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A NOTE ON THE STABLE ONE-EQUATION COUPLING OF FINITE AND BOUNDARY ELEMENTS*

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Abstract. In a recent paper [*SIAM J. Numer. Anal.*, 47 (2009), pp. 3451–3463] Sayas proved the stability of the Johnson–Nédélec coupling of finite and boundary element methods on polygonal interfaces when the direct boundary integral equation with single and double layer integral operators is used only. In this note we present two alternative proofs of this result for general Lipschitz interfaces. In particular, we prove an ellipticity estimate of the coupled bilinear form. Hence, we can use standard arguments to derive stability and error estimates for the Galerkin discretization for all pairs of finite and boundary element trial spaces.

Key words. finite elements, boundary elements, nonsymmetric coupling

AMS subject classification. 65N30

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1. Introduction.

The coupling of finite and boundary element methods is of increasing interest in many applications in engineering and science, e.g., in acoustic and electromagnetic scattering, in electromagnetism, and in elasticity, to name a few of them. In particular, boundary integral equation methods can be used to handle partial differential equations with constant coefficients in unbounded domains, while finite element methods are more favorable when dealing with partial differential equations in bounded domains with varying coefficients, or even nonlinear equations.

The first approaches to couple finite and boundary element methods are based on the use of either indirect single or double layer potentials, or the direct approach with both single and double layer integral operators; see the pioneering works of Brezzi, Johnson, and Nédélec [1, 5] and the overview by Hsiao [6]. When the stability analysis of the coupled scheme is based on the use of a Gårding inequality of the related bilinear form, the compactness of the double layer integral operator has to be assumed, which allows the consideration of smooth interface boundaries only. An alternative approach is to consider a sufficiently accurate discretization, i.e., by using a much finer boundary element mesh to approximate the Neumann data, of the boundary integral equation to ensure the ellipticity of the boundary element approximation of the related Dirichlet to Neumann map as was done by Wendland in [16]; see also [7]. While a rigorous mathematical analysis was not yet available at that time, several numerical examples indicated the stability of this coupling scheme for more general situations; see, e.g., the paper [4] by Costabel, Ervin, and Stephan.

In [2], Costabel introduced a symmetric coupling of finite and boundary element methods which allows a rigorous stability and error analysis of the coupled scheme. Moreover, preconditioned parallel iterative solution strategies are available for the symmetric coupling; see, e.g., [9]. The symmetric formulation of boundary integral equations is based on the use of a second, the so-called hypersingular boundary integral equation. Despite the fact that the bilinear form of the hypersingular boundary

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integral operator can be rewritten as a weakly singular integral due to integration by parts [11], the use of the symmetric formulation is still not popular in engineering or for more advanced applications. Hence, there is a great interest in the coupling of finite elements with the first boundary integral equation only, which is also simpler to implement.

In a recent paper [12], Sayas proved the stability of the standard finite and boundary element coupling scheme for the first time. Instead of the Galerkin discretization of the coupled bilinear form, he considered the equivalent Galerkin stability of the transposed operator. Then, by using an indirect single layer potential representation of the solution in the exterior domain, he could rewrite the transposed coupled bilinear form by using Dirichlet integrals only.

In this paper, we restrict our considerations to the case of a free space Poisson equation where we will present two alternative proofs for the stability of the finite and boundary element coupling in the case of a Lipschitz interface. In particular, we prove an ellipticity estimate of the combined bilinear form which allows us to use standard results to derive stability and related error estimates. An essential ingredient of the first approach is the use of different variational and boundary integral formulations of the Steklov–Poincaré operator [13] which is involved in the Dirichlet to Neumann map associated to the interior Dirichlet boundary value problem. The second important tool is the use of some natural Sobolev norms in $H^{\pm 1/2}(\Gamma)$ which are induced by the single layer potential and its inverse as introduced in [15]. The second approach relies on ideas used by Sayas in [12]. In particular, we prove the ellipticity of the coupled bilinear form by rewriting the boundary integral bilinear forms of the exterior problem by using appropriate Dirichlet forms.

For a review on boundary integral equation methods and results on the mapping properties of boundary integral operators, we refer, in particular, to [3, 8, 10, 14].

