(t,s)}S} \frac{\partial}{\partial t}\frac{\partial}{\partial s} F(t,s) \right\rangle = \left\langle \frac{\partial}{\partial t} F(t,s),\ \Pi_F \frac{\partial}{\partial s}\frac{\partial}{\partial t} F(t,s) \right\rangle$$

$$= \left\langle \frac{\partial}{\partial t} F(t,s),\ \underbrace{\frac{D}{ds}}_{\substack{\text{covariant derivative}\\ \text{along } s \mapsto F(t,s)}} \frac{\partial}{\partial t} F(t,s) \right\rangle = \frac{1}{2}\frac{\partial}{\partial s} \underbrace{\langle \partial_t F(t,s), \partial_t F(t,s)\rangle}_{= \|w(s)\|^2} = \langle \vec{w}(s), \vec{w}'(s)\rangle,$$

For $s=0$, $\langle \vec{w}(0), \vec{w}'(0)\rangle = \langle \vec{w}, \vec{v}\rangle = 0$.

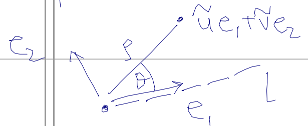
Therefore, $\langle \frac{\partial F}{\partial t}(t,0), \frac{\partial F}{\partial s}(t,0)\rangle$ is constant t.

$$\Rightarrow \left\langle \frac{\partial F}{\partial t}(1,0), \frac{\partial F}{\partial s}(1,0)\right\rangle = \lim_{t\to 0} \left\langle \frac{\partial F}{\partial t}(t,0), \frac{\partial F}{\partial s}(t,0)\right\rangle$$

$$= \lim_{t\to 0} \langle (d\exp_p)_{t\vec{w}}(t\vec{w}), (d\exp_p)_{t\vec{w}}(t\vec{v})\rangle = 0. \qquad \square$$

<u>Back to the Proposition.</u> In geodesic polar coordinates, have proved $E=1$, $F=0$.

Define normal coordinates $\tilde{u} = \rho\cos\theta$, $\tilde{v} = \rho\sin\theta$. (These are smooth coordinates near p via $\vec{y}: (\tilde{u},\tilde{v}) \mapsto \exp_p(\tilde{u}e_1 + \tilde{v}e_2)$, e_1 = unit vector in direction of L, e_2 = orthogonal unit vector.)



We have $\sqrt{EG-F^2} = \|\vec{x}_\rho \times \vec{x}_\theta\| = \left|\det\begin{pmatrix} \frac{\partial\tilde{u}}{\partial\rho} & \frac{\partial\tilde{u}}{\partial\theta} \\ \frac{\partial\tilde{v}}{\partial\rho} & \frac{\partial\tilde{v}}{\partial\theta}\end{pmatrix}\right| \underbrace{\|\vec{y}_{\tilde{u}} \times \vec{y}_{\tilde{v}}\|}_{=\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}$ coefficients of the 1st fund. form in $\tilde{u},\tilde{v}$ coordinates.

$\Longrightarrow \sqrt{G} = \rho\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}$. But at p, $\tilde{E}=\tilde{G}=1$, $\tilde{F}=0$.

$\Rightarrow \lim_{\rho\to 0}\sqrt{G} = 0$, $\lim_{\rho\to 0}(\sqrt{G})_\rho = \sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}\Big|_p = 1$. $\square$

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<u>Remark.</u> In a coordinate system with $E=1$, $F=0$, we have

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_\theta}{\sqrt{EG}}\right)_\theta + \left(\frac{G_\rho}{\sqrt{EG}}\right)_\rho\right) = -\frac{1}{\sqrt{G}}\left(\frac{G_\rho}{2\sqrt{G}}\right)_\rho = -\frac{1}{\sqrt{G}}\left((\sqrt{G})_\rho\right)_\rho$$

$$\Longrightarrow K = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}.$$

Now, in general, knowledge of the Gauss curvature (as a function of 2 variables in local coordinates) does <u>not</u> determine the 1st fund. form of a surface. <u>However</u>:

Theorem (Minding) Any two regular surfaces with the same constant Gauss curvature are locally isometric. In fact, given regular surfaces S_1, S_2 with the same constant Gauss curvature K, and given $p_1 \in S_1$, $p_2 \in S_2$ and orthonormal bases e_1, e_2 of $T_{p_1}S_1$, f_1, f_2 of $T_{p_2}S_2$, there exist neighborhoods V_1 of p_1 and V_2 of p_2, and an isometry $\phi: V_1 \to V_2$ such that $d\varphi_p(e_j) = f_j$, $j=1,2$.

Proof: Step 1 We use $K = -(\sqrt{G})_{\rho\rho}/\sqrt{G}$, i.e. $(\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0$, to determine the coefficients of the 1^{st} fund. form in geodesic polar coordinates.

Case 1. $K=0$. $(\sqrt{G})_{\rho\rho} = 0 \Rightarrow (\sqrt{G})_\rho = g(\theta)$, and $1 = \lim_{\rho\to 0} (\sqrt{G})_\rho = g(\theta)$

$\Rightarrow (\sqrt{G})_\rho = 1 \Rightarrow \sqrt{G} = c + \rho$ and $0 = \lim_{\rho\to 0}\sqrt{G} = c$

$\Rightarrow \sqrt{G} = \rho \Rightarrow G = \rho^2$ (and $E=1$, $F=0$)

Case 2. $K>0$. $\sqrt{G} = A(\theta)\cos(\sqrt{K}\rho) + B(\theta)\sin(\sqrt{K}\rho)$

From $\lim_{\rho\to 0}\sqrt{G} = 0$, we obtain $A=0$. Hence $(\sqrt{G})_\rho = B(\theta)\sqrt{K}\cos(\sqrt{K}\rho) \xrightarrow{\rho\to 0} 1$ requires $B(\theta) = \frac{1}{\sqrt{K}}$. Therefore,

$$G = \frac{1}{K}\sin^2(\sqrt{K}\rho).$$

Case 3. $K<0$. $\sqrt{G} = A(\theta)\cosh(\sqrt{-K}\rho) + B(\theta)\sinh(\sqrt{-K}\rho)$.

$\lim_{\rho\to 0}\sqrt{G} = 0$ implies $A=0$, and $1 = \lim_{\rho\to 0}(\sqrt{G})_\rho = \sqrt{-K}\,B(\theta) \Rightarrow B(\theta) = \frac{1}{\sqrt{-K}}$.

Thus, $G = \frac{1}{-K}\sinh(\sqrt{-K}\rho)$.

· Step 2. Let $A: T_{p_1}S_1 \to T_{p_2}S_2$ be the linear isometry with $Ae_j = f_j$. We then set

$$\phi := \exp_{p_2} \circ A \circ \exp_{p_1}^{-1}.$$

By step 1, the coefficients of the first fundamental form in geodesic polar coordinates on S_1, S_2 based at p_1, p_2 agree at p and $\phi(p)$ (p near p_1). $\Rightarrow \phi$ is an isometry. □

· Another application: interpretation of K using geodesic circles. Fix geodesic polar coords $\vec{x}(\rho,\theta)$. Let $L(r)$ = length(geodesic circle of radius r) = $\int_0^{2\pi} \|\vec{x}_\theta(r,\theta)\|\,d\theta = \int_0^{2\pi}\sqrt{G(r,\theta)}\,d\theta$.

Since $\sqrt{G(r,\theta)} \approx r$, this is $\approx 2\pi r$. To get a more precise result, we compute the Taylor expansion of $\sqrt{G}$. (By the proof of the above Proposition, $\sqrt{G}$ is smooth in $\rho \geq 0$!)

- $\lim_{\rho\to 0}\sqrt{G}=0$, $\lim_{\rho\to 0}(\sqrt{G})_\rho=1$.
- We have $(\sqrt{G})_{\rho\rho}=-K\sqrt{G} \xrightarrow{\rho\to 0} 0$, and $(\sqrt{G})_{\rho\rho\rho}=-K_\rho\sqrt{G}-K(\sqrt{G})_\rho \xrightarrow{\rho\to 0} -K(p)$.

$\Rightarrow \sqrt{G}=\rho+\frac{1}{3!}(-K(p))\rho^3+R(\rho,\theta)$, $\lim_{\rho\to 0} R(\rho,\theta)/\rho^3=0$.

