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Review

Numerical Inversion of the Laplace Transform: a Survey and Comparison of Methods

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A large number of different methods for numerically inverting the Laplace transform are tested and evaluated according to the criteria of applicability to actual inversion problems, applicability to various types of functions, numerical accuracy, computational efficiency, and ease of programming and implementation. The methods are presented briefly and classified theoretically into methods which compute a sample, methods which expand $f(t)$ in exponential functions, methods based on Gaussian quadrature, methods based on a bilinear transformation, and methods based on Fourier series. Extensive results are presented, especially on the numerical accuracy of the methods on a set of 16 test functions. The main conclusion is that for attaining high accuracy on a wide range of test functions, the use of Laguerre polynomials is most successful, while methods based on Chebyshev polynomials and on accelerated convergence of a Fourier series are both quite good. However, no single method gives optimum results for all purposes and all occasions; the results obtained in this comparison give some idea of which methods are likely to be suitable for special problems and circumstances.

1. INTRODUCTION

There are many problems whose solution may be found in terms of a Laplace transform which is then, however, too complicated for inversion using the techniques of complex analysis. Numerous methods have been devised for the numerical evaluation of the Laplace inversion integral. For a detailed bibliography, the reader should consult Piessens (1975) and Piessens and Dang (1976); in the following we shall only refer to papers which are of direct concern.

The purpose of this paper is to provide an evaluation of a representative sample of the existing methods, to enable a rational choice of a method to be made by users. Details of the criteria used, and the methods tested, are given in Sections 2 and 3, respectively. Our intention, in both regards, is to be as comprehensive as possible. However, we feel that an attempt to be encyclopedic will defeat the purpose of the paper because of the overwhelming amount of information which would result. In particular, in selecting the methods to be tested, we have tried to compare represen-

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tatives of every known line of attack. Where a method performs well, we then test one or more variants; but in other cases we have exercised discretion in the interest of economy.

There appear to be no comprehensive comparisons of numerical methods for Laplace transform inversion in the literature. Most papers discussing a particular method compare only a few functions and test two or three methods. The only previous paper presenting any systematic comparisons is Cost (1964), but the methods tested there perform poorly except on special types of functions.

Before launching into the main part of the paper, we define some terminology which will remain standard. The Laplace transform under consideration is denoted by $F(p)$, and is related to the (unknown) original function $f(t)$ by

$$F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt. \tag{1.1}$$

The inequality

$$\mathrm{Re}(p) > c_0 \tag{1.2}$$

specifies the region of the complex p-plane in which $F(p)$ is analytic, and we assume that $F(p)$ has some form of singularity on the line $\mathrm{Re}(p) = c_0$. The inversion integral expresses $f(t)$ as

$$f(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} F(p)\, e^{pt}\, dp, \tag{1.3}$$

where $c > c_0$.

In many cases, $f(t)$ is a real function, and then the real and imaginary parts of $F(p)$ may be simply expressed as

$$\mathrm{Re}[F(c + i\omega)] = \int_0^\infty e^{-ct} f(t) \cos(\omega t)\, dt, \tag{1.4}$$

$$\mathrm{Im}[F(c + i\omega)] = -\int_0^\infty e^{-ct} f(t) \sin(\omega t)\, dt. \tag{1.5}$$

The inversion theorems for Fourier cosine and sine transforms give, as an alternative to (1.3), the pair of relations

$$f(t) = 2\pi^{-1} e^{ct} \int_0^\infty \mathrm{Re}[F(c + i\omega)] \cos(\omega t)\, dt, \tag{1.6}$$

$$f(t) = -2\pi^{-1} e^{ct} \int_0^\infty \mathrm{Im}[F(c + i\omega)] \sin(\omega t)\, dt. \tag{1.7}$$

In all these relations, $c > c_0$.

2. Criteria Used for Comparison

In this section we discuss the criteria which we have chosen to use in evaluating a number of different numerical inversion techniques. In our view, the most important criteria are: applicability to a variety of common types of inversion problems, numerical accuracy which may be achieved on a digital computer, relative computation times, programming and implementation difficulties. Actually these criteria are not fully independent, but for convenience we discuss them separately.

2.1. *Applicability to Actual Inversion Problems*

In practice there are many problems for which numerically inverting the Laplace transform either requires a special type of method, or can be greatly facilitated by using a special type of method. In other words, for such problems an all-purpose method may be inappropriate. Examples include problems with numerical data at arbitrary points, problems with transforms in the form of rational fractions, problems with noisy data (a case we hope to treat in a later paper), or problems for which the solution is known to be of particular form. Such types of problems often can best be approached by a special method, if such a method is available.

Here we are concerned only with two general cases: a general $F(p)$ about which little is known, and for which an all-purpose method is initially most convenient to apply; and an $F(p)$ for which the form of the solution $f(t)$ is roughly known. These two cases are the basis for the two sets of results presented here. First, we present results for each numerical method for that set of parameters which gives the most accurate numerical approximation to the analytical solution. So if one knows the rough form of the solution $f(t)$ which is sought, it is appropriate to use that numerical method, and the associated parameter set, which gives the most accurate numerical approximation for a tested $F(p)$ which gives an $f(t)$ of similar form to the unknown solution. Second, we present results for each method for that set of parameters which is most successful over the whole range of functions tested. These results tell one what method to use, and what associated parameters to use, if one wants a general purpose inversion method. (There is still no single best method because of the differing success of the methods according to the other criteria below.)

Although the results here might give one confidence that a particular method will be highly successful, for a certain problem it is usually worth using more than one method (as recommended by Bellman *et al.* (1966), and others) on any unknown function as a check against peculiar behavior of the function or of the numerical method, and against programming and implementation errors.

In the troublesome case that different methods produce different results, the proper course of action will have to be determined on the basis of the features of the problem at hand. One approach would be to try the methods on a function whose analytical Laplace transform is known and which is similar in form to the troublesome function: the more successful method on the known function should then be a better bet on the unknown one.

2.2. *Applicability to Various Types of Functions*

We give results for each method on each of the test functions listed in Table I. These particular functions were selected to cover a wide range of functional forms, to cover well known and understandable functional behaviors, and because they have simple analytic solutions. (Without an analytic solution, there would be no safe way to compare the different numerical solutions.) Obviously many other functions could be used as well. We believe that the ones used here are adequate to compare the different numerical methods (already there is considerable regularity in the success of the various methods on certain types of functions). For special functions, it would be straightforward for the user to compare results from promising numerical methods on further suitable test functions.

There are three basic types of functions in the set of 16: functions which are continuous and for which $F(p) \to p^{\alpha}$ as $p \to \infty$ (functions 1–9, 13, 14, 16); functions which are continuous and for which there is no value α for which $F(p) \to p^{\alpha}$ as $p \to \infty$ (functions 11, 15); and functions which have discontinuities (functions 10, 12).

2.3. *Numerical Accuracy*

For each function and each method we present two measures for the accuracy of the numerical solution. Let $f(t)$ be the analytical solution and $f_a(t)$ be the numerical solution. The two measures are

$$L = \left(\sum_{i=1}^{30} (f(i/2) - f_a(i/2))^2/30 \right)^{1/2}, \tag{2.1}$$

$$L_e = \left(\sum_{i=1}^{30} (f(i/2) - f_a(i/2))^2 \, e^{-i/2} \Big/ \left(\sum_{i=1}^{30} e^{-i/2} \right) \right)^{1/2}. \tag{2.2}$$

Thus L gives the root-mean-square deviation between the analytical and numerical solutions for the t values 0.5, 1, 1.5,.., 15, while L_e is a similar measure weighted by the factor e^{-t}.

The reason for using these measures is that presentation of results for each t value would take up a large amount of space and make it difficult to quickly perceive the accuracy of the method. L gives a fair indication of the success of a method for large t, and L_e for relatively small t, and between them a good idea of the likely precision of the numerical solution at any t value. It might be argued that smaller t values than 0.5 should be used. However, for most methods results are more accurate for small t—L_e is smaller than L—so that results for small t are likely to be better than indicated by L_e. Exceptions to this will be noted explicitly.

All the calculations were done in double precision on a Univac 1110 (there is no complex double precision arithmetic on this machine, so all complex calculations were done using real and imaginary parts), so that the precision of any given calculation is at most roughly 10^{-17}. The results for the measures L and L_e should be seen in the light of this limitation.

TABLE I

Functions Used in Method Comparisons

$F_1(p) = (p^2 + 1)^{-1/2}$	$f_1(t) = J_0(t)$
$F_2(p) = p^{-1/2}e^{-p^{-1}}$	$f_2(t) = (\pi t)^{-1/2}\cos(2t^{1/2})$
$F_3(p) = (p + 1/2)^{-1}$	$f_3(t) = e^{-t/2}$
$F_4(p) = ((p + 0.2)^2 + 1)^{-1}$	$f_4(t) = e^{-0.2t}\sin(t)$
$F_5(p) = p^{-1}$	$f_5(t) = 1$
$F_6(p) = p^{-2}$	$f_6(t) = t$
$F_7(p) = (p + 1)^{-2}$	$f_7(t) = te^{-t}$
$F_8(p) = (p^2 + 1)^{-1}$	$f_8(t) = \sin(t)$
$F_9(p) = p^{-1/2}$	$f_9(t) = (\pi t)^{-1/2}$
$F_{10}(p) = p^{-1}e^{-5p}$	$f_{10}(t) = \theta(t - 5)$
$F_{11}(p) = p^{-1}\ln(p)$	$f_{11}(t) = -\gamma - \ln(t)$
$F_{12}(p) = (p(1 + e^{-p}))^{-1}$	$f_{12}(t) =$ square wave
$F_{13}(p) = (p^2 - 1)(p^2 + 1)^{-2}$	$f_{13}(t) = t\cos(t)$
$F_{14}(p) = (p + 1/2)^{1/2} - (p + 1/4)^{1/2}$	$f_{14}(t) = (e^{-t/4} - e^{-t/2})(4\pi t^3)^{-1/2}$
$F_{15}(p) = e^{-4p^{1/2}}$	$f_{15}(t) = 2e^{-4/t}(\pi t^3)^{-1/2}$
$F_{16}(p) = \arctan(p^{-1})$	$f_{16}(t) = t^{-1}\sin(t)$

2.4. *Computation Time Required*

The average time for the evaluation of the numerical value of a single function at 30 t values is given later in Table VI. Because of differences in execution time due to different functions and especially to different parameters, and because of the particular implementation of the numerical algorithms in Fortran (although every attempt was made to optimize each algorithm), the computation times are probably accurate to no more than 25%.

2.5. *Programming and Implementation*

Some of the numerical methods are simple to program and others fairly difficult or inconvenient. The exposition in Section 3 should give a feel for this. Ease of programming may be an important consideration if only a few functions are to be inverted; if a large number of numerical inversions are to be done, then it is worth using a difficult-to-program method if it is successful in other respects. Errors in programming can be detected by comparing results on test problems with results here or in the papers originally describing the method.