2. Nonsymmetric BEM/FEM coupling. For a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) with a Lipschitz boundary $\Gamma = \partial\Omega$, we consider the free space transmission boundary value problem

$$-\operatorname{div}[A(x)\nabla u_i(x)] \,=\, f(x) \quad \text{for } x \in \Omega, \quad -\Delta u_e(x) \,=\, 0 \quad \text{for } x \in \Omega^c := \mathbb{R}^n\backslash\overline{\Omega} \tag{2.1}$$

with the interface boundary conditions

$$u_i(x) \,=\, u_e(x), \quad n_x \cdot [A(x)\nabla u_i(x)] \,=\, \frac{\partial}{\partial n_x} u_e(x) \quad \text{for almost all } x \in \Gamma, \tag{2.2}$$

and with the radiation boundary condition

$$u(x) \,=\, \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{2.3}$$

We assume that the coefficient matrix $A(x) = [A(x)]^\top > 0$ is symmetric and uniformly positive definite; i.e.,

$$\lambda_{\min}(A) := \inf_{x\in\Omega} \min_{i=1,\ldots,n} \lambda_i(A(x)) > 0.$$

Moreover, $f \in L_2(\Omega)$ is a given function, and n_x is the exterior normal vector which is defined for almost all $x \in \Gamma$. In the two-dimensional case $n = 2$ we assume the scaling condition $\operatorname{diam}\Omega < 1$.

The variational formulation of the interior Poisson equation in (2.1) is to find $u_i \in H^1(\Omega)$ such that

$$\int_\Omega [A(x)\nabla u_i(x)] \cdot \nabla v(x) dx \;=\; \int_\Omega f(x)v(x)dx + \int_\Gamma n_x \cdot [A(x)\nabla u_i(x)]v(x)ds_x \tag{2.4}$$

is satisfied for all $v \in H^1(\Omega)$. The solution of the exterior Laplace equation in (2.1) satisfying the radiation condition (2.3) is given by the representation formula

$$u_e(x) \;=\; -\int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}u_e(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u_e(y)ds_y \quad \text{for } x \in \Omega^c, \tag{2.5}$$

where

$$U^*(x,y) \;=\; \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n=2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n=3 \end{cases}$$

is the fundamental solution of the Laplace operator. To ensure that the solution u_e as given by the representation formula (2.5) fulfills the radiation condition (2.3) also in the two-dimensional case $n=2$, we need to assume that the normal derivative

$$t(x) \;:=\; \frac{\partial}{\partial n_x}u_e(x)$$

satisfies the scaling condition (see, e.g., [14, Lemma 6.21])

$$\int_\Gamma t(x)\,ds_x \;=\; 0\,. \tag{2.6}$$

From (2.5) we obtain the boundary integral equation for almost all $x \in \Gamma$,

$$(Vt)(x) \;=\; -\frac{1}{2}u_e(x) + (Ku_e)(x), \tag{2.7}$$

where

$$(Vt)(x) = \int_\Gamma U^*(x,y)t(y)ds_y, \quad (Ku_e)(x) = \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u_e(y)ds_y$$

denote the single and double layer integral operators, respectively. When inserting the interface boundary conditions (2.2) into the variational formulation (2.4) and into the boundary integral equation (2.7), we end up with a variational problem to find $u_i \in H^1(\Omega)$ and $t \in H^{-1/2}(\Gamma)$ such that

$$\int_\Omega [A(x)\nabla u_i(x)] \cdot \nabla v(x)dx - \int_\Gamma t(x)v(x)ds_x \;=\; \int_\Omega f(x)v(x)dx \tag{2.8}$$

is satisfied for all $v \in H^1(\Omega)$, and

$$\langle Vt,\tau\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right)u_i,\tau\right\rangle_\Gamma \;=\; 0 \tag{2.9}$$

is satisfied for all $\tau \in H^{-1/2}(\Gamma)$. When choosing in (2.8) as a test function $v = 1$, this gives

$$-\int_\Gamma t(x)\, ds_x \;=\; \int_\Omega f(x)\, dx\,. \tag{2.10}$$

Hence, in the two-dimensional case $n = 2$, we have to assume the solvability condition

$$\int_\Omega f(x)\, dx \;=\; 0 \tag{2.11}$$

to ensure the required scaling condition (2.6).