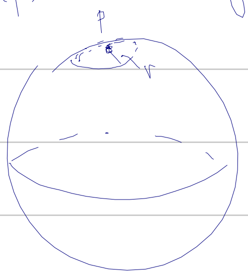
Therefore,

$$L(r)=\int_0^{2\pi} r-\frac{1}{6}K(p)r^3\,d\theta+R_1(r), \quad \lim_{r\to 0}\frac{R_1(r)}{r^3}=0$$
$$=2\pi r-\frac{\pi}{3}K(p)r^3+R_1(r).$$

Thm. $K(p)=\lim_{r\to 0}\left(\frac{3}{\pi}\,\frac{2\pi r-L(r)}{r^3}\right)$. ($L(r)=\text{length}(\exp_p(S_r))$, $S_r=\{v\in T_pS : \|v\|=r\}$.
Note: $\text{length}(S_r)=2\pi r$.)

Thus, $K(p)>0$ makes geodesic circles have smaller circumference.

Ex.



$\text{length}(\exp_p(S_r))=2\pi(\sin r)=2\pi r-2\pi\frac{1}{3!}r^3+\cdots$

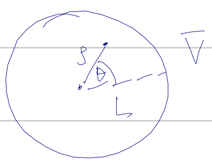
S^2 (unit sphere) $=2\pi r-\frac{\pi}{3}r^3+\cdots$,

and indeed $K(p)=1$.

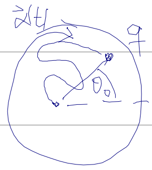
- Using geodesic polar coordinates, we can finally prove that geodesics are the shortest paths connecting two nearby points:

Prop. Let $p\in S$. Then $\exists$ nbhd. $V\subset S$ of p s.t. if $\vec{\gamma}:[0,t_1]\to V$ is a geodesic with $\vec{\gamma}(0)=p$, $\vec{\gamma}(t_1)=q\in V$, and if $\vec{\alpha}:[0,t_2]\to S$ is a regular curve with $\vec{\alpha}(0)=p$, $\vec{\alpha}(t_2)=q$, then
$$l_{\vec{\gamma}}=\text{length}(\vec{\gamma})\leq l_{\vec{\alpha}}=\text{length}(\vec{\alpha}).$$
Moreover, if $l_{\vec{\gamma}}=l_{\vec{\alpha}}$, then $\vec{\alpha}([0,t_2])=\vec{\gamma}([0,t_1])$.

Proof. Let $W\subset S$ be a normal neighborhood of p, and let $r>0$ be so small that $\overline{V}:=\exp_p(\{v\in T_pS:\|v\|\leq r\})\subset W$. Let (ρ,θ) be geodesic polar coordinates in $\overline{V}\setminus L$ centered at p.



Case 1. $\vec{\alpha}([0,t_2]) \subset V = \exp_p(\{\|v\|<\varepsilon\})$. Write $\vec{\alpha}(t) = \vec{x}(\rho(t), \theta(t))$; then



$\|\vec{\alpha}'\| = \sqrt{(\rho')^2 + G(\theta')^2} \geq \sqrt{(\rho')^2} = |\rho'|$

$\Rightarrow l_{\vec{\alpha}} = \int_0^{t_2} \|\alpha'(t)\|\, dt \geq \int_0^{t_2} |\rho'(t)|\, dt \geq |\rho(t_2) - \rho(0)| = \rho(t_2)$.

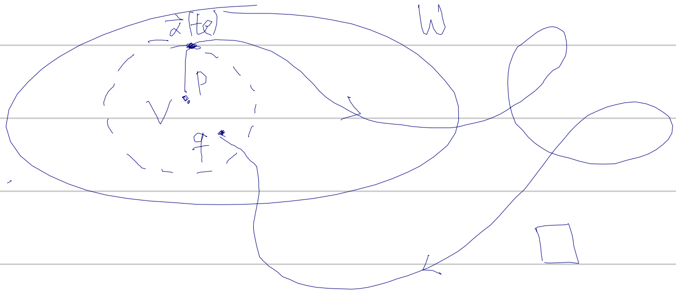
On the other hand, we can parameterize $\vec{\gamma}$ by arc length, $\vec{\gamma}(t) = \vec{x}(t, \theta_0)$, where $0 \leq t \leq t_1 = \rho(t_2)$ (where θ_0 = angular coordinate of $\vec{\alpha}(t_2)$ so that $\vec{\gamma}(t_1) = \vec{x}(\rho(t_2), \theta_0) = \vec{\alpha}(t_2) = q$ indeed); then $l_{\vec{\gamma}} = t_1 = \rho(t_2)$.

Therefore, $l_{\vec{\alpha}} \geq l_{\vec{\gamma}}$. Equality holds iff $\theta'(t) = 0$ and $\rho'(t) \geq 0$ $\forall t \in [0,t_2]$

$\Rightarrow \vec{\alpha}(t) = \vec{x}(\rho(t), \theta_0)$ is a reparameterization of $\vec{\gamma}$.

Case 2. If $\vec{\alpha}([0,t_2]) \not\subset V$, let $t_e \in [0,t_2]$ be the first time at which $\vec{\alpha}(t_e) \notin V$, so $\vec{\alpha}(t_e) \in \overline{V} \setminus V$. Let $\vec{\gamma}_e$ be the radial geodesic from p to $\vec{\alpha}(t_e)$. Then we already proved that $l(\vec{\gamma}_e) = r < l(\vec{\alpha}|_{[0,t_e]}) < l_{\vec{\alpha}}$, while l(radial geodesic $\vec{\gamma}$ from p to q) $< r$.

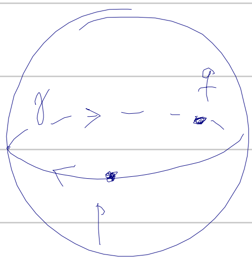
$\Rightarrow l_{\vec{\gamma}} < l_{\vec{\alpha}}$.



□

A geodesic, sufficiently extended, may not be the shortest path between its endpoints.

S^2



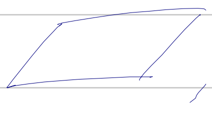
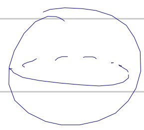
Cor. Let $\vec{\alpha}: I \to S$ be an arc length parameterized regular curve s.t. the arc length of α between any two points $\vec{\alpha}(t_1), \vec{\alpha}(t_2)$ $(t_1, t_2 \in I)$ is $\leq$ arc length of any regular curve connecting $\vec{\alpha}(t_1), \vec{\alpha}(t_2)$. Then $\vec{\alpha}$ is a geodesic.

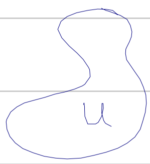
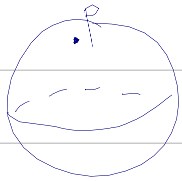
V Global differential geometry

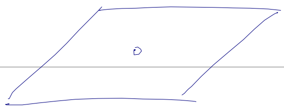
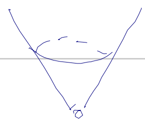
All surfaces from now on will be regular and connected.

V.1 Complete surfaces

Definition: A surface S is complete if for every point $p \in S$, any geodesic $\gamma: (-\varepsilon, \varepsilon) \to S$, $\gamma(0) = p$, can be extended to a parameterized geodesic $\gamma^*: \mathbb{R} \to S$.
(Equivalent: $\exp_p$ is defined on all of T_pS.)

Examples: Complete: $\mathbb{R}^2$ , S^2 , right cylinder 

Not complete: $U \subsetneq \mathbb{R}^2$ , $S^2 \setminus \{p\}$ ,

$\mathbb{R}^2 \setminus \{0\}$ , cone w/o vertex 

Recall:
- intrinsic distance $d(p,q) := \inf \ell(\vec{\alpha})$, $\vec{\alpha}: [0,T] \to S$ piecewise regular curve with $\vec{\alpha}(0) = p$, $\vec{\alpha}(T) = q$.
- S connected, $p, q \in S \Rightarrow d(p,q) < \infty$.

Lemma: d is a metric on S: $d(p,q) = d(q,p)$, $d(p,r) \leq d(p,q) + d(q,r)$, $d(p,q) \geq 0$, and $d(p,q) = 0$ iff $p = q$.