Implementation here refers to applying an already programmed method to a particular problem. The greatest difficulty involved in implementation is in determining $F(p)$. This is easy for real values of p, but for complex values may be very messy and a source of error. Therefore for each method it is noted (in Table VI) whether or not values of $F(p)$ for complex p are required.

2.6. *Selection of Methods for Testing*

Although we have tested a large number of numerical methods, it will be apparent that we have omitted quite a number of others. There are two reasons for this. First, a large number of numerical methods for inverting the Laplace transform are variants of other methods. When the original method is not reasonably successful we have not tested variants (for example, we have not tested Piessens' (1969) variant of the rather poor method of Bellman *et al.* (1966).) On the other hand, when a given method shows great promise, we have attempted to test a number, though not necessarily all, of its variants. This is the reason for including methods due to Dubner and Abate (1968), Siverberg (1970), and Crump (1976). Second, we have been forced to limit our testing of different methods by our own limited resources. Hopefully the results here can provide a basis for the testing and comparison of other numerical methods for inverting the Laplace transform which have been omitted by us or which are as yet undeveloped.

3. Survey of Numerical Inversion Procedures

In this section we briefly survey some of the methods which have been proposed for the numerical inversion of Laplace transforms. Questions related to the merits of these methods are left until the next section, although the actual selection of methods made here reflects to some extent our assessment of these matters. The overall organization is under six main headings which are based on theoretical considerations.

3.1. *Methods Which Compute a Sample*

An approximate inversion algorithm for the Laplace transform can be obtained by computing a sample, namely,

$$I_n(t) = \int_0^\infty \delta_n(t, u) f(u)\, du,$$

where the functions $\delta_n(t, u)$ form a delta convergent sequence (Gelfand and Shilov, 1964, p. 34), which means that they have the property that $I_n(t)$ tends to $f(t)$ with increasing n. Using the functions $(nu/t)^n \exp(-nu/t)/(n-1)!$ gives the result

$$f(t) \approx I_n(t) = (-1)^n n^{n+1}(n!)^{-1} t^{-(n+1)} F^{(n)}(n/t), \tag{3.1}$$

where $F^{(n)}(p)$ is the nth derivative of $F(p)$ with respect to p. This formula was obtained by Widder (1934), though not in the context of numerically inverting the Laplace transform. An obvious disadvantage of this method is the need to differentiate $F(p)$ repeatedly.

A slightly different formula than (3.1), but which may be obtained in a similar manner (see Cost, 1964) was proposed by ter Haar (1951):

$$f(t) \approx t^{-1} F(t^{-1}). \tag{3.2}$$

Another variant is due to Schapery (1962):

$$f(t) \approx (2t)^{-1} F((2t)^{-1}). \tag{3.3}$$

Gaver (1966) has proposed the use of the functions

$$\delta_n(t, u) = (2n)!(n!(n-1)!)^{-1} a(1 - e^{-au})^n e^{-nau},$$

where $a = \ln 2/t$, which leads to a result similar to (3.1) but involving the nth finite difference $\Delta^n F(n \ln 2/t)$. The formula, as it stands, is not convenient, because the rate of convergence of $I_n(t)$ to $f(t)$ is rather slow. However, Gaver has shown that the quantity $I_n(t) - f(t)$ may be expanded in an asymptotic expansion in inverse powers of n; consequently the result may be improved using extrapolation. The most useful extrapolation formula has been derived by Stehfest (1970, 1970a) to give the algorithm

$$\begin{aligned} f(t) &\approx \ln 2/t \sum_{n=1}^{N} K_n F(n \ln 2/t), \\ K_n &= (-1)^{n+N/2} \sum_{k=[(n+1)/2]}^{\min(n, N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!\, k!(k-1)!(n-k)!(2k-n)!}, \end{aligned} \tag{3.4}$$

where N is even.

The Gaver–Stehfest method (3.4) is a particular example of a general class of methods proposed by Zakian (1969). In our notation, Zakian chooses

$$\delta_n(t, u) = \sum_{i=1}^{N} K_i t^{-1} \exp(-a_i u/t),$$

so that

$$I_n(t) = \sum_{i=1}^{N} K_i t^{-1} F(a_i/t). \tag{3.5}$$

Zakian has proposed two methods for choosing the coefficients K_i and a_i. In the first (Zakian, 1970), he compares the Laplace transform of $\delta_n(t, u)$, which is a rational function, with the Laplace transform of $\delta(u - t)$, which is an exponential function, and chooses the coefficients so that the rational functions are equal to the classical Padé approximations for the exponential function. It has been noted by Singhal and Vlach (1975) that this method is a special case of the Gaussian quadrature method, which we outline in Section 3.3. The other proposal involves a least-squares optimization (Zakian and Gannon, 1971).

3.2. *Methods Which expand $f(t)$ in Exponential Functions*

A number of methods depend on representing $f(t)$ by exponential functions, usually by introducing $\exp(-rt)$ as a new independent variable. Bateman (1936),

Erdélyi (1943), Papoulis (1956), and Lanczos (1957) have proposed Legendre functions. Writing

$$f(t) = \sum_{n=0}^{\infty} a_n P_{2n}(e^{-rt}), \tag{3.6}$$

taking the Laplace transformation of this formula, and substituting $p = (2k + 1)\, r$, $k = 0, 1, 2, \ldots, N$ leads to the relations

$$rF[(2k + 1)\, r] = \sum_{m=0}^{k} \frac{(k - m + 1)_m}{2(k + 1/2)_{m+1}}\, a_n\,, \tag{3.7}$$

where

$$\begin{aligned} (j)_n &= 1, && n = 0 \\ &= j(j + 1) \cdots (j + n - 1), && n > 0. \end{aligned}$$

Equations (3.7) may be solved recursively for the coefficients a_m (Papoulis, 1956); the Legendre polynomials are calculated from the relations

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ (n + 1)\, P_{n+1}(x) &= (2n + 1)\, xP_n(x) - nP_{n-1}(x). \end{aligned} \tag{3.8}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form a very general set. Miller and Guy (1966) have used them to approximate $f(t)$ as

$$f(t) = \sum_{m=0}^{\infty} a_m P_m^{(0,\beta)}(2e^{-rt} - 1), \tag{3.9}$$

where α is set to zero. On taking the Laplace transform and setting $p = (\beta + k + 1)\, r$, they obtain

$$rF[(\beta + k + 1)\, r] = \sum_{m=0}^{k} \frac{(k - m + 1)_m}{(k + \beta + 1)_{m+1}}\, a_m\,,$$

which may readily be solved for the coefficients a_m. Together with the recursion relations

$$\begin{aligned} P_0^{(\alpha,\beta)}(x) &= 1, \\ P_1^{(\alpha,\beta)}(x) &= x(1 + (\alpha + \beta)/2) + (\alpha - \beta)/2, \\ 2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)\, P_{n+1}^{(\alpha,\beta)}(x) & \\ = [(2n + \alpha + \beta + 1)(\alpha^2 - \beta^2) + (2n + \alpha + \beta)_3\, x]\, P_n^{(\alpha,\beta)}(x) & \\ - 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)\, P_{n-1}^{(\alpha,\beta)}(x), & \end{aligned}$$

this specifies the method. Berger (1968) has also proposed the use of Jacobi poly-

nomials, and as well has used ultraspherical polynomials (Berger, 1966; Berger and Duangudom, 1973).

Bellman *et al.* (1966) have outlined a method which they derive from consideration of the Gauss–Legendre quadrature rule applied to the definition of $F(p)$ (Eq. (1.1)). However, it may be regarded as a special case of the Legendre method, by setting $k = 0$ and $t_i = -r^{-1} \ln((1 + x_i)/2)$, $i = 1, 2, \ldots, N$, where the x_i are the zeros of $P_N(x)$. The result is

$$f(t_i) = \sum_{k=1}^{N} w_{ik} r F(kr), \tag{3.10}$$

where the coefficients w_{ik} are defined by

$$\sum_{k=0}^{N-1} w_{ik} x^k = \frac{P_N((1 + x)/2)}{x - x_i} \left(\int_0^1 \frac{P_N((1 + x)/2)}{x - x_i} \, dx \right)^{-1}. \tag{3.11}$$

An important disadvantage of this method is that it gives the values of $f(t)$ only at a restricted set of nonequidistant points. Tables of the coefficients w_{ik} may be found in Bellman *et al.* (1966). An extension of Bellman *et al.*'s method has been presented by Piessens (1969).

Another group of methods depend on transformations of the type $\cos \theta = \exp(-rt)$, followed by an expansion in trigonometric functions. Papoulis (1956) writes

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k \sin((2k + 1)\, \theta), \\ \cos \theta &= e^{-rt}. \end{aligned} \tag{3.12}$$

Taking the Laplace transform with $p = (2k + 1)\, r$, $k = 0, 1, \ldots, N$, gives the equations

$$rF[(2k + 1)\, r] = \pi 2^{-2(n+1)} \sum_{m=0}^{k} \left[\binom{2k}{k - m} - \binom{2k}{k - m - 1} \right] a_m . \tag{3.13}$$

A number of other similar schemes have been proposed, using either a Fourier series (Doetsch, 1971; Lanczos, 1957) or an expansion in Chebyshev polynomials (Lanczos, 1957).

For certain problems, Schapery (1962; see Cost (1964) and Rizzo and Shippy (1970)) has proposed the expansion

$$f(t) = A + Bt + \sum_{k=1}^{N} a_k e^{-b_k t}, \tag{3.14}$$

where the exponent weights b_k are chosen to suit the expected form of the function $f(t)$. Laplace transformation yields

$$pF(p) = A + Bp^{-1} + \sum_{k=1}^{N} a_k (1 + b_k p^{-1})^{-1}, \tag{3.15}$$

and these equations are solved for a_k by substituting $p = b_k$, $k = 1,..., N$, and also using the identities

$$B = \lim_{p \to 0} p^2 F(p),$$
$$A = -\sum_{k=1}^{N} a_k . \tag{3.16}$$

3.3. *Gaussian Numerical Quadrature of the Inversion Integral*

Gaussian quadrature is a well-known method for the approximation of integrals, based on ensuring that the rules are exact for polynomials. Such a rule has been developed for the inversion integral, designed to invert exactly Laplace transforms of the form $p^{-s}\Phi(p^{-1})$, where $\Phi(p^{-1})$ is a polynomial in p^{-1} (Salzer, 1955, 1958, 1961; Shirtliffe and Stephenson, 1960; Piessens, 1969a, 1971). To state the rule in invariant form, it is necessary to write the inversion integral (1.3) as

$$f(t) = (2\pi i t)^{-1} \int_{c'-i\infty}^{c'+i\infty} F(z/t)\, e^z\, dz, \tag{3.17}$$

where $c' = c/t$. The approximation consists of writing

$$f(t) = \sum_{i=1}^{N} K_i t^{-1} F(a_i/t), \tag{3.18}$$

and choosing the coefficients to make the rule exact whenever $p^{s+1}F(p)$ is a polynomial in p^{-1} of degree $\leqslant 2N - 1$. Thus, if $p^{s+1}F(p)$ has a Taylor expansion in powers of p^{-1}, the method recovers a Taylor series expansion for $f(t)$. A survey of tables of the weights K_i and nodes a_i has been given by Piessens (1971), and a computer program for calculating the weights and nodes has been presented by Piessens (1973).