Since the single layer integral operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and $H^{-1/2}(\Gamma)$-elliptic, we may choose the natural density (equilibrium distribution) $t_{\mathrm{eq}} = V^{-1}1$ as a test function in (2.9),

$$\langle t, 1\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right) u_i, V^{-1}1 \right\rangle_\Gamma \;=\; 0\,.$$

By using the equilibrium condition (2.10) and the symmetry relation $KV = VK'$ (see, e.g., [8, 14, 15]) with the adjoint double layer integral operator K', we obtain

$$\begin{aligned}\int_\Omega f(x)dx &= \left\langle \left(\frac{1}{2}I - K\right) u_i, V^{-1}1 \right\rangle_\Gamma \\ &= \langle u_i, t_{\mathrm{eq}}\rangle_\Gamma - \left\langle u_i, \left(\frac{1}{2}I + K'\right) V^{-1}1 \right\rangle_\Gamma \\ &= \langle u_i, t_{\mathrm{eq}}\rangle_\Gamma - \left\langle u_i, V^{-1}\left(\frac{1}{2}I + K\right) 1 \right\rangle_\Gamma \;=\; \langle u_i, t_{\mathrm{eq}}\rangle_\Gamma\end{aligned}$$

due to $(\frac{1}{2}I + K)1 = 0$. Hence, we may introduce the splitting

$$u_i(x) \;=\; u_0 + \widetilde{u}_i(x) \quad \text{for } x \in \Omega, \tag{2.12}$$

where $\widetilde{u}_i \in H^1(\Omega)$ satisfies the scaling condition

$$\langle \widetilde{u}_i, t_{\mathrm{eq}}\rangle_\Gamma \;=\; 0, \tag{2.13}$$

and where the constant u_0 is given by

$$u_0 \;=\; \frac{1}{\langle 1, t_{\mathrm{eq}}\rangle_\Gamma} \int_\Omega f(x)\, dx \;=\; \frac{1}{\langle V t_{\mathrm{eq}}, t_{\mathrm{eq}}\rangle_\Gamma} \int_\Omega f(x)\, dx\,. \tag{2.14}$$

Due to the solvability condition (2.11) in the two-dimensional case $n = 2$, we obtain $u_0 = 0$.

Instead of the coupled variational formulation (2.8) and (2.9), we now consider a modified variational problem to find $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\int_\Omega [A(x)\nabla \widetilde{u}_i(x)] \cdot \nabla v(x) dx + \langle \widetilde{u}_i, t_{\mathrm{eq}}\rangle_\Gamma \langle v, t_{\mathrm{eq}}\rangle_\Gamma - \langle t, v\rangle_\Gamma = \int_\Omega f(x) v(x) dx \tag{2.15}$$

$$\langle Vt, \tau\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right) \widetilde{u}_i, \tau \right\rangle_\Gamma = -\langle u_0, \tau\rangle_\Gamma \tag{2.16}$$

is satisfied for all $(v, \tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$.

By construction we have seen that any solution of the coupled variational problem (2.8) and (2.9) implies a solution of the modified variational problem (2.15) and (2.16). In fact, the variational formulations (2.8)–(2.9) and (2.15)–(2.16) are equivalent in the following sense.

LEMMA 2.1. *Let* $(u_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *be a solution of the variational formulations* (2.8)–(2.9). *Then there exists a unique splitting* $u_i = u_0 + \widetilde{u}_i$ *with* $u_0 \in \mathbb{R}$, $\widetilde{u}_i \in H^1(\Omega)$ *satisfying* $\langle \widetilde{u}_i, t_{eq}\rangle_\Gamma = 0$ *such that* $(\widetilde{u}_i, t_i) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *is a solution of the modified variational formulation* (2.15)–(2.16). *Vice versa, any solution* $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *of the modified variational problem* (2.15)–(2.16) *implies a solution* $(\widetilde{u}_i + u_0, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *of the coupled variational problem* (2.8) *and* (2.9). *Moreover,* $\widetilde{u}_i$ *satisfies the scaling condition* (2.13).