Proof: Only prove last assertion. Suppose $q \neq p$. Let $V \subset S$ be a neighborhood of p with $q \notin V$ s.t. every point in V can be joined to p by a unique geodesic in V. For $\varepsilon > 0$ small, the geodesic ball $B_\varepsilon(p) \subset V$ ($B_\varepsilon(p) = \exp_p(\{v \in T_pS : \|v\| < \varepsilon\})$). By assumption, $\exists\, \vec{\alpha}: [0,T] \to S$, $\vec{\alpha}(0) = p$, $\vec{\alpha}(T) = q$, $\ell(\vec{\alpha}) < \frac{\varepsilon}{2}$. But $q \notin V$ and $\vec{\alpha}([0,T])$ is connected, hence $\exists\, t \in (0,T)$ s.t. $\vec{\alpha}(t) \in \partial B_\varepsilon(p)$. Therefore, $\ell(\vec{\alpha}) \geq \varepsilon$, $\lightning$. $\square$

The metric d interacts well with the topology on S:

04/30/2020

Prop. Fix $p_0 \in S$. Then $f: S \to \mathbb{R}$, $f(p) = d(p_0, p)$, is a continuous function.

Proof. Let $p \in S$, $\varepsilon > 0$. Fix $\varepsilon' > 0$ s.t. $\exp_p: B_{\varepsilon'}(0) \to \exp_p(B_{\varepsilon'}(0)) =: V$ is a diffeomorphism; V is open so $\exists\ \delta > 0$ and an open ball $B_\delta(p) \subset \mathbb{R}^3$ s.t. $B_\delta(p) \cap S \subset V$. Therefore, if $q \in B_\delta(p) \cap S$,

$$|d(p_0, q) - d(p_0, p)| \leq d(p, q) < \varepsilon'. \qquad \square$$

Prop. A closed surface $S \subset \mathbb{R}^3$ is complete.

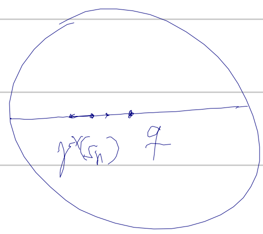
Proof. Let $\gamma: (-\varepsilon, \varepsilon) \to S$ be a geodesic; suppose $\gamma^*: I \to S$ is the maximal extension of γ as a parameterized geodesic. (i.e. $\tilde{\gamma}: J \to S$ geod., $J \supset (-\varepsilon, \varepsilon)$, $\tilde{\gamma} = \gamma$ on $(-\varepsilon, \varepsilon)$ $\Rightarrow J \subset I$, and $\tilde{\gamma} = \gamma^*$ on J.)

- Suppose $s_0 \in I$. Then γ^* agrees near s_0 with the unique geodesic through $\gamma^*(s_0)$ with tangent vector $\frac{d}{ds}\gamma^*(s_0)$. By local existence of geodesics, I is therefore open.
- Let $s_0 \in \mathbb{R}$ be such that $(s_0 - \varepsilon, s_0) \subset I$ for some $\varepsilon > 0$. Let $(s_n)_{n=1}^{\infty}$ be a sequence in $(s_0 - \varepsilon, s_0)$ with $s_n \xrightarrow{n \to \infty} s_0$. We claim that $\gamma^*(s_n)$ converges. Indeed,

$$\|\gamma^*(s_n) - \gamma^*(s_m)\|_{\mathbb{R}^3} \leq d(\gamma^*(s_n), \gamma^*(s_m)) \leq \ell(\gamma^*|_{[s_n, s_m]})$$
$$\leq C|s_m - s_n| \quad \text{where } C = \|\gamma'(0)\| \ (= \|(\gamma^*)'(s)\|\ \forall s).$$

$\Rightarrow \gamma^*(s_n)$ is a Cauchy sequence in $\mathbb{R}^3$, $\gamma^*(s_n) \xrightarrow{n \to \infty} q \in \mathbb{R}^3$. Since S is closed, $q \in S$.

- Fact: $\exists$ neighborhood $W \subset S$ of q, $\delta > 0$ s.t. $\forall\, r \in W$, $\exp_r$ is defined on $B_0(\delta) \subset T_r S$ and $W \subset \exp_r(B_0(\delta))$.



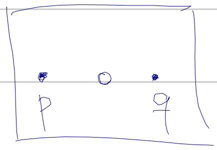
- Let n, m be s.t. $\gamma^*(s_n), \gamma^*(s_m) \in W$, $|s_n - s_m| < \delta/C$, so $d(\gamma^*(s_n), \gamma^*(s_m)) < \delta$, and let $\tilde{\gamma}$ be the unique geodesic contained in W joining $\gamma^*(s_n)$ and $\gamma^*(s_m)$; then $\tilde{\gamma}$ agrees with γ^*. Since $\exp_{\gamma^*(s_n)}$ is a diffeomorphism in $B_0(\delta)$ and $W \subset \exp_{\gamma^*(s_n)}(B_0(\delta))$, $\tilde{\gamma}$ and thus γ^* can be extended past q.

$\Rightarrow I$ is closed (contains s_0).

$\Rightarrow I \subset \mathbb{R}$ is open, closed, nonempty $\Rightarrow I = \mathbb{R}$. $\square$

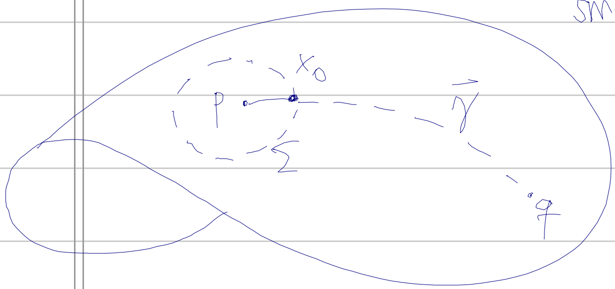
<u>Theorem</u> (Hopf–Rinow) Let S be a complete surface. Then for all $p, q \in S$, there exists a minimizing geodesic γ joining p and q. (That is, $d(p,q) = \ell(\gamma)$.)

<u>Remark.</u> · Minimizing geodesics do not always exist: $\mathbb{R}^2 \setminus \{0\}$



· If a curve C joining p and q exists with $\ell(C) = d(p,q)$, then C is a geodesic (exercise).

<u>Proof of thm.</u> Let $r = d(p,q)$. Let $x_0 \in \Sigma := \exp_p(\partial B_\varepsilon(0))$ (with $\varepsilon > 0$ sufficiently small) be so that $d(x_0, q) = \inf_{y \in \Sigma} d(y,q)$.



(x_0 exists since $d(-,q)$ is continuous on S, and Σ is compact.)

Let $\vec{\gamma}$ = arc length parameterized geodesic s.t. $\vec{\gamma}(0) = p$, $\vec{\gamma}(\varepsilon) = x_0$. Since S is complete, $\vec{\gamma}$ is defined for all real arguments. <u>Claim</u>: $\vec{\gamma}(r) = q$.

<u>Proof of claim.</u> We shall show $d(\vec{\gamma}(s), q) = r - s$ for $s \in [\varepsilon, r]$. For $s = r$, this implies the claim. Let

$$A = \{ s \in [\varepsilon, r] : d(\vec{\gamma}(s), q) = r - s \}.$$

(i) A is non-empty: $r = d(p,q) = \inf_\alpha \ell(\alpha_{p,q}) = \inf_{x \in \Sigma} \left(\inf_\alpha \ell(\alpha_{p,x}) + \inf_\alpha \ell(\alpha_{x,q}) \right)$

$$= \inf_{x \in \Sigma} \left(d(p,x) + d(x,q) \right) = \varepsilon + \inf_{x \in \Sigma} d(x,q) = \varepsilon + d(x_0, q)$$

$\Rightarrow d(\vec{\gamma}(\varepsilon), q) = d(x_0, q) = r - \varepsilon$.

(ii) A is closed: follows from the continuity of $d(-,q) : S \to \mathbb{R}$.

(iii) A is open in $[\varepsilon, r]$: need to show that if $s_0 \in A$, $s_0 < r$, then $s_0 + \delta \in A$ for small $\delta > 0$.