For $s = 0$, a more general method has been given by Singhal and Vlach (1975). They approximate the inversion integral (3.17) by replacing the exponential function by its Padé approximant

$$\xi_{M,N}(z) = \frac{\sum_{k=1}^{M} (M + N + 1 - k)! \binom{M}{k} z^k}{\sum_{k=1}^{N} (M + N + 1 - k)! \binom{N}{k} (-z)^k}, \tag{3.19}$$

with $M < N$, and closing the inversion contour in the right-hand half plane using residue theory. Since the first $M + N + 1$ terms of the Taylor series for $\xi_{M,N}$ are identical with those for e^z, it is readily shown that this method is exact for $F(p) = p^{-k}$, $k = 1, 2,..., M + N + 1$. Thus, on setting $M = N - 1$, we recover the Gaussian method for $s = 0$.

An advantage of this more general method arises out of the fact that for $M < N - 1$, the numerical inversion of p^k, $k = 0, 1,..., N - M - 2$, is identically

zero. Thus, on choosing $M \leqslant N - k - 1$, we have the following approximation to the kth derivative:

$$f^{(k)}(t) = \sum_{i=1}^{N} K_i t^{-1}(a_i/t)^k F(a_i/t). \tag{3.20}$$

This holds because the Laplace transform of $f^{(k)}(t)$ differs from $p^k F(p)$ only via a polynomial of degree $k - 1$ in p, and these terms make no contribution to the result when $M \leqslant N - k - 1$.

Schmittroth (1960) has proposed a method which uses a Gaussian rule derived by Hurwitz and Zweifel (1956) for trigonometric integrals of the type occurring in the inversion formulas of (1.6) and (1.7). Using (1.7), and writing $\phi(\omega) = -\mathrm{Im}[F(c + i\omega)]$, the essential numerical formulas are

$$f(t) = 2e^{ct}t^{-1} \sum_{n=0}^{\infty} I_n(t), \tag{3.21}$$

$$\begin{aligned} I_n(t) &= (-1)^n \int_{-0.5}^{0.5} \phi[\pi t^{-1}(\omega + n + 0.5)] \cos \pi\omega \, d\omega, \\ &\approx (-1)^n \sum_{j=1}^{N} A_j{}^N (\cos \pi\theta_j{}^N)^{-1} (\phi[\pi t^{-1}(-\theta_j{}^N + n + 0.5)] \\ &\quad + \phi[\pi t^{-1}(\theta_j{}^N + n + 0.5)]), \end{aligned} \tag{3.22}$$

where the coefficients $\theta_j{}^N$ and $A_j{}^N$ are the nodes and weights for an N-point Gaussian approximation to the integral $I_n(t)$, and are given in Hurwitz and Zweifel (1956). Since the Gaussian rule depends on approximating the function $\phi[\pi t^{-1}(\omega + n + 0.5)]$ for $-0.5 < \omega < 0.5$, it becomes increasingly accurate for fixed N as t increases. Thus the method operates in general for $t > t_{\min}$, in contrast with other methods, which operate for $t < t_{\max}$. The sequence of partial sums for the alternating series (3.21) converges rather slowly, and Schmittroth applies the averaging procedure

$$\begin{aligned} S_m{}^k &= (S_m^{k-1} + S_{m+1}^{k-1})/2, \qquad k = 1, 2, \ldots, \\ S_m{}^0 &= \sum_{n=0}^{m} I_n \, . \end{aligned} \tag{3.23}$$

to speed the convergence. The series (3.21) is summed until a relative accuracy ϵ is achieved or a limit $n_{\max}$ of terms is reached.

3.4. *Methods Which Use a Bilinear Transformation of p*

The methods of Section 3.2 consist essentially of choosing an expansion for $f(t)$, and determining the parameters by a process of collocation of the Laplace transform of the expansion with $F(p)$ at a finite set of points in the p-plane. An alternative approach is to approximate $F(p)$ directly. However, for this purpose, neither poly-

nomials in p nor in e^{-rp} are appropriate. In fact, it is more appropriate to use an expansion in inverse powers of p, and this is most readily done by introducing a new variable z as $z = (p + a)/(p + c)$, $c > a$, which maps $p = \infty$ to $z = 1$, $p = -(a + c)/2$ to $z = -1$, and the half plane $\mathrm{Re}(p) > -(a + c)/2$ to the interior of the unit circle $|z| < 1$.

An important class of applications arises by expanding the original function $f(t)$ in a series of generalized Laguerre functions. Suggestions along this line go back to Tricomi (1935) and Widder (1935): see Piessens (1975). Let

$$f(t) \approx t^{\alpha}e^{-ct} \sum_{k=0}^{N} a_k \frac{k!}{(\alpha + k)!} L_k^{\alpha}(t/T), \tag{3.24}$$

where α, c, and T are parameters. The Laguerre polynomials may be calculated from the recursion relations

$$\begin{aligned} L_0^{\alpha}(t) &= 1, \\ L_1^{\alpha}(t) &= 1 + \alpha - t, \\ nL_n^{\alpha}(t) &= (2n + \alpha - 1 - t)\, L_{n-1}^{\alpha}(t) - (n - 1 + \alpha)\, L_{n-2}^{\alpha}(t), \end{aligned} \tag{3.25}$$

and the coefficients a_k as follows. First note that the Laplace transform of (3.24) is

$$F(p) \approx \sum_{k=0}^{N} a_k (p + c - 1)^k (p + c)^{-(k+\alpha+1)},$$

which the bilinear transformation $z = (p + c - 1)/(p + c)$ turns into

$$\begin{aligned} \Phi(z) &= (p + c)^{\alpha+1} F(p) \\ &\approx \sum_{k=0}^{N} a_k z^k. \end{aligned}$$

The coefficients a_k may be determined by setting $z = e^{i\theta}$ and using standard formulas for trigonometric interpolation. Weeks (1966) set $\alpha = 0$, $c = c_0 + 1/2T$; Piessens and Branders (1971) assume that $F(p)$ has the asymptotic form $p^{-\alpha-1}$ for large p from which α is determined. In either case, the appropriate formulas for the coefficients a_k are

$$\begin{aligned} a_0 &= (N + 1)^{-1} \sum_{j=0}^{N} h(\theta_j), \\ a_k &= 2(N + 1)^{-1} \sum_{j=0}^{N} h(\theta_j) \cos(k\theta_j), \\ \theta_j &= \left(\frac{2j + 1}{N + 1}\right) \frac{\pi}{2}, \\ h(\theta) &= \mathrm{Re}[((1 + \cot(\theta/2))/(2T))^{\alpha+1} F(1/(2T) + c + i \cot(\theta/2)/2T)]. \end{aligned} \tag{3.26}$$

Laguerre polynomials have the advantage that their Laplace transforms are directly connected with a bilinear transformation. However, there are other possible expansions which may be applied to the function $\Phi(z)$. Piessens (1972) has prooposed Jacobi polynomials, but in practice restricts his attention to the special case of Chebyshev polynomials. Thus the basic approximation is

$$F(p) \approx p^{-\alpha-1} \sum_{n=0}^{N} a_n T_n(1 - bp^{-1}).$$

Inverting the series term by term, we find that the corresponding approximation to $f(t)$ is

$$f(t) \approx t^{\alpha}(\alpha!)^{-1} \sum_{n=0}^{N} a_n \phi_n(bt/2), \tag{3.27}$$

where $\phi_k(x)$ is a polynomial of degree k. The first three of these polynomials are readily found as

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= 1 - 2x(\alpha + 1)^{-1}, \\ \phi_2(x) &= 1 - 8x(\alpha + 1)^{-1} + 8x^2(\alpha + 1)^{-1}(\alpha + 2)^{-1}. \end{aligned} \tag{3.28}$$

For $n > 2$, the recurrence relations for $\phi_n(x)$ are

$$\begin{aligned} -\phi_n(x) &= (A + Bx)\,\phi_{n-1}(x) + (C + Dx)\,\phi_{n-2}(x) + E\phi_{n-3}(x), \\ A &= 2n + (n - 1)(2n - 3)(\alpha + n - 1)(n - 2)^{-1}(\alpha + n)^{-1}, \\ B &= 4(n + \alpha)^{-1}, \\ C &= 1 + A + E, \\ D &= 4(n - 1)(n - 2)^{-1}(\alpha + n)^{-1}, \\ E &= (n - 1)(\alpha - n + 1)(n - 2)^{-1}(\alpha + n)^{-1}. \end{aligned} \tag{3.29}$$

The coefficients a_n in the expansion may be expressed as finite sums using standard techniques. The result is

$$\begin{aligned} a_0 &= (N + 1)^{-1} \sum_{n=0}^{N} \Phi\left(\cos\left(\left(\frac{2n + 1}{N + 1}\right)\frac{\pi}{2}\right)\right), \\ a_k &= 2(N + 1)^{-1} \sum_{n=0}^{N} \Phi\left(\cos\left(\left(\frac{2n + 1}{N + 1}\right)\frac{\pi}{2}\right)\right) \cos\left(\left(\frac{2n + 1}{N + 1}\right)\frac{k\pi}{2}\right), \\ \Phi(u) &= (b/(1 - u))^{\alpha+1} F(b/(1 - u)). \end{aligned} \tag{3.30}$$

3.5. *Representation by Fourier Series*

If the trapezoidal rule is applied to either of (1.6) and (1.7), then the resulting numerical expressions are Fourier series. Dubner and Abate (1968) have started with (1.6) to write

$$f(t) \approx 2e^{ct}T^{-1} \sum_{k=0}^{\infty}{}' \operatorname{Re}[F(c + i\pi k/T)] \cos(\pi kt/T), \tag{3.31}$$

where T is a parameter and the prime on the summation means that the $k = 0$ term has weight one-half. Dubner and Abate derived this formula by considering the expansion of $f(t)$, $0 \leqslant t < T$, as a Fourier cosine series, and showed that the error could be made small for $0 \leqslant t < T/2$ by choosing cT sufficiently large.