Proof. The first statement follows from the construction of the modified variational formulation (2.15) and (2.16). Hence, we discuss the opposite direction only.

Let $(\widetilde{u}_i, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ be any solution of the modified variational problem (2.15)–(2.16). The choice $v = 1$ and $\tau = 0$ gives

$$\langle \widetilde{u}_i, t_{\rm eq}\rangle_\Gamma \langle 1, t_{\rm eq}\rangle_\Gamma - \langle t, 1\rangle_\Gamma \;=\; \int_\Omega f(x)dx,$$

while by choosing $v = 0$ and $\tau = t_{\rm eq} = V^{-1}1$ we obtain, by using (2.14),

$$\langle t, 1\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right)\widetilde{u}_i, t_{\rm eq}\right\rangle_\Gamma \;=\; -\langle u_0, t_{\rm eq}\rangle_\Gamma \;=\; -\int_\Omega f(x)\,dx\,.$$

Note that

$$\left\langle \left(\frac{1}{2}I - K\right)\widetilde{u}_i, t_{\rm eq}\right\rangle_\Gamma \;=\; \langle \widetilde{u}_i, t_{\rm eq}\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)\widetilde{u}_i, V^{-1}1\right\rangle_\Gamma \;=\; \langle \widetilde{u}_i, t_{\rm eq}\rangle_\Gamma\,.$$

Hence, by summing up both equations, we obtain

$$\langle \widetilde{u}_i, t_{\rm eq}\rangle_\Gamma \left[1 + \langle 1, t_{\rm eq}\rangle_\Gamma\right] \;=\; 0,$$

and, therefore,

$$\langle \widetilde{u}_i, t_{\rm eq}\rangle_\Gamma \;=\; 0$$

follows. Due to $u_0 \in \mathbb{R}$ we have $(\frac{1}{2}I - K)u_0 = u_0$. Then it is a direct consequence that $(\widetilde{u}_i + u_0, t) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ is also a solution of the coupled variational problem (2.8) and (2.9). ∎

To ensure unique solvability of the modified variational formulation (2.15)–(2.16) as well as stability of a related Galerkin scheme, we need to establish some ellipticity or coercivity estimate of an associated bilinear form. First we note that

$$\|v\|^2_{H^1(\Omega),\Gamma} \;=\; \int_\Omega |\nabla v(x)|^2 dx + [\langle v, t_{\rm eq}\rangle_\Gamma]^2$$

defines an equivalent norm on $H^1(\Omega)$ (see, e.g., [14, Theorem 2.6]), while

$$\|\tau\|^2_V \;=\; \langle V\tau, \tau\rangle_\Gamma$$

defines an equivalent norm on $H^{-1/2}(\Gamma)$. Associated to the modified variational formulation (2.15)–(2.16), we introduce the bilinear form

$$\begin{aligned}(2.17)\quad \widetilde{a}(u, t; v, \tau) := &\int_\Omega [A(x)\nabla u(x)]\cdot\nabla v(x)dx + \langle u, t_{\rm eq}\rangle_\Gamma\langle v, t_{\rm eq}\rangle_\Gamma - \langle t, v\rangle_\Gamma \\ &+ 2\left[\langle Vt, \tau\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right)u, \tau\right\rangle_\Gamma\right]\end{aligned}$$

which is bounded for all $(u,t),(v,\tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$. For $(v,\tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ we then obtain

$$\widetilde{a}(v,\tau;v,\tau) \ \geq \ \lambda_{\min}(A)\,\|v\|^2_{H^1(\Omega),\Gamma} + 2\,\|\tau\|^2_V - 2\langle Kv,\tau\rangle_\Gamma. \tag{2.18}$$

The form (2.18) is coercive satisfying a Gårding inequality when we assume that the double layer integral operator $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact. In this case, stability of the related Galerkin scheme follows asymptotically when using standard results for the Galerkin approximation of compact perturbations of elliptic operators. But the compactness of the double layer integral operator K allows the consideration of smooth interface boundaries Γ only. Note that the bilinear form (2.17) corresponds to the variational problem (2.15)–(2.16), where we multiply (2.16) with the factor 2.