Fix $\delta>0$ so small that $\exp_{\vec{\gamma}(s_0)}|_{B_0(\delta)}$ is a diffeomorphism. Let $\Sigma' := \exp_{\vec{\gamma}(s_0)}(\partial B_0(\delta))$.
Let $x_0' \in \Sigma'$ be such that $d(x_0', q) = \min_{x' \in \Sigma'} d(x', q)$. Then

$$d(x_0', q) = d(\vec{\gamma}(s_0), q) - \delta \quad \text{(same argument as before)}$$
$$= r - s_0 - \delta.$$

Moreover, $d(p, x_0') \geq d(p,q) - d(x_0', q) = r - (r - s_0 - \delta) \geq s_0 + \delta$.
But the curve $\tilde{\gamma}$ obtained by concatenating $\gamma|_{[0,s_0]}$ with the (radial) geodesic from $\vec{\gamma}(s_0)$ to x_0' has length $= s_0 + \delta \leq d(p, x_0')$. Thus, $\tilde{\gamma}$ minimizes distance between any of its points $\Rightarrow \tilde{\gamma}$ is a geodesic extending $\gamma \Rightarrow$ $\vec{\gamma}(s_0 + \delta) = x_0' \Rightarrow s_0 + \delta \in A$ indeed.

- In conclusion, $A \subset [\varepsilon, r]$ is nonempty, closed and open $\Rightarrow A = [\varepsilon, r]$. $\square$

<u>Corollary</u>. If S is complete, then $\exp_p : T_pS \to S$ is surjective.

<u>Corollary</u>. If S is complete, and $\exists\, R>0$ s.t. $d(p,q) < R$ $\forall\, p,q \in S$, then S is compact.

<u>Proof</u>. $d(p,q) < R$ $\forall\, p,q \Rightarrow \exists\, \overline{B}_r(0) \subset T_pS$ $(r<R)$ s.t. $S = \exp_p(\overline{B}_r(0))$.
$\exp_p$ is continuous and $\overline{B}_r(0)$ is compact $\Rightarrow S$ is compact. $\square$

V.2 Abstract surfaces.

We now consider surfaces without any reference to an ambient space such as $\mathbb{R}^3$.

<u>Def.</u> An <u>abstract surface</u> (or "smooth manifold of dimension 2") is a set S together with a family of injective maps $\vec{x}_\alpha : U_\alpha \to S$, $U_\alpha \subset \mathbb{R}^2$ open, such that:

(i) $S = \bigcup_\alpha \vec{x}_\alpha(U_\alpha)$;

(ii) for each pair α, β with $\vec{x}_\alpha(U_\alpha) \cap \vec{x}_\beta(U_\beta) = W \neq \emptyset$, the sets $\vec{x}_\alpha^{-1}(W), \vec{x}_\beta^{-1}(W) \subset \mathbb{R}^2$ are open, and $\vec{x}_\beta^{-1} \circ \vec{x}_\alpha$, $\vec{x}_\alpha^{-1} \circ \vec{x}_\beta$ are smooth maps (i.e. $\vec{x}_\beta^{-1} \circ \vec{x}_\alpha : \vec{x}_\alpha^{-1}(W) \to \vec{x}_\beta^{-1}(W)$ is a diffeomorphism).

• For $p \in S$, a pair $(U_\alpha, \vec{x}_\alpha)$ with $p \in U_\alpha$ is called a parameterization of S around p; $\vec{x}_\alpha$ is a coordinate neighborhood, and $\{(U_\alpha, \vec{x}_\alpha)\}$ is called a smooth structure for S.

Example: real projective plane. Let $\mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2=1\}$, and let $A : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map, $A(x,y,z) = (-x,-y,-z)$. Let

$\mathbb{RP}^2 := \mathbb{S}^2/\sim$, where $p \sim q$ iff $p=q$ or $p = A(q)$.

(That is, identify antipodal points of $\mathbb{S}^2$.)

Denote $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$, $p \mapsto \{p, A(p)\}$. Cover $\mathbb{S}^2$ with parameterizations $\vec{x}_\alpha : U_\alpha \subset \mathbb{R}^2 \to \mathbb{S}^2$ s.t. $\vec{x}_\alpha(U_\alpha) \cap A(\vec{x}_\alpha(U_\alpha)) = \phi$. Since A is a diffeomorphism and $\mathbb{S}^2$ is a regular surface, $\mathbb{RP}^2$ together with the maps $\pi \circ \vec{x}_\alpha : U_\alpha \to \mathbb{RP}^2$ is an abstract surface.

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• Let S be an abstract surface. Can define smooth functions on S as on regular surfaces. Tangent vectors are more subtle since there is no ambient vector space ($\mathbb{R}^3$) anymore; instead, we regard tangent vectors as gadgets to differentiate functions along:

Def. • Let $\vec{\alpha} : (-\varepsilon, \varepsilon) \to S$ be a smooth curve (i.e. $\vec{x}_\beta^{-1} \circ \vec{\alpha} : (-\varepsilon,\varepsilon) \to \mathbb{R}^2$ is a smooth parameterized curve) with $\vec{\alpha}(0) = p \in S$. Let

$D = \{$functions on S differentiable at $p\}$.

The tangent vector to the curve $\vec{\alpha}$ at $t=0$ is the functional

$$\vec{\alpha}'(0) : D \to \mathbb{R}, \qquad \vec{\alpha}'(0)(f) = \frac{d}{dt} f(\vec{\alpha}(t))\Big|_{t=0} \quad (f \in D).$$

• A tangent vector at p is the tangent vector of some curve through p.

In coordinates $(\vec{x}, U)$ near p, write $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$ and $f \in D$ as $f(u,v)$; WLOG $p = \vec{x}(0,0)$. Then

$$\vec{\alpha}'(0)(f) = \frac{d}{dt}(f \circ \vec{\alpha})\Big|_{t=0} = \frac{d}{dt} f(u(t), v(t))\Big|_{t=0} = \left(\frac{\partial f}{\partial u}\right)_{(0,0)} u'(0) + \left(\frac{\partial f}{\partial v}\right)_{(0,0)} v'(0)$$

$$= \left(u'(0) \left(\frac{\partial}{\partial u}\right)_{(0,0)} + v'(0) \left(\frac{\partial}{\partial v}\right)_{(0,0)} \right)(f),$$

where $\left(\frac{\partial}{\partial u}\right)_{(0,0)} : f \mapsto \frac{\partial f}{\partial u}(0,0)$. ($\left(\frac{\partial}{\partial u}\right)_{(0,0)}, \left(\frac{\partial}{\partial v}\right)_{(0,0)}$ are the tangent vectors at p of the coordinate curves, analogous to $\vec{x}_u, \vec{x}_v$!)

Cor. $T_pS := \{$tangent vectors of S at $p\}$ is a 2-dimensional vector space.

Def. Let S_1, S_2 be abstract surfaces, $\varphi : S_1 \to S_2$ smooth. Given $p \in S_1$, $\vec{v} \in T_pS_1$, let $\vec{\alpha} : (-\varepsilon, \varepsilon) \to S_1$ be a smooth curve with $\vec{\alpha}(0) = p$, $\vec{\alpha}'(0) = \vec{v}$. Then

$$d\varphi_p(\vec{v}) = \frac{d}{dt}\varphi(\vec{\alpha}(t))\Big|_{t=0} \in T_{\varphi(p)}S_2$$ is well-defined (independent of $\vec{\alpha}$) and called the differential of φ at p.

Def. • A 2-dimensional Riemannian manifold (or "geometric surface") is an abstract surface S together with a choice of a positive definite inner product $\langle \cdot, \cdot \rangle_p : T_pS \times T_pS \to \mathbb{R}$ which varies smoothly with p in the following sense: for any parameterization $\vec{x} : U \to S$,

$$E(u,v) = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle_p, \quad F(u,v) = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle_p, \quad G(u,v) = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle_p$$

are differentiable functions in U.

• $\langle \cdot, \cdot \rangle$ is called a (Riemannian) metric on S.

(Open problem (Schlaefli 1862). S = geometric surface, $p \in S$. Show that there exists a neighborhood $U \subset S$ of p and a smooth map $\varphi : U \to \mathbb{R}^3$ with injective differential s.t. the first fundamental form of $\varphi(U)$ agrees with the Riemannian metric on S.)

All notions depending only on the 1st fundamental form can be defined on geometric surfaces: Christoffel symbols, covariant derivatives, parallel transport, geodesics, Gauss curvature, intrinsic distance, completeness.

The hyperbolic plane $\mathbb{H}$.

Let $S = \mathbb{R}^2$ with coordinates (u,v). Define an inner product at $q=(u,v)$ by

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle_q = E = 1, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle_q = F = 0, \quad \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle_q = G = e^{2u}.$$

Def. $\mathbb{H} = (S, \langle \cdot, \cdot \rangle)$ is called the hyperbolic plane. Gauss curvature:

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \right) = -\frac{1}{2e^u}\left(\frac{2e^{2u}}{e^u}\right)_u = -1.$$

(Lobachevsky geometry: satisfies Euclid's postulates except for #5.)