There is no theoretical reason for using the real part of $F(p)$ as in (3.31); Silverberg (1970) and Durbin (1974) have proposed that the trapezoidal rule be applied to (2.3) directly, which is equivalent to averaging (1.6) and (1.7). The resulting approximation is

$$\begin{aligned} f(t) \approx e^{ct}T^{-1} \sum_{k=0}^{\infty}{}' &(\operatorname{Re}[F(c + i\pi k/T)] \cos(\pi kt/T) \\ &- \operatorname{Im}[F(c + i\pi k/T)] \sin(\pi kt/T)). \end{aligned} \tag{3.32}$$

In both cases, it may be shown that the error is essentially bounded by $\exp[(c_0 - c)\,T]$, becoming worse as t approaches $T/2$ for (3.31) and T for (3.32). However, for large cT, this becomes small much more rapidly than the usual T^{-2} law for the trapezoidal rule, and this is connected with the fact that both methods are Fourier representations of a function closely related to $f(t)$.

Both of the above methods involve an infinite sum, and this is approximated, in the original papers, by truncating after N terms. In the interest of efficiency, a number of alternative proposals have been made. One of them is the use of fast Fourier transform techniques (Cooley and Tukey 1965; Cooley *et al.*, 1970); however, this is a computational variant rather than one which is theoretically different. Crump (1976) has proposed the use of convergence speeding techniques, choosing the epsilon algorithm (MacDonald, 1964). The algorithm is essentially

$$\epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + (\epsilon_s^{(m+1)} - \epsilon_s^{(m)})^{-1}, \tag{3.33}$$

where $\epsilon_{-1}^{(m)} = 0$, and $\epsilon_0^{(m)}$ is the mth partial sum of either (3.31) or (3.32).

3.6. *Padé Approximation*

A considerable number of papers have been published on the use of Padé approximation to the Laplace transform, whereby $F(p)$ is replaced by Padé approximants $\xi_{n,m}(p)$ in the inversion integral (see Longman, 1975, and references therein). Unfortunately, the construction of the rational functions $\xi_{n,m}(p)$ requires a knowledge of the Taylor expansion of $F(p)$ about the origin, and this makes it impossible to

implement a procedure which uses only values of $F(p)$. In addition, it has been reported (Longman, 1973) that such methods may perform erratically even on functions of the same form, but differing in some parameter value.

4. Results

4.1. *Methods Tested*

In Table II are listed the numerical methods for which numerical results are reported for the test functions in Table I, along with the references from which the equations used in programming each method were obtained, and the numbers of the equations in this paper where the methods are specified. In the next few paragraphs we comment on some methods which were tested but for which results are not listed, and on some methods which were not tested.

Method 1 is presented to indicate the accuracy obtainable using crude analytical approximations. The methods due to ter Haar (1951)—(3.2)—and Schapery (1962

TABLE II

Methods for Which Results Are Given

Type of method	Method number	Method name	Source of equations	Equation number
Computes a sample	1	Widder	Cost (1964)	(3.1), $n = 1$
	2	Gaver–Stehfest	Stehfest (1970, 1970a)	(3.4)
Expands $f(t)$ in exponential functions	3	Legendre polynomials	Papoulis (1956)	(3.6)–(3.8)
	4	Bellman *et al.*	Bellman *et al.* (1966)	(3.10)–(3.11)
	5	Trigonometric	Papoulis (1956)	(3.12)–(3.13)
	6	Schapery	Cost (1964)	(3.14)–(3.16)
Gaussian quadrature	7	Gaussian quadrature	Piessens (1971)	(3.18)
	8	Schmittroth	Schmittroth (1960)	(3.21)–(3.23)
Bilinear transformation	9	Laguerre–Weeks	Weeks (1966)	(3.24)–(3.26)
	10	Laguerre–Piessens–Branders	Piessens and Branders (1971)	(3.24)–(3.26)
	11	Chebyshev	Piessens (1972)	(3.27)–(3.29)
Fourier series	12	Dubner–Abate	Dubner and Abate (1968)	(3.31)
	13	Silverberg–Durbin	Silverberg (1970)	(3.32)
	14	Crump	Crump (1976)	(3.32)–(3.33)

—(3.3)—usually give somewhat worse results than method 1. On the other hand, using higher derivatives gives higher accuracy, but tests to $n = 3$ (and $n = 8$ on function 3) indicate that convergence is slow; in any case, calculation of higher derivatives analytically is out of the question for most practical problems.

As noted previously, the method presented by Zakian (1970) is a special case of the Gaussian quadrature method (method 7); we have not tested the variant presented by Zakian and Gannon (1971), which does not seem to promise any dramatic improvement.

As well as Legendre polynomials (method 3), we also tested a very similar method based on using the shifted Legendre polynomials. The results were also fairly similar, being slightly better on some functions and slightly worse on others. This comparison gives us some confidence that most variants of methods (such as using different sets of orthogonal polynomials of the same general type) are unlikely to have a dramatic effect on the accuracy obtained.

The use of Jacobi polynomials and of ultraspherical polynomials was not tested, because published results (Miller and Guy, 1966; Berger, 1966, 1968) do not indicate any great improvement over Legendre polynmials, which themselves give only mediocre results.

Because method 4 of Bellman *et al.* (1966) gave such poor results, its extension given by Piessens (1969) was not tested.

The Gaussian quadrature method of Salzer (1955, 1958, 1961) was not tested, since it is a special case of the more general approach of Piessens (1971): Salzer's method will give the same results as Piessens' method for $s = 0$, and worse results for $s \neq 0$. Similarly, the more general Gaussian quadrature method of Singhal and Vlach (1975) is identical to Piessens' method for the results obtained here: Singhal and Vlach's formulation is advantageous when $f^{(k)}(t)$ is desired.

Veillon (1974) provides an improvement to the method of Dubner and Abate (1968)—method 12— in the form of a program for automatically calculating the optimum value of the parameter c. This was not tested independently since the search for optimum parameter values was part of the procedure adopted here anyway. However, Veillon's improvement would be worth considering if method 12 were found suitable for other reasons.

The method of Padé approximation of $F(p)$ was tested using the algorithm found in Longman (1971), but the conclusion was that no useful and comprehensive comparison was possible with the present set of functions and methods. In the first place, seven of the test functions are in the form of rational functions, and Padé approximation therefore gives exact analytical (and numerical) results. Of course, such exactness is not indicative of the promise of the Padé method on functions which differ even slightly from rational fractions, whereas the accuracy of other methods on such rational fractions is usually a good indication of their accuracy on more complex problems of the same type. (Similar comments apply to several of the methods when applied to one or two of the functions, especially function 5.) Second, four of the test functions cannot be expanded in a suitable Taylor series for application of the Padé method. Of the remaining five test functions for which a Taylor series or

nonexact Padé approximation can be achieved, no satisfactory results could be obtained, apparently because of enormous roundoff error which occurred before convergence could be obtained. In any case, we have not included the Padé approach approach in the comparison here. Our conclusion is that it should be used on all problems which are exact rational fractions, but otherwise only after satisfactory results have been obtained (such as we have not obtained) on suitable test functions.

4.2. *Parameters*

An essential part of testing the methods is determining the optimum values of the variable parameters. For each method we followed this general procedure:

(a) If papers describing the method specified an optimum value of a parameter under certain conditions, we adopted that value if appropriate. For example, Crump (1976, p. 94) notes that $T = 0.8t_{\max}$ gives fairly optimal results, so we have adopted $T = 12$ for method 14.

(b) A reasonable spread of values for the remaining parameters were used in preliminary tests, constrained by suggestions in papers and computational effort required (that is, methods requiring large amounts of computation time were usually not tested with as great a number of different parameter values).

(c) Assuming that best results on most of the test functions were obtained in a relatively small range of parameter values (which was usually the case), final tests were made covering these values more closely, again subject to limitations of computational time. The ranges used in these final tests are listed in Table III.

The "optimum" parameter values for any given test function are taken to be the ones, of those tested under stage (c), that give the highest accuracy according to the measures L or L_e. Slightly better results, obviously, usually could be obtained if a finer mesh of parameter values were tested.

The "optimum" parameter values for the set of 16 test functions are determined in the following way: for each parameter, the "optimum" value is taken to be the median of the optimum parameter values for the 32 measures of best fit (16 individual test functions, a value of L and L_e for each), excluding those measures for which the accuracy is very poor (defined arbitrarily as L or $L_e > 10^{-3}$) or always very good (for example, L or $L_e < 10^{-8}$ for all parameter values). For example, if $N = 60$ is the "optimum" number of terms in a Chebyshev approximation, then half of the cases in which L or L_e is minimized will be for $N \geqslant 60$ and half for $N \leqslant 60$, excluding those cases for which results are poor anyway, or for which results are always excellent.

Note that accuracy approaching the accuracy obtained with "optimum" parameter values in many cases may be achieved with other parameter values. In particular, if a much smaller number of terms than the "optimum" number still achieves comparable or satisfactory accuracy, then the smaller number of terms often might be preferred in the interests of computational efficiency. For example, with $N = 301$ method 14 gives $L = 1.4 \times 10^{-11}$ for function 4, while with $N = 101$, $L = 4.2 \times 10^{-6}$ which is satisfactory for many purposes.

TABLE III

Parameter Values Used for Each Numerical Method Tested

Method	Parameter	Value(s)
1	n	1
2	N	8, 10,..., 32
3	r	0.1, 0.2,..., 0.5
	N	10, 15, 20, 25
4	N	4, 7, 10, 15
5	r	0.1, 0.2,..., 0.5
	N	10, 15, 20, 25
6	b_k	$0.01 \times 2^{k-1}$
	N	5, 10,..., 25
7	N	2, 4, 6, 8
8	N	20
	$n_{\max}$	100
	ϵ	10^{-6}
	$c - c_0$	0.2, 0.4, 0.6, 0.8
9	N	20, 40,..., 300
	$c - c_0$	8/15, 10/15, 12/15
10	N	20, 40,..., 300
	$k_{\max}$	N
	$c - c_0$	1/4, 1/2, 1
11	N	10, 20,... until $N \geqslant 60$ and L and L_e begin increasing
	b	0.075, 0.125, 0.2
12	T	30
	N	200, 400,..., 5000
	$c - c_0$	4/30, 8/30, 12/30
13	T	15
	N	200, 400,..., 5000
	$c - c_0$	2/15, 4/15, 6/15
14	T	12
	N	21, 41,..., 301
	$c - c_0$	1/2, 1, 3/2

It should be remembered that the parameter ranges in Table III are only those surveyed under stage (c). In preliminary tests under stage (b), other parameter values were tried, but often not for the full set of functions. For example, for method 8, values of $N = 10$, 20, and 30 were tried initially, then values of $N = 15$, 20, and 25; since $N = 20$ so consistently gave best results and because the lengthy computational time dictated a limited parameter range in the final tests, only this value was used in the final tests using different values of $c - c_0$.