Instead of a Gårding inequality as in (2.18), we will prove an ellipticity estimate for the bounded bilinear form

$$\begin{aligned} a(u,t;v,\tau) := &\int_\Omega [A(x)\nabla u(x)]\cdot\nabla v(x)dx + \langle u,t_{\text{eq}}\rangle_\Gamma\langle v,t_{\text{eq}}\rangle_\Gamma - \langle t,v\rangle_\Gamma \\ &+ \langle Vt,\tau\rangle_\Gamma + \left\langle \left(\frac{1}{2}I - K\right)u,\tau\right\rangle_\Gamma \end{aligned} \tag{2.19}$$

from which we can derive stability and error estimates in a standard way. In what follows we state the main result of this paper.

THEOREM 2.2. *Let $\lambda_{\min}(A) > \frac{1}{4}$ be satisfied. The bilinear form as defined in (2.19) is then $H^1(\Omega)\times H^{-1/2}(\Gamma)$-elliptic; i.e.,*

$$a(v,\tau;v,\tau) \ \geq \ \frac{1}{2}\Big[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\Big]\Big[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|^2_V\Big] \tag{2.20}$$

for all $(v,\tau) \in H^1(\Omega)\times H^{-1/2}(\Gamma)$.

Two different proofs of Theorem 2.2 will be given in sections 4 and 5, respectively.

Remark 2.1. As a corollary of Theorem 2.2 we conclude unique solvability and stability of any coupled conformal finite and boundary element discretization scheme. In addition, convergence follows from the regularity of the solution $(\widetilde{u}_i,t)$ and from the approximation properties of both the finite and boundary element trial spaces.

Remark 2.2. The consideration of the potential equation in (2.1) motivates the use of the natural density t_{eq} to define the Sobolev norm $\|\cdot\|_{H^1(\Omega),\Gamma}$ in $H^1(\Omega)$. Instead, one may use any other appropriate linear functional to define an equivalent norm in $H^1(\Omega)$, or weighted Sobolev norms, as, e.g., in [12, sect. 4]. Moreover, when considering the Yukawa equation instead of the potential equation, as was done in [12], the consideration of the natural density is not required. This remains true when considering the potential equation in Ω with some additional Dirichlet boundary conditions at some interior boundary.

3. Dirichlet to Neumann maps. The first proof of Theorem 2.2 is essentially based on different representations of the Dirichlet to Neumann map which is related to the solution of the interior Dirichlet boundary value problem

$$-\Delta u(x) \ = \ 0 \quad \text{for } x\in\Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma \tag{3.1}$$

whose weak solution $u \in H^1(\Omega)$, $u(x) = g(x)$ for $x \in \Gamma$, is the unique solution of the variational problem

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx \ = \ 0 \quad \text{for all } v \in H^1_0(\Gamma). \tag{3.2}$$

Since $u \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem (3.1), we can define the related normal derivative $\lambda := n_x \cdot \nabla u$ in the sense of $H^{-1/2}(\Gamma)$; i.e., $\lambda \in H^{-1/2}(\Gamma)$ satisfies

$$\int_\Gamma \lambda(x)v(x)ds_x = \int_\Omega \nabla u(x) \cdot \nabla v(x)dx \quad \text{for all } v \in H^1(\Omega). \tag{3.3}$$

Hence, by solving the Dirichlet boundary value problem (3.1) and by defining the normal derivative λ via (3.3), we obtain the Dirichlet to Neumann map $\lambda = Sg$ in the sense of $H^{-1/2}(\Gamma)$, where the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is defined implicitly by solving the Dirichlet boundary value problem (3.1).

Instead of the variational definition (3.3) of the Steklov–Poincaré operator S, we now consider equivalent definitions of the Steklov–Poincaré operator which are based on the use of boundary integral equations. In particular, the symmetric boundary integral representation of the Steklov–Poincaré operator is given by

$$S = D + \left(\frac{1}{2}I + K'\right) V^{-1} \left(\frac{1}{2}I + K\right) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)\,, \tag{3.4}$$

where K' is the adjoint double layer integral operator, and D is the hypersingular boundary integral operator. In the proof of Theorem 2.2 we will replace the Dirichlet to Neumann map $\lambda = Sg$ which is originally defined via the variational formulation (3.3) by the equivalent symmetric boundary integral representation (3.4).