· Upperhalf plane model. Define $\Phi : \mathbb{H} \to \mathbb{R}^2_+ = \{(x,y) : y > 0\}$,

$$\Phi(u,v) = (v, e^{-u});$$

the diffeomorphism Φ induces an inner product on $\mathbb{R}^2_+$ via

$$\langle d\Phi_q(v_1), d\Phi_q(v_2) \rangle_{\Phi(q)} := \langle v_1, v_2 \rangle_q .$$

Have $(x,y) = (v, e^{-u})$, so $(u,v) = (-\log y, x)$ (i.e. $\Phi^{-1}(x,y) = (-\log y, x)$).

$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} = -\frac{1}{y}\frac{\partial}{\partial u} = -e^u \frac{\partial}{\partial u}$, so

$$\begin{cases} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = e^{2u} = y^{-2} \\ \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = 0 \\ \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = e^{2u}\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = y^{-2}. \end{cases}$$

$(\mathbb{R}^2_+$, this metric) is called the Poincaré half plane.

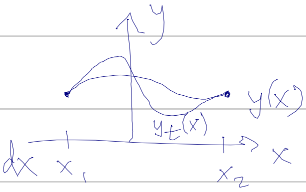
· $\mathbb{H}$ is complete.

· Curves $x = \text{const.}$ are geodesics. Find other geodesics $\vec{\gamma}$ of the form $y = y(x)$, so $\vec{\gamma}(t) = (x(t), y(x(t)))$.

$x' > 0. \Rightarrow l(\vec{\gamma}) = \int \sqrt{E \cdot 1^2 + G \cdot (y')^2}\, x'(t)\, dt = \int \sqrt{\frac{1}{y^2} + \frac{(y')^2}{y^2}}\, dx = \int \frac{\sqrt{1+(y')^2}}{y}\, dx.$

Let $y_t(x)$ be a variation of $y(x)$ s.t. $y_t(x_1) = y(x_1)$, $y_t(x_2) = y(x_2)$ $\forall t$, $y_0(x) = y(x)$.

Then the variation of length of $\vec{\gamma}_t(x) = (x, y(x))$ is



$$\frac{d}{dt}\int_{x_1}^{x_2} \frac{\sqrt{1+(y_t')^2}}{y_t}\, dx = \int_{x_1}^{x_2} \left(-\frac{\sqrt{1+(y_t')^2}}{y_t^2}\frac{\partial y_t}{\partial t} + \frac{y_t'}{y_t\sqrt{1+(y_t')^2}}\frac{\partial y_t'}{\partial t} \right) dx$$

$$(\text{integrate by parts}) = \underbrace{\left. \frac{\partial y_t}{\partial t}\frac{y_t'}{y_t\sqrt{1+(y_t')^2}} \right|_{x_1}^{x_2}}_{=0!} + \int_{x_1}^{x_2} \left(-\frac{\partial y_t}{\partial t}\right)\left(\frac{\sqrt{1+(y_t')^2}}{y_t^2} + \frac{\partial}{\partial x}\frac{y_t'}{y_t\sqrt{1+(y_t')^2}} \right) dx.$$

Set $t=0$. $\frac{d}{dt}\int_{x_1}^{x_2} \frac{\sqrt{1+(y_t')^2}}{y_t}\,dx\Big|_{t=0} = 0$ for all variations $x \mapsto \frac{\partial y_t}{\partial t}(0,x)$ if and only if $y(x)$ satisfies the (Euler–Lagrange) equation

$$\frac{\sqrt{1+(y')^2}}{y^2} + \frac{d}{dx}\frac{y'}{y\sqrt{1+(y')^2}} = 0.$$

(Geodesics are locally length-minimizing, hence must satisfy this ODE when of the form $x \mapsto (x, y(x))$.)

$$\left(\frac{z}{\sqrt{1+z^2}}\right)' = \frac{1}{(1+z^2)^{3/2}}$$

$\Updownarrow$

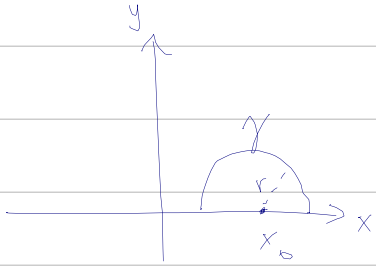
$$y^{-2}\sqrt{1+(y')^2} + \frac{y''}{y}\frac{1}{(1+(y')^2)^{3/2}} - \frac{(y')^2}{y^2\sqrt{1+(y')^2}} = 0$$

$\Leftrightarrow$ $(1+(y')^2)^2 + yy'' - (y')^2(1+(y')^2) = 0$

$\Leftrightarrow$ $(y')^2 + yy'' = -1$

$\Leftrightarrow$ $(y^2)'' = 2yy'' + 2(y')^2 = -2$

$\Leftrightarrow$ $y^2 = -(x-x_0)^2 + r^2$ for some $r, x_0 \in \mathbb{R}$.



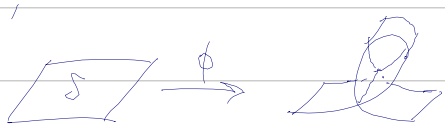
Conclusion: Geodesic in upper half plane are circles orthogonal to the x-axis, and vertical lines.

Hilbert's Theorem (1901). A complete geometric surface S with constant negative Gauss curvature cannot be isometrically immersed in $\mathbb{R}^3$.

Def. A smooth map $\phi: S \to \mathbb{R}^3$ from an abstract surface into $\mathbb{R}^3$ is called an immersion if $d\phi_p: T_pS \to \mathbb{R}^3$ is injective for all $p \in S$. If in addition S has a metric $\langle -,- \rangle$ and

$$\langle d\phi_p(v), d\phi_p(w)\rangle_{\varphi(p)} = \langle v, w\rangle_p, \quad v, w \in T_pS,$$

then ϕ is said to be an isometric immersion.



- For example, $\mathbb{H}$ is not isometric to a surface in $\mathbb{R}^3$ even if we allow self-intersections.
- Neat fact: For every $g \geq 2$, there exist compact genus g abstract surfaces with constant Gauss curvature -1 metrics. (These cannot be isometrically embedded (or immersed) into $\mathbb{R}^3$ since compact regular surfaces always have points with $K > 0$.)

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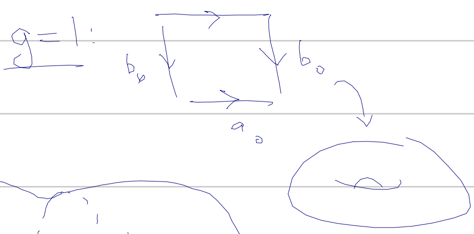
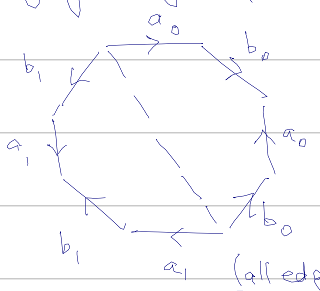
- Why? Note first that if S is one of our genus g surfaces diffeomorphic to

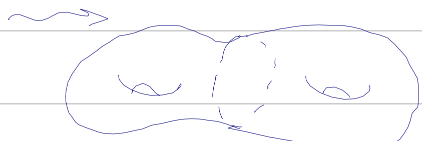
then by Gauss–Bonnet, $2\pi\chi(S) = 2\pi(1-g) = \iint_S K\,d\sigma < 0$,
so $g \geq 2$ is necessary.

Sufficiency of $g \geq 2$? Seek special S; starting point: the g-holed torus arises by identifying edges of a $4g$-gon: $\underline{g=1}$:



$\underline{g=2}$





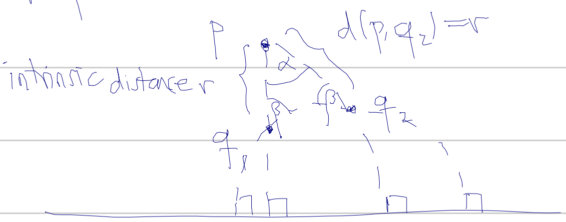
(all edges have the same length; all interior angles are equal)

$\rightsquigarrow$ Question: Do there exist regular n-gons R $(n = 4g)$ in $\mathbb{H}$ with geodesic edges?