A few comments need to be made concerning specific methods. Method 4 gives results only at a set of specified t values (t_i, $i = 1,..., N$), distributed approximately uniformly in $\ln(t)$. These results were used directly to form a measure $L_4 = (\sum_{i=1}^{N} (f(t_i) - f_a(t_i))^2/N)^{1/2}$, which because of the distribution of t values we liken to our measures L_e; no attempt was made to interpolate to achieve results at the normal specified t values. Concerning method 10, Piessens and Branders (1971) allow the sum in (3.24) to be terminated at a value of k less than N, thereby obtaining more accurate results. However, since these authors give no rule to indicate when this sum should be terminated and because we feel that no more than two free parameters can be usefully handled in applying a method in practice, we have not incorporated this flexibility in our parameter testing. For method 13, Silverberg (1970) and Durbin (1974)[1] note that results are accurate in the t range $(0, 2T)$. However, using $t = 7.5$ gave poor results on most functions and we therefore settled on $T = 15$. Finally, several of the methods (8, 9, 12, 13, 14) specified a different way of determining the position of the real part c of the contour in the complex plane, for example by specifying a fixed value of c rather than of $c - c_0$. We have in all cases tested a range of values of $c - c_0$, both for consistency and because in any case the value c_0 could have been shifted originally by choosing a suitably altered test function.

It can be argued that Laplace transform inversion is still more an art than a science. At least one example of the subtleties faced in our testing is in order. By running a single precision version of our program for method 13 we were able to reproduce the results of Durbin (1974) for the function $F(p) = p^{-1}e^{-10p}$, which also show in this situation that method 13 is more accurate than method 12. However, for our test function $F_{10}(p) = p^{-1}e^{-5p}$ we find that method 12 is more accurate than 13. The difference in results seems to depend on the value of the parameter T that is used: the success of method 13 on this problem requires apparently that T be less than twice the distance from the origin to the discontinuity.

4.3. *Results*

Results are presented in Tables IV and V. In Table IV are listed the "optimum" parameter values for L_e for each function, and the values of L_e and L at these parameter values (sometimes the minimum L is at a different set of parameter values than L_e, but we have ignored this difference to save space). In Table V are listed the "optimum" parameter values for the set of 16 test functions, and the values of L_e and L at these values.

The average computation time (CAU time on the Univac 1110) for obtaining $f(t)$ at the 30 t values, averaged over the range of parameter values tested and over the 16 test functions, is listed in Table VI, along with some specific requirements of some of the methods in terms of programming implementation.

[1] F. Durbin has brought to our attention his use of acceleration factors which speed the convergence of his method. However, we have chosen not to incorporate these and other unpublished refinements to the various methods; instead we prefer to implement each method more or less in the manner that might be followed by a careful user following the existing literature.

TABLE IV

For Each of 16 Individual Test Functions and Each of 14 Numerical Methods, the "Optimum" Parameter Values for Minimizing the Measure L_e, and the Values of L_e and L at These Parameter Values ($3.0(-1) \equiv 3.0 \times 10^{-1}$)

Method	$F_1(p) = (p^2+1)^{-1/2}$, $f_1(t) = J_0(t)$ Parameter values	L_e	L	$F_2(p) = p^{-1/2}e^{-p^{-1}}$, $f_2(t) = (\pi t)^{-1/2}\cos(2t^{1/2})$ Parameter values	L_e	L
1		3.0(−1)	2.3(−1)		1.5(−1)	1.4(−1)
2	$N = 24$	1.2(−3)	5.3(−2)	$N = 18$	1.9(−7)	1.6(−5)
3	$N = 15, r = 0.2$	5.0(−3)	1.0(−1)	$N = 15, r = 0.4$	6.1(−2)	5.7(−2)
4		diverges		$N = 4$	5.3(−1)	
5	$N = 15, r = 0.3$	1.3(−2)	1.5(−1)	$N = 15, r = 0.2$	2.5(−1)	2.5(−1)
6	$N = 20$	2.0(−2)	1.8(−1)	$N = 10$	1.1(+0)	1.3(+0)
7	$N = 6$	1.1(−6)	3.7(−4)	$N = 4$	2.6(−11)	2.5(−10)
8	$c - c_0 = 0.4$	2.0(−7)	2.7(−7)	$c - c_0 = 0.6$	1.3(−7)	2.1(−6)
9	$N = 180$, $c - c_0 = 8/15$	3.1(−16)	3.1(−14)	$N = 280$, $c - c_0 = 8/15$	4.2(−4)	1.1(−2)
10	$N = 100$, $c - c_0 = 1/2$	2.9(−16)	1.7(−15)	$N = 40$, $c - c_0 = 1/4$	2.1(−18)	1.7(−17)
11	$N = 110$, $b = 0.125$	9.8(−8)	2.5(−5)	$N = 40$, $b = .075$	8.5(−16)	2.8(−14)
12	$N = 4800$, $c - c_0 = 8/30$	7.6(−8)	9.2(−6)	$N = 4800$, $c - c_0 = 4/30$	1.2(−3)	4.2(−3)
13	$N = 5000$, $c - c_0 = 4/15$	2.8(−4)	4.6(−4)	$N = 4600$, $c - c_0 = 2/15$	7.6(−3)	4.6(−3)
14	$N = 301$, $c - c_0 = 1$	2.0(−10)	5.7(−11)	$N = 281$, $c - c_0 = 1$	1.5(−10)	4.3(−10)

Method	$F_3(p) = (p+1/2)^{-1}$, $f_3(t) = e^{-t/2}$ Parameter values	L_e	L	$F_4(p) = ((p+.2)^2+1)^{-1}$, $f_4(t) = e^{-0.2t}\sin(t)$ Parameter values	L_e	L
1		1.4(−1)	5.8(−2)		2.9(−1)	2.1(−1)
2	$N = 18$	8.3(−8)	1.8(−6)	$N = 26$	3.3(−4)	1.3(−2)
3	$N = 15, r = 0.1$	1.2(−7)	1.4(−7)	$N = 20, r = 0.1$	1.2(−3)	6.0(−3)
4	$N = 15$	1.1(−3)		$N = 15$	2.5(−2)	
5	$N = 15, r = 0.3$	1.5(−2)	1.5(−2)	$N = 20, r = 0.1$	8.5(−4)	6.1(−3)
6	$N = 25$	5.9(−5)	2.1(−4)	$N = 25$	2.6(−2)	1.1(−1)
7	$N = 4$	1.2(−8)	2.9(−6)	$N = 6$	2.0(−7)	7.3(−5)
8	$c - c_0 = 0.6$	2.4(−7)	9.3(−8)	$c - c_0 = 0.6$	1.5(−7)	8.9(−8)

TABLE IV—*Continued*

Method	Parameter values	L_e	L	Parameter values	L_e	L
9	$N = 20$, $c - c_0 = 10/15$	4.0(−18)	7.2(−18)	$N = 100$, $c - c_0 = 10/15$	3.1(−17)	1.7(−15)
10	$N = 40$, $c - c_0 = 1/4$	4.3(−18)	1.8(−18)	$N = 120$, $c - c_0 = 1/2$	2.5(−17)	9.5(−17)
11	$N = 60$, $b = 0.125$	2.0(−12)	4.7(−10)	$N = 110$, $b = 0.125$	1.5(−8)	4.8(−6)
12	$N = 4800$, $c - c_0 = 12/30$	4.0(−8)	2.8(−8)	$N = 4800$, $c - c_0 = 8/30$	1.5(−7)	4.6(−6)
13	$N = 4800$, $c - c_0 = 4/15$	1.9(−4)	5.7(−5)	$N = 2200$, $c - c_0 = 6/15$	2.4(−6)	1.8(−6)
14	$N = 301$, $c - c_0 = 1$	1.3(−10)	3.8(−11)	$N = 301$, $c - c_0 = 1.5$	8.4(−12)	1.4(−11)
	$F_5(p) = p^{-1}$ $f_5(t) = 1$			$F_6(p) = p^{-2}$ $f_6(t) = t$		
Method	Parameter values	L_e	L	Parameter values	L_e	L
1		2.9(−19)	1.1(−18)		1.6(+0)	8.9(+0)
2	$N = 8$	2.3(−15)	3.3(−15)	$N = 18$	3.2(−9)	1.4(−8)
3	$N = 10$, $r = 0.2$	2.1(−13)	2.1(−13)	$N = 25$, $r = 0.1$	5.4(−2)	3.2(−1)
4	$N = 4$	1.2(−15)		$N = 15$	8.8(−2)	
5	$N = 15$, $r = 0.3$	1.5(−2)	1.5(−2)	$N = 20$, $r = 0.4$	8.2(−2)	1.5(+0)
6	$N = 5$	1.0(−18)	1.7(−18)	$N = 5$	0	0
7	$N = 2$	1.8(−16)	1.8(−16)	$N = 2$	9.9(−17)	5.4(−16)
8	$c - c_0 = 0.6$	4.1(−7)	4.2(−6)	$c - c_0 = 0.8$	5.4(−7)	4.5(−6)
9	$N = 20$, $c - c_0 = 8/15$	1.4(−17)	1.3(−15)	$N = 20$, $c - c_0 = 8/15$	2.6(−17)	2.9(−15)
10	$N = 20$, $c - c_0 = 1/4$	7.3(−18)	2.6(−17)	$N = 20$, $c - c_0 = 1/4$	2.2(−17)	1.8(−16)
11	$N = 10$, $b = 0.075$	2.5(−17)	2.3(−16)	$N = 10$, $b = 0.075$	3.0(−17)	3.7(−16)
12	$N = 4800$, $c - c_0 = 12/30$	1.2(−7)	8.2(−6)	$N = 4800$, $c - c_0 = 12/30$	3.0(−7)	5.3(−5)
13	$N = 4800$, $c - c_0 = 4/15$	2.5(−4)	1.4(−4)	$N = 800$, $c - c_0 = 6/15$	1.9(−4)	2.5(−4)
14	$N = 281$, $c - c_0 = 1.5$	3.9(−10)	5.6(−9)	$N = 301$, $c - c_0 = 1.5$	2.1(−11)	2.5(−10)

