18.950/9501 (S20): HOMEWORK 1

The book references are to do Carmo, *Differential Geometry of Curves and Surfaces*.

Due: Thursday, Feb 13, in class.[1]

Exercise 1. Chapter 1–3, Problem 10 (page 11).

Exercise 2. Let $\alpha\colon I \to \mathbb{R}^3$ be a smooth parameterized curve, and let $[a,b] \subset I$ be a closed interval. For every $n \in \mathbb{N}$ and every partition $P = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 < t_1 < \ldots < t_n = b$, of $[a,b]$, define

$$\ell(\alpha, P) := \sum_{i=1}^{n} \|\alpha(t_i) - \alpha(t_{i-1})\|.$$

Define the norm $|P|$ of the partition P as

$$|P| = \max_{i=1,\ldots,n} (t_i - t_{i-1}).$$

(1) Show that $\ell(\alpha, P) \leq \ell := \int_a^b \|\alpha'(t)\|\, dt$.
(2) Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$, then $\ell - \epsilon \leq \ell(\alpha, P)$.
(3) Conclude that the arc length $\ell = \int_a^b \|\alpha'(t)\|\, dt$ is given by

$$\ell = \lim_{\substack{P \text{ partition} \\ |P| \to 0}} \ell(\alpha, P).$$

Exercise 3. Let $\alpha\colon I = (a,b) \to \mathbb{R}^3$ denote a regular parameterized curve which is not necessarily by arc length. Let $t_0 \in I$, and let $s(t) = \int_{t_0}^{t} \|\alpha'(t')\|\, dt'$ denote the arc length.

(1) Show that $s(t)$ has a smooth inverse $t(s)$, defined for $s \in (s(a), s(b))$. Prove that the curve $\beta(s) := \alpha(t(s))$ is a regular curve parameterized by arc length.
(2) Show that $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -\alpha' \cdot \alpha''/|\alpha'|^4$.
(3) Show that the curvature of α is given by

$$k(t) = |\alpha'(t) \times \alpha''(t)|/|\alpha'(t)|^3.$$

(4) If $k(t) \neq 0$, show that the torsion at t is equal to

$$\tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2}.$$

Exercise 4. Chapter 1–5, Problem 10 (page 24).

Exercise 5. Chapter 1–6, Problem 3 (page 30).

Exercise 6. Compute the Frenet frame, curvature, and torsion of the following curves:

(1) $\alpha(t) = (e^t \cos t, e^t \sin t, e^t)$, $t \in (-\infty, \infty)$.
(2) $\alpha(t) = (\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{1}{\sqrt{2}}t)$, $t \in (-1, 1)$.

Date: January 30, 2020.

[1] See the course website, https://math.mit.edu/~phintz/18.950-S20/, for homework policies.