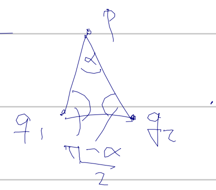
Answer: Yes, for $g \geq 2$ (so $n \geq 8$), and in fact for $n \geq 5$.

Proof. (i) Claim: Let $p \in \mathbb{H}$, $\alpha \in (0,\pi)$, $\beta_0 \in (0, \frac{\pi-\alpha}{2})$. Then $\exists$ isosceles geodesic triangle with base angles β_0 and angle α at p.

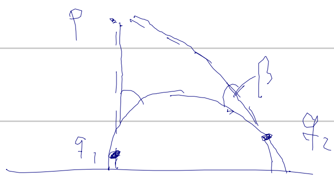
Proof: upper half plane:



As $r \to 0$, $\beta \to$ euclidean value $\frac{\pi-\alpha}{2}$



As $r \to \infty$, $\beta \to 0$:



$\square$

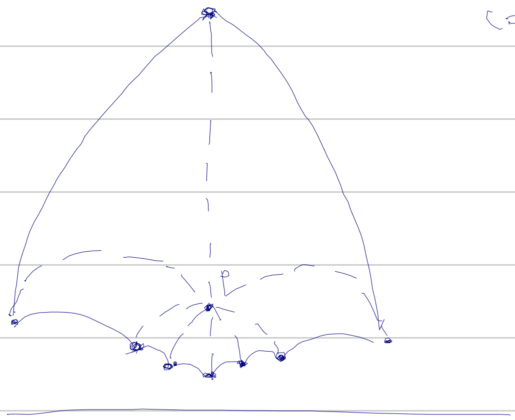
(ii) Can rotate this construction around p (by Minding's theorem).

(iii) Apply the construction n times with $\alpha = \frac{2\pi}{n}$, $\beta = \frac{\pi}{n}$

(so $\beta < \frac{\pi - \alpha}{2}$ holds since $\frac{\pi}{n} < \frac{\pi(1-\frac{2}{n})}{2} = \pi \, \frac{n-2}{2n}$

$\iff 2 < n-2 \iff n > 4$).

$n=8$:



Interior angles at the points $q_1, \dots, q_{2n}$ (which get identified with one another!) add up to $2n \cdot \frac{\pi}{2} = 2\pi$

$\Rightarrow$ after doing identification of edges as above, get smooth genus g surface with $K \equiv -1$. $\square$

V.3 First and second variation of length

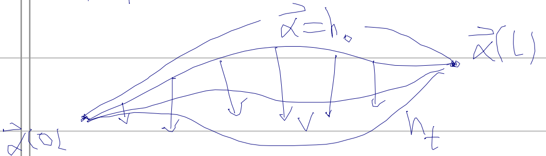
Goal: Bonnet's theorem. If a complete surface S has $K \geq \delta > 0$, then S is compact, and $\operatorname{diam}(S) = \sup_{p,q \in S} d(p,q) \leq \frac{\pi}{\sqrt{\delta}}$.

Key tool: formulas for arc length under small variations of a curve.

Def. • Let $\vec{\alpha} : [0,L] \to S$ be a regular arc length parameterized curve. A variation of $\vec{\alpha}$ is a smooth map $h : [0,L] \times (-\varepsilon, \varepsilon) \to S$ s.t. $h(s,0) = \vec{\alpha}(s) \ \forall s$.

• The curves $h_t(s) := h(s,t)$ are the curves of variation of h.

• h is proper if $h(0,t) = \vec{\alpha}(0)$ and $h(L,t) = \vec{\alpha}(L) \ \forall t$.



• The vector field $V(s) = \frac{\partial h}{\partial t}(s,0)$ is the variational vector field of h.

(If h is proper, then $V(0) = V(L) = 0$.)

Prop. If V is a vector field along $\vec{\alpha}$, $\exists$ variation $h:[0,L]\times(-\varepsilon,\varepsilon)\to S$ of $\vec{\alpha}$ with variational vector field V. If $V(0)=V(L)=0$, then h can be chosen to be proper.

Proof Set $h(s,t)=\exp_{\vec{\alpha}(s)}(tV(s))$. For $|t|<\varepsilon$, ε small, this is well-defined, and $\frac{\partial h}{\partial t}(s,0)=\frac{d}{dt}\exp_{\vec{\alpha}(s)}(tV(s))\Big|_{t=0}=V(s)$ indeed. $\square$

Want to compare arclengths of th curves of variation. Let

$$L(t)=\int_0^L \left\|\frac{\partial h}{\partial s}(s,t)\right\| ds, \qquad t\in(-\varepsilon,\varepsilon).$$

Prop. Let h be a proper variation of $\vec{\alpha}:[0,L]\to S$; let $V(s)=\frac{\partial h}{\partial t}(s,0)$. Then

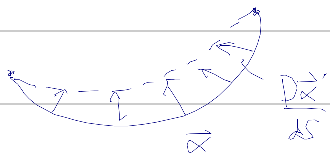
$$L'(0)=-\int_0^L \left\langle \frac{D\vec{\alpha}'}{ds}(s), V(s)\right\rangle ds.$$

Proof We have

$$L'(t)=\int_0^L \frac{d}{dt}\sqrt{\left\langle \frac{\partial h}{\partial s},\frac{\partial h}{\partial s}\right\rangle}\, ds$$

$$=\int_0^L \frac{\left\langle \frac{D}{dt}\frac{\partial h}{\partial s},\frac{\partial h}{\partial s}\right\rangle}{\left|\frac{\partial h}{\partial s}\right|}\, ds$$

$$=\int_0^L \frac{\left\langle \frac{D}{ds}\frac{\partial h}{\partial t},\frac{\partial h}{\partial s}\right\rangle}{\left|\frac{\partial h}{\partial s}\right|}\, ds.$$



"Moving $\vec{\alpha}$ towards its acceleration vector makes it shorter."

At $t=0$, $\left|\frac{\partial h}{\partial s}\right|=1$ (since $\vec{\alpha}$ is arclength parameterized), so

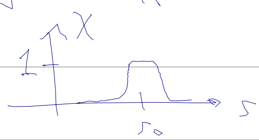
$$L'(0)=\int_0^L \left\langle \frac{D}{ds}\frac{\partial h}{\partial t},\frac{\partial h}{\partial s}\right\rangle ds=\int_0^L \frac{d}{ds}\left\langle \frac{\partial h}{\partial t},\frac{\partial h}{\partial s}\right\rangle-\left\langle \frac{\partial h}{\partial t},\frac{D}{ds}\frac{\partial h}{\partial s}\right\rangle ds$$

$$=\underbrace{\left\langle \frac{\partial h}{\partial t},\frac{\partial h}{\partial s}\right\rangle\Big|_0^L}_{=0 \text{ since } \frac{\partial h}{\partial t}(0,t)=\frac{\partial h}{\partial t}(L,t)=0\ \forall t}-\int_0^L\left\langle V,\frac{D\vec{\alpha}'}{ds}\right\rangle ds. \qquad \square$$

Cor. An arc length parameterized curve $\vec{\alpha}:[0,L]\to S$ is a geodesic iff $L'(0)=0$ for all proper variations h of $\vec{\alpha}$.

Pf. If $\frac{D\vec{\alpha}'}{ds}(s_0)\neq 0$, $s_0\in(0,L)$, can find a vector field V along $\vec{\alpha}(s)$, nonzero only near $s=s_0$, s.t. $\left\langle \frac{D\vec{\alpha}'}{ds},V\right\rangle\geq 0\ \forall s$, $\left\langle \frac{D\vec{\alpha}'}{ds}(s_0),V(s_0)\right\rangle>0$. (E.g. $V(s)=\chi(s)\frac{D\vec{\alpha}'}{ds}(s)$.)

Let $h=$ proper variation with var. v.f. $V\Rightarrow L'(0)<0$. $\square$



(Cf. characterization of minimal surfaces as critical points of area.)

From now on, study variations of arc length parameterized geodesics $\vec{\alpha}:[0,L]\to S$; restrict to proper orthogonal variations, i.e. $\langle \vec{\alpha}'(s), V(s)\rangle = 0 \ \forall s$.

Goal: formula for $L''(0)$. (Is $L(0)$ a local minimum? Maximum? Saddle point?)