4. Proof of Theorem 2.2. To prove the ellipticity estimate of Theorem 2.2 we first consider the form (2.19); i.e.,

$$\begin{aligned} a(v,\tau;v,\tau) &= \int_\Omega [A(x)\nabla v(x)] \cdot \nabla v(x)dx + [\langle v, t_{\mathrm{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right) v, \tau \right\rangle_\Gamma \\ &\geq \lambda_{\min}(A) \int_\Omega |\nabla v(x)|^2 dx + [\langle v, t_{\mathrm{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right) v, \tau \right\rangle_\Gamma . \end{aligned} \tag{4.1}$$

For $v \in H^1(\Omega)$ we consider the splitting $v = v_\Gamma + \widetilde{v}$, where v_Γ is the harmonic extension of $v_{|\Gamma}$; i.e., $v_\Gamma \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem

$$-\Delta v_\Gamma(x) = 0 \quad \text{for } x \in \Omega, \quad v_\Gamma(x) = v(x) \quad \text{for } x \in \Gamma.$$

In particular, $v_\Gamma \in H^1(\Omega)$, $v_\Gamma(x) = v(x)$ for $x \in \Gamma$, solves

$$\int_\Omega \nabla v_\Gamma(x) \cdot \nabla z(x)dx = 0 \quad \text{for all } z \in H^1_0(\Omega).$$

By construction we also have $\widetilde{v} \in H^1_0(\Omega)$. Hence, we obtain

$$\int_\Omega |\nabla v(x)|^2 dx = \int_\Omega |\nabla(v_\Gamma(x) + \widetilde{v}(x))|^2 dx = \int_\Omega |\nabla v_\Gamma(x)|^2 dx + \int_\Omega |\nabla \widetilde{v}(x)|^2 dx\,. \tag{4.2}$$

By using Green's first formula we further conclude that

$$\int_\Omega |\nabla v_\Gamma(x)|^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} v_\Gamma(x) v_\Gamma(x) ds_x = \int_\Gamma (Sv_\Gamma)(x) v_\Gamma(x) ds_x,$$

where $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the Steklov–Poincaré operator which realizes the Dirichlet to Neumann map when considering an interior Dirichlet boundary value problem for the Laplace equation. Hence, we can rewrite (4.1) as

$$\begin{aligned}(4.3)\qquad a(v,\tau;v,\tau) \geq \lambda_{\min}(A)\int_\Omega |\nabla\widetilde{v}(x)|^2dx + [\langle v, t_{\mathrm{eq}}\rangle_\Gamma]^2 \\ + \lambda_{\min}(A)\,\langle Sv_\Gamma, v_\Gamma\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma .\end{aligned}$$

By definition we have $v_\Gamma(x) = v(x)$ for $x \in \Gamma$. Therefore, by using the symmetric boundary integral representation (3.4) of the Steklov–Poincaré operator S, we further obtain

$$\begin{aligned}&\lambda_{\min}(A)\,\langle Sv, v\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma \\ &\quad = \lambda_{\min}(A)\left[\langle Dv, v\rangle_\Gamma + \left\langle V^{-1}\left(\frac{1}{2}I + qK\right)v, \left(\frac{1}{2}I + K\right)v\right\rangle_\Gamma\right] \\ &\qquad + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma \\ &\quad = \lambda_{\min}(A)\langle Dv, v\rangle_\Gamma + \lambda_{\min}(A)\left\|\left(\frac{1}{2}I + K\right)v\right\|^2_{V^{-1}} + \|\tau\|^2_V - \left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma ,\end{aligned}$$

where

$$\|\cdot\|_{V^{-1}} := \sqrt{\langle V^{-1}\cdot,\cdot\rangle_\Gamma}$$

defines an equivalent norm in $H^{1/2}(\Gamma)$. With the bound (see, e.g., [15])

$$\left|\left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma\right| \leq \|\left(\frac{1}{2}I + K\right)v\|_{V^{-1}}\|\tau\|_V,$$