TABLE IV—*Continued*

Method	$F_7(p) = (p+1)^{-2}$, $f_7(t) = te^{-t}$ Parameter values	L_e	L	$F_8(p) = (p^2+1)^{-1}$, $f_8(t) = \sin(t)$ Parameter values	L_e	L
1		9.0(−2)	4.7(−2)		4.5(−1)	6.9(−1)
2	$N = 22$	4.1(−7)	2.0(−6)	$N = 26$	3.7(−3)	2.4(−1)
3	$N = 15$, $r = 0.1$	4.2(−9)	2.8(−9)	$N = 20$, $r = 0.1$	1.0(−2)	1.4(−1)
4	$N = 15$	2.2(−4)		$N = 7$	6.3(−2)	
5	$N = 20$, $r = 0.4$	3.3(−5)	1.0(−4)	$N = 20$, $r = 0.1$	8.6(−3)	1.3(−1)
6	$N = 20$	1.5(−4)	5.4(−4)	$N = 25$	8.9(−2)	6.1(−1)
7	$N = 6$	1.4(−8)	6.1(−7)	$N = 6$	1.7(−6)	6.3(−4)
8	$c - c_0 = 0.8$	8.7(−8)	3.8(−8)	$c - c_0 = 0.6$	2.0(−7)	3.0(−7)
9	$N = 20$, $c - c_0 = 10/15$	1.9(−18)	9.4(−19)	$N = 100$, $c - c_0 = 10/15$	1.5(−16)	2.8(−14)
10	$N = 20$, $c - c_0 = 1$	3.9(−18)	1.9(−18)	$N = 120$, $c - c_0 = 1/2$	5.5(−17)	9.4(−16)
11	$N = 120$, $b = 0.075$	8.5(−11)	2.4(−8)	$N = 120$, $b = 0.125$	3.5(−8)	9.7(−6)
12	$N = 4800$, $c - c_0 = 12/30$	5.6(−8)	2.1(−8)	$N = 4800$, $c - c_0 = 12/30$	3.1(−7)	1.6(−5)
13	$N = 800$, $c - c_0 = 6/15$	7.5(−5)	2.5(−5)	$N = 2200$, $c - c_0 = 6/15$	3.3(−6)	1.6(−5)
14	$N = 301$, $c - c_0 = 1.5$	1.4(−12)	4.5(−13)	$N = 301$, $c - c_0 = 1$	2.5(−11)	2.6(−11)

Method	$F_9(p) = p^{-1/2}$, $f_9(t) = (\pi t)^{-1/2}$ Parameter values	L_e	L	$F_{10}(p) = p^{-1}e^{-5p}$, $f_{10}(t) = \theta(t-5)$ Parameter values	L_e	L
1		7.1(−2)	3.3(−2)		2.1(−1)	2.6(−1)
2	$N = 16$	4.8(−9)	2.5(−9)	$N = 32$	1.2(−2)	9.1(−2)
3	$N = 25$, $r = 0.2$	3.4(−2)	9.1(−2)	$N = 20$, $r = 0.2$	1.3(−2)	3.8(−2)
4	$N = 4$	5.3(−1)		$N = 4$	3.0(−2)	
5	$N = 15$, $r = 0.2$	2.5(−1)	2.5(−1)	$N = 20$, $r = 0.2$	1.3(−2)	3.9(−2)
6	$N = 10$	1.1(+0)	1.2(+0)	$N = 25$	4.1(−2)	1.3(−1)
7	$N = 2$	1.7(−16)	8.0(−17)	$N = 2$	3.6(−2)	6.7(−1)
8	$c - c_0 = 0.4$	1.8(−7)	2.6(−7)	$c - c_0 = 0.8$	1.2(−1)	8.4(−1)
9	$N = 280$, $c - c_0 = 8/15$	4.2(−4)	1.1(−2)	$N = 300$, $c - c_0 = 8/15$	8.3(−3)	7.5(−2)
10	$N = 20$, $c - c_0 = 1/4$	1.0(−17)	2.5(−17)	$N = 240$, $c - c_0 = 1/2$	7.0(−2)	2.1(+0)
11	$N = 10$, $b = 0.075$	2.7(−17)	1.4(−16)	$N = 60$, $b = 0.075$	3.1(−2)	1.2(+0)

TABLE IV—*Continued*

Method	Parameter values	L_e	L	Parameter values	L_e	L
12	$N = 4800$, $c - c_0 = 4/30$	1.3(−3)	4.5(−3)	$N = 5000$, $c - c_0 = 8/30$	8.2(−5)	5.3(−4)
13	$N = 4800$, $c - c_0 = 2/15$	8.4(−3)	4.0(−3)	$N = 600$, $c - c_0 = 2/15$	6.9(−2)	2.2(−1)
14	$N = 301$, $c - c_0 = 1$	1.0(−10)	9.8(−11)	$N = 81$, $c - c_0 = 0.5$	6.5(−2)	2.1(+0)
	$F_{11}(p) = p^{-1}\ln(p)$ $f_{11}(t) = -\gamma - \ln(t)$			$F_{12}(p) = (p(1 + e^{-p}))^{-1}$ $f_{12}(t) =$ square wave		
Method	Parameter values	L_e	L	Parameter values	L_e	L
1		4.2(−1)	4.2(−1)		3.2(−1)	3.5(−1)
2	$N = 16$	1.8(−8)	1.9(−8)	$N = 28$	6.1(−2)	2.9(−1)
3	$N = 20$, $r = 0.3$	1.4(−2)	5.0(−2)	$N = 25$, $r = 0.3$	4.4(−2)	3.0(−1)
4	$N = 15$	8.9(−2)		$N = 15$	2.2(−1)	
5	$N = 15$, $r = 0.2$	9.2(−2)	9.2(−2)	$N = 20$, $r = 0.4$	5.8(−2)	3.3(−1)
6	$N = 20$	1.7(−1)	1.4(−1)	$N = 20$	2.1(−1)	3.6(−1)
7	$N = 8$	4.4(−2)	3.7(−2)	$N = 8$	6.0(−2)	2.9(−1)
8	$c - c_0 = 0.8$	8.9(−7)	7.0(−5)	$c - c_0 = 0.6$	3.0(−3)	2.1(−2)
9	$N = 280$, $c - c_0 = 8/15$	2.5(−5)	5.8(−4)	$N = 280$, $c - c_0 = 10/15$	2.3(−2)	8.8(−1)
10	$N = 300$, $c - c_0 = 1/2$	1.5(−3)	9.9(−3)	$N = 200$, $c - c_0 = 1/2$	4.2(−2)	3.6(−1)
11	$N = 10$, $b = 0.075$	6.3(−1)	1.6(+0)	$N = 20$, $b = 0.125$	2.7(−1)	1.5(+0)
12	$N = 4800$, $c - c_0 = 8/30$	1.5(−4)	1.8(−3)	$N = 5000$, $c - c_0 = 8/30$	9.6(−4)	1.7(−3)
13	$N = 4600$, $c - c_0 = 4/15$	1.6(−3)	3.7(−3)	$N = 4800$, $c - c_0 = 4/15$	6.7(−4)	5.9(−4)
14	$N = 281$, $c - c_0 = 1$	2.1(−10)	1.7(−10)	$N = 281$, $c - c_0 = 0.5$	5.7(−4)	1.4(−2)
	$F_{13}(p) = (p^2 - 1)(p^2 + 1)^{-2}$ $f_{13}(t) = t\cos(t)$			$F_{14}(p) = (p + 1/2)^{1/2} - (p + 1/4)^{1/2}$ $f_{14}(t) = (e^{-t/4} - e^{-t/2})(4\pi t^3)^{-1/2}$		
Method	Parameter values	L_e	L	Parameter values	L_e	L
1		1.1(+0)	6.0(+0)		1.1(−2)	3.7(−3)
2	$N = 28$	3.9(−2)	2.8(+0)	$N = 16$	1.6(−8)	6.4(−8)
3	$N = 25$, $r = 0.2$	7.1(−2)	3.9(+0)	$N = 15$, $r = 0.2$	7.4(−3)	4.9(−3)
4	$N = 15$	1.2(−1)		$N = 4$	6.6(−2)	
5	$N = 25$, $r = 0.4$	1.5(−1)	6.3(+0)	$N = 15$, $r = 0.2$	3.2(−2)	3.2(−2)

TABLE IV—*Continued*

Method	Parameter values	L_e	L	Parameter values	L_e	L
6	$N = 10$	4.8(−1)	6.1(+0)	$N = 15$	2.0(+0)	1.6(+0)
7	$N = 6$	5.4(−5)	2.0(−2)	$N = 4$	6.9(−11)	1.7(−8)
8	$c - c_0 = 0.8$	3.7(−7)	3.9(−6)	$c - c_0 = 0.4$	2.4(−8)	8.4(−9)
9	$N = 220$, $c - c_0 = 10/15$	2.2(−16)	4.5(−14)	$N = 280$, $c - c_0 = 8/15$	3.6(−5)	5.1(−5)
10	$N = 120$, $c - c_0 = 1/2$	2.2(−17)	2.4(−15)	$N = 40$, $c - c_0 = 1/4$	1.4(−18)	6.3(−19)
11	$N = 100$, $b = 0.2$	5.1(−7)	1.8(−4)	$N = 40$, $b = 0.2$	2.6(−13)	3.3(−11)
12	$N = 4800$, $c - c_0 = 12/30$	3.5(−7)	3.6(−5)	$N = 4800$, $c - c_0 = 4/30$	1.1(−4)	6.7(−5)
13	$N = 800$, $c - c_0 = 6/15$	1.4(−4)	2.0(−4)	$N = 4600$, $c - c_0 = 2/15$	7.5(−4)	2.8(−4)
14	$N = 301$ $c - c_0 = 1.5$	1.0(−11)	3.1(−10)	$N = 301$, $c - c_0 = 1$	2.1(−11)	6.1(−12)

	$F_{15}(p) = e^{-4p^{1/2}}$ $f_{15}(t) = 2e^{-4/t}(\pi t^3)^{-1/2}$			$F_{16}(p) = \arctan(p^{-1})$ $f_{16}(t) = t^{-1}\sin(t)$		
Method	Parameter values	L_e	L	Parameter values	L_e	L
1		1.6(−2)	1.1(−2)		2.5(−1)	1.5(−1)
2	$N = 24$	3.6(−7)	1.8(−6)	$N = 24$	4.1(−4)	1.9(−2)
3	$N = 20, r = 0.1$	4.3(−5)	5.4(−5)	$N = 20, r = 0.1$	6.2(−4)	8.3(−3)
4	$N = 15$	4.0(−4)		$N = 4$	9.8(−3)	
5	$N = 20, r = 0.1$	1.3(−5)	4.9(−5)	$N = 15, r = 0.3$	1.3(−2)	5.2(−2)
6	$N = 25$	2.8(−4)	4.6(−4)	$N = 25$	1.2(−2)	7.2(−2)
7	$N = 8$	2.3(−6)	9.6(−5)	$N = 8$	3.7(−3)	3.4(−3)
8	$c - c_0 = 0.8$	8.1(−9)	2.7(−7)	$c - c_0 = 0.2$	7.6(−5)	2.7(−2)
9	$N = 300$, $c - c_0 = 8/15$	6.7(−18)	1.0(−15)	$N = 100$, $c - c_0 = 8/15$	1.4(−11)	3.9(−9)
10	$N = 300$, $c - c_0 = 1$	3.3(−8)	1.0(−5)	$N = 220$, $c - c_0 = 1/4$	1.1(−13)	3.3(−11)
11	$N = 20$, $b = 0.125$	4.1(−3)	3.4(−2)	$N = 80$, $b = 0.2$	1.1(−8)	3.5(−6)
12	$N = 1000$, $c - c_0 = 12/30$	1.8(−11)	5.1(−9)	$N = 5000$, $c - c_0 = 4/30$	2.1(−3)	7.0(−1)
13	$N = 400$, $c - c_0 = 6/15$	3.5(−8)	2.8(−8)	$N = 4800$, $c - c_0 = 2/15$	5.4(−2)	8.9(+0)
14	$N = 261$, $c - c_0 = 1.5$	1.0(−13)	1.8(−11)	$N = 281$, $c - c_0 = 0.5$	1.1(−10)	4.4(−9)