Prop. Let $h:[0,L]\times(-\varepsilon,\varepsilon)\to S$ be a proper orthogonal variation of an arc length parameterized geodesic $\vec{\gamma}:[0,L]\to S$. Let $V=\frac{\partial h}{\partial t}(s,0)$. Then

$$L''(0)=\int_0^L \left(\left\| \frac{D}{ds}V(s)\right\|^2 - K(s)\|V(s)\|^2 \right) ds. \quad (*)$$

Proof. Recall:

$$L'(t)=\int_0^L \frac{\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\rangle}{\|\frac{\partial h}{\partial s}\|}\, ds \qquad \left(\text{using } \frac{D}{dt}\frac{\partial h}{\partial s}=\frac{D}{ds}\frac{\partial h}{\partial t}\right)$$

$$\Rightarrow L''(t)=\int_0^L \frac{\frac{d}{dt}\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\rangle}{\|\frac{\partial h}{\partial s}\|} - \frac{\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\rangle^2}{\|\frac{\partial h}{\partial s}\|^3}\, ds.$$

- At $t=0$, we have

$$\left\|\frac{\partial h}{\partial s}\right\|=1 \ (\vec{\gamma}=\text{arc length parametrized}), \quad \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle=0 \ (V\perp\vec{\gamma}'(s)),$$

so $0=\frac{d}{ds}\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\rangle = \langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\rangle + \langle \frac{\partial h}{\partial t}, \underbrace{\frac{D}{ds}\frac{\partial h}{\partial s}}_{=0}\rangle$

$$\Rightarrow L''(0)=\int_0^L \frac{d}{dt}\left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle ds.$$

- Compute (at $t=0$)

$$\frac{d}{dt}\left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle = \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{D}{dt}\frac{\partial h}{\partial s}\right\rangle$$

$$= \left\langle \left(\frac{D}{dt}\frac{D}{ds}-\frac{D}{ds}\frac{D}{dt}\right)\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\|\frac{D}{ds}V\right\|^2.$$

- 3rd term: as in $(*)$
- 2nd term: $= \frac{d}{ds}\left\langle \frac{D}{dt}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle$ (since $\vec{\gamma}$ = geodesic),

so $\int_0^L (\text{—})\, ds = \left\langle \frac{D}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle\Big|_0^L = 0$ $\left(\frac{\partial h}{\partial t}(0,t)=0, \frac{\partial h}{\partial t}(L,t)=0\right.$ since h is proper$\left.\right)$.

- 1st term:

Lemma Let $W(s,t)$ be a smooth vector field on S along h (i.e. $W(s,t)\in T_{h(s,t)}S$). Then $\frac{D}{\partial t}\frac{D}{\partial s}W - \frac{D}{\partial s}\frac{D}{\partial t}W = \underbrace{K(s,t)}_{\text{Gauss curvature of } S \text{ at } h(s,t).}\left(\frac{\partial h}{\partial s}\times\frac{\partial h}{\partial t}\right)\times W$

Proof Direct calculation. □

$\Rightarrow$ 1st term at $t=0$: $\left\langle K(s,0)\left(\frac{\partial h}{\partial s}\times\frac{\partial h}{\partial t}\right)\times\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle$

$$= -K(s,0)\left\langle \|V(s)\|^2\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle = -K(s,0)\|V(s)\|^2. \qquad \square$$

05/12/2020

<u>Proof of Bonnet's theorem</u> $(K \geq \delta > 0 \Rightarrow \operatorname{diam}(S) \leq \pi/\sqrt{\delta}.)$

Given any 2 points $p, q \in S$, there exists (by Hopf–Rinow) a minimal geodesic γ on S joining p to q. We shall prove that $l := d(p,q) \leq \pi/\sqrt{\delta}$.

- Assuming that $l > \pi/\sqrt{\delta}$, we consider a variation of the geodesic $\gamma : [0, l] \to S$ defined as follows: let $w_0 \in T_{\gamma(0)}S$ be a unit vector $w_0 \perp \gamma'(0)$, and let $w(s)$, $s \in [0, l]$, be the parallel transport of w_0 along γ. (In particular, $\|w(s)\| = 1$, $\langle w(s), \gamma'(s)\rangle = 0 \ \forall s$.) Consider the vector field $V(s) = f(s) w(s)$, with f to be chosen with $f(0) = 0$, $f(l) = 0$. For a proper variation of γ with variational vector field V, we then have

$$L''(0) = \int_0^l \left\|\frac{D}{ds}V\right\|^2 - K(s)\|V(s)\|^2 \, ds = \int_0^l f'(s)^2 - K(s) f(s)^2 \, ds.$$

- We claim that for $f(s) = \sin\left(\frac{\pi}{l} s\right)$, this is negative; indeed,

$$L''(0) = \int_0^l \frac{\pi^2}{l^2}\cos^2\left(\frac{\pi}{l}s\right) - K \sin^2\left(\frac{\pi}{l}s\right) ds$$

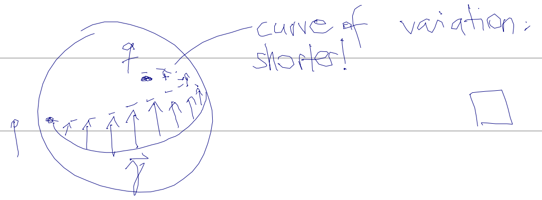
$$< \int_0^l \frac{\pi^2}{l^2}\left(\cos^2\left(\frac{\pi}{l}s\right) - \sin^2\left(\frac{\pi}{l}s\right)\right) ds$$

$K \geq \delta > \frac{\pi^2}{l^2}$

$$= \frac{\pi^2}{l^2}\int_0^l \cos\left(\frac{2\pi}{l}s\right) ds = \frac{\pi^2}{l^2} \cdot \frac{l}{2\pi} \sin\left(\frac{2\pi s}{l}\right)\Big|_0^l = 0.$$

- On the other hand, <u>any</u> curve joining p to q has length $\geq l = L(0)$, hence $L(0)$ must be a local minimum for <u>all</u> variations $\Rightarrow L''(0) \geq 0$. Contradiction!

$\Longrightarrow l \leq \pi/\sqrt{\delta}$, finishing the proof. $\square$



<u>Remark</u> regarding the choice of f: using the same ideas and techniques that we used in the discussion of geodesics on $\mathbb{H}$, convince yourself that the functional $f \longmapsto \int_0^l f'(s)^2 - K(s) f(s)^2 ds$ has f as a critical point (for all variations of f) if $f''(s) + K(s) f(s) = 0$; this suggest trying $\sin$ & $\cos$ for f. Needing to satisfy $f(0) = 0$, $f(l) = 0$ gives the f above. (<u>Note</u>: It is not necessary to find the "best" f: any f for which $L''(0) < 0$ serves our purpose in the proof!)

End of semester fun:

VI. Lorentzian geometry in 2 dimensions

Def. A (2-dimensional) spacetime is an abstract surface S together with a bilinear form $\langle\cdot,\cdot\rangle_p : T_pS \times T_pS \to \mathbb{R}$ of signature $(1,1)$ which depends smoothly on p: its coefficients $E(u,v), F(u,v), G(u,v)$ in local coordinates (u,v) depend smoothly on u,v.

Example: Minkowski space: $S = \mathbb{R}^2$, coordinates t, x,
$E = -1$, $F = 0$, $G = 1$. So $\langle a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}, a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\rangle = -a^2 + b^2$.

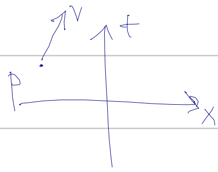
Def. S = spacetime, $p \in S$.

(i) $v \in T_pS, v \neq 0$ is called
- timelike if $\langle v,v\rangle_p < 0$
- lightlike if $\langle v,v\rangle_p = 0$ (& gravitational waves)
- spacelike if $\langle v,v\rangle_p > 0$.

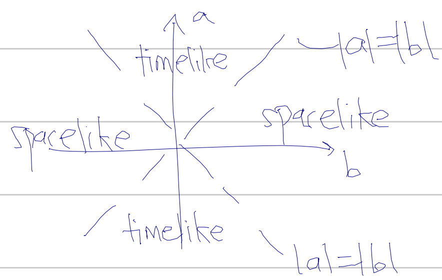
(ii) A regular curve $\alpha: I \to S$ is called timelike, ... if $\alpha'(t)$ is timelike, ... for all $t \in I$.

Physics: light travels along lightlike curves, massive objects along timelike curves.

Example Minkowski space, $v = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$.