we conclude that

$$\begin{aligned}&\lambda_{\min}(A)\langle Sv, v\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \left\langle \left(\frac{1}{2}I + K\right)v, \tau\right\rangle_\Gamma \\ &\quad \geq \lambda_{\min}(A)\langle Dv, v\rangle_\Gamma + \lambda_{\min}(A)\left\|\left(\frac{1}{2}I + K\right)v\right\|^2_{V^{-1}} \\ &\qquad + \|\tau\|^2_V - \left\|\langle\left(\frac{1}{2}I + K\right)v\right\|_{V^{-1}}\|\tau\|_V \\ &\quad = \lambda_{\min}(A)\langle Dv, v\rangle_\Gamma + \left(\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma^2}\right)\left\|\left(\frac{1}{2}I + K\right)v\right\|^2_{V^{-1}} + \left(1 - \frac{1}{2}\gamma^2\right)\|\tau\|^2_V \\ &\qquad + \frac{1}{2}\left(\gamma\,\|\tau\|_V - \frac{1}{\gamma}\left\|\left(\frac{1}{2}I + K\right)v\right\|_{V^{-1}}\right)^2 \\ &\quad = \lambda_{\min}(A)\langle Dv, v\rangle_\Gamma + \left(\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma^2}\right)\left\|\left(\frac{1}{2}I + K\right)v\right\|^2_{V^{-1}} + \left(1 - \frac{1}{2}\gamma^2\right)\|\tau\|^2_V \\ &\quad \geq \left(1 - \frac{1}{2}\gamma_*^2\right)\left[\langle Dv, v\rangle_\Gamma + \left\|\left(\frac{1}{2}I + K\right)v\right\|^2_{V^{-1}} + \|\tau\|^2_V\right]\end{aligned}$$

if

$$\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma_*^2} \;=\; 1 - \frac{1}{2}\gamma_*^2 \tag{4.4}$$

is satisfied. From (4.4) we find

$$\gamma_*^2 \;=\; 1 - \lambda_{\min}(A) + \sqrt{(1-\lambda_{\min}(A))^2 + 1},$$

and, therefore,

$$1 - \frac{1}{2}\gamma_*^2 \;=\; \frac{1}{2}\left[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\right] \;>\; 0$$

if we assume $\lambda_{\min}(A) > \frac{1}{4}$. Hence, we finally obtain

$$\begin{aligned}
a(v,\tau;v,\tau) &\geq \lambda_{\min}(A)\int_\Omega |\nabla\widetilde{v}(x)|^2dx + [\langle v, t_{\text{eq}}\rangle_\Gamma]^2 \\
&\quad + \frac{1}{2}\left[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\right]\left[\langle Sv_\Gamma, v_\Gamma\rangle_\Gamma + \|\tau\|_V^2\right] \\
&\geq \frac{1}{2}\left[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\right] \\
&\quad \cdot \left[\int_\Omega |\nabla\widetilde{v}(x)|^2dx + \int_\Omega |\nabla v_\Gamma(x)|^2dx + [\langle v, t_{\text{eq}}\rangle_\Gamma]^2 + \|\tau\|_V^2\right] \\
&= \frac{1}{2}\left[(1+\lambda_{\min}(A)) - \sqrt{(1-\lambda_{\min}(A))^2+1}\right]\left[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|_V^2\right],
\end{aligned}$$

as stated in Theorem 2.2.

5. An alternative proof. While the proof in section 4 is based on an equivalent boundary integral representation of the Steklov–Poincaré operator which is related to the interior problem, we now consider equivalent representations of the boundary integral bilinear forms in the exterior by using related Dirichlet forms. In fact, this approach follows the idea of Sayas in [12], but in contrast to [12] we prove the ellipticity of the original coupled bilinear form. For this we consider the estimate (4.1); i.e.,

$$a(v,\tau;v,\tau) \geq \lambda_{\min}(A)\int_\Omega |\nabla v(x)|^2dx + [\langle v, t_{\text{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \left\langle v, \left(\frac{1}{2}I + K'\right)\tau\right\rangle_\Gamma . \tag{5.1}$$