TABLE V

For the Set of 16 Test Functions as a Whole, "Optimum" Parameter Values for Each of 14 Numerical Methods, and the Values of L_e and L for Each Test Function at These Parameter Values ($3.0(-1) \equiv 3.0 \times 10^{-1}$)

Method	Parameter values	$F_1(p) = (p^2+1)^{-1/2}$, $f_1(t) = J_0(t)$		$F_2(p) = p^{-1/2}e^{-p^{-1}}$, $f_2(t) = (\pi t)^{-1/2}\cos(2t^{1/2})$	
		L_e	L	L_e	L
1		3.0(−1)	2.3(−1)	1.5(−1)	1.4(−1)
2	$N = 18$	4.3(−3)	9.0(−2)	1.9(−7)	1.6(−5)
3	$N = 15, r = 0.1$	2.8(−2)	5.8(−2)	6.4(−2)	3.8(−2)
4	$N = 15$	diverges		9.5(−1)	
5	$N = 20, r = 0.1$	3.2(−2)	3.6(−2)	5.5(−1)	2.8(−1)
6	$N = 25$	2.0(−2)	1.8(−1)	2.8(+2)	2.4(+2)
7	$N = 6$	1.1(−6)	3.7(−4)	8.0(−7)	3.1(−7)
8	$c - c_0 = 0.6$	3.2(−7)	4.2(−6)	1.3(−7)	2.1(−6)
9	$N = 200$, $c - c_0 = 8/15$	3.4(−16)	3.3(−14)	3.3(−3)	2.0(−2)
10	$N = 140$, $c - c_0 = 1/2$	3.0(−16)	1.0(−15)	1.7(−17)	8.7(−16)
11	$N = 60, b = 0.125$	1.0(−4)	1.7(−2)	5.0(−13)	9.3(−11)
12	$N = 4800$, $c - c_0 = 12/30$	1.2(−7)	6.9(−6)	2.5(−3)	1.4(−1)
13	$N = 4600$, $c - c_0 = 4/15$	2.9(−4)	5.1(−4)	8.6(−3)	1.6(−2)
14	$N = 301, c - c_0 = 1$	2.0(−10)	5.7(−11)	2.1(−10)	9.4(−11)

Method	$F_3(p) = (p + 1/2)^{-1}$, $f_3(t) = e^{-t/2}$		$F_4(p) = ((p+0.2)^2+1)^{-1}$, $f_4(t) = e^{-0.2t}\sin(t)$	
	L_e	L	L_e	L
1	1.4(−1)	5.8(−2)	2.9(−1)	2.1(−1)
2	8.3(−8)	1.8(−6)	2.7(−3)	4.0(−2)
3	1.2(−7)	1.4(−7)	7.9(−3)	1.8(−2)
4	1.1(−3)		2.5(−2)	
5	3.4(−2)	1.7(−2)	8.5(−4)	6.1(−3)
6	5.9(−5)	2.1(−4)	2.6(−2)	1.1(−1)
7	3.8(−7)	3.1(−7)	2.0(−7)	7.3(−5)
8	2.4(−7)	9.3(−8)	1.5(−7)	8.9(−8)
9	7.1(−17)	8.7(−17)	2.6(−16)	4.8(−15)
10	1.3(−17)	8.1(−18)	3.2(−17)	7.2(−17)
11	2.0(−12)	4.7(−10)	7.4(−5)	1.5(−2)
12	4.0(−8)	2.8(−8)	1.6(−7)	1.1(−6)
13	3.8(−4)	1.2(−4)	1.7(−4)	8.6(−5)
14	1.3(−10)	3.8(−11)	1.2(−11)	8.7(−12)

TABLE V—*Continued*

Method	$F_5(p) = p^{-1}$ $f_5(t) = 1$ L_e	L	$F_6(p) = p^{-2}$ $f_6(t) = t$ L_e	L
1	2.9(−19)	1.1(−18)	1.6(+0)	8.9(+0)
2	4.1(−9)	7.2(−9)	3.2(−9)	1.4(−8)
3	3.9(−9)	2.8(−9)	5.0(−1)	6.4(−1)
4	2.8(−9)		8.8(−2)	
5	3.4(−2)	1.7(−2)	1.4(−1)	4.1(−1)
6	2.5(−18)	2.8(−18)	0	0
7	4.0(−7)	4.0(−7)	1.9(−8)	1.1(−7)
8	4.1(−7)	4.2(−6)	8.3(−7)	3.8(−6)
9	3.8(−16)	2.7(−14)	2.3(−14)	3.5(−12)
10	3.6(−17)	1.3(−15)	1.3(−16)	2.1(−15)
11	1.4(−11)	3.0(−9)	1.1(−11)	2.7(−9)
12	1.2(−7)	8.2(−6)	3.0(−7)	5.3(−5)
13	5.5(−4)	6.2(−4)	1.0(−2)	1.3(−2)
14	5.4(−10)	1.6(−10)	2.1(−9)	1.3(−9)

Method	$F_7(p) = (p + 1)^{-2}$ $f_7(t) = te^{-t}$ L_e	L	$F_8(p) = (p^2 + 1)^{-1}$ $f_8(t) = \sin(t)$ L_e	L
1	9.0(−2)	4.7(−2)	4.5(−1)	6.9(−1)
2	2.2(−6)	1.6(−5)	1.7(−2)	4.2(−1)
3	4.2(−9)	2.8(−9)	5.7(−2)	2.7(−1)
4	2.2(−4)		9.1(−2)	
5	2.0(−4)	1.1(−4)	8.6(−3)	1.3(−1)
6	1.5(−4)	5.4(−4)	8.9(−2)	6.1(−1)
7	1.4(−8)	6.1(−7)	1.7(−6)	6.3(−4)
8	3.9(−7)	1.2(−7)	2.0(−7)	3.0(−7)
9	4.4(−15)	1.4(−15)	6.4(−16)	8.3(−14)
10	2.5(−17)	1.0(−17)	5.5(−17)	7.7(−16)
11	1.1(−6)	2.7(−4)	3.3(−4)	5.4(−2)
12	5.6(−8)	2.1(−8)	3.1(−7)	1.6(−5)
13	4.4(−3)	1.4(−3)	2.2(−4)	2.3(−4)
14	4.0(−10)	1.3(−10)	2.5(−11)	2.6(−11)

TABLE V—*Continued*

Method	$F_9(p) = p^{-1/2}$ $f_9(t) = (\pi t)^{-1/2}$		$F_{10}(p) = p^{-1}e^{-5p}$ $f_{10}(t) = \theta(t-5)$	
	L_e	L	L_e	L
1	7.1(−2)	3.3(−2)	2.1(−1)	2.6(−1)
2	4.0(−8)	1.5(−8)	2.7(−2)	6.3(−2)
3	6.2(−2)	3.6(−2)	4.5(−2)	5.0(−2)
4	9.4(−1)		1.2(−1)	
5	5.5(−1)	2.8(−1)	1.8(−2)	3.2(−2)
6	2.8(+2)	2.4(+2)	4.1(−2)	1.3(−1)
7	8.6(−7)	4.0(−7)	6.7(−1)	2.3(+1)
8	2.8(−7)	3.8(−6)	1.2(−1)	8.4(−1)
9	3.3(−3)	2.0(−2)	1.4(−2)	6.6(−2)
10	4.8(−17)	2.0(−15)	7.2(−2)	2.1(+0)
11	1.4(−11)	2.5(−9)	4.4(−2)	5.1(+0)
12	2.5(−3)	1.3(−1)	1.6(−4)	9.2(−4)
13	8.6(−3)	1.6(−2)	8.1(−2)	3.4(−1)
14	1.0(−10)	9.8(−11)	1.2(−1)	3.9(+1)

Method	$F_{11}(p) = p^{-1}\ln(p)$ $f_{11}(t) = -\gamma - \ln(t)$		$F_{12}(p) = (p(1+e^{-p}))^{-1}$ $f_{12}(t) =$ square wave	
	L_e	L	L_e	L
1	4.2(−1)	4.2(−1)	3.2(−1)	3.5(−1)
2	2.7(−8)	1.1(−8)	1.1(−1)	3.1(−1)
3	1.6(−2)	1.8(−2)	1.6(−1)	3.1(−1)
4	8.9(−2)		2.2(−1)	
5	2.2(−1)	1.1(−1)	8.1(−2)	2.9(−1)
6	1.2(+0)	1.0(+0)	2.1(−1)	3.6(−1)
7	4.6(−2)	4.6(−2)	6.8(−2)	3.1(−1)
8	1.9(−6)	4.3(−6)	3.0(−3)	2.1(−2)
9	1.6(−4)	1.5(−3)	2.9(−2)	1.1(−1)
10	5.9(−3)	3.2(−2)	7.3(−2)	4.6(−1)
11	3.9(+3)	5.5(+5)	8.8(+0)	1.2(+3)
12	2.5(−4)	1.3(−2)	1.2(−3)	1.4(−3)
13	1.6(−3)	3.7(−3)	9.4(−4)	1.0(−3)
14	4.7(−10)	2.0(−10)	7.7(−4)	1.5(−3)

TABLE V—*Continued*

Method	$F_{13}(p) = (p^2 - 1)(p^2 + 1)^{-2}$ $f_{13}(t) = t\cos(t)$		$F_{14}(p) = (p + 1/2)^{1/2} - (p + 1/4)^{1/2}$ $f_{14}(t) = (e^{-t/4} - e^{-t/2})(4\pi t^3)^{-1/2}$	
	L_e	L	L_e	L
1	1.1(+0)	6.0(+0)	1.1(−2)	3.7(−3)
2	1.8(−1)	4.9(+0)	1.7(−7)	5.8(−8)
3	1.4(+0)	3.6(+0)	7.8(−3)	4.6(−3)
4	1.2(−1)		1.2(−1)	
5	5.3(−1)	2.6(+0)	6.9(−2)	3.5(−2)
6	4.8(−1)	6.1(+0)	3.3(+1)	2.7(+1)
7	5.4(−5)	2.0(−2)	1.1(−7)	4.8(−8)
8	9.0(−7)	2.6(−6)	2.8(−8)	1.7(−8)
9	9.0(−14)	1.4(−11)	3.6(−4)	1.3(−4)
10	2.6(−17)	7.0(−16)	3.7(−18)	1.2(−17)
11	6.6(−3)	1.8(+0)	8.4(−13)	1.3(−10)
12	3.5(−7)	3.6(−5)	1.6(−4)	6.1(−4)
13	8.1(−3)	9.2(−3)	8.4(−4)	3.5(−4)
14	7.9(−10)	8.6(−10)	2.1(−11)	6.1(−12)