T_pS :



Timelike: $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$ $(|v|<1)$

Lightlike: $\frac{\partial}{\partial t} - \frac{\partial}{\partial x}$ ("speed of light: 1")

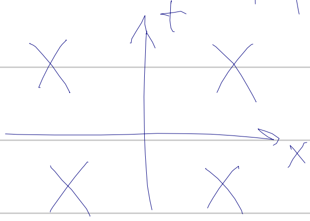
Spacelike: $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}$ ("faster-than-light")

Def. $L_pM := \{v \in T_pS : \langle v,v\rangle_p = 0\} \subset T_pS$ is the light cone at $p \in S$.

- Useful way to visualize properties of spacetime: draw light cones at a few points

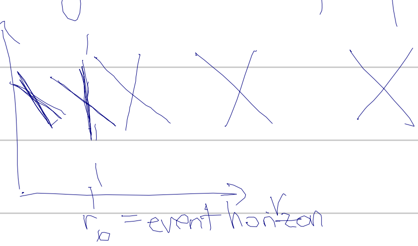
Examples

(1) Minkowski:



(2) Toy black hole:

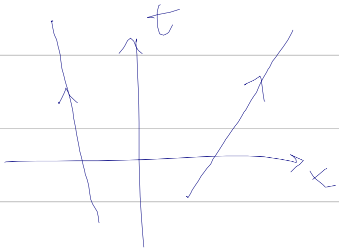
(Physics: light (or observer) traveling into $r < r_0$ as t increases cannot enter $r > r_0$ ever again.)



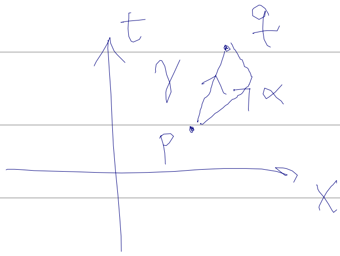
Rmk. Specifying the light cone at $p \in S$ determines $\langle \cdot, \cdot \rangle_p$ up to a conformal factor (i.e. if $\langle v,v \rangle_p = 0$ iff $\langle v,v \rangle'_p = 0$ $\forall v \in T_pS$,
then $\exists \lambda \in \mathbb{R}, \lambda \neq 0$, s.t. $\langle v,w \rangle_p = \lambda \langle v,w \rangle'_p$ $\forall v,w \in T_pS$).

• Intrinsic geometry notions apply: parallel transport, geodesics, Gauss curvature, etc.
"Freely falling observers" move through S along timelike geodesics — geodesics $\gamma : I \to S$ (thus with vanishing acceleration $\frac{D\gamma'}{ds} = 0$) s.t. $\gamma'(s)$ is timelike.

Ex. Minkowski:



• "Length" of curves?
Def. If $\alpha : [0,L] \to S$ is a smooth timelike curve, then the proper time along α is
$$\tau(\alpha) := \int_0^L \sqrt{-\langle \alpha'(t), \alpha'(t) \rangle}\, dt \quad (= \text{time elapsed on watch}).$$
Fact: if in a normal neighborhood $V \subset S$ of p, $\gamma : [0,L] \to S$ is a timelike geodesic connecting p and $q \in V$, then $\tau(\gamma) \geq \tau(\alpha)$ for any other timelike curve $\alpha : [0,L'] \to S$, $\alpha(0) = p$, $\alpha(L') = q$.

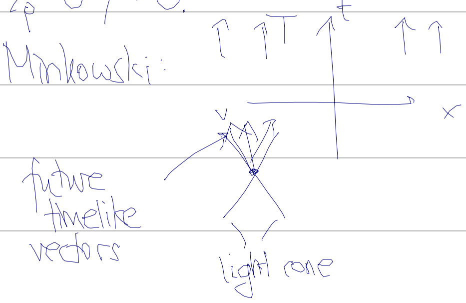


⇝ Twin paradox: $\tau(\gamma) > \tau(\alpha)$: when α and γ meet again at q, less time has elapsed for α than for γ. (More generally, the twin paradox merely gives a fancy name to the fact that different curves with the same start and end points typically have different proper length.)

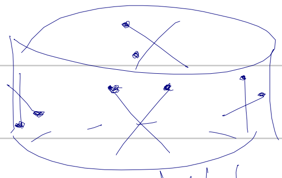
Rmk. The funny signs are due to the negative sign of the "squared length" of timelike vectors.

Def. A <u>time orientation</u> on S is a smooth vector field T on S s.t. $T(p) \in T_pS$ is timelike for all $p \in S$. Given T, we call a timelike vector $v \in T_pS$ future/past timelike if $\langle v, T \rangle_p < 0$ / > 0.

Ex. (1) Minkowski:



(2)

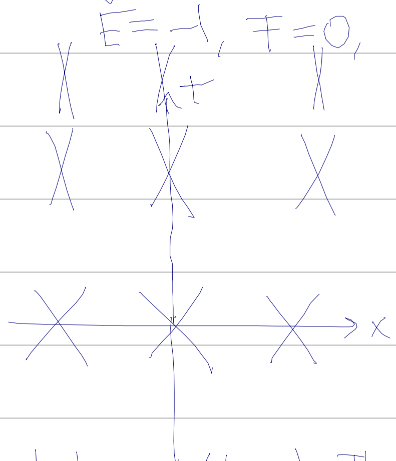


Not time orientable!

Examples (1) expanding universe (de Sitter space): $S = \mathbb{R}^2_{t}$, coordinates (t, x),

$$E = -1, \quad F = 0, \quad G = e^{2t}. \qquad (\text{Gauss curvature } K \equiv -1.)$$

time orientation: $\uparrow$



- Examples of timelike geodesics:

$$\gamma(s) = (s, x_0), \quad x_0 \in \mathbb{R}.$$

- A lightlike geodesic:

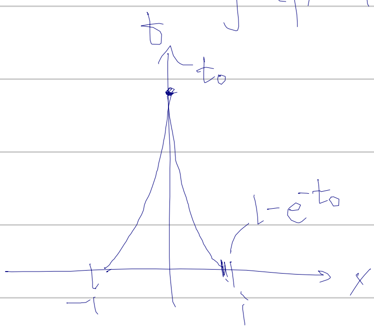
$$\tilde{\gamma}(z) = (z, x_0 + e^{-z})$$

- Let $p = (t_0, 0)$. Then the <u>causal past</u>

$$J^-(p) = \{ q \in S : \exists \text{ future timelike or lightlike curve from } q \text{ to } p \}$$

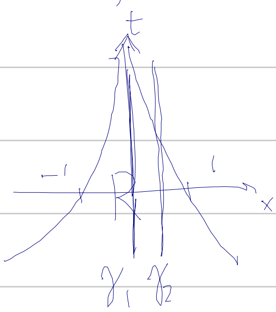
can be computed explicitly:

$$J^-(p) = \{ (t, x) : t \le t_0, \ |x| \le e^{-t} - e^{-t_0} \}$$



As $t_0 \to \infty$, this converges to the set

$$R = \{ (t, x) : |x| \le e^{-t} \}$$



Consider $\gamma_1(s) = (s, 0)$ and

$$\gamma_2(s) = (s, \tfrac{1}{2}).$$

Both are geodesics. However, if $s_0 > \log 2$ and $s \in \mathbb{R}$ is <u>arbitrary</u>, there does not exist a future timelike or lightlike curve from $\gamma_2(s_0)$ to $\gamma_1(s)$! "A stellar explosion at $\gamma_2(s_0)$ will never be detected by the observer γ_1!" (This is a perfectly sensible statement often made mysterious by claiming that the spacetime expands faster than the speed of light.)

(2) Big bang: FLRW spacetime, $E(t,x)=-1$, $F(t,x)=0$, $G(t,x)=t^{2/3}$, $S=\{(t,x): t>0\}$.

Gauss curvature: $K(t,x)=\frac{2}{9t^2} \nearrow \infty$ as $t\to 0$. S is past geodesically incomplete; in fact, any past timelike curve starting at some $p\in S$ has finite proper time.

- <u>General relativity</u>: 4-dim. spacetimes, with coefficients of metric satisfying a second order partial differential equation ("Einstein field equation").

  "Einstein vacuum equation": $\forall\, p\in S,\ v\in T_pS, v\neq 0,$

$Ric_p(v,v) = $ (average of the Gauss curvatures $K(p)$ of all surfaces $\{\exp_p(tv+sw): t,s\in(-\varepsilon,\varepsilon)\}\subset S$ (where $w\neq 0$, $w\perp v$))

$=0.$