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Eigenvalue Problems for the Curl Operator with non Classical Boundary Conditions

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The Green's formula for the **curl** operator

$$\int_D \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v} - \mathbf{curl}\,\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial D} (\mathbf{u}\times\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}S\ ,$$

illustrates that this first order operator is in fact symmetric once suitable boundary conditions are imposed. Starting from this observation we examine the **curl** operator as an unbounded operator in $\boldsymbol{L}^2(D)$ and ask which boundary conditions will render this operator self-adjoint.

This question is important in connection with so-called stable force-free magentic fields, that is, current distributions that are not affected by their own induced magnetic fields. Stable force-free magnetic fields have to satisfy

$$\exists\alpha\in\mathbb{R}\backslash\{0\}:\quad \mathbf{curl}\,\mathbf{H} = \alpha\mathbf{H}\ . \tag{1}$$

This is an eigenvalue problem for the **curl** operator.

More precisely, we seek self-adjoint extensions $\mathbf{curl}_s : D_s \subset \boldsymbol{L}^2(D) \mapsto \boldsymbol{L}^2(D)$ of the minimal closed **curl** operator with domain $\boldsymbol{H}_0(\mathbf{curl}, D)$. We adopt the calculus of differential forms on domains D in 3-dimensional space. In this framework, **curl** is treated as operator $\star\mathsf{d}$ on 1-forms on D. Here $\star$ is a Hodge operator arising from a metric on D, and d denotes the exterior derivative.

We resort to the connection between self-adjoint extensions of symmetric operators and symplectic geometry. In the case of **curl**, every self-adjoint extension is described as a complete Lagrangian subspace of the trace space $\boldsymbol{H}(\mathbf{curl}, D)/\boldsymbol{H}_0(\mathbf{curl}, D)$ of 1-forms on ∂D equipped with the symplectic product

$$[\omega,\eta] := \int_{\partial D} \omega\wedge\eta\ .$$

It is important to notice that this symplectic pairing is completely metric free.

We identify two important classes of self-adjoint extensions of **curl** through their definiting boundary conditions.

1. operators defined on 1-forms with closed traces

$$\mathcal{D}_s := \{\mathbf{u}\in\boldsymbol{H}(\mathbf{curl}, D):\ \mathrm{curl}_\Gamma(\gamma_{\mathbf{t}}\mathbf{u}) = 0 \text{ on } \partial D,\ \int_\gamma \mathbf{u}\cdot\mathrm{d}\vec{s} = 0\ \forall\gamma\in\Gamma\}\ ,$$

2. **curl** operators acting on 1-forms with co-closed traces

$$\mathcal{D}_s := \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, D) : \ \mathrm{div}_\Gamma(\gamma_{\mathbf{t}}\mathbf{u}) = 0 \text{ on } \partial D, \int_\gamma \mathbf{u} \cdot \mathrm{d}\vec{s} = 0 \ \forall \gamma \in \Gamma\} \ .$$

In each case Γ comprises a subset of g generators (1-cycles on ∂D) of the homology group $\mathbb{H}_1(\partial D)$ of order $2g$.

We show that all these self-adjoint **curl** operators possess a complete orthonormal eigensystem. This connects our research with (1).

Related publications.

Report to appear in 2008.