Method	$F_{15}(p) = e^{-4p^{1/2}}$ $f_{15}(t) = 2e^{-4/t}(\pi t^3)^{-1/2}$		$F_{16}(p) = \arctan(p^{-1})$ $f_{16}(t) = t^{-1}\sin(t)$	
	L_e	L	L_e	L
1	1.6(−2)	1.1(−2)	2.5(−1)	1.5(−1)
2	4.0(−6)	2.3(−6)	1.7(−3)	3.5(−2)
3	8.7(−5)	9.0(−5)	5.6(−3)	1.9(−2)
4	4.0(−4)		1.4(−2)	
5	1.3(−5)	4.9(−5)	3.4(−2)	2.1(−2)
6	2.8(−4)	4.6(−4)	1.2(−2)	7.2(−2)
7	1.3(−5)	2.4(−4)	2.2(+1)	7.1(+3)
8	8.4(−9)	4.3(−8)	2.6(−2)	9.7(+0)
9	2.2(−15)	1.1(−14)	1.0(−10)	4.0(−8)
10	1.9(−6)	3.2(−5)	3.3(−12)	1.2(−9)
11	4.4(−2)	6.8(+0)	3.7(−5)	5.4(−3)
12	1.8(−11)	5.1(−9)	1.4(−1)	5.1(+1)
13	1.9(−6)	1.5(−6)	2.1(+0)	6.0(+2)
14	6.7(−13)	2.8(−13)	5.6(−8)	2.3(−5)

TABLE VI

Approximate Execution Times and Implementation Requirements for the Methods Tested

Method	Approximate execution time	Implementation requirements	Programming advantages
1	0.0079	Knowledge of $F^{(k)}(p)$	Easy to program
2	0.043		
3	0.031		
4	0.043		
5	0.081		
6	0.14		
7	0.059	$F(p)$ in complex plane; knowledge of α such that $\lim_{p\to\infty} F(p) \sim p^{-\alpha}$	
8	20.	$F(p)$ in complex plane	
9	2.4	$F(p)$ in complex plane	
10	2.6	$F(p)$ in complex plane; knowledge of α such that $\lim_{p\to\infty} F(p) \sim p^{-\alpha}$	
11	8.9	Knowledge of α such that $\lim_{p\to\infty} F(p) \sim p^{-\alpha}$	
12	50.	$F(p)$ in complex plane	Easy to program; more terms give more accuracy
13	61.	$F(p)$ in complex plane	Easy to program; more terms give more accuracy
14	27.	$F(p)$ in complex plane	More terms give more accuracy

Computation time in many applications depends primarily on the number of evaluations of $F(p)$ required, and in this context it should be noted that methods1, 2, 7, and 8 require a fixed number of function evaluations for each value of t required, whereas the other methods provide for a range of t values for each set of function evaluations. Hence the computation times for methods 1, 2, 7, and 8 would be considerably reduced or incrased for respectively fewer or greater than the 30 t values used in calculating the entries in Table VI.

The execution times for methods 9 to 13 can be reduced by use of the fast Fourier transform, especially if the number of terms in the series whose sum approximates $f(t)$ is large. In addition, use of the FFT reduces the roundoff error involved, though we do not estimate this improvement to make any noticeable change to the values we present for L and L_e. The results given by Durbin (1974) with and without use of the FFT support this conclusion.

Tables IV–VI thus provide a basis for comparing the 14 methods tested.

5. Conclusion

In Section 4 are presented results of tests of a number of different numerical methods for inverting the Laplace transform. These results may be evaluated according to the criteria given in Section 2: applicability to actual inversion problems, applicability to various types of functions, numerical accuracy, computational efficiency, and ease of programming and implementation. Unfortunately there is no simple and clear-cut conclusion to be drawn from these results: no one method is definitely superior by all criteria. The following conclusions which we draw should be seen in the light of this limitation.

First, if at all possible, more than one numerical inversion method should be used on any unknown function. This is because every method breaks down on some functions, and therefore verification by different methods can greatly increase confidence in any results achieved.

Second, if a general-purpose method is desired for which high accuracy on many different types of functions is desired, then:

(a) the use of Laguere polynomials (methods 9 and 10) gives exceptional accuracy on a wide range of functions;

(b) the use of Chebyshev polynomials (method 11) or the Fourier series approach with accelerated convergence through the epsilon algorithm (method 14) give excellent accuracy on a wide range of functions;

(c) the following methods give good accuracy on a fairly wide range of functions: Gaver–Stehfest (method 2), Gaussian quadrature (methods 7 and 8), and the Fourier series approach without accelerated convergence (methods 12 and 13);

(d) the following methods seldom give high accuracy: Widder (method 1) and expanding $f(t)$ in exponential functions (methods 3–6).

Third, if a method is required for a function whose functional behavior is well known and understood, comments (a)–(d) plus the results in Table IV should give a good idea of which methods are most appropriate.

Fourth, the following comments apply to special situations:

(a) if $F(p)$ is in the form of a rational fraction, the method of Padé approximation of $F(p)$ gives highly accurate results;

(b) if $F(p)$ is known only on the real axis, or if determination of $F(p)$ for complex p is very diffiicult, Chebyshev polynomials (method 11) give excellent accuracy on a wide range of functions and Gaver–Stehfest (method 2) gives good accuracy on a fairly wide range of functions, each through use of values of $F(p)$ on the real axis only;

(c) for functions with discontinuities, the use of a Fourier series approach (methods 12–14) gives the greatest promise of accuracy.

Fifth, in particular circumstances a special weighting of different criteria (such as computation time or ease of programming) may dictate the choice of method or

methods. The results in Tables III–VI should be useful in choosing between the methods tested here, or in evaluating other methods which may seem appropriate to the task at hand.

REFERENCES

H. BATEMAN, *Duke Math. J.* **2** (1936), 569–577.

R. BELLMAN, R. E. KALABA, AND J. A. LOCKETT, "Numerical Inversion of the Laplace Transform: Applications to Biology, Economics, Engineering, and Physics," American Elsevier, New York, 1966.

B. S. BERGER, *Math. Comp.* **20** (1966), 418–421.

B. S. BERGER, *J. Appl. Mech.* **35** (1968), 837–839.

B. S. BERGER AND S. DUANGUDOM, *J. Appl. Mech.* **40** (1973), 1110–1112.

J. W. COOLEY, P. A. W LEWIS, AND P. D. WELCH, *J. Sound Vib.* **12** (1970), 315–337.

J. W. COOLEY AND J. W. TUKEY, *Math. Comp.* **19** (1965), 297–301.

T. L. COST, Approximate Laplace transform inversions in viscoelastic stress analysis, *AIAA J.* **2** (1964), 2157–2166.

K. S. CRUMP, *J. ACM* **23** (1976), 89–96.

G. DOETSCH, "Guide to the Applications of the Laplace and Z-Transforms," Van Nostrand Reinhold, London, 1971.

H. DUBNER AND J. ABATE, *J. ACM* **15** (1968), 115–123.

F. DURBIN, *Comput. J.* **17** (1974), 371–376.

A. ERDÉLYI, *Philos. Mag.* **34** (1943), 533–537.

D. P. GAVER, JR., *Operational Res.* **14** (1966), 444–459.

I. M. GELFAND AND G. E. SHILOV, "Generalized Functions," Academic Press, New York, 1964.

H. HURWITZ, JR. AND P. F. ZWEIFEL, *Math. Tables Other Aids Comput.* **10** (1956), 140–149.

C. LANCZOS, "Applied Analysis," Pitman, London, 1957.

I. M. LONGMAN, *Intern. J. Comput. Math. B* **3** (1971), 53–64.

I. M. LONGMAN, Use of Padé Table for Approximate Laplace Transform Inversion, *in* "Padé Approximants and Their Applications" (P. R. Graves-Morris, Ed.), pp. 131–133, Academic Press, London, 1973.

I. M. LONGMAN, *J. Comp. Appl. Math.* **1** (1975), 17–23.

J. R. MACDONALD, *J. Appl. Phys.* **35** (1964), 3034–3041.

M. K. MILLER AND W. T. GUY, JR., *SIAM J. Numer. Anal.* **3** (1966), 624–635.

A. PAPOULIS, *Quart. Appl. Math.* **14** (1956), 405–414.

R. PIESSENS, *IEEE Trans. Automatic Control AC-14* (1969), 299–301.

R. PIESSENS, *Nordisk Tidskr. Informationsbehandling* **9** (1969a), 351–361.

R. PIESSENS, *J. Engirg. Math.* **5** (1971), 1–9.

R. PIESSENS, *J. Inst. Math. Appl.* **10** (1972), 185–192.

R. PIESSENS, *Comm. ACM* **16** (1973), 486–487.

R. PIESSENS, *J. Comput. Appl. Math.* **1** (1975), 115–128.

R. PIESSENS AND M. BRANDERS, *Proc. IEE* **118** (1971), 1517–1522.

R. PIESSENS AND N. D. P. DANG, *J. Comput. Appl. Math.* **2** (1976), 225–228.

F. J. RIZZO AND D. J. SHIPPY, *AIAA J.* **8** (1970), 2004–2009.

H. E. SALZER, *Math. Tables Other Aids Comput.* **9** (1955), 164–177.

H. E. SALZER, *J. Mathematical Phys.* **37** (1958), 89–109.

H. E. SALZER, *J. Mathematical Phys.* **40** (1961), 72–86.

R. A. SCHAPERY, *Proc. 4th U.S. Nat. Congr. Appl. Mech., ASME* **2** (1962), 1075–1085.

L. A. SCHMITTROTH, *Comm. ACM* **3** (1960), 171–173.

C. J. SHIRTLIFFE AND D. G. STEPHENSON, *J. Mathematical Phys.* **40** (1961), 135–141.

M. SILVERBERG, *Electron Lett.* **6** (1970), 105–106.
K. SINGHAL AND J. VLACH, *J. Franklin Inst.* **299** (1975), 109–126.
H. STEHFEST, *Comm. ACM* **13** (1970), 47–49.
H. STEHFEST, *Comm. ACM* **13** (1970), 624.
D. TER HAAR, *J. Polymer Sci.* **6** (1951), 247.
F. TRICOMI, *R. C. Accad. Nat. dei Lincei* **21** (1935), 232–239.
F. VEILLON, *Comm. ACM* **17** (1974), 587–589.
W. T. WEEKS, *J. ACM* **13** (1966), 419–426.
D. V. WIDDER, *Amer. Math. Soc. Trans.* **36** (1934), 107–200.
D. V. WIDDER, *Duke Math. J.* **1** (1935), 126–136.
V. ZAKIAN, *Electron. Lett.* **5** (1969), 120–121.
V. ZAKIAN, *Electron. Lett.* **6** (1970), 677–679.
V. ZAKIAN AND D. R. GANNON, *Electron. Lett.* **7** (1971), 70–71.