For an arbitrary but fixed $\tau \in H^{-1/2}(\Gamma)$ we define the harmonic function

$$u_\tau(x) \;=\; (\widetilde{V}\tau)(x) \;=\; \int_\Gamma U^*(x,y)\tau(y)ds_y \quad \text{for } x \in \mathbb{R}^d\backslash\Gamma$$

for which we have the usual boundary traces for almost all $x \in \Gamma$ as (see, e.g., [14])

$$\gamma_0 u_\tau(x) \;=\; (V\tau)(x), \quad \gamma_1^{\text{int}}u_\tau(x) \;=\; \left(\frac{1}{2}I + K'\right)\tau(x),$$

$$\gamma_1^{\text{ext}}u_\tau(x) \;=\; \left(-\frac{1}{2}I + K'\right)\tau(x).$$

Hence, by using Green's first formula we can write

$$\left\langle v, \left(\frac{1}{2}I + K'\right)\tau\right\rangle_\Gamma = \int_\Gamma \gamma_1^{\text{int}} u_\tau(x)\gamma_0 v(x) ds_x = \int_\Omega \nabla u_\tau(x)\cdot\nabla v(x)dx\,.$$

As in the proof of the ellipticity of the single layer potential V (see, e.g., [14]) we also have

$$\langle V\tau,\tau\rangle_\Gamma = \int_\Omega |\nabla u_\tau(x)|^2dx + \int_{\Omega^c} |\nabla u_\tau(x)|^2 dx,$$

where we have to assume an appropriate far field behavior of u_τ. Hence, we can write the bilinear form (5.1) as

$$\begin{aligned} a(v,\tau;v,\tau) \geq \lambda_{\min}(A)\int_\Omega |\nabla v(x)|^2dx + [\langle v, t_{\text{eq}}\rangle_\Gamma]^2 \\ + \int_\Omega |\nabla u_\tau(x)|^2dx + \int_{\Omega^c}|\nabla u_\tau(x)|^2dx - \int_\Omega \nabla u_\tau(x)\cdot\nabla v(x)dx. \end{aligned}$$

By using

$$\int_\Omega \nabla u_\tau(x)\cdot\nabla v(x)dx \leq \frac{1}{2}\gamma^2\,\|\nabla u_\tau\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{1}{\gamma^2}\,\|\nabla v\|^2_{L_2(\Omega)}$$

for some $\gamma > 0$, we further conclude that

$$\begin{aligned} a(v,\tau;v,\tau) &\geq [\langle v,t_{\text{eq}}\rangle_\Gamma]^2 + \left[\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma^2}\right]\|\nabla v\|^2_{L_2(\Omega)} \\ &\quad + \left[1-\frac{1}{2}\gamma^2\right]\|\nabla u_\tau\|^2_{L_2(\Omega)} + \|\nabla u_\tau\|^2_{L_2(\Omega^c)} \\ &\geq \left(1-\frac{1}{2}\gamma_*^2\right)\left[[\langle v,t_{\text{eq}}\rangle_\Gamma]^2 + \|\nabla v\|^2_{L_2(\Omega)} + \|\nabla u_\tau\|^2_{L_2(\Omega)} + \|\nabla u_\tau\|^2_{L_2(\Omega^c)}\right] \\ &= \left(1-\frac{1}{2}\gamma_*^2\right)\left[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|^2_V\right] \end{aligned}$$

if

$$\lambda_{\min}(A) - \frac{1}{2}\frac{1}{\gamma_*^2} = 1 - \frac{1}{2}\gamma_*^2$$

is satisfied. The assertion of Theorem 2.2 now follows as in section 4.

Remark 5.1. Both proofs of Theorem 2.2 presented in this paper are in some sense equivalent to each other. This relies on different representations of boundary integral operators by using Dirichlet integrals which are related to homogeneous partial differenial equations in both the interior domain Ω and in the exterior domain Ω^c, respectively. In the first proof (presented in section 4) we used the Steklov–Poincaré operator which is related to the Laplace equation in the interior domain, while in the second proof (given in this section) we have replaced all boundary integral operator bilinear forms by appropriate Dirichlet integrals of the Laplace operator in both the interior and in the exterior domain.

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